A SURVEY ON LOEWNER CHAINS, APPROXIMATION RESULTS, AND RELATED PROBLEMS FOR UNIVALENT MAPPINGS ON THE UNIT BALL IN $\mathbb{C}^N$

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In this paper we survey recent results on Loewner theory and approximation properties of univalent mappings on the Euclidean unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$. We present recent applications of the Andersén-Lempert theorem, concerning the locally uniform approximation of biholomorphic mappings of starlike domains in $\mathbb{C}^n$ onto Runge domains by automorphisms of $\mathbb{C}^n$ ($n \geq 2$), in the study of the following geometric properties of univalent mappings on the Euclidean unit ball $\mathbb{B}^n$: convexity, starlikeness, spirallikeness, parametric representation and embedding into univalent subordination chains. We consider density results and characterizations of these geometric properties through automorphisms of $\mathbb{C}^n$, when $n \geq 2$. We also consider density results for families of normalized starlike, spirallike, convex, and mappings with parametric representation on $\mathbb{B}^n$, by smooth quasiconformal diffeomorphisms of $\mathbb{C}^n$. On the other hand, we consider the following question: under which conditions is a normalized univalent mapping on $\mathbb{B}^n$ ($n \geq 2$), up to a normalized automorphism of $\mathbb{C}^n$, spirallike (or starlike, respectively convex)? This question was first studied by Arosio, Bracci and Wold for convex mappings, by using a smooth boundary assumption, while the authors of the present paper extended their work under lower boundary assumptions, by considering also the case of spirallike mappings.

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1. INTRODUCTION

The first extension of the theory of Loewner chains on the Euclidean unit ball in $\mathbb{C}^n$ is due to Pfaltzgraff [30]. Next, Poreda [33] studied the parametric representation on the unit polydisc in $\mathbb{C}^n$. Moreover, Poreda [34] obtained certain results regarding the subordination chains and the Loewner differential equation on the unit ball of a complex Banach space.

The results in [14], related to the compactness of the Carathéodory family and the compact family of mappings that have parametric representation, with respect to an arbitrary norm on $\mathbb{C}^n$, motivated further investigations of the

Loewner chains in several complex variables. The authors in [10] and [15] studied the Loewner differential equations with respect to a linear operator A, observing that the spectrum of A plays an essential role in the development of the theory. Further results in this direction were obtained by Arosio [2] and Voda [40].

A general extension of the theory of Loewner chains given in [3, 6] led to the interesting connection between the subordination chains and the Runge property, pointed out by Arosio, Bracci and Wold in [4]. Meanwhile, the interest in the study of extremal problems associated with Loewner chains in $\mathbb{C}^n$ grew, see [16]. Schleißinger [38] used the Runge property to prove certain Kirwan–Pell type results conjectured in [16]. Further results in this direction, using control theory, were obtained in [7, 17, 18, 37].

On the other hand, Arosio, Bracci and Wold used the Runge property in [5] to apply the Andersén-Lempert theorem [1] in the study of the mappings that embed into certain Loewner chains. The Andersén-Lempert theorem says that every biholomorphic mapping from a starlike domain in $\mathbb{C}^n$ onto a Runge domain in $\mathbb{C}^n$ can be approximated locally uniformly by automorphisms of $\mathbb{C}^n$, for $n \geq 2$. Taking into account the above ideas, one may ask: what happens when the biholomorphic mapping approximated by automorphisms of $\mathbb{C}^n$ has certain geometric properties, like starlikeness, spirallikeness, convexity, parametric representation, etc.? In this survey we present some answers to this question, mainly given in [5, 22–24].

2. PRELIMINARIES

Let $\mathbb{C}^n$ be the space of n complex variables $z = (z_1, \dots, z_n)$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^n z_j\overline{w}_j$ and the Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$. Also, let $\mathbb{B}^n$ be the Euclidean unit ball in $\mathbb{C}^n$ and let $\mathbb{B}^1 = \mathbb{U}$ be the unit disc. $\mathbb{U}^n$ denotes the unit polydisc in $\mathbb{C}^n$. Let $L(\mathbb{C}^n)$ be the space of linear operators from $\mathbb{C}^n$ into $\mathbb{C}^n$ with the standard operator norm, and let I_n be the identity in $L(\mathbb{C}^n)$.

We denote by $H(\mathbb{B}^n)$ the family of holomorphic mappings from $\mathbb{B}^n$ into $\mathbb{C}^n$ with the standard topology of locally uniform convergence. If $f \in H(\mathbb{B}^n)$, we say that f is normalized if $f(0) = 0$ and $Df(0) = I_n$. Let $S(\mathbb{B}^n)$ be the subset of $H(\mathbb{B}^n)$ consisting of all normalized univalent (biholomorphic) mappings on $\mathbb{B}^n$.

We use the following notations related to an operator $A \in L(\mathbb{C}^n)$ (cf. [36]):

$$\begin{aligned} m(A) &= \min\{\Re\langle A(z), z\rangle : \|z\| = 1\}, \\ k_+(A) &= \max\{\Re\lambda : \lambda \in \sigma(A)\}, \end{aligned}$$

where $\sigma(A)$ is the spectrum of A, and $k_+(A)$ is the upper exponential index (Lyapunov index) of A. In all of the results presented in this survey that involve an operator $A \in L(\mathbb{C}^n)$, we have the assumption that $m(A) > 0$. Note that the condition $k_+(A) < 2m(A)$ implies $m(A) > 0$.

