FRONTS, PULSES AND PERIODIC TRAVELLING WAVES IN TWO-COMPONENT SHALLOW WATER MODELS

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This survey is devoted to the travelling-wave solutions to some two-component partial differential equations modelling shallow water waves on irrotational flows as well as on shear flows. Qualitative informations about the travelling-wave solutions are obtained from a general ordinary differential equation for each model considered. The existence and the profile of the travelling waves depend on the values of the constants of integration, and on the existence, the sign and order of multiplicity of the roots of some polynomials of degree 3, 4, 5, 6, depending on the model; fronts, pulses, anti-pulses, multi-pulses, periodic travelling waves will arise. By comparing the effects of the vorticity on the pulse waves in the models with and without vorticity, we find that the right-going waves propagating in the same direction as the underlying shear flow have a higher amplitude and narrower wavelength and the right-going waves for which the underlying shear flow propagates in the opposite direction are wider, their amplitude decreases.

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1. **INTRODUCTION**

One fascinating feature of the study of water waves is that their motion can exhibit elementary patterns, such as, fronts, pulses or periodic wave trains. Mathematical understanding of these elementary patterns is essential to gain fundamental insights into the more complex patterns. We review recent works on the most important models that appear in the literature - the Green–Naghdi model, the two-component Camassa–Holm model, a new two-component model of Green–Naghdi type, the Zakharov–Itō model and the Kaup–Boussinesq model - for the description of waves in shallow water, waves whose wavelength is very much longer than the water depth, propagating on irrotational flows as well as on shear flows. Although the irrotational models (with considerable advantages in their mathematical analysis) of wave motion yield many practical results that tell us about the nature of water waves, in reality there is always some vorticity present in actual wave motion: for wind-driven waves, waves riding upon a sheared current, or waves near a ship or pier. We pay special attention to the case when the vorticity is present but has a constant value. For waves which are long compared with the water depth, the choice of constant vorticity is not just a mathematical simplification but it is also physically reasonable, since, in this case, the non-zero mean vorticity is more important than its specific distribution - see the discussion in [20]. In particular, constant vorticity gives a good description of tidal currents [20]. While vorticity cannot be generated in an incompressible inviscid flow, it can be prescribed by thinking of the vorticity beneath the wave as injected by a current. The problem of understanding the effects of a current on the dynamics of water waves is very difficult both theoretically as well as experimentally and numerically see the discussions in [14, 17, 18, 45, 60, 64] and the references therein. The water waves propagating in the presence of constant vorticity have to be two-dimensional - see [13, 65], the vorticity has only one non-zero component which points in the horizontal direction orthogonal to the direction of wave propagation; in contrast, within the setting of irrotational flows, for two-dimensional wave trains entering a still water region with a flat bed there is no restriction concerning the direction of wave propagation. At least in the absence of flow reversal, the vorticity does not destroy the symmetry of the surface travelling waves - see [15, 34].

Most of the studies devoted to travelling waves are focused on a particular sub-class of solutions: the solitary waves - pulses - [5, 7, 12, 20, 55, 63]. These localized travelling waves, whose shapes do not change as they propagate along with a constant velocity, are less ubiquitous than the periodic wave trains but nevertheless represent observable and beautiful wave patterns. By applying a unified procedure, the most general ordinary differential equation describing the whole family of travelling wave solutions to each two-component model above, was derived in [21, 22]. The equations describing the solitary wave solutions are obtained by choosing the constants of integration appropriately. Some of the general equations can be solved analytically to obtain the explicit solutions, but a description of the travelling wave profiles for all models above can be made by performing a phase-plane analysis [21, 22]. The existence and the profile of the travelling waves depend on the values of the constants of integration, and on the existence, the sign and order of multiplicity of the roots of some polynomials of degree 3, 4, 5, 6, depending on the model. A closed curve in the phase-plane yields a periodic travelling wave solution, a homoclinic orbit gives a pulse type solution and a heteroclinic orbit in the phase-plane provides a front type solution. For certain values of the constants, all models possess pulses. For the Kaup–Boussinesq system, interesting analytical multi-pulse travelling wave solutions are found. For the Zakharov–Itō system pulse and anti-pulse solutions are obtained. The two-component Camassa-Holm (CH2) model, with or without vorticity, possesses front wave solutions too. The front wave solutions decay algebraically in the far field. If we compare the effects of the vorticity on the pulse waves in the CH2 model and $CH2_{\Omega}$ model, we find that the right-going waves propagating in the same direction as the underlying shear flow ($\Omega > 0$) have a higher amplitude and narrower wavelength and the right-going waves for which the underlying shear flow propagates in the opposite direction ($\Omega < 0$) are wider, their amplitude decreases.

2. SHALLOW-WATER APPROXIMATION

We consider the wave motion in a single layer of incompressible fluid that occupies a domain with a free upper surface and a flat bottom. The fluid is inviscid but may be rotational. The effects of surface tension are ignored, therefore the evolution of waves from their initial profile is governed by the balance between gravity and the inertia of the system. The fundamental governing equations and boundary conditions are: the Euler equation, the equation for incompressible fluids, free surface and bottom kinematic conditions, and a dynamic condition constant pressure at the free surface. All these together constitute the classical water-wave problem. In the absence of non-uniform currents in the water, the assumption of irrotational flow is realistic. For these flows, the following equation: the curl of the velocity field is zero, has to be additionally taken into account. But in order to incorporate the ubiquitous effects of currents and wave-current interactions, the vorticity - the curl of the velocity field - is very important.

In order to derive approximations to the governing equations it is useful to write them in non-dimensional form. The water-wave problem has the following non-dimensional form (see, for example, [42], [14]):

$$
\begin{aligned}
u_t + uu_x + vu_z &= -p_x \\
\delta^2(v_t + uv_x + vv_z) &= -p_z \\
u_x + v_z &= 0 \\
u_z - \delta^2 v_x &= \Omega \\
v = \epsilon(\eta_t + u\eta_x) &\quad \text{on } z = 1 + \epsilon\eta(x,t) \\
p = \epsilon\eta &\quad \text{on } z = 1 + \epsilon\eta(x,t) \\
v = 0 &\quad \text{on } z = 0.
\end{aligned}
\tag{1}
$$

Here (x, z) are the non-dimensional Cartesian coordinates, the x-axis being in the direction of wave propagation and the z-axis pointing vertically upwards. $(u(x, z, t), v(x, z, t))$ is the non-dimensional velocity field of the fluid and $p(x, z, t)$ denotes the non-dimensional pressure. $\Omega(x, z, t)$ is the non-dimensional vorticity. For example, in the case of irrotational flows $\Omega(x, z, t) = 0$, and in the case of constant vorticity flows $\Omega(x, z, t) =$const. $\eta(x, t)$ is the non-dimensional fluid surface displacement from the undisturbed fluid level. The dimensionless form of the water-wave problem involves two fundamental parameters

$$\epsilon := \frac{a}{h_0}, \quad \delta := \frac{h_0}{\lambda},$$

where a is the typical amplitude of the wave, h_0 is the undisturbed depth of the fluid and λ is the wavelength. The amplitude parameter ϵ is associated with the nonlinearity of the wave. The long-wave (or shallowness) parameter δ is associated with the dispersion of the wave, it measures the deviation of the pressure, in the water below the wave, away from the hydrostatic pressure distribution. The role of δ independent of ϵ is useful in the description of arbitrary amplitude shallow-water waves, that is, $\delta \to 0$, for arbitrary fixed ϵ. For $\delta = 0$, the leading-order equations become

$$\begin{aligned} u_t + uu_x + vu_z &= -p_x \\ p_z &= 0 \\ u_x + v_z &= 0 \\ u_z &= \Omega \\ v = \epsilon(\eta_t + u\eta_x) & \quad \text{on } z = 1 + \epsilon\eta(x, t) \\ p = \epsilon\eta & \quad \text{on } z = 1 + \epsilon\eta(x, t) \\ v = 0 & \quad \text{on } z = 0. \end{aligned} \tag{2}$$

