ON A SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH p-LAPLACIAN OPERATORS AND INTEGRAL BOUNDARY CONDITIONS

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We study the existence and nonexistence of positive solutions for a system of nonlinear Riemann-Liouville fractional differential equations with parameters and p-Laplacian operators, supplemented with integral boundary conditions which contain fractional derivatives. The proof of our main existence theorems is based on the Guo-Krasnosel'skii fixed point theorem.

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1. INTRODUCTION

Fractional differential equations describe many phenomena in several fields of engineering and scientific disciplines such as physics, biophysics, chemistry, biology (for example, the primary infection with HIV), economics, control theory, signal and image processing, thermoelasticity, aerodynamics, viscoelasticity, electromagnetics, and rheology (see the books [3, 4, 19, 20, 26–28], and the papers [2, 5–8, 24, 25, 30]). Fractional differential equations are also regarded as a better tool for the description of hereditary properties of various materials and processes than the corresponding integer order differential equations.

In this paper we consider the system of nonlinear ordinary fractional differential equations with r_1-Laplacian and r_2-Laplacian operators

$$(1)\qquad \begin{cases} D_{0+}^{\alpha_1}(\varphi_{r_1}(D_{0+}^{\beta_1}x(t))) + \lambda f(t,x(t),y(t)) = 0, \ \ t\in(0,1), \\ D_{0+}^{\alpha_2}(\varphi_{r_2}(D_{0+}^{\beta_2}y(t))) + \mu g(t,x(t),y(t)) = 0, \ \ t\in(0,1), \end{cases}$$

with the integral boundary conditions

$$
(2)\qquad \begin{cases} x^{(j)}(0)=0,\ j=0,\dots,n-2;\ \ D_{0+}^{\beta_1}x(0)=0,\\ D_{0+}^{\gamma_0}x(1)=\displaystyle\sum_{i=1}^{p}\int_0^1 D_{0+}^{\gamma_i}x(t)\,\mathrm{d}H_i(t),\\ y^{(j)}(0)=0,\ j=0,\dots,m-2;\ \ D_{0+}^{\beta_2}y(0)=0,\\ D_{0+}^{\delta_0}y(1)=\displaystyle\sum_{i=1}^{q}\int_0^1 D_{0+}^{\delta_i}y(t)\,\mathrm{d}K_i(t), \end{cases}
$$

where $\alpha_1,\ \alpha_2\in(0,1]$, $\beta_1\in(n-1,n]$, $\beta_2\in(m-1,m]$, $n,\ m\in\mathbb{N}$, $n,\ m\geq 3$, $p,q\in\mathbb{N}$, $\gamma_i\in\mathbb{R}$ for all $i=0,1,\dots,p$, $0\leq\gamma_1<\gamma_2<\cdots<\gamma_p\leq\gamma_0<\beta_1-1$, $\gamma_0\geq 1$, $\delta_i\in\mathbb{R}$ for all $i=0,1,\dots,q$, $0\leq\delta_1<\delta_2<\cdots<\delta_q\leq\delta_0<\beta_2-1$, $\delta_0\geq 1$, $r_1,\ r_2>1$, $\varphi_{r_i}(s)=|s|^{r_i-2}s$, $\varphi_{r_i}^{-1}=\varphi_{\varrho_i}$, $\frac{1}{r_i}+\frac{1}{\varrho_i}=1$, $i=1,2$, $\lambda,\ \mu>0$, $f,\ g\in C([0,1]\times[0,\infty)\times[0,\infty),[0,\infty))$, the integrals from (2) are Riemann-Stieltjes integrals with H_i, $i=1,\dots,p$ and K_i, $i=1,\cdots,q$ functions of bounded variation, and D_{0+}^{k} denotes the Riemann-Liouville derivative of order k (for $k=\alpha_1,\ \beta_1,\ \alpha_2,\ \beta_2,\ \gamma_i$ for $i=0,1,\dots,p$, δ_i for $i=0,1,\dots,q$).

Under some assumptions on the functions f and g, we present intervals for the parameters λ and μ such that positive solutions of problem (1),(2) exist. By a positive solution of problem (1),(2) we mean a pair of functions $(x,y)\in(C([0,1],[0,\infty)))^2$, satisfying the system (1) and the boundary conditions (2) with $x(t)>0$ for all $t\in(0,1]$, or $y(t)>0$ for all $t\in(0,1]$. The nonexistence of positive solutions for the above problem is also investigated. The problem (1),(2) is a generalization of the problem studied in [22]. Indeed, if $p=1$, $q=1$, $\gamma_0=p_1$, $\gamma_1=q_1$, $\delta_0=p_2$, $\delta_1=q_2$, H_1 is a step function given by $H_1(t)=\{0,\ t\in[0,\xi_1);\ a_1,\ t\in[\xi_1,\xi_2);\ a_1+a_2,\ t\in[\xi_2,\xi_3);\dots;\sum_{i=1}^{N}a_i,\ t\in[\xi_N,1]\}$, and K_1 is a step function given by $K_1(t)=\{0,\ t\in[0,\eta_1);\ b_1,\ t\in[\eta_1,\eta_2);\ b_1+b_2,\ t\in[\eta_2,\eta_3);\dots;\sum_{i=1}^{M}b_i,\ t\in[\eta_M,1]\}$, then the boundary conditions (2) become the multi-point boundary conditions (BC) from [22]. Systems with fractional differential equations without p-Laplacian operators, subject to various multi-point or Riemann-Stieltjes integral boundary conditions were studied in the last years in [10] – [18], [21], [23], [29], [31] – [33].

The paper is organized as follows. In Section 2, we investigate two nonlocal boundary value problems for fractional differential equations with p-Laplacian operators, and we present some properties of the associated Green functions. Section 3 contains the main existence theorems for the positive solutions with respect to a cone for our problem (1),(2), based on the Guo-Krasnosel'skii fixed point theorem (see [9]). In Section 4, we study the nonexistence of positive solutions of (1),(2), and in Section 5, an example is given to illustrate our results.

2. PRELIMINARY RESULTS

First we consider the nonlinear fractional differential equation

$$D_{0+}^{\alpha_1}(\varphi_{r_1}(D_{0+}^{\beta_1}x(t))) + h(t) = 0, \ \ t \in (0,1), \tag{3}$$

with the boundary conditions

$$\begin{cases} x^{(j)}(0) = 0, \ j = 0, \dots, n-2; \ \ D_{0+}^{\beta_1}x(0) = 0, \\ D_{0+}^{\gamma_0}x(1) = \displaystyle\sum_{i=1}^{p} \int_0^1 D_{0+}^{\gamma_i}x(t)\,\mathrm{d}H_i(t), \end{cases} \tag{4}$$

where $\alpha_1 \in (0,1]$, $\beta_1 \in (n-1,n]$, $n \in \mathbb{N}$, $n \geq 3$, $p \in \mathbb{N}$, $\gamma_i \in \mathbb{R}$ for all $i = 0,1,\dots,p$, $0 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_p \leq \gamma_0 < \beta_1 - 1$, $\gamma_0 \geq 1$, H_i, $i = 1,\dots,p$ are bounded variation functions, and $h \in C[0,1]$.

If we denote by $\varphi_{r_1}(D_{0+}^{\beta_1}x(t)) = u(t)$, then problem (3),(4) is equivalent to the following two boundary value problems

$$D_{0+}^{\alpha_1}u(t) + h(t) = 0, \ \ 0 < t < 1; \ \ u(0) = 0, \tag{5}$$

and

$$\begin{cases} D_{0+}^{\beta_1}x(t) = \varphi_{\varrho_1}u(t), \ \ 0 < t < 1; \\ x^{(j)}(0) = 0, \ j = 0, \dots, n-2; \ \ D_{0+}^{\gamma_0}x(1) = \displaystyle\sum_{i=1}^{p} \int_0^1 D_{0+}^{\gamma_i}x(t)\,\mathrm{d}H_i(t). \end{cases} \tag{6}$$

For the first problem (5), the function

$$u(t) = -I_{0+}^{\alpha_1}h(t) = -\frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}h(s)\,\mathrm{d}s, \ \ t \in [0,1], \tag{7}$$

is the unique solution $u \in C[0,1]$ of (5).