Remark 2.1. If $A \in L(\mathbb{C}^n)$ with $m(A) > 0$, then ([10, Lemma 2.1]; see also [15]):

$$\|e^{tA}u\| \geq e^{m(A)t} \text{ and } \|e^{-tA}u\| \leq e^{-m(A)t}, \quad t \geq 0, \quad \|u\| = 1.$$

Definition 2.2. Let $A \in L(\mathbb{C}^n)$ be such that $m(A) \geq 0$. The following subfamily of $H(\mathbb{B}^n)$ is a generalization to $\mathbb{C}^n$ of the Carathéodory family on $\mathbb{U}$ (see e.g. [39]):

$$\mathcal{N}_A = \big\{h \in H(\mathbb{B}^n) : h(0) = 0,\, Dh(0) = A,\, \Re\langle h(z), z\rangle \geq 0,\, z \in \mathbb{B}^n\big\}.$$

This family plays basic roles in geometric function theory in higher dimensions (see e.g. [19, 39]). The compactness of the family $\mathcal{N}_A$, which was proved in [15] and essentially in [14] (cf. [24]), plays an essential role in the Loewner theory in $\mathbb{C}^n$ (cf. [30, 34]), especially in the the study of extremal problems (see [7, 16–18, 37, 38]).

Remark 2.3. Let $A \in L(\mathbb{C}^n)$ be such that $m(A) \geq 0$. Then $\mathcal{N}_A$ is a compact subset of $H(\mathbb{B}^n)$.

Next, we recall the definition of spirallikeness with respect to a given operator $A \in L(\mathbb{C}^n)$ with $m(A) > 0$ (see [39]).

Definition 2.4. Let $A \in L(\mathbb{C}^n)$ be such that $m(A) > 0$. A mapping $f \in S(\mathbb{B}^n)$ is said to be spirallike with respect to A (denoted by $f \in \widehat{S}_A(\mathbb{B}^n)$) if $f(\mathbb{B}^n)$ is a spirallike domain with respect to A, i.e. $e^{-tA}f(\mathbb{B}^n) \subseteq f(\mathbb{B}^n)$, for all $t \geq 0$.

The following result due to Suffridge [39] provides a necessary and sufficient condition of spirallikeness for locally biholomorphic mappings on $\mathbb{B}^n$.

PROPOSITION 2.5. *Let $A \in L(\mathbb{C}^n)$ be such that $m(A) > 0$, and let $f \in H(\mathbb{B}^n)$ be a normalized and locally biholomorphic mapping. Then $f \in \widehat{S}_A(\mathbb{B}^n)$ iff there exists $h \in \mathcal{N}_A$ such that $Df(z)h(z) = Af(z)$, for all $z \in \mathbb{B}^n$.*

We denote by $S^*(\mathbb{B}^n)$ the family of normalized starlike (i.e. I_n-spirallike) mappings on $\mathbb{B}^n$. Moreover, $K(\mathbb{B}^n)$ denotes the family of mappings in $S(\mathbb{B}^n)$ with convex image.

3. CARATHÉODORY MAPPINGS AND SUBORDINATION CHAINS

In this section we present two important tools that are used in the proofs of the results presented in the following sections: the Carathéodory mapping and the subordination chain. Also, we point out an essential relation between them.

Definition 3.1. Let $J \subseteq [0, \infty)$ be an interval. A mapping $h : \mathbb{B}^n \times J \to \mathbb{C}^n$ is called a Carathéodory mapping on J with values in $\mathcal{N}_A$ if the following conditions hold:

(i) $h(\cdot, t) \in \mathcal{N}_A$, for all $t \in J$;

(ii) $h(z, \cdot)$ is measurable on J, for all $z \in \mathbb{B}^n$.

Let $\mathcal{C}(J, \mathcal{N}_A)$ be the family of Carathéodory mappings on J with values in $\mathcal{N}_A$.

A mapping $h \in \mathcal{C}([0, \infty), \mathcal{N}_A)$ is also called a Herglotz vector field (cf. [6], [10]).

Next, we recall the notion of normalized subordination chain (see [15]; cf. [30]).

Definition 3.2. A mapping $f : \mathbb{B}^n \times [0, \infty) \to \mathbb{C}^n$ is called a subordination chain if $f(\cdot, t) \in H(\mathbb{B}^n)$, $f(0, t) = 0$, for $t \geq 0$, and for all $t \geq s \geq 0$ there is a holomorphic Schwarz mapping $v_{s,t} : \mathbb{B}^n \to \mathbb{B}^n$, called the transition mapping associated with f, such that $f(z, s) = f(v_{s,t}(z), t)$ for $z \in \mathbb{B}^n$.

A subordination chain f is said to be univalent if $f(\cdot, t)$ is a univalent mapping on $\mathbb{B}^n$, for all $t \geq 0$.

A subordination chain f is said to be A-normalized if $Df(0, t) = e^{tA}$ for $t \geq 0$, where $A \in L(\mathbb{C}^n)$ with $m(A) > 0$. Moreover, f is said to be normal if $\{e^{-tA} f(\cdot, t)\}$ is a normal family in $H(\mathbb{B}^n)$. An I_n-normalized univalent subordination chain is said to be a Loewner chain (or a normalized univalent subordination chain).

If $f : \mathbb{B}^n \times [0, \infty) \to \mathbb{C}^n$ is a univalent subordination chain, $\bigcup_{t \geq 0} f(\mathbb{B}^n, t)$ denoted by $R(f)$ is called the Loewner range of f. We note that every normal A-normalized univalent subordination chain has range $\mathbb{C}^n$, for $A \in L(\mathbb{C}^n)$ with $m(A) > 0$ (cf. the proof of [10, Theorem 3.1]).