3. TWO-COMPONENT SHALLOW WATER MODELS

3.1. Irrotational case

In this case, $\Omega = 0$, the system of equations (2) reduces to

$$u = u(x, t), \quad v = -zu_x, \quad p = \epsilon\eta(x, t) \tag{3}$$

and

$$\begin{aligned} u_t + uu_x + H_x &= 0 \\ H_t + (Hu)_x &= 0, \end{aligned} \tag{4}$$

where

$$H(x,t) := 1 + \epsilon\eta(x,t). \tag{5}$$

The hyperbolic partial differential equations (4) are the so-called classical shallow water equations (see, for example, [59]). The system (4) is Hamiltonian with respect to three distinct Hamiltonian structures ([10, 50, 52]) which are compatible and thus, the system is completely integrable [53]. This system provides a good approximation to the exact solution of the water-wave problem [2].

The classical shallow water equations do not take into account any dispersive effect. Higher-order corrections to these equations can be obtained through different routes. The relevant system to model highly nonlinear weakly dispersive waves propagating in shallow water is the following system:

$$\text{(GN)}\begin{cases} u_t + u\,u_x + H_x = \frac{1}{3H}\left[H^3\left(u\,u_{xx} + u_{xt} - u_x^2\right)\right]_x \\ H_t + (Hu)_x = 0. \end{cases} \tag{6}$$

In 1953, Serre [56] derived this system by assuming that the horizontal component of fluid velocity is independent of the vertical coordinate z - the first two conditions in (3) are satisfied, but the pressure depends on z and δ^2 - and by integrating the Euler equations over z in the interval $[0, H(x,t)]$. More then ten years later Su and Gardner [58] obtained the system (6) by depth-averaging the two-dimensional irrotational water-wave problem, by using asymptotic expansion in the small shallowness parameter δ and by retaining terms as far as $O(\delta^4)$. Green and Naghdi considered in [29] the three-dimensional water-wave problem with a free surface and a variable bottom, and without imposing the condition that the fluid motion should be irrotational. The model equations were not derived by a formal asymptotic expansion, but instead by imposing the condition that the horizontal velocity is independent of the vertical coordinate z, that the vertical velocity has only a linear dependence on z and by using the mass conservation equation and the energy equation in integral form plus invariance under rigid-body translation. For one horizontal x-coordinate and for a flat bottom, the equations obtained have the form (6). In the literature, the equations (6) are referred to as the Serre equations, or the Su-Gardner equations but usually they are called the Green-Naghdi (GN) equations. Throughout this work we will call them the GN equations.

The GN equations are mathematically well-posed in the sense that they admit solutions over the relevant time scale for any initial data reasonably smooth (see [1, 48]). The solution of the GN equations provides a good approximation of the solution of the full water-wave problem, the difference between both solutions remaining of order $O(\delta^4)$ as long as the wave does not exhibit any kind of singularity such as wave breaking (see [2, 48]). The GN equations have a Hamiltonian formulation (see [33]). The GN equations can be obtained by a variational approach in the Lagrangian formalism [35]. In the shallow-water regime, for a velocity field with a horizontal component which is independent of the vertical coordinate z and a vertical component which has only a linear dependence on z, the Lagrangian used in the variational derivation is [35]:

$$\begin{aligned} \mathfrak{L}(u,H) &= E_{kinetic}(u,H) - E_{potential}(H) \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\left[Hu^2 + \frac{1}{3}H^3u_x^2 - (H-1)^2\right]\,\mathrm{d}x. \end{aligned} \tag{7}$$

We stress that there is no limitation on the amplitude assumed in the derivation of the GN equations. They have exact solitary wave solutions [56,58] - see Section 4.1 for their explicit expressions. The small-amplitude solitary wave solutions of the GN equations are linearly stable [46, 47]. The GN equations have also the periodic cnoidal wave solutions [9, 23, 35, 56]. The stability of periodic cnoidal waves is further investigated in [9]: it is established that the waves with sufficiently small amplitude are stable and the waves with sufficiently large amplitude are unstable.

An asymptotic reduction of the GN system for small-amplitude shallow water waves is obtained by Constantin and Ivanov [16] by using the expansion of the variables with respect to ϵ and δ^2 and by keeping the leading order terms, the following integrable two-component Camassa-Holm (CH2) system*:

$$\text{(CH2)}\begin{cases} u_t + 3uu_x - u_{txx} - 2u_xu_{xx} - uu_{xxx} + HH_x = 0 \\ H_t + (Hu)_x = 0. \end{cases} \tag{8}$$

For $H = 0$, this system reduces to the celebrated peakon equation derived by Camassa and Holm [8]. The system (8) can be obtained within the shallow water regime by a variational approach in the Lagrangian formalism [36]. The Lagrangian used in the variational derivation is [36]:

$$\begin{aligned} \mathfrak{L}(u,H) &= E_{kinetic}(u) - E_{potential}(H) \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\left[u^2 + u_x^2 - (H-1)^2\right]\,\mathrm{d}x. \end{aligned} \tag{9}$$

The mathematical properties, such as well-posedness and wave breaking, the geometric aspects and the travelling wave solutions of the CH2 system were further investigated in many works (see, for example, [16, 24, 25, 30, 51]).

*The system (8) with a minus sign in front of the HH_x term appears originally in [54] as a tri-Hamiltonian system. Alternative derivations of the system (8) with the minus sign in front of the HH_x term are provided in [11, 27, 49, 57].

By a variational approach in the Lagrangian formalism, in [37] is derived a new two-component (N2C) system:

$$(10)\qquad (\text{N2C})\begin{cases} u_t + 3uu_x + HH_x &= \left[H^2\left(u\,u_{xx} + u_{xt} - \frac{1}{2}u_x^2\right)\right]_x \\ H_t + (Hu)_x &= 0. \end{cases}$$

In the shallow-water regime, for a velocity field with a horizontal component which is independent of the vertical coordinate z and a vertical component which has only a linear dependence on z, the Lagrangian used to obtain (10) is [37]:

$$\begin{aligned} \mathfrak{L}(u,H) &= E_{kinetic}(u,H) - E_{potential}(H) \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\left[u^2 + H^2u_x^2 - (H-1)^2\right]\,\mathrm{d}x. \end{aligned}\qquad (11)$$

The system (10) has a noncanonical Hamiltonian formulation [37]. Its exact solitary-wave solutions differ from the classical sech^2-form [37] - see Section 4.1 for their explicit expressions.

3.2. **Constant vorticity case**

In this case, $\Omega =$ const in (2) , the water waves travelling over the constant shear current $U(z) = \Omega\, z$. For $\Omega > 0$, the underlying shear flow is propagating in the positive direction of the $x-$coordinate, for $\Omega < 0$ it propagates in the negative direction.
The leading-order equations (2) reduce to

$$(12)\qquad u = \Omega\, z + u(x,t), \quad v = -zu_x, \quad p = \epsilon\eta(x,t)$$

and

$$(13)\qquad \begin{aligned} u_t + uu_x + H_x &= 0 \\ H_t + (Hu)_x + \Omega\, HH_x &= 0, \end{aligned}$$

with the same H defined in (5).