For the second problem (6), if

$$\Delta_1 = \frac{\Gamma(\beta_1)}{\Gamma(\beta_1-\gamma_0)} - \sum_{i=1}^{p} \frac{\Gamma(\beta_1)}{\Gamma(\beta_1-\gamma_i)} \times \int_0^1 s^{\beta_1-\gamma_i-1}\,\mathrm{d}H_i(s) \neq 0,$$

then by Lemma 2.2 from [1], we deduce that the function

$$x(t) = -\int_0^1 G_1(t,s)\varphi_{\varrho_1}u(s)\,\mathrm{d}s, \ \ t \in [0,1], \tag{8}$$

is the unique solution $x \in C[0,1]$ of problem (6). Here the Green function G_1 is given by

$$G_1(t,s) = g_1(t,s) + \frac{t^{\beta_1-1}}{\Delta_1}\sum_{i=1}^{p}\left(\int_0^1 g_{2i}(\tau,s)\,\mathrm{d}H_i(\tau)\right), \ \ t,s \in [0,1], \tag{9}$$

with

$$(10)\quad g_1(t,s)=\frac{1}{\Gamma(\beta_1)}\begin{cases} t^{\beta_1-1}(1-s)^{\beta_1-\gamma_0-1}-(t-s)^{\beta_1-1}, \\ \quad 0\le s\le t\le 1, \\ t^{\beta_1-1}(1-s)^{\beta_1-\gamma_0-1},\ \ 0\le t\le s\le 1, \end{cases}$$
$$g_{2i}(t,s)=\frac{1}{\Gamma(\beta_1-\gamma_i)}\begin{cases} t^{\beta_1-\gamma_i-1}(1-s)^{\beta_1-\gamma_0-1}-(t-s)^{\beta_1-\gamma_i-1}, \\ \quad 0\le s\le t\le 1, \\ t^{\beta_1-\gamma_i-1}(1-s)^{\beta_1-\gamma_0-1},\ \ 0\le t\le s\le 1, \end{cases}$$
$$i=1,\dots,p.$$

Therefore by (7) and (8) we obtain the following lemma.

LEMMA 1. *If* $\Delta_1\neq 0$*, then the function*

$$(11)\qquad x(t)=\int_0^1 G_1(t,s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1}h(s))\,\mathrm{d}s,\ \ t\in[0,1],$$

is the unique solution $x\in C[0,1]$ *of problem (3),(4).*

Next we consider the nonlinear fractional differential equation

$$(12)\qquad D_{0+}^{\alpha_2}(\varphi_{r_2}(D_{0+}^{\beta_2}y(t)))+k(t)=0,\ \ t\in(0,1),$$

with the boundary conditions

$$(13)\qquad \begin{cases} y^{(j)}(0)=0,\ j=0,\dots,m-2;\ \ D_{0+}^{\beta_2}y(0)=0, \\ D_{0+}^{\delta_0}y(1)=\displaystyle\sum_{i=1}^{q}\int_0^1 D_{0+}^{\delta_i}y(t)\,\mathrm{d}K_i(t), \end{cases}$$

where $\alpha_2\in(0,1]$, $\beta_2\in(m-1,m]$, $m\in\mathbb{N}$, $m\ge 3$, $q\in\mathbb{N}$, $\delta_i\in\mathbb{R}$ for all $i=0,\dots,q$, $0\le\delta_1<\delta_2<\dots<\delta_q\le\delta_0<\beta_2-1$, $\delta_0\ge 1$, K_i, $i=1,\dots,q$ are bounded variation functions, and $k\in C[0,1]$.

We denote by

$$\Delta_2=\frac{\Gamma(\beta_2)}{\Gamma(\beta_2-\delta_0)}-\sum_{i=1}^{q}\frac{\Gamma(\beta_2)}{\Gamma(\beta_2-\delta_i)}\int_0^1 s^{\beta_2-\delta_i-1}\,\mathrm{d}K_i(s),$$

and by G_2, g_3, g_{4i}, $i=1,\dots,q$ the following functions

$$(14)\qquad G_2(t,s)=g_3(t,s)+\frac{t^{\beta_2-1}}{\Delta_2}\sum_{i=1}^{q}\left(\int_0^1 g_{4i}(\tau,s)\,\mathrm{d}K_i(\tau)\right),\ \ t,s\in[0,1],$$

with

$$(15)\quad g_3(t,s)=\frac{1}{\Gamma(\beta_2)}\begin{cases} t^{\beta_2-1}(1-s)^{\beta_2-\delta_0-1}-(t-s)^{\beta_2-1}, \\ \quad 0\le s\le t\le 1, \\ t^{\beta_2-1}(1-s)^{\beta_2-\delta_0-1},\ \ 0\le t\le s\le 1, \end{cases}$$
$$g_{4i}(t,s)=\frac{1}{\Gamma(\beta_2-\delta_i)}\begin{cases} t^{\beta_2-\delta_i-1}(1-s)^{\beta_2-\delta_0-1}-(t-s)^{\beta_2-\delta_i-1}, \\ \quad 0\le s\le t\le 1, \\ t^{\beta_2-\delta_i-1}(1-s)^{\beta_2-\delta_0-1},\ \ 0\le t\le s\le 1. \end{cases}$$

for $i=1,\dots,q$. In a similar manner as above, we obtain the following result.

LEMMA 2. *If $\Delta_2 \neq 0$, then the function*

$$y(t)=\int_0^1 G_2(t,s)\varphi_{\varrho_2}(I_{0+}^{\alpha_2}k(s))\,\mathrm{d}s,\ \ t\in[0,1], \tag{16}$$

is the unique solution $y\in C[0,1]$ of problem (12),(13).

By using the properties of the functions g_1, g_{2i}, $i=1,\dots,p$, g_3, g_{4i}, $i=1,\dots,q$ given by (10) and (15) (see [1] and [12]), we obtain the following properties of the Green functions G_1 and G_2 that will be used in the next sections.

LEMMA 3. *Assume that $H_i:[0,1]\to\mathbb{R}$, $i=1,\dots,p$, and $K_i:[0,1]\to\mathbb{R}$, $i=1,\dots,q$ are nondecreasing functions and Δ_1, $\Delta_2>0$. Then the Green functions G_1 and G_2 given by (9) and (14) have the properties:*

a) *G_1, $G_2:[0,1]\times[0,1]\to[0,\infty)$ are continuous functions;*

b) *$G_1(t,s)\le J_1(s)$ for all t, $s\in[0,1]$, where $J_1(s)=h_1(s)+\frac{1}{\Delta_1}\sum_{i=1}^{p}\int_0^1 g_{2i}(\tau,s)\,\mathrm{d}H_i(\tau)$, and $h_1(s)=\frac{1}{\Gamma(\beta_1)}[(1-s)^{\beta_1-\gamma_0-1}-(1-s)^{\beta_1-1}]$, $s\in[0,1]$;*

c) *$G_1(t,s)\ge t^{\beta_1-1}J_1(s)$ for all t, $s\in[0,1]$;*

d) *$G_2(t,s)\le J_2(s)$ for all t, $s\in[0,1]$, where $J_2(s)=h_3(s)+\frac{1}{\Delta_2}\sum_{i=1}^{q}\int_0^1 g_{4i}(\tau,s)\,\mathrm{d}K_i(\tau)$, and $h_3(s)=\frac{1}{\Gamma(\beta_2)}[(1-s)^{\beta_2-\delta_0-1}-(1-s)^{\beta_2-1}]$, $s\in[0,1]$;*

e) *$G_2(t,s)\ge t^{\beta_2-1}J_2(s)$ for all t, $s\in[0,1]$.*

3. EXISTENCE OF POSITIVE SOLUTIONS

In this section we present sufficient conditions on the functions f, g, and intervals for the parameters λ, μ such that positive solutions with respect to a cone for our problem (1),(2) exist.

We present now the assumptions that we will use in the sequel.

(H1) α_1, $\alpha_2\in(0,1]$, $\beta_1\in(n-1,n]$, $\beta_2\in(m-1,m]$, n, $m\in\mathbb{N}$, n, $m\ge 3$, p, $q\in\mathbb{N}$, $\gamma_i\in\mathbb{R}$ for all $i=0,1,\dots,p$, $0\le\gamma_1<\gamma_2<\dots<\gamma_p\le\gamma_0<\beta_1-1$, $\gamma_0\ge 1$, $\delta_i\in\mathbb{R}$ for all $i=0,1,\dots,q$, $0\le\delta_1<\delta_2<\dots<\delta_q\le\delta_0<\beta_2-1$, $\delta_0\ge 1$, H_i, $i=1,\dots,p$, K_i, $i=1,\dots,q$ are nondecreasing functions, λ, $\mu>0$, $\Delta_1>0$, $\Delta_2>0$, $r_i>1$, $\varphi_{r_i}(s)=|s|^{r_i-2}s$, $\varphi_{r_i}^{-1}=\varphi_{\varrho_i}$, $\varrho_i=\frac{r_i}{r_i-1}$, $i=1,2$.