In this survey, the following family of univalent mappings on $\mathbb{B}^n$ that can be embedded as the first elements into univalent subordination chains with range $\mathbb{C}^n$ has a central role.

Definition 3.3. Let $A \in L(\mathbb{C}^n)$ be such that $m(A) > 0$. We denote by $S^1_A(\mathbb{B}^n)$ the family of all mappings $f \in S(\mathbb{B}^n)$ for which there is an A-normalized univalent subordination chain L with range $R(L) = \mathbb{C}^n$ such that $f = L(\cdot, 0)$. If $A = I_n$, the family $S^1_{I_n}(\mathbb{B}^n)$ is denoted by $S^1(\mathbb{B}^n)$ (see e.g. [5]).

Definition 3.4. (see e.g. [6,10]) Let $A \in L(\mathbb{C}^n)$ be such that $m(A) > 0$ and let $h \in \mathcal{C}([0,\infty), \mathcal{N}_A)$. Let $f : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ be such that $f(\cdot, t) \in H(\mathbb{B}^n)$, $f(0,t) = 0$, for $t \geq 0$, and $f(z,\cdot)$ is locally absolutely continuous on $[0,\infty)$ locally uniformly with respect to $z \in \mathbb{B}^n$. If f satisfies the Loewner differential equation

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad z \in \mathbb{B}^n, \text{ a.e. } t \geq 0, \tag{3.1}$$

then f is called a standard solution of (3.1) associated to h.

In the next proposition, we point out the connection between a standard solution (and thus implicitly a Carathéodory mapping) and a subordination chain (see [2, 3, 10, 19, 24, 40]).

Proposition 3.5. *Let $A \in L(\mathbb{C}^n)$ be such that $m(A) > 0$. Then the following statements hold:*

(i) *If f is an A-normalized univalent subordination chain, then there exists $h \in \mathcal{C}([0,\infty), \mathcal{N}_A)$ such that f is a standard solution of* (3.1) *associated to h.*

(ii) *Conversely, let $h \in \mathcal{C}([0,\infty), \mathcal{N}_A)$. Then there exists an A-normalized univalent subordination chain f that is a standard solution of* (3.1) *associated to h. Moreover, if g is another standard solution of* (3.1) *associated to h, then g is a subordination chain and there exists a holomorphic mapping $\Phi : R(f) \to \mathbb{C}^n$ such that $g = \Phi \circ f$.*

Remark 3.6. Let $A \in L(\mathbb{C}^n)$ be such that $k_+(A) < 2m(A)$ and $h \in \mathcal{C}([0,\infty), \mathcal{N}_A)$. Then there exists a unique normal A-normalized univalent subordination chain f that is a standard solution of (3.1) associated to h, and it is called the canonical solution of (3.1) (see [10] and [20]).

4. REACHABLE FAMILIES AND A-PARAMETRIC REPRESENTATION

We consider the definition of the reachable families of the Loewner equation. These families have been studied from a control-theoretic point of view in e.g. [16], [17], [18] and [37].

Definition 4.1. Let $J = [0, T]$ with $T > 0$ or $J = [0,\infty)$ and let $A \in L(\mathbb{C}^n)$ be such that $m(A) > 0$. For every $h \in \mathcal{C}(J, \mathcal{N}_A)$, let $v = v(z, t; h)$ be the unique locally absolutely continuous solution on J of the initial value problem

$$\frac{\partial v}{\partial t} = -h(v,t), \text{ a.e. } t \in J, \quad v(z, 0; h) = z, \tag{4.1}$$

for all $z \in \mathbb{B}^n$. Also, for $T \in (0,\infty)$, let

$$\widetilde{\mathcal{R}}_T(\mathrm{id}_{\mathbb{B}^n}, \mathcal{N}_A) = \big\{e^{TA}v(\cdot,T;h) : \ h \in \mathcal{C}([0,T],\mathcal{N}_A)\big\},$$

be the *time-T-reachable family* of (4.1).

Next, we present the family of mappings that have parametric representation (see [14] and [15]), which can be viewed as the time-infinity-reachable family (see e.g. [17]).

Definition 4.2. Let $A \in L(\mathbb{C}^n)$ be such that $m(A) > 0$. Also, let $f \in H(\mathbb{B}^n)$ be a normalized mapping. We say that f has A-parametric representation if there exists a mapping $h \in \mathcal{C}([0,\infty),\mathcal{N}_A)$ such that

$$f = \lim_{t\to\infty} e^{tA}v(\cdot,t)$$

locally uniformly on $\mathbb{B}^n$, where $v(z,\cdot)$ is the unique locally absolutely continuous solution on $[0,\infty)$ of the initial value problem

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -h(v,t), \text{ a.e. } t \geq 0, \quad v(z,0) = z,$$

for all $z \in \mathbb{B}^n$. Let $S^0_A(\mathbb{B}^n)$ be the family of mappings which have A-parametric representation on $\mathbb{B}^n$ (see [15]). In the case $A = I_n$, let $S^0(\mathbb{B}^n) = S^0_{I_n}(\mathbb{B}^n)$ be the family of mappings with the usual parametric representation on $\mathbb{B}^n$ (see [14]).

The following proposition given in [24] (see [17], in the case $k_+(A) < 2m(A)$) establishes a connection between reachable families and univalent subordination chains and also the family of mappings that have parametric representation.

