ON THE SEARCH FOR EXTREME POINTS OF THE CLOSED CONVEX HULL OF THE FAMILY OF CONVEX MAPPINGS OF THE BALL

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Almost half a century ago, the extreme points of the closed convex hulls of various geometric families of analytic functions of the unit disk were completely classified, paving the way for solutions to linear extremal problems over the families. Little is known about such extreme points for the analogous families defined on the Euclidean unit ball $\mathbb{B} \subseteq \mathbb{C}^n$ for $n \geq 2$. Only recently were any extreme points found for the closed convex hull of the family of convex mappings of $\mathbb{B}$. We will survey the techniques used to find this set of extreme points, demonstrate that the set is incomplete, and give some related results that may help to guide future work.

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1. PROLOGUE

Prior to the settlement by de Branges [7] of the Bieberbach conjecture [5] on the bounds of the Taylor coefficients of members of the schlicht class

$$\mathcal{S} = \mathcal{S}(\mathbb{D}) = \{f \in H(\mathbb{D}) : f \text{ is univalent, } f(0) = 0, \text{ and } f'(0) = 1\},$$

where $H(\mathbb{D})$ denotes the space of functions analytic in the unit disk $\mathbb{D} \subseteq \mathbb{C}$, significant attention turned to various compact subfamilies of $\mathcal{S}$ defined using some geometric constraint on a function's range. For such a family $\mathcal{F} \subseteq \mathcal{S}$ that is invariant under rotations, the problem of maximizing the modulus of a coefficient in the series expansions of members of $\mathcal{F}$ is equivalent to maximizing the value of a real-linear functional on $H(\mathbb{D})$ over $\mathcal{F}$. Since the maximum value of such a functional will be attained at an extreme point of the closed convex hull of $\mathcal{F}$, identification of these extreme points is of interest.

Taking a step back, let X be a locally convex topological vector space (such as $H(\mathbb{D})$ under the topology of uniform convergence on compact sets), and let $A \subseteq X$. The *convex hull* of A is the set $\operatorname{co} A$ consisting of all $x \in X$ of the form $x = \sum_{k=1}^{n} \lambda_k a_k$ with $a_1, \dots, a_n \in A$ and $\lambda_1, \dots, \lambda_n \geq 0$ such that $\sum_{k=1}^{n} \lambda_k = 1$. The *closed convex hull* of A is the closure of $\operatorname{co} A$ in X and is denoted $\overline{\operatorname{co}}\, A$. If $E \subseteq X$, then an *extreme point* of E is any $x \in E$ such that the equality $x = (1-t)a + tb$ for $a, b \in E$ and $t \in (0,1)$ implies $a = b = x$. By Milman's theorem, extreme points of $\overline{\operatorname{co}}\, A$ are also extreme points of A when A and $\overline{\operatorname{co}}\, A$ are both compact, but the converse clearly does not hold in general. (See [22].)

Let us now consider the particular family

$$\mathcal{K} = \mathcal{K}(\mathbb{D}) = \{f \in \mathcal{S} : f(\mathbb{D}) \text{ is convex}\},$$

which is well known to be compact. (See [10].) The complete classification of the extreme points of $\overline{\operatorname{co}}\, \mathcal{K}$ was given by Brickman, MacGregor, and Wilken [4]. Here, we frame their work in a manner that helps motivate what follows. (See also [12, 13].) Let

$$\mathcal{R} = \mathcal{R}(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : f(0) = 0,\ f'(0) = 1, \text{ and } \operatorname{Re} \frac{f(z)}{z} > \frac{1}{2} \text{ for } z \in \mathbb{D} \right\}.$$

A classical result of Marx [11] and Strohhäcker [28] gives that $\mathcal{K} \subseteq \mathcal{R}$.

For a Borel subset E of a topological space X, let $P(E)$ be the set of all regular Borel probability measures μ on X such that $\mu(X \setminus E) = 0$. One may use the well-known Herglotz representation for analytic functions with positive real part in $\mathbb{D}$ (see [10]) to see that the association of $f \in \mathcal{R}$ to $\mu \in P(\partial\mathbb{D})$ through the representation

$$f(z) = \int_{\partial\mathbb{D}} \frac{z}{1 - uz}\, \mathrm{d}\mu(u), \qquad z \in \mathbb{D}, \tag{1.1}$$

establishes a bijection between $\mathcal{R}$ and $P(\partial\mathbb{D})$. As $P(\partial\mathbb{D})$ can be viewed as a subset of the dual space of the Banach space of continuous complex-valued functions on $\partial\mathbb{D}$ by the Riesz representation theorem [27], it has an induced weak-$*$ topology. With this and the topology of uniform convergence on compact sets applied to $\mathcal{R}$, this bijection is actually an affine homeomorphism. The extreme points of the compact, convex set $P(\partial\mathbb{D})$ are the Dirac (point mass) measures δ_α for $\alpha \in \partial\mathbb{D}$, and therefore the extreme points of $\mathcal{R}$ (itself now seen to be a compact, convex set) are the half-plane mappings

$$z \mapsto \frac{z}{1 - \alpha z}, \qquad z \in \mathbb{D}, \tag{1.2}$$

for $\alpha \in \partial\mathbb{D}$. Now $\overline{\operatorname{co}}\, \mathcal{K} \subseteq \mathcal{R}$, the extreme points of $\mathcal{R}$ lie in $\mathcal{K}$, and $\mathcal{R}$ is the closed convex hull of its extreme points by the Krein–Milman theorem. We conclude that $\overline{\operatorname{co}}\, \mathcal{K} = \mathcal{R}$, which completely exposes the extreme points of $\overline{\operatorname{co}}\, \mathcal{K}$.