(H2) The functions f, $g:[0,1]\times[0,\infty)\times[0,\infty)\to[0,\infty)$ are continuous.

For $[c_1, c_2] \subset [0,1]$ with $0 < c_1 < c_2 \le 1$, we introduce the following extreme limits

$$f_0^s = \limsup_{\substack{x+y\to 0+\\ x,y\ge 0}} \max_{t\in[0,1]} \frac{f(t,x,y)}{(x+y)^{r_1-1}},\quad g_0^s = \limsup_{\substack{x+y\to 0+\\ x,y\ge 0}} \max_{t\in[0,1]} \frac{g(t,x,y)}{(x+y)^{r_2-1}},$$
$$f_0^i = \liminf_{\substack{x+y\to 0+\\ x,y\ge 0}} \min_{t\in[c_1,c_2]} \frac{f(t,x,y)}{(x+y)^{r_1-1}},\quad g_0^i = \liminf_{\substack{x+y\to 0+\\ x,y\ge 0}} \min_{t\in[c_1,c_2]} \frac{g(t,x,y)}{(x+y)^{r_2-1}},$$
$$f_\infty^s = \limsup_{\substack{x+y\to \infty\\ x,y\ge 0}} \max_{t\in[0,1]} \frac{f(t,x,y)}{(x+y)^{r_1-1}},\quad g_\infty^s = \limsup_{\substack{x+y\to \infty\\ x,y\ge 0}} \max_{t\in[0,1]} \frac{g(t,x,y)}{(x+y)^{r_2-1}},$$
$$f_\infty^i = \liminf_{\substack{x+y\to \infty\\ x,y\ge 0}} \min_{t\in[c_1,c_2]} \frac{f(t,x,y)}{(x+y)^{r_1-1}},\quad g_\infty^i = \liminf_{\substack{x+y\to \infty\\ x,y\ge 0}} \min_{t\in[c_1,c_2]} \frac{g(t,x,y)}{(x+y)^{r_2-1}}.$$

By using Lemma 1 and Lemma 2 (the relations (11) and (16)), (x, y) is a solution of the following nonlinear system of integral equations

$$\begin{cases} x(t) = \lambda^{\varrho_1-1} \displaystyle\int_0^1 G_1(t,s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1} f(s,x(s),y(s)))\,\mathrm{d}s, & t\in[0,1],\\ y(t) = \mu^{\varrho_2-1} \displaystyle\int_0^1 G_2(t,s)\varphi_{\varrho_2}(I_{0+}^{\alpha_2} g(s,x(s),y(s)))\,\mathrm{d}s, & t\in[0,1], \end{cases}$$

if and only if (x, y) is a solution of problem (1),(2).

We consider the Banach space $X = C[0,1]$ with the supremum norm $\|\cdot\|$, and the Banach space $Y = X \times X$ with the norm $\|(x,y)\|_Y = \|x\| + \|y\|$. We define the cones

$$P_1 = \{x \in X,\ \ x(t) \ge t^{\beta_1-1}\|x\|,\ \ \forall\, t\in[0,1]\} \subset X,$$
$$P_2 = \{y \in X,\ \ y(t) \ge t^{\beta_2-1}\|y\|,\ \ \forall\, t\in[0,1]\} \subset X,$$

and $P = P_1 \times P_2 \subset Y$.

We define now the operators Q_1, $Q_2 : Y \to X$ and $Q : Y \to Y$ by

$$Q_1(x,y)(t) = \lambda^{\varrho_1-1}\int_0^1 G_1(t,s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1} f(s,x(s),y(s)))\,\mathrm{d}s,\ \ t\in[0,1],$$
$$Q_2(x,y)(t) = \mu^{\varrho_2-1}\int_0^1 G_2(t,s)\varphi_{\varrho_2}(I_{0+}^{\alpha_2} g(s,x(s),y(s)))\,\mathrm{d}s,\ \ t\in[0,1],$$

and $Q(x,y) = (Q_1(x,y), Q_2(x,y))$, $(x,y) \in Y$. Then (x,y) is a solution of problem (1),(2) if and only if (x,y) is a fixed point of operator Q.

In a similar manner as we proved Lemma 3.1 from [22], we obtain the next lemma.

Lemma 4. *If* $(H1) - (H2)$ *hold, then* $Q : P \to P$ *is a completely continuous operator.*

For $[c_1, c_2] \subset [0,1]$ with $0 < c_1 < c_2 \le 1$, we denote by

$$A = \frac{1}{(\Gamma(\alpha_1+1))^{\varrho_1-1}} \int_{c_1}^{c_2} (s-c_1)^{\alpha_1(\varrho_1-1)} J_1(s)\,\mathrm{d}s,$$
$$B = \frac{1}{(\Gamma(\alpha_1+1))^{\varrho_1-1}} \int_0^1 s^{\alpha_1(\varrho_1-1)} J_1(s)\,\mathrm{d}s,$$
$$C = \frac{1}{(\Gamma(\alpha_2+1))^{\varrho_2-1}} \int_{c_1}^{c_2} (s-c_1)^{\alpha_2(\varrho_2-1)} J_2(s)\,\mathrm{d}s,$$
$$D = \frac{1}{(\Gamma(\alpha_2+1))^{\varrho_2-1}} \int_0^1 s^{\alpha_2(\varrho_2-1)} J_2(s)\,\mathrm{d}s,$$

where J_1 and J_2 are defined in Lemma 3.

First, for f_0^s, g_0^s, f_∞^i, $g_\infty^i \in (0,\infty)$ and numbers α_1', $\alpha_2' \ge 0$, $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2 > 0$ such that $\alpha_1' + \alpha_2' = 1$ and $\widetilde{\alpha}_1 + \widetilde{\alpha}_2 = 1$, we define the numbers L_1, L_2, L_3, L_4, L_2', L_4' by

$$L_1 = \frac{1}{f_\infty^i}\left(\frac{\alpha_1'}{\gamma\gamma_1 A}\right)^{r_1-1}, \quad L_2 = \frac{1}{f_0^s}\left(\frac{\widetilde{\alpha}_1}{B}\right)^{r_1-1}, \quad L_3 = \frac{1}{g_\infty^i}\left(\frac{\alpha_2'}{\gamma\gamma_2 C}\right)^{r_2-1},$$
$$L_4 = \frac{1}{g_0^s}\left(\frac{\widetilde{\alpha}_2}{D}\right)^{r_2-1}, \quad L_2' = \frac{1}{f_0^s B^{r_1-1}}, \quad L_4' = \frac{1}{g_0^s D^{r_2-1}},$$

where $\gamma_1 = c_1^{\beta_1-1}$, $\gamma_2 = c_1^{\beta_2-1}$, $\gamma = \min\{\gamma_1, \gamma_2\}$.

THEOREM 1. *Assume that* $(H1)$ *and* $(H2)$ *hold,* $[c_1, c_2] \subset [0,1]$ *with* $0 < c_1 < c_2 \le 1$, α_1', $\alpha_2' \ge 0$, $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2 > 0$ *such that* $\alpha_1' + \alpha_2' = 1$, $\widetilde{\alpha}_1 + \widetilde{\alpha}_2 = 1$.

1) *If* f_0^s, g_0^s, f_∞^i, $g_\infty^i \in (0,\infty)$, $L_1 < L_2$ *and* $L_3 < L_4$, *then for each* $\lambda \in (L_1, L_2)$ *and* $\mu \in (L_3, L_4)$ *there exists a positive solution* $(x(t), y(t))$, $t \in [0,1]$ *for* (1),(2).

2) *If* $f_0^s = 0$, g_0^s, f_∞^i, $g_\infty^i \in (0,\infty)$ *and* $L_3 < L_4'$, *then for each* $\lambda \in (L_1, \infty)$ *and* $\mu \in (L_3, L_4')$ *there exists a positive solution* $(x(t), y(t))$, $t \in [0,1]$ *for* (1),(2).