Proposition 4.3. *Let $T > 0$ and $A \in L(\mathbb{C}^n)$ be such that $m(A) > 0$. Then the following statements hold:*

(i) *$\varphi \in \widetilde{\mathcal{R}}_T(\mathrm{id}_{\mathbb{B}^n}, \mathcal{N}_A)$ if and only if there is an A-normalized univalent subordination chain L such that $L(\cdot,0) = \varphi$, and $L(\cdot,t) = e^{tA}\mathrm{id}_{\mathbb{B}^n}$, for $t \geq T$ (see [24, Proposition 3.3]).*

(ii) *$\widetilde{\mathcal{R}}_T(\mathrm{id}_{\mathbb{B}^n}, \mathcal{N}_A)$ is a subfamily of $S^0_A(\mathbb{B}^n)$ and it is compact in $H(\mathbb{B}^n)$ (see [24, Remark 3.2, Proposition 3.4]).*

Remark 4.4. Let $A \in L(\mathbb{C}^n)$ be such that $k_+(A) < 2m(A)$ and let $\varphi \in S(\mathbb{B}^n)$. Then $\varphi \in S^0_A(\mathbb{B}^n)$ if and only if there exists a normal A-normalized univalent subordination chain L such that $L(\cdot,0) = \varphi$ (see [10], [14] and [15]). This implies, in view of Proposition 3.5 and Remark 3.6, that $f \in S^1_A(\mathbb{B}^n)$ if and only if there exists a normalized automorphism Φ of $\mathbb{C}^n$ such that $\Phi \circ f \in S^0_A(\mathbb{B}^n)$.

5. DENSITY RESULTS WITH AUTOMORPHISMS OF $\mathbb{C}^n$

We consider the following families:

$$\mathcal{A}(\mathbb{C}^n) = \big\{\Phi : \Phi \text{ is a normalized automorphism of } \mathbb{C}^n\big\},$$
$$\mathcal{A}(\mathbb{B}^n) = \big\{\Phi\big|_{\mathbb{B}^n} : \Phi \in \mathcal{A}(\mathbb{C}^n)\big\},$$
$$S_R(\mathbb{B}^n) = \big\{f \in S(\mathbb{B}^n) : f(\mathbb{B}^n) \text{ is Runge}\big\},$$

where $A \in L(\mathbb{C}^n)$ is such that $m(A) > 0$. For $n = 1$ we note that $S^0(\mathbb{U}) = S^1(\mathbb{U}) = S_R(\mathbb{U}) = S(\mathbb{U})$ and $\mathcal{A}(\mathbb{U}) = \{\mathrm{id}_{\mathbb{U}}\}$. In higher dimensions, the following relations between the families $S_A^1(\mathbb{B}^n)$, $\mathcal{A}(\mathbb{B}^n)$ and $S_R(\mathbb{B}^n)$ hold (see [5] and [24, Proposition 4.6]).

Proposition 5.1. *Let $n \geq 2$ and $A \in L(\mathbb{C}^n)$ be such that $m(A) > 0$. Then*

$$\mathcal{A}(\mathbb{B}^n) \subset S_A^1(\mathbb{B}^n) \subset S_R(\mathbb{B}^n) = \overline{\mathcal{A}(\mathbb{B}^n)}.$$

For the first inclusion, one can construct an A-normalized subordination chain with range $\mathbb{C}^n$ using a normalized automorphism (see [5], for $A = I_n$). The second inclusion is due to [21, Theorem 5.1] and [9, Satz 17-19] (see also [4]). Finally, the above equality is a consequence of the Andersén-Lempert theorem [1], which states that if a biholomorphic mapping on a starlike domain has a Runge image then it can be approximated locally uniformly by automorphisms of $\mathbb{C}^n$ (this result was extended for certain spirallike domains by Hamada [21], by applying [13, Theorem 1.1]).

Using Proposition 5.1, we make the next remark.

Remark 5.2. Let $A \in L(\mathbb{C}^n)$ be such that $m(A) > 0$.

(i) $\widehat{S}_A(\mathbb{B}^n) \subsetneq S_A^1(\mathbb{B}^n)$, because every $f \in \widehat{S}_A(\mathbb{B}^n)$ embeds as a first element into an A-normalized subordination chain L given by $L(z,t) = e^{tA}f(z)$, $z \in \mathbb{B}^n, t \geq 0$, which has the range $\mathbb{C}^n$ (see [10, Lemma 2.1]; see also Remark 2.1). In particular, $\widehat{S}_A(\mathbb{B}^n) \subseteq S_R(\mathbb{B}^n)$ (cf. [21, Theorem 3.1]), $S^*(\mathbb{B}^n) \subseteq S_R(\mathbb{B}^n)$ (cf. [11, Proposition 1]) and also $K(\mathbb{B}^n) \subseteq S_R(\mathbb{B}^n)$.

(ii) By Proposition 4.3 (i) and a similar argument, we deduce that

$$\widetilde{\mathcal{R}}_T(\mathrm{id}_{\mathbb{B}^n}, \mathcal{N}_A) \subseteq S_A^1(\mathbb{B}^n) \subseteq S_R(\mathbb{B}^n), \quad \forall T \in [0, \infty).$$

By Definition 4.2 and Proposition 4.3 (ii), $\overline{S_A^0(\mathbb{B}^n)} = \overline{\bigcup_{T>0} \widetilde{\mathcal{R}}_T(\mathrm{id}_{\mathbb{B}^n}, \mathcal{N}_A)}$, and thus $S_A^0(\mathbb{B}^n) \subseteq S_R(\mathbb{B}^n)$ (cf. [38, Theorem 2.3]).