2. BACKGROUND ON $\mathcal{K}(\mathbb{B})$

Geometric function theory in higher dimensions is unsurprisingly more complicated than its one-variable counterpart. Our generalization of the unit disk $\mathbb{D}$ is the unit ball $\mathbb{B} = \mathbb{B}_n \subseteq \mathbb{C}^n$ with respect to the Euclidean norm $\|\cdot\|$. (The lack of a Riemann mapping theorem in higher dimensions makes this a nontrivial choice.) The space of holomorphic mappings from $\mathbb{B}$ into $\mathbb{C}^m$, $m \in \mathbb{N}$, is written $H(\mathbb{B}, \mathbb{C}^m)$. Notably, this space is locally convex in the topology of uniform convergence on compact sets. The Fréchet derivative of $f \in H(\mathbb{B}, \mathbb{C}^m)$ at a point $z \in \mathbb{B}$ is the linear operator $Df(z) \colon \mathbb{C}^n \to \mathbb{C}^m$.

Since the family

$$\mathcal{S}(\mathbb{B}) = \{f \in H(\mathbb{B}, \mathbb{C}^n) : f \text{ is biholomorphic, } f(0) = 0, \text{ and } Df(0) = I\},$$

where I is the identity operator, is not compact when $n \geq 2$ (a fundamental difference from the one-variable case), it is especially important to consider various compact subfamilies of $\mathcal{S}(\mathbb{B})$. Such a compact subfamily is (see [10], for instance)

$$\mathcal{K}(\mathbb{B}) = \{f \in \mathcal{S}(\mathbb{B}) : f(\mathbb{B}) \text{ is convex}\},$$

the natural generalization of the one-variable family $\mathcal{K}$, and its members are called *convex mappings*. As above, we wonder, what are the extreme points of $\overline{\mathrm{co}}\, \mathcal{K}(\mathbb{B})$? A complete answer likely would reveal maxima of real-linear functionals defined on $H(\mathbb{B}, \mathbb{C}^n)$ over $\mathcal{K}(\mathbb{B})$. For instance, since $\mathcal{K}(\mathbb{B})$ is invariant under unitary rotations, we may look to find the maximum modulus of a Taylor coefficient in the expansions of coordinate functions of members of $\mathcal{K}(\mathbb{B})$.

The remainder of this note is a discussion of mathematical developments, some recent and some not so much, directed towards answering this question. Rather than progress chronologically, we choose to take advantage of more recently found results that provide context or clarity to concepts established earlier. While complete proofs will not be given, as the reader may consult the cited references, some sketches of proofs of what we feel to be the more important results are provided to give the reader a sense of the underlying work.

Before beginning our exploration, we give some useful properties of $\mathcal{K}(\mathbb{B})$. For the first, we recall that a polynomial $P \colon \mathbb{C}^n \to \mathbb{C}^m$ is *homogeneous of degree* $k \in \mathbb{N} \cup \{0\}$ if $P(\lambda z) = \lambda^k P(z)$ for all $z \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$. The norm of such a P is

$$\|P\| = \sup_{u \in \partial\mathbb{B}} \|P(u)\| = \sup_{z \in \mathbb{C}^n \setminus \{0\}} \frac{\|P(z)\|}{\|z\|^k}.$$

In the case $m = 1$, we write $\mathcal{P}_k(n)$ for the space of complex-valued homogeneous polynomials of degree k, which is a Banach space in the above norm.

Any $f \in H(\mathbb{B}, \mathbb{C}^m)$ has a series expansion of the form $f = \sum_{k=0}^{\infty} P_k$, where each P_k is a homogeneous polynomial of degree k. If $f \in \mathcal{K}(\mathbb{B})$, then $P_0 = 0$, $P_1 = I$, and $\|P_k\| \leq 1$ for all $k \geq 2$, as shown in [21]. Note that the analogous result in dimension $n = 1$ can easily be derived from (1.1) by expanding the integrand and moving the integral inside the series. Applying the norm to the series expansion of $f \in \mathcal{K}(\mathbb{B})$ yields our second property, the useful growth bound

$$\|f(z)\| \leq \frac{\|z\|}{1 - \|z\|}, \qquad z \in \mathbb{B}, \tag{2.1}$$

whose development historically preceded the bound on terms of the homogeneous polynomial expansion. From this bound, we see that $\mathcal{K}(\mathbb{B})$ is a normal family, a result already implied by its compactness noted above.