3) *If* $g_0^s = 0$, f_0^s, f_∞^i, $g_\infty^i \in (0,\infty)$ *and* $L_1 < L_2'$, *then for each* $\lambda \in (L_1, L_2')$ *and* $\mu \in (L_3, \infty)$ *there exists a positive solution* $(x(t), y(t))$, $t \in [0,1]$ *for* (1),(2).

4) *If* $f_0^s = g_0^s = 0$, f_∞^i, $g_\infty^i \in (0,\infty)$, *then for each* $\lambda \in (L_1, \infty)$ *and* $\mu \in (L_3, \infty)$ *there exists a positive solution* $(x(t), y(t))$, $t \in [0,1]$ *for* (1),(2).

5) *If* f_0^s, $g_0^s \in (0,\infty)$ *and at least one of* f_∞^i, g_∞^i *is* ∞, *then for each* $\lambda \in (0, L_2)$ *and* $\mu \in (0, L_4)$ *there exists a positive solution* $(x(t), y(t))$, $t \in [0,1]$ *for* (1),(2).

6) *If* $f_0^s = 0$, $g_0^s \in (0,\infty)$ *and at least one of* f_∞^i, g_∞^i *is* ∞, *then for each* $\lambda \in (0,\infty)$ *and* $\mu \in (0, L_4')$ *there exists a positive solution* $(x(t), y(t))$, $t \in [0,1]$ *for* (1),(2).

7) *If* $f_0^s \in (0,\infty)$, $g_0^s = 0$ *and at least one of* f_∞^i, g_∞^i *is* ∞, *then for each* $\lambda \in (0, L_2')$ *and* $\mu \in (0,\infty)$ *there exists a positive solution* $(x(t), y(t))$, $t \in [0,1]$ *for* (1),(2).

8) *If* $f_0^s = g_0^s = 0$ *and at least one of* f_∞^i, g_∞^i *is* ∞, *then for each* $\lambda \in (0,\infty)$ *and* $\mu \in (0,\infty)$ *there exists a positive solution* $(x(t), y(t))$, $t \in [0,1]$ *for* (1),(2).

Proof. We consider the above cone $P \subset Y$ and the operators Q_1, Q_2 and Q. Because the proofs of the above cases are similar, in what follows we will prove one of them, namely Case 2). We have $f_0^s = 0$, $g_0^s, f_\infty^i, g_\infty^i \in (0,\infty)$ and $L_3 < L_4'$. Let $\lambda \in (L_1, \infty)$ and $\mu \in (L_3, L_4')$. We consider the numbers $\widetilde{\alpha}_2' \in (D(\mu g_0^s)^{\varrho_2 - 1}, 1)$ and $\widetilde{\alpha}_1' = 1 - \widetilde{\alpha}_2'$. The choise of $\widetilde{\alpha}_2'$ is possible because $\mu < 1/(g_0^s D^{r_2-1})$. Let $\varepsilon > 0$ such that $\varepsilon < f_\infty^i$, $\varepsilon < g_\infty^i$ and

$$\frac{1}{f_\infty^i - \varepsilon}\left(\frac{\alpha_1'}{\gamma\gamma_1 A}\right)^{r_1-1} \le \lambda \le \frac{1}{\varepsilon}\left(\frac{\widetilde{\alpha}_1'}{B}\right)^{r_1-1},$$
$$\frac{1}{g_\infty^i - \varepsilon}\left(\frac{\alpha_2'}{\gamma\gamma_2 C}\right)^{r_2-1} \le \mu \le \frac{1}{g_0^s + \varepsilon}\left(\frac{\widetilde{\alpha}_2'}{D}\right)^{r_2-1}.$$

By using (H2) and the definitions of f_0^s and g_0^s, we deduce that there exists $R_1 > 0$ such that

$$f(t,x,y) \le \varepsilon(x+y)^{r_1-1}, \ g(t,x,y) \le (g_0^s + \varepsilon)(x+y)^{r_2-1},$$

for all $t \in [0,1]$ and $x, y \ge 0$, $x + y \le R_1$.

We define the set $\Omega_1 = \{(x,y) \in Y, \ \|(x,y)\|_Y < R_1\}$. Now let $(x,y) \in P \cap \partial\Omega_1$, that is $(x,y) \in P$ with $\|(x,y)\|_Y = R_1$ or equivalently $\|x\| + \|y\| = R_1$. Then $x(t) + y(t) \le R_1$ for all $t \in [0,1]$, and by Lemma 3, we obtain

$$\begin{aligned}
&Q_1(x,y)(t) \le \lambda^{\varrho_1-1}\int_0^1 J_1(s)\varphi_{\varrho_1}\left(\frac{1}{\Gamma(\alpha_1)}\int_0^s (s-\tau)^{\alpha_1-1} f(\tau, x(\tau), y(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\le \lambda^{\varrho_1-1}\int_0^1 J_1(s)\varphi_{\varrho_1}\left(\frac{1}{\Gamma(\alpha_1)}\int_0^s (s-\tau)^{\alpha_1-1}\varepsilon(x(\tau)+y(\tau))^{r_1-1}\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\le \lambda^{\varrho_1-1}\varepsilon^{\varrho_1-1}\int_0^1 J_1(s)\varphi_{\varrho_1}\left(\frac{1}{\Gamma(\alpha_1)}\int_0^s (s-\tau)^{\alpha_1-1}(\|x\|+\|y\|)^{r_1-1}\,\mathrm{d}\tau\right)\mathrm{d}s \\
&= \lambda^{\varrho_1-1}\varepsilon^{\varrho_1-1}(\|x\|+\|y\|)\int_0^1 J_1(s)\varphi_{\varrho_1}\left(\frac{1}{\Gamma(\alpha_1)}\int_0^s (s-\tau)^{\alpha_1-1}\,\mathrm{d}\tau\right)\mathrm{d}s \\
&= \lambda^{\varrho_1-1}\varepsilon^{\varrho_1-1}\|(x,y)\|_Y\int_0^1 J_1(s)\varphi_{\varrho_1}\left(\frac{s^{\alpha_1}}{\Gamma(\alpha_1+1)}\right)\mathrm{d}s \\
&= \lambda^{\varrho_1-1}\varepsilon^{\varrho_1-1}\|(x,y)\|_Y\int_0^1 J_1(s)\frac{1}{(\Gamma(\alpha_1+1))^{\varrho_1-1}}s^{\alpha_1(\varrho_1-1)}\,\mathrm{d}s \\
&= \lambda^{\varrho_1-1}\varepsilon^{\varrho_1-1}B\|(x,y)\|_Y \le \widetilde{\alpha}_1'\|(x,y)\|_Y, \ \forall t \in [0,1].
\end{aligned}$$

Therefore we have $\|Q_1(x,y)\| \le \widetilde{\alpha}_1'\|(x,y)\|_Y$.

In a similar manner, we conclude

$$Q_2(x,y)(t) \le \mu^{\varrho_2-1}(g_0^s+\varepsilon)^{\varrho_2-1}D\|(x,y)\|_Y \le \widetilde{\alpha}_2'\|(x,y)\|_Y, \ \ \forall\, t\in[0,1].$$

Hence we get $\|Q_2(x,y)\| \le \widetilde{\alpha}_2'\|(x,y)\|_Y$.

Then for $(x,y)\in P\cap\partial\Omega_1$, we deduce

(17)
$$\|Q(x,y)\|_Y = \|Q_1(x,y)\| + \|Q_2(x,y)\| \le \widetilde{\alpha}_1'\|(x,y)\|_Y + \widetilde{\alpha}_2'\|(x,y)\|_Y = \|(x,y)\|_Y.$$

Next, by the definitions of f_∞^i and g_∞^i there exists $\overline{R}_2>0$ such that

$$f(t,x,y) \ge (f_\infty^i-\varepsilon)(x+y)^{r_1-1}, \ \ g(t,x,y) \ge (g_\infty^i-\varepsilon)(x+y)^{r_2-1},$$

for all $t\in[c_1,c_2]$ and $x,y\ge 0$, $x+y\ge\overline{R}_2$.