Returning to the non-dimensional system (1), one observes that the component v of the velocity and the pressure p are proportional to ϵ, that is, to the wave amplitude. By introduction a suitable scaling around the shear flow and by truncating asymptotic expansions of the variables to the first order in ϵ and δ^2, Ivanov [39] obtained the following system:
(14)

$$(\text{CH2}_\Omega)\begin{cases} u_t + 3uu_x - u_{txx} - 2u_xu_{xx} - uu_{xxx} + HH_x - \Omega\, u_x = 0 \\ H_t + (Hu)_x = 0. \end{cases}$$

This is an integrable bi-Hamiltonian system [39]. Wave-breaking criteria and a sufficient condition guaranteeing the existence of a global solution are presented in [31]. The well-posedness of this system was studied in [26]. For $\Omega = 0$, we get the integrable CH2 system (8).

By a further rescaling of the spatial variable x, of the time variable t and of the z-independent part of the horizontal component of the velocity $u(x,t)$, Ivanov [39] proceeded with the derivation in a direction that leads to the following system:

$$(15) \qquad (\mathrm{ZI}_\Omega) \begin{cases} u_t + u_{xxx} + 3uu_x + HH_x - \Omega\, u_x = 0 \\ H_t + (Hu)_x = 0, \end{cases}$$

which matches the Zakharov–Itō (ZI) system [38, 66] with constant vorticity. This system represents a two-component generalization of the classical KdV equation. It has a Lax pair [62], a bi-Hamiltonian structure and an integrable hierarchy [4].

Another system that describes motion of shallow water in the lowest order in small parameters controlling weak dispersion and weak nonlinearity effects, is the Kaup–Bousinesq (KB) system. In the presence of a linear shear current this system has the form [39]:

$$(16) \qquad (\mathrm{KB}_\Omega) \begin{cases} u_t + \left(\frac{1}{2}\, u^2 + H\right)_x = 0 \\ H_t - \frac{1}{4}\, u_{xxx} + \frac{\mathcal{C}(\Omega)}{2}\, [(H - 1)\, u]_x = 0, \end{cases}$$

with

$$(17) \qquad \mathcal{C}(\Omega) := 1 + \frac{1}{4}\left(\Omega + \sqrt{4 + \Omega^2}\right)^2.$$

The system (16) is integrable iff $\Omega = 0$, see [39, 44]. In the original derivation proposed by Kaup in [44] as an early example of a coupled pair of equations that admits an inverse-scattering formalism, the second term in the second equation of the system appears with '+' sign. However, this yields a linearly ill-posed model, see [3]. The inverse scattering for the KB equations was developed further in [32]. For other studies on the KB system see, for example, the papers [19, 28, 40, 43] and the references therein.

4. TRAVELLING WAVE SOLUTIONS

In this chapter we present the most general ordinary differential equations describing the whole family of travelling wave solutions to the systems (6), (8), (10) and to the systems (14), (15), (16) which constant vorticity.

We look for right-going waves travelling at a constant speed $c > 0$, whose profile are steady relative to a frame of reference moving with velocity c in the $x-$direction. Combining the independent variables x and t into one variable

$\xi := x - ct$, we suppose that

$$H(x,t) = H(\xi), \qquad u(x,t) = u(\xi), \qquad \xi := x - ct. \tag{18}$$

We can distinguish between different shapes of the travelling waves. Wave trains are spatially *periodic* travelling waves with a period $L > 0$, if

$$H(\xi + L) = H(\xi), \quad u(\xi + L) = u(\xi), \quad \text{for all } \xi.$$

Fronts and *pulses* are travelling waves that are asymptotically constant, that is, they converge to rest states

$$\lim_{\xi \to \pm\infty} H(\xi) = H_{\pm}, \quad \lim_{\xi \to \pm\infty} u(\xi) = u_{\pm}.$$

For fronts we have $H_- \neq H_+$, $u_- \neq u_+$, and for pulses $H_- = H_+$, $u_- = u_+$.

We substitute the Ansatz (18) into the systems (6), (8), (10), (14), (15), (16). We observe that, for the first five systems, the second equation is the same. This equation becomes

$$\left(-cH + Hu\right)' = 0, \tag{19}$$

with prime denoting the usual derivative operation with respect to ξ, which yields:

$$u = \frac{cH - \mathcal{K}_1}{H}, \qquad \mathcal{K}_1 \in \mathbb{R}, \tag{20}$$

where $\mathcal{K}_1 \in \left\{\mathcal{K}_1^{\mathrm{GN}},\ \mathcal{K}_1^{\mathrm{CH2}},\ \mathcal{K}_1^{\mathrm{N2C}}, \mathcal{K}_1^{\mathrm{CH2}\Omega}, \mathcal{K}_1^{\mathrm{ZI}\Omega}\right\}$ is an integration constant and $c \in \left\{c^{\mathrm{GN}},\ c^{\mathrm{CH2}},\ c^{\mathrm{N2C}},\ c^{\mathrm{CH2}\Omega}, c^{\mathrm{ZI}\Omega}\right\}$ the constant wave speed corresponding to each model, respectively.

After substituting (18) into the first equations of the systems (6), (8), (10), (14), (15), by using the expression (20) for u along with its derivatives, that is,

$$u' = \frac{\mathcal{K}_1 H'}{H^2}, \qquad u'' = \frac{\mathcal{K}_1 H''}{H^2} - \frac{2\mathcal{K}_1 (H')^2}{H^3},$$

we integrate these equations once. Then, we multiply the equations obtained by $2u' = \dfrac{2\mathcal{K}_1 H'}{H^2}$ and we integrate once again; after each integration, the integration constants corresponding to each model are denoted by $\mathcal{K}_2 \in \left\{\mathcal{K}_2^{\mathrm{GN}},\ \mathcal{K}_2^{\mathrm{CH2}},\ \mathcal{K}_2^{\mathrm{N2C}}, \mathcal{K}_2^{\mathrm{CH2}\Omega}, \mathcal{K}_2^{\mathrm{ZI}\Omega}\right\}$ and $\mathcal{K}_3 \in \left\{\mathcal{K}_3^{\mathrm{GN}},\ \mathcal{K}_3^{\mathrm{CH2}},\ \mathcal{K}_2^{\mathrm{N2C}}, \mathcal{K}_3^{\mathrm{CH2}\Omega}, \mathcal{K}_3^{\mathrm{ZI}\Omega}\right\}$, respectively. Finally, we obtain the following first order implicit ODEs, which describe all possible travelling waves of the models under consideration [21, 22]: for the GN model:

$$(H')^2 = -\frac{3}{\mathcal{K}_1^2} H^3 + \frac{\mathcal{K}_3 + 2c\mathcal{K}_2}{\mathcal{K}_1^3} H^2 - \frac{2\mathcal{K}_2}{\mathcal{K}_1^2} H + 3, \tag{21}$$

for the CH2 model:

$$(22)\quad (H')^2 = H^2\Big[-\frac{1}{\mathcal{K}_1^2}H^4 + \frac{\mathcal{K}_3 + 2c\mathcal{K}_2}{\mathcal{K}_1^3}H^3 + \frac{c^2 - 2\mathcal{K}_2}{\mathcal{K}_1^2}H^2 - \frac{2c}{\mathcal{K}_1}H + 1\Big],$$

for the N2C model:

$$(23)\quad (H')^2 = -\frac{1}{\mathcal{K}_1^2}H^4 + \frac{\mathcal{K}_3 + 2c\mathcal{K}_2}{\mathcal{K}_1^3}H^3 + \frac{c^2 - 2\mathcal{K}_2}{\mathcal{K}_1^2}H^2 - \frac{2c}{\mathcal{K}_1}H + 1,$$

for the $\mathrm{CH2}_\Omega$ model:

$$(24)\quad (H')^2 = H^2\Big[-\frac{1}{\mathcal{K}_1^2}H^4 + \Big(\frac{c^2\Omega + 2c\mathcal{K}_2 + \mathcal{K}_3}{\mathcal{K}_1^3}\Big)H^3 + \Big(\frac{c^2 - 2\Omega c - 2\mathcal{K}_2}{\mathcal{K}_1^2}\Big)H^2 - \Big(\frac{2c - \Omega}{\mathcal{K}_1}\Big)H + 1\Big],$$

and for the ZI_Ω model:

$$(25)\quad (H')^2 = \mathcal{K}_1 H\cdot\Big[-\frac{1}{\mathcal{K}_1^2}H^4 + \Big(\frac{c^2\Omega + 2c\mathcal{K}_2 + \mathcal{K}_3}{\mathcal{K}_1^3}\Big)H^3 + \Big(\frac{c^2 - 2\Omega c - 2\mathcal{K}_2}{\mathcal{K}_1^2}\Big)H^2 - \Big(\frac{2c - \Omega}{\mathcal{K}_1}\Big)H + 1\Big].$$

