q-STIRLING RECURRENCES

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The q-Stirling numbers have many properties similar to those of the classical Stirling numbers. In this note, we introduce two recurrence relations for the q-Stirling numbers.

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1. INTRODUCTION

The q-Stirling numbers of the first kind $s_q[n,k]$ and the second kind $S_q[n,k]$ are a natural extension of the classical Stirling numbers. The q-Stirling numbers of the first kind are the coefficients in the expansion

$$(x)_{n,q} = \sum_{k=0}^{n} s_q[n,k]x^k,$$

where

$$(x)_{n,q} = \prod_{k=0}^{n-1} (x - [k]_q),$$

with $(x)_{0,q} = 1$. The q-Stirling numbers of the second kind are the coefficients of $(x)_{k,q}$ in the reverse relation

$$x^n = \sum_{k=0}^{n} S_q[n,k](x)_{k,q},$$

For $q \neq 1$ the q-number $[n]_q$ is defined by

$$[n]_q = 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q}.$$

It is clear that

$$\lim_{q\to 1} [n]_q = n.$$

On the other hand, the q-Stirling numbers can be defined by the recurrence relations

$$s_q[n,k] = s_q[n-1,k-1] - [n-1]_q s_q[n-1,k]$$

and

$$S_q[n,k] = S_q[n-1,k-1] + [k]_q S_q[n-1,k],$$

with boundary conditions

$$s_q[n,0] = S_q[n,0] = \delta_{n,0} \qquad \text{and} \qquad s_q[0,k] = S_q[0,k] = \delta_{0,k},$$

where $\delta_{i,j}$ is the usual Kronecker delta function. There is a long history of studying q-Stirling numbers [1, 2, 4, 5, 6, 9, 10, 11]. New and interesting combinatorial interpretations of these numbers have recently been given by Cai and Readdy [3].

Recall that the q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \dfrac{[n]_q!}{[k]_q![n-k]_q!}, & \text{for } k \in \{0,\ldots,n\}, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$[n]_q! = [n]_q[n-1]_q \cdots [1]_q \ ,$$

is q-factorial, with $[0]_q! = 1$. In a recent paper [8], the author prove that the q-Stirling numbers of both kinds can be expressed in terms of the q-binomial coefficients,

$$s_q[n+1,n+1-k] = (1-q)^{-k} \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{j+1}{2}} \binom{n-j}{k-j} \begin{bmatrix} n \\ j \end{bmatrix}_q,$$

$$S_q[n+1+k,n+1] = (1-q)^{-k} \sum_{j=0}^{k} (-q)^j \binom{n+k}{k-j} \begin{bmatrix} n+j \\ j \end{bmatrix}_q,$$

and vice versa

$$q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{j=0}^{k} (1-q)^j \binom{n-j}{k-j} s_q[n+1,n+1-j],$$

$$q^k \begin{bmatrix} n+k \\ k \end{bmatrix}_q = \sum_{j=0}^{k} (q-1)^j \binom{n+k}{k-j} S_q[n+1+j,n+1].$$

The last two identities can be seen as recurrence relations for the q-Stirling numbers and can be rewritten as:

$$q^{\binom{n-k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{j=k}^{n} (1-q)^{n-j} \binom{j}{k} s_q[n+1, j+1],$$

$$q^{n-k} \begin{bmatrix} n \\ n-k \end{bmatrix}_q = \sum_{j=k}^{n} (q-1)^{j-k} \binom{n}{j} S_q[j+1, k+1].$$

The first identity is a horizontal recurrence relation for the q-Stirling numbers of the first kind. The second is a vertical recurrence relation for the q-Stirling numbers of the second kind.

In this paper, motivated by these identities, we shall prove two recurrence relations for the q-Stirling numbers of both kinds. These relations does not involve q-binomial coefficients.

THEOREM 1.1. *Let k and n be non-negative integers such that $k \leqslant n$. Then*

$$s_q[n+1, k+1] = \sum_{j=k}^{n} (-1)^{j-k} q^{n-j} \binom{j}{k} s_q[n, j].$$

THEOREM 1.2. *Let k and n be non-negative integers such that $k \leqslant n$. Then*

$$S_q[n+1, k+1] = \sum_{j=k}^{n} q^{n-j} \binom{n}{j} S_q[j, k].$$

Two known recurrence relations involving the classical Stirling numbers of both kinds can be easily derived as the limiting case $q \to 1$ of these results.