We consider $R_2=\max\{2R_1,\overline{R}_2/\gamma\}$ and we define the set $\Omega_2=\{(x,y)\in Y,\ \|(x,y)\|_Y<R_2\}$. Then for $(x,y)\in P\cap\partial\Omega_2$, we obtain

$$\begin{aligned}&x(t)+y(t) \ge \min\nolimits_{t\in[c_1,c_2]} t^{\beta_1-1}\|x\| + \min\nolimits_{t\in[c_1,c_2]} t^{\beta_2-1}\|y\|\\ &= c_1^{\beta_1-1}\|x\| + c_1^{\beta_2-1}\|y\| = \gamma_1\|x\|+\gamma_2\|y\| \ge \gamma\|(x,y)\|_Y\\ &= \gamma R_2 \ge \overline{R}_2, \ \ \forall\, t\in[c_1,c_2].\end{aligned}$$

Then, by Lemma 3, we conclude

$$\begin{aligned}&Q_1(x,y)(c_1) \ge \lambda^{\varrho_1-1}\int_0^1 c_1^{\beta_1-1}J_1(s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1}f(s,x(s),y(s)))\,\mathrm{d}s\\
&\ge \lambda^{\varrho_1-1}\gamma_1\int_{c_1}^{c_2} J_1(s)\varphi_{\varrho_1}\bigg(\frac{1}{\Gamma(\alpha_1)}\int_{c_1}^s (s-\tau)^{\alpha_1-1}f(\tau,x(\tau),y(\tau))\,\mathrm{d}\tau\bigg)\mathrm{d}s\\
&\ge \lambda^{\varrho_1-1}\gamma_1\int_{c_1}^{c_2} J_1(s)\varphi_{\varrho_1}\bigg(\frac{1}{\Gamma(\alpha_1)}\int_{c_1}^s (s-\tau)^{\alpha_1-1}(f_\infty^i-\varepsilon)(x(\tau)+y(\tau))^{r_1-1}\,\mathrm{d}\tau\bigg)\mathrm{d}s\\
&\ge \lambda^{\varrho_1-1}\gamma_1\int_{c_1}^{c_2} J_1(s)\varphi_{\varrho_1}\bigg(\frac{1}{\Gamma(\alpha_1)}\int_{c_1}^s (s-\tau)^{\alpha_1-1}(f_\infty^i-\varepsilon)(\gamma\|(x,y)\|_Y)^{r_1-1}\mathrm{d}\tau\bigg)\mathrm{d}s\\
&= \lambda^{\varrho_1-1}\gamma_1(f_\infty^i-\varepsilon)^{\varrho_1-1}\gamma\|(x,y)\|_Y\int_{c_1}^{c_2} J_1(s)\varphi_{\varrho_1}\bigg(\frac{1}{\Gamma(\alpha_1)}\int_{c_1}^s (s-\tau)^{\alpha_1-1}\,\mathrm{d}\tau\bigg)\mathrm{d}s\\
&= \lambda^{\varrho_1-1}\gamma_1(f_\infty^i-\varepsilon)^{\varrho_1-1}\gamma\|(x,y)\|_Y\int_{c_1}^{c_2} J_1(s)\varphi_{\varrho_1}\bigg(\frac{(s-c_1)^{\alpha_1}}{\Gamma(\alpha_1+1)}\bigg)\mathrm{d}s\\
&= \gamma\gamma_1\lambda^{\varrho_1-1}(f_\infty^i-\varepsilon)^{\varrho_1-1}\|(x,y)\|_Y\int_{c_1}^{c_2} J_1(s)\frac{1}{(\Gamma(\alpha_1+1))^{\varrho_1-1}}(s-c_1)^{\alpha_1(\varrho_1-1)}\mathrm{d}s\\
&= \gamma\gamma_1\lambda^{\varrho_1-1}(f_\infty^i-\varepsilon)^{\varrho_1-1}A\|(x,y)\|_Y \ge \alpha_1'\|(x,y)\|_Y.\end{aligned}$$

Therefore we obtain $\|Q_1(x,y)\| \ge Q_1(x,y)(c_1) \ge \alpha_1'\|(x,y)\|_Y$.

In a similar manner, we deduce

$$Q_2(x,y)(c_1) \ge \gamma\gamma_2\mu^{\varrho_2-1}(g_\infty^i-\varepsilon)^{\varrho_2-1}C\|(x,y)\|_Y \ge \alpha_2'\|(x,y)\|_Y,$$

and then $\|Q_2(x,y)\| \ge Q_2(x,y)(c_1) \ge \alpha_2'\|(x,y)\|_Y$.

Then for $(x,y) \in P \cap \partial\Omega_2$, we obtain

(18) $\|Q(x,y)\|_Y = \|Q_1(x,y)\| + \|Q_2(x,y)\| \geq (\alpha_1' + \alpha_2')\|(x,y)\|_Y = \|(x,y)\|_Y.$

By using Lemma 4, the relations (17), (18) and the Guo-Krasnosel'skii fixed point theorem, we conclude that the operator Q has a fixed point $(x,y) \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$, so $x(t) \geq t^{\beta_1-1}\|x\|$, $y(t) \geq t^{\beta_2-1}\|y\|$ for all $t \in [0,1]$ and $R_1 \leq \|x\| + \|y\| \leq R_2$. If $\|x\| > 0$ then $x(t) > 0$ for all $t \in (0,1]$, and if $\|y\| > 0$ then $y(t) > 0$ for all $t \in (0,1]$. Hence $(x(t), y(t)),\ t \in [0,1]$ is a positive solution of problem (1),(2). □

In what follows, for f_0^i, g_0^i, f_∞^s, $g_\infty^s \in (0,\infty)$ and numbers α_1', $\alpha_2' \geq 0$, $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2 > 0$ such that $\alpha_1' + \alpha_2' = 1$ and $\widetilde{\alpha}_1 + \widetilde{\alpha}_2 = 1$, we define the numbers $\widetilde{L}_1$, $\widetilde{L}_2$, $\widetilde{L}_3$, $\widetilde{L}_4$, $\widetilde{L}_2'$ and $\widetilde{L}_4'$ by

$$\widetilde{L}_1 = \frac{1}{f_0^i}\left(\frac{\alpha_1'}{\gamma\gamma_1 A}\right)^{r_1-1}, \quad \widetilde{L}_2 = \frac{1}{f_\infty^s}\left(\frac{\widetilde{\alpha}_1}{B}\right)^{r_1-1}, \quad \widetilde{L}_3 = \frac{1}{g_0^i}\left(\frac{\alpha_2'}{\gamma\gamma_2 C}\right)^{r_2-1},$$
$$\widetilde{L}_4 = \frac{1}{g_\infty^s}\left(\frac{\widetilde{\alpha}_2}{D}\right)^{r_2-1}, \quad \widetilde{L}_2' = \frac{1}{f_\infty^s B^{r_1-1}}, \quad \widetilde{L}_4' = \frac{1}{g_\infty^s D^{r_2-1}}.$$

THEOREM 2. *Assume that* $(H1)$ *and* $(H2)$ *hold,* $[c_1,c_2] \subset [0,1]$ *with* $0 < c_1 < c_2 \leq 1$, α_1', $\alpha_2' \geq 0$, $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2 > 0$ *such that* $\alpha_1' + \alpha_2' = 1$, $\widetilde{\alpha}_1 + \widetilde{\alpha}_2 = 1$.

1) If f_0^i, g_0^i, f_∞^s, $g_\infty^s \in (0,\infty)$, $\widetilde{L}_1 < \widetilde{L}_2$ *and* $\widetilde{L}_3 < \widetilde{L}_4$, *then for each* $\lambda \in (\widetilde{L}_1, \widetilde{L}_2)$ *and* $\mu \in (\widetilde{L}_3, \widetilde{L}_4)$ *there exists a positive solution* $(x(t), y(t)),\ t \in [0,1]$ *for* (1),(2).

2) If f_0^i, g_0^i, $f_\infty^s \in (0,\infty)$, $g_\infty^s = 0$ *and* $\widetilde{L}_1 < \widetilde{L}_2'$, *then for each* $\lambda \in (\widetilde{L}_1, \widetilde{L}_2')$ *and* $\mu \in (\widetilde{L}_3, \infty)$ *there exists a positive solution* $(x(t), y(t)),\ t \in [0,1]$ *for* (1),(2).

3) If f_0^i, g_0^i, $g_\infty^s \in (0,\infty)$, $f_\infty^s = 0$ *and* $\widetilde{L}_3 < \widetilde{L}_4'$, *then for each* $\lambda \in (\widetilde{L}_1, \infty)$ *and* $\mu \in (\widetilde{L}_3, \widetilde{L}_4')$ *there exists a positive solution* $(x(t), y(t)),\ t \in [0,1]$ *for* (1),(2).