For the KB_Ω system, after substituting (18) into (16), we integrate once and we get

$$(26)\qquad H = \mathcal{K}_1 + cu - \frac{1}{2}u^2,$$

$$(27)\qquad -cH - \frac{1}{4}u'' + \frac{\mathcal{C}(\Omega)}{2}(H-1)u = \mathcal{K}_2,$$

where c is a c^{KB_Ω} and $\mathcal{K}_1$, $\mathcal{K}_2$ are the integration constants corresponding to this model, that is, $\mathcal{K}_1^{\mathrm{KB}_\Omega}$, $\mathcal{K}_2^{\mathrm{KB}_\Omega}$. We replace (26) into (27), and we get a differential equation in u only. We multiply this equation by $2u'$, we integrate once again and we obtain now an ODE for the variable u [22]:

$$(28)\quad (u')^2 = -\frac{\mathcal{C}(\Omega)}{2}u^4 + \frac{4c\big[1 + \mathcal{C}(\Omega)\big]}{3}u^3 + 2\big[(\mathcal{K}_1 - 1)\mathcal{C}(\Omega) - 2c^2\big]u^2 - 8(c\mathcal{K}_1 + \mathcal{K}_2)u + \mathcal{K}_3,$$

$\mathcal{K}_3$ being an integration constant corresponding to this model, i.e. $\mathcal{K}_3^{\mathrm{KB}_\Omega}$.

The equations (20) with (21), (22), (23), (24), (25), respectively, and (26) with (28), completely characterize the travelling wave solutions of the model partial differential equations considered. We see that the real-valued solutions exist only if the right-hand side of each equation (21) - (25) and (28) is non-negative. This depends on the values of the parameters and the roots of the polynomial functions on the right-hand side of each equation (21) - (25) and (28), and gives also bounds on the wave height $H(\xi)$ and on $u(\xi)$, respectively, $\forall\,\xi \in \mathbb{R}$. At this point, each problem involves four interdependent parameters: the propagation wave speed c and the integration constants $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$. For different values of the integration constants $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$, different shapes of the travelling waves propagating at the speed c arise.

4.1. **Fronts, pulses, anti-pulses and multi-pulses**

Let us now look for the solitary wave solutions that are asymptotically stable at $\pm\infty$.

By imposing the conditions:

$$H(\xi) \to 1, \quad H'(\xi) \to 0, \quad H''(\xi) \to 0, \tag{29}$$

$$u(\xi) \to 0, \quad u'(\xi) \to 0, \quad u''(\xi) \to 0, \tag{30}$$

we can determine the integration constants $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$ and the equations which govern the corresponding solitary waves [21, 22]:

$$\mathcal{K}_1^{\mathrm{GN}} = c^{\mathrm{GN}}, \quad \mathcal{K}_2^{\mathrm{GN}} = 3(c^{\mathrm{GN}})^2 + \frac{3}{2}, \quad \mathcal{K}_3^{\mathrm{GN}} = -3(c^{\mathrm{GN}})^3 + 3\,c^{\mathrm{GN}},$$

$$(H')^2 = \frac{3}{(c^{\mathrm{GN}})^2}\,(H - 1)^2\,\big[(c^{\mathrm{GN}})^2 - H\big], \tag{31}$$

$$\mathcal{K}_1^{\mathrm{CH2}} = c^{\mathrm{CH2}}, \quad \mathcal{K}_2^{\mathrm{CH2}} = \frac{1}{2}, \quad \mathcal{K}_3^{\mathrm{CH2}} = c^{\mathrm{CH2}},$$

$$(H')^2 = \frac{1}{(c^{\mathrm{CH2}})^2}\,H^2\,(H - 1)^2\,\big[(c^{\mathrm{CH2}})^2 - H^2\big], \tag{32}$$

$$\mathcal{K}_1^{\mathrm{N2C}} = c^{\mathrm{N2C}}, \quad \mathcal{K}_2^{\mathrm{N2C}} = \frac{1}{2}, \quad \mathcal{K}_3^{\mathrm{N2C}} = c^{\mathrm{N2C}}$$

$$(H')^2 = \frac{1}{(c^{\mathrm{N2C}})^2}\,(H - 1)^2\,\big[(c^{\mathrm{N2C}})^2 - H^2\big], \tag{33}$$

$$\mathcal{K}_1^{\mathrm{CH2}\Omega} = c^{\mathrm{CH2}\Omega}, \quad \mathcal{K}_2^{\mathrm{CH2}\Omega} = \frac{1}{2}, \quad \mathcal{K}_3^{\mathrm{CH2}\Omega} = c^{\mathrm{CH2}\Omega}$$

$$(H')^2 = \frac{H^2}{(c^{\mathrm{CH2}\Omega})^2}\cdot(H - 1)^2\cdot(c^{\mathrm{CH2}\Omega}\,\mathfrak{c}^+ - H)\cdot(H - c^{\mathrm{CH2}\Omega}\,\mathfrak{c}^-), \tag{34}$$

$$\mathcal{K}_1^{\mathrm{ZI}\Omega} = c^{\mathrm{ZI}\Omega}, \quad \mathcal{K}_2^{\mathrm{ZI}\Omega} = \frac{1}{2}, \quad \mathcal{K}_3^{\mathrm{ZI}\Omega} = c^{\mathrm{ZI}\Omega}$$

$$(35)\qquad (H')^2 = \frac{H}{c^{\mathrm{ZI}_\Omega}}(H - 1)^2 \cdot (c^{\mathrm{ZI}_\Omega}\,\mathfrak{c}^+ - H)\cdot(H - c^{\mathrm{ZI}_\Omega}\,\mathfrak{c}^-),$$

$$\mathcal{K}_1^{\mathrm{KB}_\Omega} = 1,\quad \mathcal{K}_2^{\mathrm{KB}_\Omega} = -c^{\mathrm{KB}_\Omega},\quad \mathcal{K}_3^{\mathrm{KB}_\Omega} = 0$$

$$(36)\qquad (u')^2 = u^2\Big[-\frac{\mathcal{C}(\Omega)}{2}u^2 + \frac{4\,c\,[1 + \mathcal{C}(\Omega)]}{3}u - 4(c^{\mathrm{KB}_\Omega})^2\Big].$$

In (34) and (35), $\mathfrak{c}^{\pm}$ given by:

$$\mathfrak{c}^{\pm} := \frac{1}{2}\left(\Omega \pm \sqrt{4 + \Omega^2}\right),$$

are the speeds to the right, $\mathfrak{c}^+ > 0$, and to the left, $\mathfrak{c}^- < 0$ of the linear shallow water waves on the constant shear current $U(z) = \Omega\, z$, see [6, 7, 41, 61].