Taking into account the density result given in Proposition 5.1 and the above remark, the authors in [22] and [24] considered the following question:

Question 5.3. Let $\mathcal{F} \subseteq S(\mathbb{B}^n)$ be one of the subfamilies of $S_R(\mathbb{B}^n)$ given in Remark 5.2. Then does the equality $\mathcal{F} = \overline{\mathcal{F} \cap \mathcal{A}(\mathbb{B}^n)}$ hold when $n \geq 2$?

Clearly, when $n = 1$ this equality does not hold, because $\mathcal{A}(\mathbb{B}^1) = \{\mathrm{id}_{\mathbb{B}^1}\}$. Moreover, we shall point out that the equality may fail if we replace the Euclidean unit ball $\mathbb{B}^n$ with the unit polydisc $\mathbb{U}^n$, even when $n \geq 2$.

The following results were obtained in [24]. Theorems 5.4 and 5.5 below provide positive answers to the above question.

THEOREM 5.4. *Let $n \geq 2$ and $A \in L(\mathbb{C}^n)$ be such that $m(A) > 0$. Then*

$$\widehat{S}_A(\mathbb{B}^n) = \overline{\widehat{S}_A(\mathbb{B}^n) \cap \mathcal{A}(\mathbb{B}^n)}.$$

In particular,

$$S^*(\mathbb{B}^n) = \overline{S^*(\mathbb{B}^n) \cap \mathcal{A}(\mathbb{B}^n)}.$$

THEOREM 5.5. *If $n \geq 2$, then $K(\mathbb{B}^n) = \overline{K(\mathbb{B}^n) \cap \mathcal{A}(\mathbb{B}^n)}$.*

The proofs of the above theorems are based on the density of the automorphisms previously mentioned and the analytic characterizations of the spirallike mappings, respectively the convex mappings (see e.g. [19, Chapter 6] and [39]). In contrast with Theorem 5.5, the following proposition from [24] holds in view of the characterization of the convex mappings on the polydisc (see e.g. [19, Theorem 6.3.2]).

PROPOSITION 5.6. $\overline{K(\mathbb{U}^n) \cap \mathcal{A}(\mathbb{U}^n)} = \{\mathrm{id}_{\mathbb{U}^n}\} \subsetneq K(\mathbb{U}^n)$, *for all $n \geq 1$.*

Taking into account Theorems 5.4 and 5.5, the analogous density result for the family $S_A^0(\mathbb{B}^n)$ has been considered in [22] and [24], for $n \geq 2$ and $A \in L(\mathbb{C}^n)$ with $m(A) > 0$ (cf. [26] and [38], for $A = I_n$). However, in this general case, we do not know if $S_A^0(\mathbb{B}^n)$ is a compact subfamily of $H(\mathbb{B}^n)$ (see [40] for a related discussion), therefore we present the next density result for the closure of this family.

THEOREM 5.7. *Let $n \geq 2$ and $A \in L(\mathbb{C}^n)$ be such that $m(A) > 0$. Then*

$$\overline{S_A^0(\mathbb{B}^n)} = \overline{S_A^0(\mathbb{B}^n) \cap \mathcal{A}(\mathbb{B}^n)}.$$

Remark 5.8. If $k_+(A) < 2m(A)$, then $S_A^0(\mathbb{B}^n)$ is compact (see [15, Theorem 2.15] and [10, Remark 2.8]; see also [14] for $A = I_n$), and thus

$$S_A^0(\mathbb{B}^n) = \overline{S_A^0(\mathbb{B}^n) \cap \mathcal{A}(\mathbb{B}^n)}, \quad \forall\, n \geq 2.$$

The above result was first obtained in [24, Theorem 4.11] in the case A is a nonresonant operator (see e.g. [2], [40]), by using the one-to-one correspondence between the mappings in $\widehat{S}_A(\mathbb{B}^n)$ and the mappings in $\mathcal{N}_A$ (see [24, Remark 2.15]):

$$\mathcal{N}_A = \Big\{h \in H(\mathbb{B}^n) : h(z) = \big(Df(z)\big)^{-1} Af(z), z \in \mathbb{B}^n, \text{ for some } f \in \widehat{S}_A(\mathbb{B}^n)\Big\}.$$

The above equality is based on a result due to Voda [40, Remark 3.2] (cf. [39] for $A = I_n$).

In [22], the authors managed to prove the above theorem in the general case, when $A \in L(\mathbb{C}^n)$ satisfies $m(A) > 0$, by adapting a variational result due to Bracci, Graham, Hamada, Kohr [7] for A-normalized univalent subordination chains (see [18, Theorem 4.1] for A-normalized univalent subordination chains with $k_+(A) < 2m(A)$; see also [22, Theorem 3.1]), which essentially says the following: if f embeds as a first element of an A-normalized univalent subordination chain L that satisfies certain regularity assumptions, then, for every fixed $T > 0$, $L(\cdot, 0) + g$ embeds as a first element of an A-normalized univalent subordination chain $\tilde{L}$ such that $\tilde{L}(\cdot, t) = L(\cdot, t)$, $t \geq T$, for all $g \in H(\mathbb{B}^n)$ with $g(0) = 0$, $Dg(0) = 0$, $\sup_{z\in\mathbb{B}^n} \|g(z)\| \leq \varepsilon$, $\sup_{z\in\mathbb{B}^n} \|Dg(z)\| \leq \varepsilon$ and sufficiently small $\varepsilon > 0$.