4) If f_0^i, $g_0^i \in (0,\infty)$, $f_\infty^s = g_\infty^s = 0$, *then for each* $\lambda \in (\widetilde{L}_1, \infty)$ *and* $\mu \in (\widetilde{L}_3, \infty)$ *there exists a positive solution* $(x(t), y(t)),\ t \in [0,1]$ *for* (1),(2).

5) If f_∞^s, $g_\infty^s \in (0,\infty)$ *and at least one of* f_0^i, g_0^i *is* ∞, *then for each* $\lambda \in (0, \widetilde{L}_2)$ *and* $\mu \in (0, \widetilde{L}_4)$ *there exists a positive solution* $(x(t), y(t)),\ t \in [0,1]$ *for* (1),(2).

6) If $f_\infty^s \in (0,\infty)$, $g_\infty^s = 0$ *and at least one of* f_0^i, g_0^i *is* ∞, *then for each* $\lambda \in (0, \widetilde{L}_2')$ *and* $\mu \in (0,\infty)$ *there exists a positive solution* $(x(t), y(t)),\ t \in [0,1]$ *for* (1),(2).

7) If $f_\infty^s = 0$, $g_\infty^s \in (0,\infty)$ *and at least one of* f_0^i, g_0^i *is* ∞, *then for each* $\lambda \in (0,\infty)$ *and* $\mu \in (0, \widetilde{L}_4')$ *there exists a positive solution* $(x(t), y(t)),\ t \in [0,1]$ *for* (1),(2).

8) *If* $f_\infty^s = g_\infty^s = 0$ *and at least one of* f_0^i, g_0^i *is* ∞, *then for each* $\lambda \in (0,\infty)$ *and* $\mu \in (0,\infty)$ *there exists a positive solution* $(x(t), y(t))$, $t \in [0,1]$ *for* (1),(2).

Proof. We consider the cone $P \subset Y$ and the operators Q_1, Q_2 and Q defined at the beginning of this section. Because the proofs of the above cases are similar, in what follows we will prove one of them, namely Case 6). We consider $f_\infty^s \in (0,\infty)$, $g_\infty^s = 0$, and $f_0^i = \infty$. Let $\lambda \in (0, \widetilde{L}_2')$ and $\mu \in (0,\infty)$. We choose $\widetilde{\alpha}_1' \in (B(\lambda f_\infty^s)^{\varrho_1 - 1}, 1)$ and $\widetilde{\alpha}_2' = 1 - \widetilde{\alpha}_1'$, and let $\varepsilon > 0$ such that

$$\varepsilon \left(\frac{1}{\gamma\gamma_1 A}\right)^{r_1-1} \le \lambda \le \frac{1}{f_\infty^s + \varepsilon}\left(\frac{\widetilde{\alpha}_1'}{B}\right)^{r_1-1}, \quad \mu \le \frac{1}{\varepsilon}\left(\frac{\widetilde{\alpha}_2'}{D}\right)^{r_2-1}.$$

By (H2) and the definition of f_0^i we deduce that there exists $R_3 > 0$ such that

$$f(t,x,y) \ge \frac{1}{\varepsilon}(x+y)^{r_1-1}, \ \forall\, t \in [c_1, c_2], \ x, \ y \ge 0, \ x+y \le R_3.$$

We denote by $\Omega_3 = \{(x,y) \in Y, \ \|(x,y)\|_Y < R_3\}$. Let $(x,y) \in P$ with $\|(x,y)\|_Y = R_3$, that is $\|x\| + \|y\| = R_3$. Because $x(t) + y(t) \le \|x\| + \|y\| = R_3$ for all $t \in [0,1]$, then by Lemma 3 we obtain

$$\begin{aligned}
&Q_1(x,y)(c_1) \ge \lambda^{\varrho_1-1}\int_0^1 c_1^{\beta_1-1} J_1(s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1} f(s,x(s),y(s)))\,\mathrm{d}s \\
&\ge \lambda^{\varrho_1-1} c_1^{\beta_1-1}\int_{c_1}^{c_2} J_1(s)\varphi_{\varrho_1}\left(\frac{1}{\Gamma(\alpha_1)}\int_{c_1}^{s}(s-\tau)^{\alpha_1-1} f(\tau,x(\tau),y(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\ge \lambda^{\varrho_1-1} c_1^{\beta_1-1}\int_{c_1}^{c_2} J_1(s)\varphi_{\varrho_1}\left(\frac{1}{\Gamma(\alpha_1)}\int_{c_1}^{s}(s-\tau)^{\alpha_1-1}\frac{1}{\varepsilon}(x(\tau)+y(\tau))^{r_1-1}\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\ge \lambda^{\varrho_1-1} c_1^{\beta_1-1}\int_{c_1}^{c_2} J_1(s)\varphi_{\varrho_1}\left(\frac{1}{\Gamma(\alpha_1)}\int_{c_1}^{s}(s-\tau)^{\alpha_1-1}\frac{1}{\varepsilon}(\gamma\|(x,y)\|_Y)^{r_1-1}\,\mathrm{d}\tau\right)\mathrm{d}s \\
&= \lambda^{\varrho_1-1} c_1^{\beta_1-1}\left(\tfrac{1}{\varepsilon}\right)^{\varrho_1-1}\gamma\|(x,y)\|_Y\int_{c_1}^{c_2} J_1(s)\frac{1}{(\Gamma(\alpha_1+1))^{\varrho_1-1}}(s-c_1)^{\alpha_1(\varrho_1-1)}\,\mathrm{d}s \\
&= \gamma\gamma_1\lambda^{\varrho_1-1}\left(\tfrac{1}{\varepsilon}\right)^{\varrho_1-1} A\|(x,y)\|_Y \ge \|(x,y)\|_Y.
\end{aligned}$$

Hence we get $\|Q_1(x,y)\| \ge Q_1(x,y)(c_1) \ge \|(x,y)\|_Y$ and then

$$\|Q(x,y)\|_Y \ge \|Q_1(x,y)\| \ge \|(x,y)\|_Y. \tag{19}$$

For the second part of the proof, we consider the functions f^*, g^* : $[0,1] \times [0,\infty) \to [0,\infty)$ defined by $f^*(t,u) = \max_{0 \le x+y \le u} f(t,x,y)$, $g^*(t,u) = \max_{0 \le x+y \le u} g(t,x,y)$, for all $t \in [0,1]$ and $u \in [0,\infty)$. Then

$$f(t,x,y) \le f^*(t,u), \ g(t,x,y) \le g^*(t,u), \ \forall\, t \in [0,1], \ x, y \ge 0, \ x+y \le u.$$

The functions $f^*(t,\cdot)$, $g^*(t,\cdot)$ are nondecreasing for every $t \in [0,1]$ and they satisfy the conditions

$$\limsup_{u\to\infty} \max_{t\in[0,1]} \frac{f^*(t,u)}{u^{r_1-1}} = f_\infty^s, \quad \lim_{u\to\infty} \max_{t\in[0,1]} \frac{g^*(t,u)}{u^{r_2-1}} = 0.$$

Therefore, for $\varepsilon > 0$ there exists $\overline{R}_4 > 0$ such that for all $u \geq \overline{R}_4$ and $t \in [0,1]$ we have

$$\frac{f^*(t,u)}{u^{r_1-1}} \leq \limsup_{u\to\infty} \max_{t\in[0,1]} \frac{f^*(t,u)}{u^{r_1-1}} + \varepsilon = f^s_\infty + \varepsilon,$$
$$\frac{g^*(t,u)}{u^{r_2-1}} \leq \lim_{u\to\infty} \max_{t\in[0,1]} \frac{g^*(t,u)}{u^{r_2-1}} + \varepsilon = \varepsilon,$$

and so $f^*(t,u) \leq (f^s_\infty + \varepsilon)u^{r_1-1}$ and $g^*(t,u) \leq \varepsilon u^{r_2-1}$.