The right-hand side of the equations (31) – (36) has to be non-negative, thus, for each model, we get the necessary condition for the existence of the solitary wave solutions:

$$(37)\qquad 0 \le H(\xi) \le (c^{\mathrm{GN}})^2,$$

$$(38)\qquad 0 \le H(\xi) \le c^{\mathrm{CH2}},$$

$$(39)\qquad 0 \le H(\xi) \le c^{\mathrm{N2C}},$$

$$(40)\qquad 0 \le H(\xi) \le c^{\mathrm{CH2}_\Omega}\,\mathfrak{c}^+,$$

$$(41)\qquad 0 \le H(\xi) \le c^{\mathrm{ZI}_\Omega}\,\mathfrak{c}^+,$$

$$(42)\qquad u_-^{\mathrm{KB}_\Omega} \le u(\xi) \le u_+^{\mathrm{KB}_\Omega},$$

respectively. In the last inequality, $u_-^{\mathrm{KB}_\Omega}$ and $u_+^{\mathrm{KB}_\Omega}$ are the two real roots of the second order polynomial in the brackets in (36). These roots exist iff

$$\frac{16\,c^2\left(1 + \mathcal{C}(\Omega)\right)^2}{9} - 8\,c^2\,\mathcal{C}(\Omega) \ge 0,$$

which yields

$$\mathcal{C}(\Omega) \le \frac{1}{2} \quad \text{or} \quad \mathcal{C}(\Omega) \ge 2.$$

Hence, with the notation (17) in view, we get for the KB_Ω model, the following restriction on the constant vorticity:

$$(43)\qquad \Omega \ge \frac{3}{2}.$$

By looking at the relations (38) and (40), we can compare the solitary waves in the CH2 model and $\mathrm{CH2}_\Omega$ model. For the right-going waves propagating in the same direction as the underlying shear flow, that is, $\Omega > 0$ yields $\mathfrak{c}^+ > 1$, thus, they have a higher amplitude and narrower wavelength. For the right-going waves for which the underlying shear flow propagates in the opposite direction $\Omega < 0$ yields $\mathfrak{c}^+ < 1$), hence, they are wider, their amplitude decreases.

According to the asymptotic behaviour of H, it follows that the right-going solitary waves exist only if

$$(44) \qquad 1 \le c\,, \qquad c \in \{c^{\mathrm{GN}},\ c^{\mathrm{CH2}},\ c^{\mathrm{N2C}}\}$$

$$(45) \qquad 1 \le c\,\mathfrak{c}^+\,, \quad c \in \{c^{\mathrm{CH2}_\Omega},\ c^{\mathrm{ZI}_\Omega}\}.$$

Some of the equations (31)–(36) can be solved explicitly. For the GN model, the explicit pulse-type wave solution of the equation (31) is [56, 58]:

$$H(\xi) = 1 + [(c^{GN})^2 - 1]\ \mathrm{sech}^2\left[\frac{\sqrt{3}}{2}\frac{\sqrt{(c^{GN})^2-1}}{c^{GN}}\xi\right].$$

For the N2C model, the explicit pulse-type wave solution of the equation (33) is [35]:

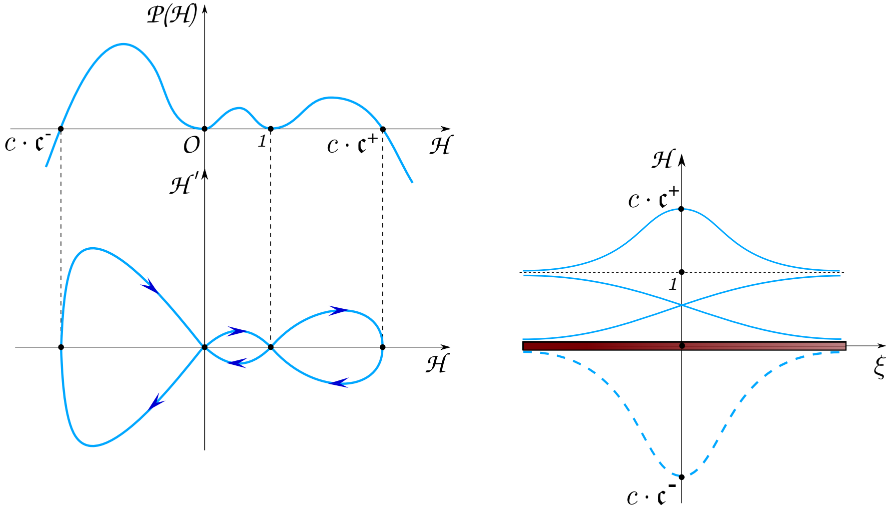
$$H(\xi) = 1 + \frac{(c^{N2C})^2 - 1}{1+\frac{(c^{N2C})^2+1}{2}\cosh\left[\frac{\sqrt{(c^{N2C})^2-1}}{c^{N2C}}\ \xi\right]+\frac{(c^{N2C})^2-1}{2}\sinh\left[\frac{\sqrt{(c^{N2C})^2-1}}{c^{N2C}}\ \xi\right]}.$$

For the equations that cannot be solved explicitly, we can give a description of the solitary wave profiles by performing a graphical phase-plane analysis [21, 22]. For the CH2 and $\mathrm{CH2}_\Omega$ models, the polynomials in H on the right-hand side of the equations (32) and (34) have two double roots, 0 and 1. In the phase-plane (H, H'), the homoclinic orbits, that start in 0 or in 1, lead to the pulse-type solutions and the heteroclinic orbits, connecting 0 and 1, to the front wave solutions, see Figures 1 in the cases: (a) $c\,\mathfrak{c}^+ > 1$ and (b) $c\,\mathfrak{c}^+ = 1$. For the CH2 model, $\Omega = 0$ yields $\mathfrak{c}^\pm = \pm 1$. We highlight the fact that two fronts tend only algebraically to the equilibrium state $H = 1$, that is, $H(\xi) \approx 1 + \xi^{-a}$ as $\xi \to \infty$, $a > 1$ being a parameter. Indeed, for $c\,\mathfrak{c}^+ = 1$, the root $H = 1$ becomes triple, thus, locally (34) becomes: $(1 - H)' \sim (1 - H)^{\frac{3}{2}}$. After integrating this relation we obtain the desired conclusion $1 - H \sim \frac{1}{\xi^2}$ as $\xi \to \infty$. By a similar reasoning, since $H = 0$ is double root in (34), the decay to $H = 0$ is exponential .

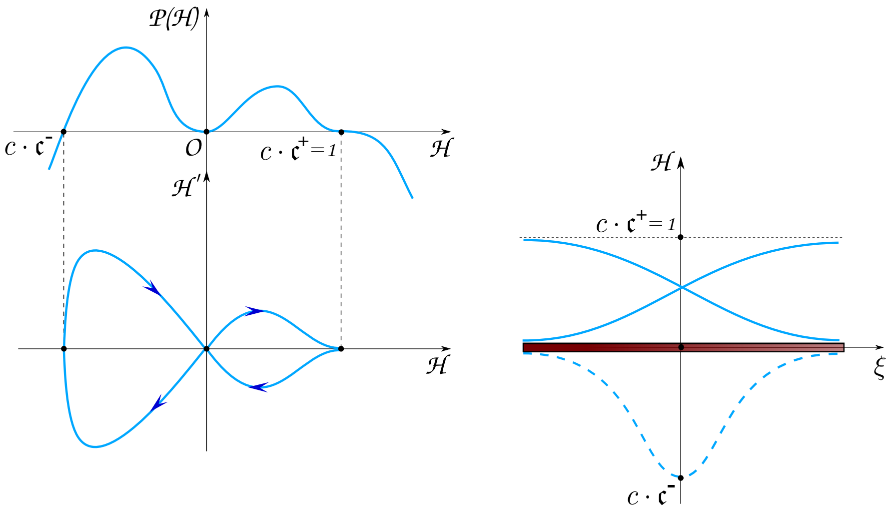
For the ZI_Ω we get in the case $c\,\mathfrak{c}^+ > 1$ a pulse and anti-pulse [22]. The phase-portrait and the solitary waves profiles are depicted in Figure 2. We highlight the fact that two fronts tend only algebraically to the equilibrium state $H = 1$.