COROLLARY 1.3. *Let k and n be non-negative integers such that $k \leqslant n$. Then*

1. $$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix} = \sum_{j=k}^{n} \binom{j}{k} \begin{bmatrix} n \\ j \end{bmatrix}$$
2. $$\begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} = \sum_{j=k}^{n} \binom{n}{j} \begin{Bmatrix} j \\ k \end{Bmatrix}$$

Recall that the unsigned Stirling number of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}$ is defined as the number of permutations of n elements with k disjoint cycles. The Stirling numbers of the second kind $\begin{Bmatrix} n \\ k \end{Bmatrix}$ count the number of ways to partition a set of n objects into k non-empty subsets.

2. PROOF OF THEOREM 1.1

In order to prove this theorem, we consider that the q-Stirling numbers of the first kind are specializations of elementary symmetric functions [5], i.e.,

$$s_q[n+1,n+1-k] = (-1)^k e_k([1]_q,[2]_q,\dots,[n]_q).$$

According to Merca [7, Theorem 1], we can write

$$\begin{aligned} &e_k([1]_q,[2]_q,\dots,[n]_q) \\ &\quad = e_k(1,1+q,1+q+q^2,\dots,1+q+\dots+q^{n-1}) \\ &\quad = e_k(1,1+q,1+q(1+q),\dots,1+q(1+q+\dots+q^{n-2})) \\ &\quad = \sum_{j=0}^{k} \binom{n-j}{k-j} e_j(q,q(1+q),\dots,q(1+q+\ldots+q^{n-2})) \\ &\quad = \sum_{j=0}^{k} q^j \binom{n-j}{k-j} e_j(1,1+q,\dots,1+q+\ldots+q^{n-2}) \\ &\quad = \sum_{j=0}^{k} q^j \binom{n-j}{k-j} e_j([1]_q,[2]_q,\dots,[n-1]_q). \end{aligned}$$

In terms of the q-Stirling numbers of the first kind, this relation can be written as

$$s_q[n+1,n+1-k] = \sum_{j=0}^{k} (-1)^{k-j} q^j \binom{n-j}{k-j} s_q[n,n-j].$$

By this identity, with k replaced by $n-k$, we obtain

$$\begin{aligned} s_q[n+1,k+1] &= \sum_{j=0}^{n-k} (-1)^{n-k-j} q^j \binom{n-j}{n-k-j} s_q[n,n-j] \\ &= \sum_{j=k}^{n} (-1)^{n-j} q^{j-k} \binom{n+k-j}{n-j} s_q[n,n+k-j] \\ &= \sum_{j=k}^{n} (-1)^{j-k} q^{n-j} \binom{j}{k} s_q[n,j]. \end{aligned}$$

The proof is complete.

3. PROOF OF THEOREM 1.2

The proof of this theorem is quite similar to the proof of Theorem 1.1. It is well know that the q-Stirling numbers of the second kind are specializations of complete homogeneous symmetric functions [5], i.e.,

$$S_q[n+k,n] = h_k([1]_q,[2]_q,\dots,[n]_q).$$

Considering [7, Theorem 1], we have

$$\begin{aligned}
&h_k([1]_q,[2]_q,\dots,[n+1]_q)\\
&\quad= h_k(1,1+q,1+q+q^2,\dots,1+q+\dots+q^n)\\
&\quad= h_k(1,1+q,1+q(1+q),\dots,1+q(1+q+\dots+q^{n-1}))\\
&\quad= \sum_{j=0}^{k}\binom{n+k}{k-j} h_j(q,q(1+q),\dots,q(1+q+\ldots+q^{n-1}))\\
&\quad= \sum_{j=0}^{k} q^j\binom{n+k}{k-j} h_j(1,1+q,\dots,1+q+\ldots+q^{n-1})\\
&\quad= \sum_{j=0}^{k} q^j\binom{n+k}{k-j} h_j([1]_q,[2]_q,\dots,[n]_q).
\end{aligned}$$

Thus we deduce that

$$S_q[n+1+k,n+1] = \sum_{j=0}^{k} q^j\binom{n+k}{k-j} S_q[n+j,n].$$

By this relation, with n replaced by $n-k$, we obtain

$$S_q[n+1,n-k+1] = \sum_{j=0}^{k} q^j\binom{n}{k-j} S_q[n-k+j,n-k].$$

The proof follows easily replacing k by $n-k$ in the last relation.

4. CONCLUDING REMARKS

A q-analogue of some Stirling identities have been discovered and proved in this paper considering that the q-Stirling numbers are specializations of complete and elementary symmetric functions.

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