Using the above approach for A-normalized univalent subordination chains, we were able to prove Theorem 5.7, and also we obtained the following density result for reachable families [22, Theorem 3.3], which was initially proved in [24, Theorem 4.9] when A is a nonresonant linear operator.

Theorem 5.9. *Let $n \geq 2$, $T > 0$ and $A \in L(\mathbb{C}^n)$ be such that $m(A) > 0$. Then*
$$\widetilde{\mathcal{R}}_T(\mathrm{id}_{\mathbb{B}^n}, \mathcal{N}_A) = \overline{\widetilde{\mathcal{R}}_T(\mathrm{id}_{\mathbb{B}^n}, \mathcal{N}_A) \cap \mathcal{A}(\mathbb{B}^n)}.$$

6. DENSITY RESULTS WITH SMOOTH QUASICONFORMAL DIFFEOMORPHISMS OF $\mathbb{C}^n$

Next, we consider the families:
$$\mathcal{Q}(\mathbb{C}^n) = \big\{\Phi : \Phi \text{ is a quasiconformal homeomorphism from } \mathbb{C}^n \text{ onto } \mathbb{C}^n\big\},$$
$$\mathcal{Q}(\mathbb{B}^n) = \big\{\Phi\big|_{\mathbb{B}^n} : \Phi \in \mathcal{Q}(\mathbb{C}^n)\big\},$$
$$\mathcal{Q}^\infty(\mathbb{C}^n) = \big\{\Phi \in \mathcal{Q}(\mathbb{C}^n) : \Phi \text{ is a smooth diffeomorphism}\big\},$$
$$\mathcal{Q}^\infty(\mathbb{B}^n) = \big\{\Phi\big|_{\mathbb{B}^n} : \Phi \in \mathcal{Q}^\infty(\mathbb{C}^n)\big\},$$
where $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ is said to be smooth if it is of class C^∞ on $\mathbb{C}^n$. A homeomorphism $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ is said to be quasiconformal (see e.g. [25]; see also e.g. [19] and the references therein) if it is differentiable a.e., ACL (absolutely continuous on lines) and there exists $K > 0$ such that
$$\|D(\Phi; z)\|^{2n} \leq K|\det D(\Phi; z)|, \quad \text{for a.e. } z \in \mathbb{C}^n,$$
with $D(\Phi; z)$ denoting the real Jacobian matrix of Φ.

The authors in [22] studied the following question: do Theorems 5.4, 5.5, 5.7 and 5.9 hold if we replace $\mathcal{A}(\mathbb{B}^n)$ with $\mathcal{Q}^\infty(\mathbb{B}^n)$? A clue regarding the answer is given by the following subfamily of $S^0(\mathbb{B}^n)$, which was studied in e.g. [25] (see also [19] and the references therein):
$$\mathcal{R}(\mathbb{B}^n) = \Big\{f \in H(\mathbb{B}^n) : f(0) = 0, Df(0) = I_n, \|Df(z) - I_n\| \leq 1, \, z \in \mathbb{B}^n\Big\}.$$

We note that $\mathcal{R}(\mathbb{B}^n) \not\subseteq \mathcal{Q}(\mathbb{B}^n)$ (see [22, Remark 1.3 (i)]). However, for $f \in \mathcal{R}(\mathbb{B}^n)$ and $r \in (0,1)$, the mapping f_r given by $f_r(z) = \frac{1}{r}f(rz)$, $z \in \mathbb{B}^n$, belongs to $\mathcal{R}(\mathbb{B}^n) \cap \mathcal{Q}(\mathbb{B}^n)$, by [25, Corollary 4.2]. Hence, $\mathcal{R}(\mathbb{B}^n) = \overline{\mathcal{R}(\mathbb{B}^n) \cap \mathcal{Q}(\mathbb{B}^n)}$. In fact, taking a look at the proof of [22, Proposition 1.2], we have

$$\mathcal{R}(\mathbb{B}^n) = \overline{\mathcal{R}(\mathbb{B}^n) \cap \mathcal{A}(\mathbb{B}^n) \cap \mathcal{Q}(\mathbb{B}^n)}, \ n \geq 2.$$

In order to find the answer to our question, we shall take a closer look at the family $\mathcal{A}(\mathbb{B}^n) \cap \mathcal{Q}^\infty(\mathbb{B}^n)$. One may suspect that $\mathcal{A}(\mathbb{C}^n) \cap \mathcal{Q}^\infty(\mathbb{C}^n)$ is sufficiently large to ensure the density of $\mathcal{A}(\mathbb{B}^n) \cap \mathcal{Q}^\infty(\mathbb{B}^n)$ in $S_R(\mathbb{B}^n)$, when $n \geq 2$. Unfortunately, this is not the case, in view of the result of Marden and Rickman [28]:

Remark 6.1. If $n \geq 2$, then $\mathcal{A}(\mathbb{C}^n) \cap \mathcal{Q}^\infty(\mathbb{C}^n) = \{\mathrm{id}_{\mathbb{C}^n}\}$.