For the KB_Ω model, with a constant vorticity $\Omega > \frac{3}{2}$, and u in the interval

$$(46) \qquad u_{\min} < u < u_{\max}\,,$$

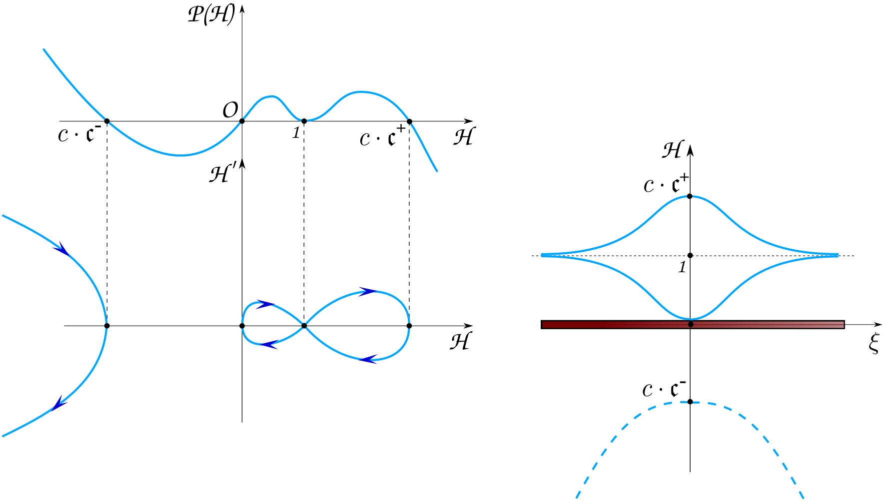


(a)

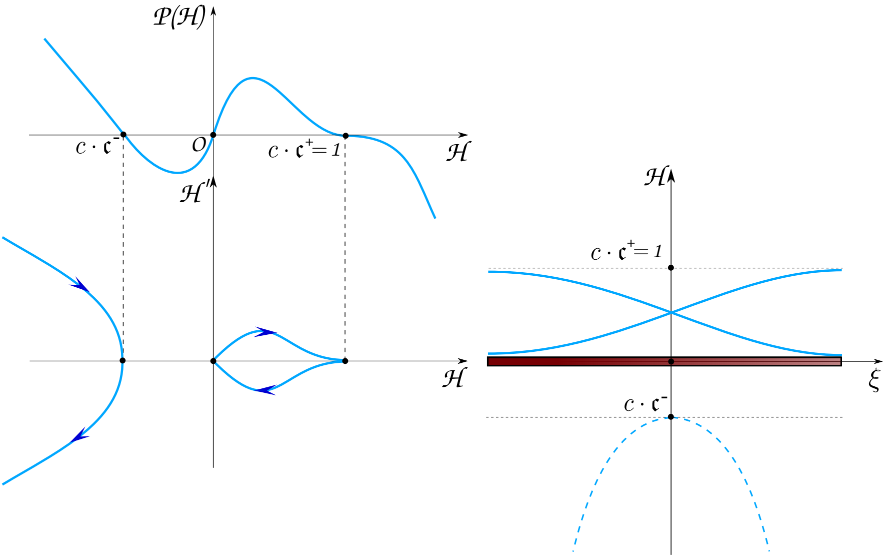


(b)

Figure 1 – (see [22]). The graph of the polynomial $\mathcal{P}(H)$, the phase-plane (H, H') and the solitary wave profiles for the CH2$_\Omega$ model, in the cases: a) $c\,\mathfrak{c}^+ > 1$ and b) $c\,\mathfrak{c}^+ = 1$. The physically admissible solutions are those which lie above the solid bottom. For the CH2 model, $\Omega = 0$ yields $\mathfrak{c}^\pm = \pm 1$.



(a)



(b)

Figure 2 – (see [22]). The graph of the polynomial $\mathcal{P}(H)$, the phase-plane (H, H') and the solitary wave profiles for the ZI_{Ω} model, in the cases: a) $c\,\mathfrak{c}^{+} > 1$ and b) $c\,\mathfrak{c}^{+} = 1$. The physically admissible solutions are those which lie above the solid bottom.

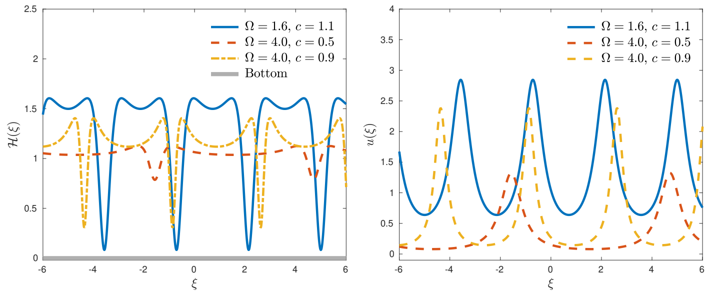


Figure 3 – (see [22]). The graph of the functions $H(\xi)$ (48) and $u(\xi)$ (47), for different values of the constant vorticity Ω and of the speed of propagation c.

$$u_{\min} := \frac{12\,c}{2\left[1 + \mathcal{C}(\Omega)\right] + \sqrt{2}\sqrt{2 - 5\,\mathcal{C}(\Omega) + 2\,\mathcal{C}(\Omega)^2}} > 0$$

$$u_{\max} := \frac{12\,c}{2\left[1 + \mathcal{C}(\Omega)\right] - \sqrt{2}\sqrt{2 - 5\,\mathcal{C}(\Omega) + 2\,\mathcal{C}(\Omega)^2}} > 0\,,$$

the explicit solution of the equation (36) is [22]:

$$u(\xi) = \frac{6\sqrt{2}\,c}{\sqrt{2}\,[1 + \mathcal{C}(\Omega)] + \sqrt{2 - 5\,\mathcal{C}(\Omega) + 2\,\mathcal{C}(\Omega)^2}\,\sin[2\,c\,\xi]}\,, \tag{47}$$

and, by (26), the function H has the expression [22]:

(48)

$$H(\xi) = 1 + \frac{6\sqrt{2}\,c^2}{\sqrt{2}\,[1 + \mathcal{C}(\Omega)] + \sqrt{2 - 5\,\mathcal{C}(\Omega) + 2\,\mathcal{C}(\Omega)^2}\,\sin[2\,c\,\xi]} - \frac{36\,c^2}{\left[\sqrt{2}\,[1 + \mathcal{C}(\Omega)] + \sqrt{2 - 5\,\mathcal{C}(\Omega) + 2\,\mathcal{C}(\Omega)^2}\,\sin[2\,c\,\xi]\right]^2}\,.$$

These are multi-pulse travelling wave solutions to the KB_Ω system, solutions which consist of an arbitrary number of crests and troughs. The number of crests and troughs increases with higher value of the constant vorticity Ω or at higher speed of propagation c, see Figure 3. We point out that the velocity u has the expression (47) only in the interval (46) which is situated above $\xi = 0$; the same the wave height H has the expression (48) only in some interval $H_{\min} < H < H_{\max}$. A multi-pulse travelling wave solution with two troughs is plotted in Figure 4. The multi-pulse travelling wave solutions to the KB system was found numerically by Chen [19].