We consider $R_4 = \max\{2R_3, \overline{R}_4\}$ and we denote by

$$\Omega_4 = \{(x,y) \in Y,\ \|(x,y)\|_Y < R_4\}.$$

Let $(x,y) \in P \cap \partial\Omega_4$. By the definitions of f^* and g^* we conclude

$$f(t,x(t),y(t)) \leq f^*(t, \|(x,y)\|_Y),\ \ g(t,x(t),y(t)) \leq g^*(t, \|(x,y)\|_Y),\ \ \forall\, t \in [0,1].$$

Then for all $t \in [0,1]$ we obtain

$$\begin{aligned}
&Q_1(x,y)(t) \leq \lambda^{\varrho_1-1} \int_0^1 J_1(s)\varphi_{\varrho_1}\left(\frac{1}{\Gamma(\alpha_1)} \int_0^s (s-\tau)^{\alpha_1-1} f(\tau, x(\tau), y(\tau))\, \mathrm{d}\tau\right) \mathrm{d}s \\
&\leq \lambda^{\varrho_1-1} \int_0^1 J_1(s)\varphi_{\varrho_1}\left(\frac{1}{\Gamma(\alpha_1)} \int_0^s (s-\tau)^{\alpha_1-1} f^*(\tau, \|(x,y)\|_Y)\, \mathrm{d}\tau\right) \mathrm{d}s \\
&\leq \lambda^{\varrho_1-1} \int_0^1 J_1(s)\varphi_{\varrho_1}\left(\frac{1}{\Gamma(\alpha_1)} (f^s_\infty + \varepsilon)\|(x,y)\|_Y^{r_1-1} \left(\int_0^s (s-\tau)^{\alpha_1-1}\, \mathrm{d}\tau\right)\right) \mathrm{d}s \\
&= \lambda^{\varrho_1-1}(f^s_\infty + \varepsilon)^{\varrho_1-1} B\|(x,y)\|_Y \leq \widetilde{\alpha}'_1 \|(x,y)\|_Y,\ \ \forall\, t \in [0,1],
\end{aligned}$$

and so $\|Q_1(x,y)\| \leq \widetilde{\alpha}'_1 \|(x,y)\|_Y$.

In a similar manner, we deduce

$$Q_2(x,y)(t) \leq \mu^{\varrho_2-1}\varepsilon^{\varrho_2-1} D\|(x,y)\|_Y \leq \widetilde{\alpha}'_2 \|(x,y)\|_Y,\ \ \forall\, t \in [0,1],$$

and then $\|Q_2(x,y)\| \leq \widetilde{\alpha}'_2 \|(x,y)\|_Y$.

Therefore for $(x,y) \in P \cap \partial\Omega_2$ it follows that

$$\|Q(x,y)\|_Y = \|Q_1(x,y)\| + \|Q_2(x,y)\| \leq (\widetilde{\alpha}'_1 + \widetilde{\alpha}'_2)\|(x,y)\|_Y = \|(x,y)\|_Y. \tag{20}$$

By using Lemma 4, the relations (19), (20) and the Guo-Krasnosel'skii fixed point theorem, we conclude that Q has a fixed point $(x,y) \in P \cap (\overline{\Omega}_4 \setminus \Omega_3)$, which is a positive solution for problem (1),(2). □

4. NONEXISTENCE OF POSITIVE SOLUTIONS

In this section we present intervals for λ and μ for which there exist no positive solutions of problem (1),(2). By using similar arguments as those used in the proofs of Theorems 4.1-4.4 from [22], we obtain the following theorems for our problem (1),(2).

THEOREM 3. *Assume that (H1) and (H2) hold. If there exist positive numbers M_1, M_2 such that*

(21)
$$f(t,x,y) \le M_1(x+y)^{r_1-1}, \ \ g(t,x,y) \le M_2(x+y)^{r_2-1}, \ \ \forall\, t\in[0,1], \ x,\, y\ge 0,$$

then there exist positive constants λ_0 and μ_0 such that for every $\lambda\in(0,\lambda_0)$ and $\mu\in(0,\mu_0)$ the boundary value problem (1),(2) has no positive solution.

In the proof of Theorem 3 we define $\lambda_0 = \frac{1}{M_1(2B)^{r_1-1}}$ and $\mu_0 = \frac{1}{M_2(2D)^{r_2-1}}$, where

$$B = \frac{1}{(\Gamma(\alpha_1+1))^{\varrho_1-1}}\int_0^1 s^{\alpha_1(\varrho_1-1)}J_1(s)\,\mathrm{d}s,$$
$$D = \frac{1}{(\Gamma(\alpha_2+1))^{\varrho_2-1}}\int_0^1 s^{\alpha_2(\varrho_2-1)}J_2(s)\,\mathrm{d}s.$$

Remark 1. a) In the proof of Theorem 3 we can also define

$$\lambda_0 = \frac{1}{M_1}\left(\frac{a_1}{B}\right)^{r_1-1} \quad \text{and} \quad \mu_0 = \frac{1}{M_2}\left(\frac{a_2}{D}\right)^{r_2-1}$$

with a_1, $a_2>0$ and $a_1+a_2=1$.

b) If f_0^s, g_0^s, f_∞^s, $g_\infty^s<\infty$, then there exist positive constants M_1, M_2 such that relation (21) holds, and then we obtain the conclusion of Theorem 3.

THEOREM 4. *Assume that (H1) and (H2) hold. If there exist positive numbers c_1, c_2 with $0<c_1<c_2\le 1$ and $m_1>0$ such that*

$$f(t,x,y)\ge m_1(x+y)^{r_1-1}, \ \ \forall\, t\in[c_1,c_2], \ x,\, y\ge 0, \tag{22}$$

then there exists a positive constant $\widetilde{\lambda}_0$ such that for every $\lambda>\widetilde{\lambda}_0$ and $\mu>0$, the boundary value problem (1),(2) has no positive solution.

In the proof of Theorem 4, we define $\widetilde{\lambda}_0 = \frac{1}{m_1(\gamma\gamma_1 A)^{r_1-1}}$, where

$$A = \frac{1}{(\Gamma(\alpha_1+1))^{\varrho_1-1}}\int_{c_1}^{c_2}(s-c_1)^{\alpha_1(\varrho_1-1)}J_1(s)\,\mathrm{d}s.$$

THEOREM 5. *Assume that (H1) and (H2) hold. If there exist positive numbers c_1, c_2 with $0<c_1<c_2\le 1$ and $m_2>0$ such that*

$$g(t,x,y)\ge m_2(x+y)^{r_2-1}, \ \ \forall\, t\in[c_1,c_2], \ x,\, y\ge 0, \tag{23}$$

then there exists a positive constant $\widetilde{\mu}_0$ such that for every $\mu>\widetilde{\mu}_0$ and $\lambda>0$, the boundary value problem (1),(2) has no positive solution.

In the proof of Theorem 5 we define $\widetilde{\mu}_0 = \frac{1}{m_2(\gamma\gamma_2 C)^{r_2-1}}$, where

$$C = \frac{1}{(\Gamma(\alpha_2+1))^{\varrho_2-1}}\int_{c_1}^{c_2}(s-c_1)^{\alpha_2(\varrho_2-1)}J_2(s)\,\mathrm{d}s.$$

THEOREM 6. *ssume that (H1) and (H2) hold. If there exist positive numbers c_1, c_2 with $0 < c_1 < c_2 \leq 1$ and m_1, $m_2 > 0$ such that*

(24)
$$f(t,x,y) \geq m_1(x+y)^{r_1-1},\ \ g(t,x,y) \geq m_2(x+y)^{r_2-1},\ \ \forall\, t \in [c_1,c_2],\ x,\, y \geq 0,$$

then there exist positive constants $\hat{\lambda}_0$ and $\hat{\mu}_0$ such that for every $\lambda > \hat{\lambda}_0$ and $\mu > \hat{\mu}_0$, the boundary value problem (1),(2) has no positive solution.

In the proof of Theorem 6, we define

$$\hat{\lambda}_0 = \frac{1}{m_1(2\gamma\gamma_1 A)^{r_1-1}} \ \text{ and } \ \hat{\mu}_0 = \frac{1}{m_2(2\gamma\gamma_2 C)^{r_2-1}}.$$

Remark 2. a) If for c_1, c_2 with $0 < c_1 < c_2 \leq 1$, we have f_0^i, $f_\infty^i > 0$ and $f(t,x,y) > 0$ for all $t \in [c_1,c_2]$ and x, $y \geq 0$ with $x+y > 0$, then the relation (22) holds and we obtain the conclusion of Theorem 4.

b) If for c_1, c_2 with $0 < c_1 < c_2 \leq 1$, we have g_0^i, $g_\infty^i > 0$ and $g(t,x,y) > 0$ for all $t \in [c_1,c_2]$ and x, $y \geq 0$ with $x+y > 0$, then the relation (23) holds and we obtain the conclusion of Theorem 5.

c) If for c_1, c_2 with $0 < c_1 < c_2 \leq 1$, we have f_0^i, f_∞^i, g_0^i, $g_\infty^i > 0$ and $f(t,x,y) > 0$, $g(t,x,y) > 0$ for all $t \in [c_1,c_2]$ and x, $y \geq 0$ with $x+y > 0$, then the relation (24) holds and we obtain the conclusion of Theorem 6.