However, it turns out that $\mathcal{A}(\mathbb{B}^n) \subset \mathcal{Q}^\infty(\mathbb{B}^n)$. To point this out, we consider the family

$$\mathcal{I}^\infty(\mathbb{B}^n) = \big\{\Phi\big|_{\mathbb{B}^n} : \Phi \text{ is a smooth diffeomorphism from } \mathbb{C}^n \text{ onto } \mathbb{C}^n \text{ such that } \Phi\big|_{\mathbb{C}^n\setminus K} = \mathrm{id}_{\mathbb{C}^n\setminus K}, K \text{ is a compact subset of } \mathbb{C}^n\big\}.$$

Clearly, $\mathcal{I}^\infty(\mathbb{B}^n) \subset \mathcal{Q}^\infty(\mathbb{B}^n)$. A result given in [29] by Palais implies that (cf. [22, Proposition 2.25]) $\mathcal{A}(\mathbb{B}^n) \subset \mathcal{I}^\infty(\mathbb{B}^n)$.

We conclude that the following theorems hold (see [22]), even when $n = 1$.

THEOREM 6.2. *Let $A \in L(\mathbb{C}^n)$ be such that $m(A) > 0$. Then*

$$\widehat{S}_A(\mathbb{B}^n) = \overline{\widehat{S}_A(\mathbb{B}^n) \cap \mathcal{I}^\infty(\mathbb{B}^n)} = \overline{\widehat{S}_A(\mathbb{B}^n) \cap \mathcal{Q}^\infty(\mathbb{B}^n)}.$$

In particular,

$$S^*(\mathbb{B}^n) = \overline{S^*(\mathbb{B}^n) \cap \mathcal{I}^\infty(\mathbb{B}^n)} = \overline{S^*(\mathbb{B}^n) \cap \mathcal{Q}^\infty(\mathbb{B}^n)}.$$

Also,

$$K(\mathbb{B}^n) = \overline{K(\mathbb{B}^n) \cap \mathcal{I}^\infty(\mathbb{B}^n)} = \overline{K(\mathbb{B}^n) \cap \mathcal{Q}^\infty(\mathbb{B}^n)}.$$

THEOREM 6.3. *Let $A \in L(\mathbb{C}^n)$ be such that $m(A) > 0$. Then*

$$\overline{S_A^0(\mathbb{B}^n)} = \overline{S_A^0(\mathbb{B}^n) \cap \mathcal{I}^\infty(\mathbb{B}^n)} = \overline{S_A^0(\mathbb{B}^n) \cap \mathcal{Q}^\infty(\mathbb{B}^n)}$$

and

$$\widetilde{\mathcal{R}}_T(\mathrm{id}_{\mathbb{B}^n}, \mathcal{N}_A) = \overline{\widetilde{\mathcal{R}}_T(\mathrm{id}_{\mathbb{B}^n}, \mathcal{N}_A) \cap \mathcal{I}^\infty(\mathbb{B}^n)} = \overline{\widetilde{\mathcal{R}}_T(\mathrm{id}_{\mathbb{B}^n}, \mathcal{N}_A) \cap \mathcal{Q}^\infty(\mathbb{B}^n)}, \ \forall T > 0.$$

7. SPIRALSHAPELIKE, STARSHAPELIKE AND CONVEXSHAPELIKE MAPPINGS

If $A \in L(\mathbb{C}^n)$ satisfies $k_+(A) < 2m(A)$, then, in view of Remark 4.4 (see [10] and [20], for $A = I_n$), for every $f \in S_A^1(\mathbb{B}^n)$ there exists $\Phi \in \mathcal{A}(\mathbb{C}^n)$

such that $\Phi \circ f \in S_A^0(\mathbb{B}^n)$. Taking into account this characterization, we consider the following question: under which conditions for a mapping $f \in S(\mathbb{B}^n)$ ($n \geq 2$) does there exist $\Phi \in \mathcal{A}(\mathbb{C}^n)$ such that $\Phi \circ f \in \widehat{S}_A(\mathbb{B}^n)$ (or $\Phi \circ f \in S^*(\mathbb{B}^n)$, respectively $\Phi \circ f \in K(\mathbb{B}^n)$)? In other words, under which conditions is a normalized univalent mapping on $\mathbb{B}^n$ ($n \geq 2$), up to a normalized automorphism of $\mathbb{C}^n$, spirallike (or starlike, respectively convex)? This question was first studied by Arosio, Bracci and Wold in [5], for convex mappings. They found a necessary and sufficient condition, under a smooth boundary assumption, for this characterization. The authors extended their work in [23], under lower boundary assumptions, considering also the spirallike mappings.

If $m > 0$ is not an integer, then we say that a mapping f is of class C^m if f has continuous partial derivatives up to order $[m]$ and the partial derivatives of order $[m]$ are Hölder continuous with exponent $m-[m]$ ($[m]$ denotes the integral part of m). We say that a domain $D \subseteq \mathbb{C}^n$ has C^m boundary if ∂D admits a defining function of class C^m (cf. [12,31]). Also, we say that a domain $D \subset \mathbb{C}^n$ is strictly pseudoconvex if there is a C^2 strictly plurisubharmonic function r on a neighborhood U of ∂D such that $D \cap U = \{z \in U : r(z) < 0\}$. For the definition and basic properties of polynomially convex sets, see e.g. [35, Chapter VI]. We note that if K is a compact set in $\mathbb{C}^n$ that has a Stein and Runge neighborhood basis, then K is polynomially convex (see e.g. [35, Chapter VI, Theorem 1.8]). If $n = 1$, a compact $K \subset \mathbb{C}$ is polynomially convex if and only if $\mathbb{C} \setminus K$ is connected.

Definition 7.1 (see [23, Definition 2.2]). Let $A \in L(\mathbb{C}^n)$ be such that $m(A) > 0$ and let $D \subseteq \mathbb{C}^n$ be a domain such that $0 \in D$.