For the following values of the integration constants:

$$\mathcal{K}_1^{\mathrm{KB}_\Omega} = 2, \quad \mathcal{K}_2^{\mathrm{KB}_\Omega} = -2\,c^{\mathrm{KB}_\Omega}, \quad \mathcal{K}_3^{\mathrm{KB}_\Omega} = 0,$$

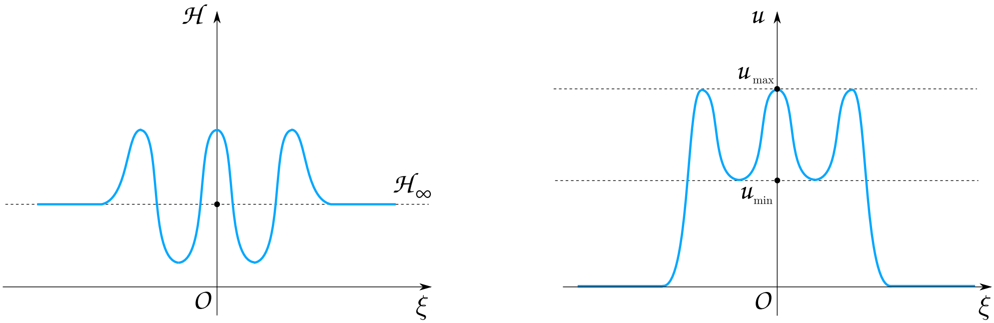


Figure 4 – (see [22]). A multi-pulse travelling wave solution with two troughs.

which will mean that the the solitary wave solution is tends to the constant state

(49) $$H(\xi) \to 2, \quad H'(\xi) \to 0, \quad H''(\xi) \to 0,$$

(50) $$u(\xi) \to 0, \quad u'(\xi) \to 0, \quad u''(\xi) \to 0,$$

the equation (28) becomes:

(51)
$$(u')^2 = u^2 \left[-\frac{\mathcal{C}(\Omega)}{2} u^2 + \frac{4c\left[1 + \mathcal{C}(\Omega)\right]}{3} u + 2\mathcal{C}(\Omega) - 4(c^{\mathrm{KB}_\Omega})^2 \right].$$

This equation can be solved analytically. If

(52) $$2c^2 < \mathcal{C}(\Omega), ^\dagger$$

the solution of the differential equation (51) has the explicit expression [22]:

(53) $$u(\xi) = \frac{e^{-\sqrt{2\left[\mathcal{C}(\Omega) - 2c^2\right]}\,\xi}}{\dfrac{\left(e^{-\sqrt{2\left[\mathcal{C}(\Omega) - 2c^2\right]}\,\xi} - \dfrac{4c\left(1+\mathcal{C}(\Omega)\right)}{3}\right)^2}{8\left[\mathcal{C}(\Omega) - 2c^2\right]} + \dfrac{\mathcal{C}(\Omega)}{2}},$$

and, taking into account (26), the function H is given by:

(54) $$H(\xi) = 2 + c\,u(\xi) - \frac{1}{2}u^2(\xi).$$

This is a one-trough travelling wave solution: see Figure 5 for different values of the constant vorticity Ω and of the speed of propagation c.

† For $\Omega = 0$, the condition (52) becomes $c^2 < 1$.

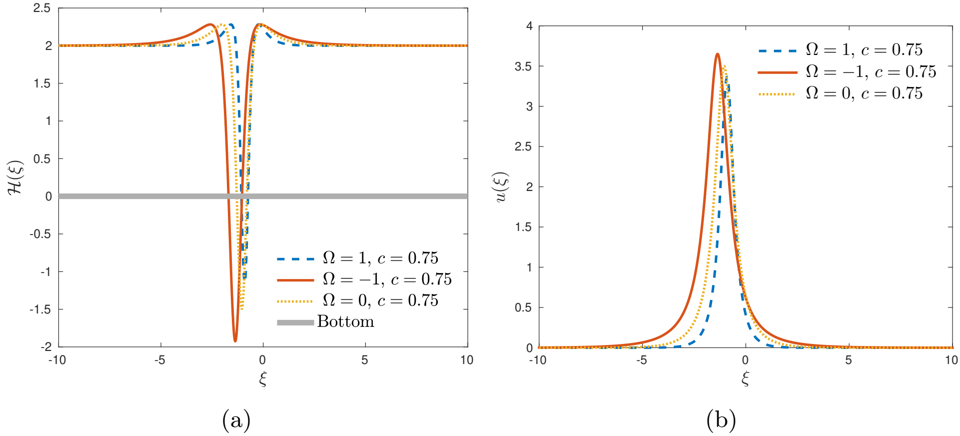


Figure 5 – (see [22]). One-trough travelling wave solution (53) – (54) to the KB_Ω model, for different values of the constant vorticity Ω and of the speed of propagation c.

4.2. **Periodic travelling wave solutions**

We now return to the general equations (21) - (25) and (28), which encompass all possible travelling waves of the six models under consideration. These equations have the general form:

$$(H')^2 \;=\; \mathcal{P}(H), \tag{55}$$

where $\mathcal{P}(H) \in \left\{\mathcal{P}^{\mathrm{GN}}(H),\ \mathcal{P}^{\mathrm{CH2}}(H),\ \mathcal{P}^{\mathrm{N2C}}(H),\ \mathcal{P}^{\mathrm{CH2}_\Omega}(H),\ \mathcal{P}^{\mathrm{ZI}_\Omega}(H)\right\}$ are polynomials in H, and

$$(u')^2 \;=\; \mathcal{P}(u), \tag{56}$$

with $\mathcal{P}(u) = \mathcal{P}^{\mathrm{KB}_\Omega}(u)$ polynomial in u. The existence and the behaviour (e.g. periodic, with decay to a constant state) of the solutions of these equations are based on the qualitative analysis of the real roots and the signs of the polynomials above; see [21, 22] for many situations that can encounter. We mention that, for the GN model, the equation (21) can also be solved analytically [56] (see also [9, 23]):

$$H(\xi) \;=\; H_2 \;+\; (H_3 - H_2)\ \mathrm{cn}^2\left[\frac{\sqrt{3}}{2}\ \frac{\sqrt{H_3 - H_1}}{\mathcal{K}_1^{GN}}\ \ \xi; k\right] \tag{57}$$

where $0 < H_1 < H_2 < H_3$ are the roots of the cubic polynomial in (21), $\mathrm{cn}(\cdot\,, k)$ is the cn-Jacobi elliptic function with the elliptic modulus k, $0 < k^2 < 1$,

$$k^2 \;:=\; \frac{H_3 - H_2}{H_3 - H_1}. \tag{58}$$

If the cubic polynomial (21) has one real root, denoted $H_0 > 0$, and two complex conjugate roots, $\mathcal{P}^{GN}(H) = -(H - H_0)(H^2 + pH + q)$, $p, q \in \mathbf{R}$, then, the equation (21) has the following periodic solution [35]:

$$(59) \qquad H(\xi) = H_0 - \sqrt{H_0^2 + pH_0 + q}\; \frac{1 - \mathrm{cn}\left[\frac{\sqrt{3}(H_0^2+pH_0+q)^{\frac{1}{4}}}{\mathcal{K}_1^{GN}}\,\xi; k\right]}{1 + \mathrm{cn}\left[\frac{\sqrt{3}(H_0^2+pH_0+q)^{\frac{1}{4}}}{\mathcal{K}_1^{GN}}\,\xi; k\right]}$$

with the elliptic modulus k, $0 < k^2 < 1$,

$$(60) \qquad k^2 := \frac{1}{2}\left(1 + \frac{H_0 + \frac{p}{2}}{\sqrt{H_0^2 + pH_0 + q}}\right).$$

We return to the general study of the polynomials in H and u, respectively. If $\mathcal{P}(H)$, $\mathcal{P}(u)$, respectively, have only one real root, then no bounded solutions exists. We restrict ourselves to bounded solutions because they are physically more acceptable. If H, u, are bounded smooth solutions of the equations (55), (56), respectively, let

$$H_{\min} = \inf_{\xi \in \mathbb{R}} H(\xi), \quad H_{\max} = \sup_{\xi \in \mathbb{R}} H(\xi).$$

$$u_{\min} = \inf_{\xi \in \mathbb{R}} u(\xi), \quad u_{\max} = \sup_{\xi \in \mathbb{R}} u(\xi).$$

Then, we use the fact that H, u are continuous and $H' \to 0$, $u' \to 0$ as $\xi \to H_{\min}$ or $\xi \to H_{\max}$, to obtain that the infimum and the supremum of the smooth solutions H, u, are zeros of $\mathcal{P}(H)$, $\mathcal{P}(u)$, respectively.