5. AN EXAMPLE

Let $\alpha_1 = 1/4$, $\alpha_2 = 2/3$, $n = 3$, $\beta_1 = 8/3$, $m = 4$, $\beta_2 = 7/2$, $p = 2$, $q = 1$, $\gamma_0 = 3/2$, $\gamma_1 = 1/3$, $\gamma_2 = 6/5$, $\delta_0 = 4/3$, $\delta_1 = 5/4$, $r_1 = 5$, $\varrho_1 = 5/4$, $\varphi_{r_1}(s) = s|s|^3$, $\varphi_{\varrho_1}(s) = s|s|^{-3/4}$, $r_2 = 3$, $\varrho_2 = 3/2$, $\varphi_{r_2} = s|s|$, $\varphi_{\varrho_2} = s|s|^{-1/2}$, $H_1(t) = \{0,\ t \in [0,1/2);\ 2,\ t \in [1/2,1]\}$, $H_2(t) = t/4$ for all $t \in [0,1]$, and $K_1(t) = \{0,\ t \in [0,2/3);\ 1/2,\ t \in [2/3,1]\}$.

We consider the system of fractional differential equations

$$(25)\quad \begin{cases} D_{0+}^{1/4}(\varphi_5(D_{0+}^{8/3}x(t))) + \lambda(t+1)^a(x^5(t)+y^5(t)) = 0, & t \in (0,1), \\ D_{0+}^{2/3}(\varphi_3(D_{0+}^{7/2}y(t))) + \mu(2-t)^b\left(e^{(x(t)+y(t))^2}-1\right) = 0, & t \in (0,1), \end{cases}$$

with the nonlocal boundary conditions

$$(26)\quad \begin{cases} x(0) = x'(0) = 0,\ D_{0+}^{8/3}x(0) = 0, \\ D_{0+}^{3/2}x(1) = 2D_{0+}^{1/3}x\left(\frac{1}{2}\right) + \frac{1}{4}\displaystyle\int_0^1 D_{0+}^{6/5}x(t)\,\mathrm{d}t, \\ y(0) = y'(0) = y''(0) = 0,\ D_{0+}^{7/2}y(0) = 0,\ D_{0+}^{4/3}y(1) = \frac{1}{2}D_{0+}^{5/4}y\left(\frac{2}{3}\right), \end{cases}$$

where a, $b > 0$.

Here we have $f(t,x,y)=(t+1)^a(x^5+y^5)$, $g(t,x,y)=(2-t)^b\left(e^{(x+y)^2}-1\right)$, $\forall\, t\in[0,1]$, $x,\ y\geq 0$. Then we obtain $\Delta_1\approx 0.32923823>0$, $\Delta_2\approx 2.18703744>0$, and so the assumptions $(H1)$ and $(H2)$ are satisfied. In addition, we deduce

$$g_1(t,s)=\frac{1}{\Gamma(8/3)}\begin{cases} t^{5/3}(1-s)^{1/6}-(t-s)^{5/3}, & 0\leq s\leq t\leq 1,\\ t^{5/3}(1-s)^{1/6}, & 0\leq t\leq s\leq 1,\end{cases}$$

$$g_{21}(t,s)=\frac{1}{\Gamma(7/3)}\begin{cases} t^{4/3}(1-s)^{1/6}-(t-s)^{4/3}, & 0\leq s\leq t\leq 1,\\ t^{4/3}(1-s)^{1/6}, & 0\leq t\leq s\leq 1,\end{cases}$$

$$g_{22}(t,s)=\frac{1}{\Gamma(22/15)}\begin{cases} t^{7/15}(1-s)^{1/6}-(t-s)^{7/15}, & 0\leq s\leq t\leq 1,\\ t^{7/15}(1-s)^{1/6}, & 0\leq t\leq s\leq 1,\end{cases}$$

$$g_3(t,s)=\frac{1}{\Gamma(7/2)}\begin{cases} t^{5/2}(1-s)^{7/6}-(t-s)^{5/2}, & 0\leq s\leq t\leq 1,\\ t^{5/2}(1-s)^{7/6}, & 0\leq t\leq s\leq 1,\end{cases}$$

$$g_{41}(t,s)=\frac{1}{\Gamma(9/4)}\begin{cases} t^{5/4}(1-s)^{7/6}-(t-s)^{5/4}, & 0\leq s\leq t\leq 1,\\ t^{5/4}(1-s)^{7/6}, & 0\leq t\leq s\leq 1,\end{cases}$$

$$G_1(t,s)=g_1(t,s)+\frac{t^{5/3}}{\Delta_1}\left(2g_{21}\left(\frac{1}{2},s\right)+\frac{1}{4}\int_0^1 g_{22}(\tau,s)\,d\tau\right),$$

$$G_2(t,s)=g_3(t,s)+\frac{t^{5/2}}{2\Delta_2}g_{41}\left(\frac{2}{3},s\right),$$

$$h_1(s)=\frac{1}{\Gamma(8/3)}[(1-s)^{1/6}-(1-s)^{5/3}],$$

$$h_3(s)=\frac{1}{\Gamma(7/2)}[(1-s)^{7/6}-(1-s)^{5/2}].$$

For the functions J_1 and J_2 we obtain

$$J_1(s)=\begin{cases} \frac{1}{\Gamma(8/3)}[(1-s)^{1/6}-(1-s)^{5/3}] \\ \quad +\frac{1}{\Delta_1}\left\{\frac{1}{2^{1/3}\Gamma(7/3)}[(1-s)^{1/6}-(1-2s)^{4/3}]\right. \\ \quad \left. +\ \frac{1}{4\Gamma(37/15)}[(1-s)^{1/6}-(1-s)^{22/15}]\right\}, & 0\leq s<\frac{1}{2},\\ \frac{1}{\Gamma(8/3)}[(1-s)^{1/6}-(1-s)^{5/3}]+\frac{1}{\Delta_1}\left\{\frac{1}{2^{1/3}\Gamma(7/3)}(1-s)^{1/6}\right. \\ \quad \left. +\ \frac{1}{4\Gamma(37/15)}[(1-s)^{1/6}-(1-s)^{22/15}]\right\}, & \frac{1}{2}\leq s\leq 1,\end{cases}$$

$$J_2(s)=\begin{cases} \frac{1}{\Gamma(7/2)}[(1-s)^{7/6}-(1-s)^{5/2}] \\ \quad +\frac{1}{2\Delta_2\Gamma(9/4)3^{5/4}}[2^{5/4}(1-s)^{7/6}-(2-3s)^{5/4}], & 0\leq s<\frac{2}{3},\\ \frac{1}{\Gamma(7/2)}[(1-s)^{7/6}-(1-s)^{5/2}]+\frac{2^{1/4}}{\Delta_2\Gamma(9/4)3^{5/4}}(1-s)^{7/6}, & \frac{2}{3}\leq s\leq 1.\end{cases}$$

Now we choose $c_1=1/4$ and $c_2=3/4$, and then we deduce $\gamma_1=(1/4)^{5/3}$, $\gamma_2=(1/4)^{5/2}$, $\gamma=\gamma_2$. In addition, we have $f_0^s=0$, $f_\infty^i=\infty$, $g_0^s=2^b$, $g_\infty^i=\infty$, $C\approx 0.0323028$, $D\approx 0.0584635$.

By Theorem 1, 6), for any $\lambda \in (0,\infty)$ and $\mu \in (0, L_4')$ with $L_4' = 1/(g_0^s D^2)$, the problem (25),(26) has a positive solution $(x(t), y(t))$, $t \in [0,1]$. For example, if $b = 1$ we obtain $L_4' \approx 146.285$.

We can also use Theorem 5, because $g(t,x,y) \geq (5/4)^b (x+y)^2$ for all $t \in [1/4, 3/4]$ and x, $y \geq 0$, that is $m_2 = (5/4)^b$. If $b = 1$, we deduce $\widetilde{\mu}_0 = 1/(m_2(\gamma\gamma_2 C)^2) \approx 8.03912 \times 10^8$, and then we conclude that for every $\lambda > 0$ and $\mu > \widetilde{\mu}_0$, the boundary value problem (25),(26) has no positive solution.

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