(i) D is said to be strictly A-spirallike if $e^{-tA}\overline{D} \subset D$, $t > 0$ (see [11], for $A = I_n$).

(ii) D is said to be (strictly) A-spiralshapelike if there exists $\Phi \in \mathcal{A}(\mathbb{C}^n)$ such that $\Phi(D)$ is (strictly) A-spirallike (see [4, Definition 3.2], for $A = I_n$).

(iii) D is said to be convexshapelike if there exists $\Phi \in \mathcal{A}(\mathbb{C}^n)$ such that $\Phi(D)$ is convex (see [5]).

(iv) A mapping $f : \mathbb{B}^n \to \mathbb{C}^n$ is said to be strictly A-spirallike, (strictly) A-spiralshapelike, respectively convexshapelike, if $f \in S(\mathbb{B}^n)$ and $f(\mathbb{B}^n)$ has the corresponding property.

If $A = I_n$ in the above definitions, then we replace "spiral" with "star" and we omit the operator. We note that every convex domain is strictly starlike ([23, Remark 2.5(ii)]).

Remark 7.2. Let $A \in L(\mathbb{C}^n)$ be such that $m(A) > 0$.

i) Let $f : \mathbb{B}^n \to \mathbb{C}^n$ be a locally biholomorphic and normalized mapping. In view of the proof of [25, Theorem 3.2], we have the following sufficient condition for strict spirallikeness: if there exists $c > 0$ such that

$$\Re\langle (Df(z))^{-1} Af(z), z\rangle \geq c\|z\|^2,\ z \in \mathbb{B}^n,$$

then f is strictly A-spirallike. In general, this condition is not necessary for strict spirallikeness (see [23, Example 2.6]).

ii) Let $D \subset \mathbb{C}^n$ be a bounded domain with $0 \in D$. By [23, Remark 2.8 and Proposition 3.4], if D is strictly A-spiralshapelike and pseudoconvex or D is A-spiralshapelike and strictly pseudoconvex with C^2 boundary, then $\overline{D}$ is polynomially convex.

By [5] and [23] the following theorem holds. One of the key ingredients of the proof is the Andersén-Lempert theorem [1]. The proof is given, under the assumption of smooth boundary (in order to use the Fefferman mapping theorem), by Arosio, Bracci and Wold [5] for convexshapelike mappings, but it can be adapted for spiralshapelike mappings, under a lower regularity assumption of the boundary (see [23, Theorem 3.7]), by using the extension theorem due to Pinchuk et al. (see [27] and [32] ; see also [8, Main theorem], [12, Theorem 1.7] and [31, Theorem 3]) and Remark 7.2 i).

Theorem 7.3. *Let $n \geq 2$, $A \in L(\mathbb{C}^n)$ with $m(A) > 0$ and let $f \in S(\mathbb{B}^n)$ be such that $f(\mathbb{B}^n)$ is a bounded strictly pseudoconvex domain. If $f(\mathbb{B}^n)$ has C^m boundary with $m > 2$, then the following conditions are equivalent:*

(i) *f is (strictly) A-spiralshapelike;*

(ii) *$\overline{f(\mathbb{B}^n)}$ is polynomially convex;*

(iii) *f is (strictly) starshapelike.*

Moreover, if $f(\mathbb{B}^n)$ has C^m boundary with $m > 2+\frac{1}{2}$, then the above conditions are equivalent to the following condition:

(iv) *f is convexshapelike.*

Corollary 7.4 (see [5], for $A = I_n$, and [23]). *Let $n \geq 2$ and $A \in L(\mathbb{C}^n)$ with $m(A) > 0$. If $f \in S(\mathbb{B}^n)$ is such that $f(\mathbb{B}^n)$ is a bounded strictly pseudoconvex domain which has C^m boundary with $m > 2$ and $\overline{f(\mathbb{B}^n)}$ is polynomially convex, then $f \in S_A^1(\mathbb{B}^n)$.*

Under even lower regularity of the boundary, we have the following counterexamples given in [23] (cf. [11]).

Example 7.5. i) Let $f \in S(\mathbb{B}^2)$ be given by $f(z) = (z_1 + \frac{1}{2}z_1^2, z_2)$, $z = (z_1, z_2) \in \mathbb{B}^2$. Then f is strictly starlike, but not convexshapelike.

ii) Let $\varphi(\zeta) = \frac{1-\sqrt{\left(\frac{1-\zeta}{1+\zeta}\right)^2+1}}{1+\sqrt{\left(\frac{1-\zeta}{1+\zeta}\right)^2+1}}$, $\zeta \in \mathbb{U}$, where we choose the branch of the square root on $\mathbb{C} \setminus (-\infty, 0]$ with $\sqrt{1} = 1$, and $f \in S(\mathbb{B}^2)$ be given by $f(z) = \left(\frac{\varphi(z_1)-\varphi(0)}{\varphi'(0)}, z_2\right)$, $z = (z_1, z_2) \in \mathbb{B}^2$. Then f is starlike, but $\overline{f(\mathbb{B}^2)}$ is not polynomially convex. In particular, f is not strictly starshapelike.

Remark 7.6. In view of [5, Example 4.2] and Remark 7.2 (ii), there exist normalized Fatou-Bieberbach mappings on $\mathbb{C}^2$ which restricted to $\mathbb{B}^2$ are not A-spiralshapelike for any $A \in L(\mathbb{C}^n)$ with $m(A) > 0$.

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