By Viète formulas:

– the cubic polynomial $\mathcal{P}^{\mathrm{GN}}(H)$ has at least one positive root, because its leading coefficient is smaller than zero and its constant term is greater than zero;

– the forth order polynomial $\mathcal{P}^{\mathrm{N2C}}(H)$ has at least one positive root and one negative root, because its leading coefficient is smaller than zero and its constant term is greater than zero. The sixth order polynomials $\mathcal{P}^{\mathrm{CH2}}(H)$ and $\mathcal{P}^{\mathrm{CH2}_\Omega}(H)$ have 0 as double root and they are written as a factorization into H^2 and a forth order polynomial with the same form as the polynomial $\mathcal{P}^{\mathrm{N2C}}(H)$;

– the fifth order polynomial $\mathcal{P}^{\mathrm{ZI}_\Omega}(H)$, which has 0 as single root and is written as a factorization into $\mathcal{K}_1\, H$ and a forth order polynomial having the leading coefficient smaller than zero and the constant term greater than zero, has at least one positive root and one negative root;

– the sum of the four roots of the polynomial $\mathcal{P}^{\mathrm{KB}_\Omega}(u)$ is higher than zero for the right-going travelling waves. The leading coefficient of the polynomial $\mathcal{P}^{\mathrm{KB}_\Omega}(u)$ is smaller than zero but the sign of the constant term depends on the

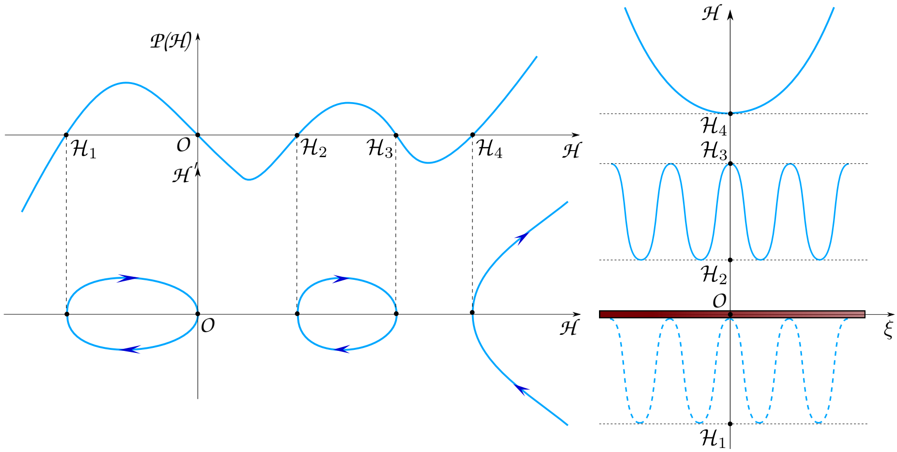


Figure 6 – (see [22]). The graph of the polynomial $\mathcal{P}(H)$, the phase-plane (H, H') and the solitary wave profiles for the ZI model with constant vorticity in the case the constant $\mathcal{K}_1 > 0$ and the polynomial in brackets from (25) has four real roots: $H_1 < 0$ and $0 < H_2 < H_3 < H_4$.

value of the constant $\mathcal{K}_3$. We conclude that, if real roots exist, at least one of them has to be positive.

The existence and order of multiplicity of the roots of the polynomials $\mathcal{P}(H)$, $\mathcal{P}(u)$, respectively, lead to different profiles for the travelling wave solutions of the model equations. By an analysis in the phase-plane (H, H') (we will write the analysis only for H, for u it will be the same), we can summarize them as follows:

• if the polynomial $\mathcal{P}(H)$ has two simple roots, let us note them $H_1 > 0$ and $H_2 > 0$, and $\mathcal{P}(H) > 0$ for $H_1 < H < H_2$, then the orbit in the phase plane (H, H') is a closed curve and there exists a physically acceptable smooth periodic travelling wave solution to the equation (55), with $H_1 = \min_{\xi \in \mathbb{R}} H(\xi)$ and $H_2 = \max_{\xi \in \mathbb{R}} H(\xi)$. This situation can be seen in Figure 6 and Figure 7.

• if the polynomial $\mathcal{P}(H)$ has a double root denoted $H_1 > 0$, a simple root denoted $H_2 > 0$, and $\mathcal{P}(H) > 0$ for $H_1 < H < H_2$, then there exists a homoclinic orbit in the the phase plane (H, H') that starts in H_1 and there exists a physically acceptable smooth pulse type solution of the equation (55), with $H_1 = \inf_{\xi \in \mathbb{R}} H(\xi)$, $H \to H_1$ as $\xi \to \pm\infty$ and $H_2 = \max_{\xi \in \mathbb{R}} H(\xi)$. This situation can be seen in Figure 7.

• if the polynomial $\mathcal{P}(H)$ has two double roots denoted $H_1 > 0$ and $H_2 > 0$, and $\mathcal{P}(H) > 0$ for $H_1 < H < H_2$, then there exist a heteroclinic orbit in the the phase plane (H, H') connecting H_1 and H_2 and there exists a physically acceptable smooth front type solution of the equation (55), with

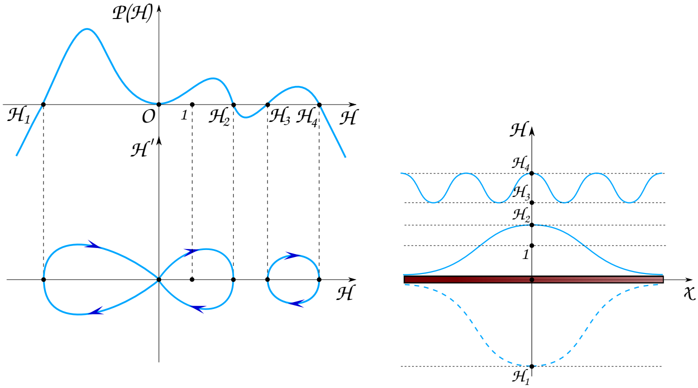


Figure 7 – (see [22]). The graph of the polynomial $\mathcal{P}(H)$, the phase-plane (H, H') and the solitary wave profiles for CH2 model in the case the polynomial in brackets from (22) has four real roots: $H_1 < 0$ and $0 < H_2 < H_3 < H_4$.

$H_1 = \inf_{\xi\in\mathbb{R}} H(\xi)$, $H \to H_1$ as $\xi \to -\infty($ or $\infty)$ and $H_2 = \sup_{\xi\in\mathbb{R}} H(\xi)$, $H \to H_2$ as $\xi \to \infty($or $-\infty$, respectively).

• if the polynomial $\mathcal{P}(H)$ has an odd number of positive simple roots, we denote by H_{big} the largest root, then no bounded travelling wave solutions will exist for $H > H_{\text{big}}$; see, for example, Figure 6.

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