Our next observation deals with the Koebe transform. If $\operatorname{Aut}\mathbb{B}$ denotes the group of biholomorphic automorphisms of $\mathbb{B}$, then for $\varphi \in \operatorname{Aut}\mathbb{B}$, the *Koebe transform* of $f \in \mathcal{S}(\mathbb{B})$ with respect to φ is the mapping formed by normalizing $f \circ \varphi$ to lie in $\mathcal{S}(\mathbb{B})$. That is, the mapping

$$\Lambda_\varphi[f](z) = D\varphi(0)^{-1} Df(\varphi(0))^{-1}[f(\varphi(z)) - f(\varphi(0))], \qquad z \in \mathbb{B}.$$

For any $\varphi \in \operatorname{Aut}\mathbb{B}$, it is geometrically evident that $\Lambda_\varphi[\mathcal{K}(\mathbb{B})] \subseteq \mathcal{K}(\mathbb{B})$. That is to say, $\mathcal{K}(\mathbb{B})$ is a *linear-invariant family*, a concept introduced in one variable by Pommerenke [23, 24] and first extended to higher dimensions in [2, 19].

Lastly, for $f \in \mathcal{K}(\mathbb{B})$, let

$$S = \left\{ u \in \partial\mathbb{B} : \lim_{\mathbb{B} \ni z \to u} \|f(z)\| = \infty \right\}.$$

Recently, Bracci and Gaussier [3] showed that S may have up to, but no more than, two elements and f extends to a homeomorphism from $\overline{\mathbb{B}} \setminus S$ onto $\overline{f(\mathbb{B})}$. The case $S = \varnothing$ coincides with f bounded. We refer to a point in S as an *infinite boundary singularity* of f.

3. THE MAPPINGS F_Q

It is only natural to propose the generalization of the half-plane mappings (1.2) to $\mathbb{B}$ as candidates for extreme points of $\overline{\operatorname{co}}\,\mathcal{K}(\mathbb{B})$; that is, the mappings

$$z \mapsto \frac{z}{1 - \langle z, \alpha \rangle}, \qquad z \in \mathbb{B}, \tag{3.1}$$

for $\alpha \in \partial\mathbb{B}$. Here, $\langle \cdot, \cdot \rangle$ is the standard Hermitian inner product in $\mathbb{C}^n$. These mappings, however, fail to be extreme when $n \geq 2$.

To see this, we begin with a result we will use again later. See [14, 16]. For $n \geq 2$ and $z \in \mathbb{C}^n$, we write $z = (z_1, \hat{z})$ for $\hat{z} \in \mathbb{C}^{n-1}$, and we let $e_1, \ldots, e_n$ be the canonical basis vectors of $\mathbb{C}^n$.

THEOREM 3.1. *Let $n \geq 2$ and $P \in \mathcal{P}_k(n-1)$ for $k \geq 2$. The mapping $f(z) = z + P(\hat{z})e_1$ lies in $\mathcal{K}(\mathbb{B})$ if and only if*

$$\|P\| \leq \frac{(k-1)^{(k-3)/2}}{k(k-2)^{(k-2)/2}}. \tag{3.2}$$

Let $n \geq 2$. We now see that if $Q \in \mathcal{P}_2(n-1)$ with $\|Q\| \leq 1/2$, then $f(z) = z + Q(\hat{z})e_1$ is a mapping in $\mathcal{K}(\mathbb{B})$. For $r \in (0,1)$, let $\varphi_r \in \operatorname{Aut}\mathbb{B}$ be given by

$$\varphi_r(z) = \left(\frac{z_1 - r}{1 - rz_1}, \frac{\sqrt{1-r^2}\,\hat{z}}{1 - rz_1} \right). \tag{3.3}$$

(Background on $\operatorname{Aut}\mathbb{B}$ can be found in [10, 26].) From above, we know that $\Lambda_{\varphi_r}[f] \in \mathcal{K}(\mathbb{B})$ for all $r \in (0,1)$, and hence compactness gives that

$$F_Q(z) = \lim_{r \to 1^-} \Lambda_{\varphi_r}[f](z) = \frac{z}{1-z_1} + \frac{Q(\hat{z})}{(1-z_1)^2}e_1, \qquad z \in \mathbb{B}, \tag{3.4}$$

lies in $\mathcal{K}(\mathbb{B})$. (That the limit is uniform on compact sets follows from Vitali's theorem in several complex variables.) We see that mappings of the form (3.1) fail to be extreme points of $\mathcal{K}(\mathbb{B})$ due to the simple observation

$$\frac{z}{1-z_1} = \frac{F_Q(z) + F_{-Q}(z)}{2}, \qquad z \in \mathbb{B},$$

for any $Q \in \mathcal{P}_2(n-1)$ with $0 < \|Q\| \leq 1/2$.

One may be tempted to mimic the above technique using homogeneous polynomials of degree $k > 2$ with norm bounded by (3.2). Oddly, only homogeneous polynomials of degree 2 produce interesting results. Insight into this phenomenon is given in [20].

It is now only natural to consider the mappings F_Q, for $Q \in \mathcal{P}_2(n-1)$ with $\|Q\| \leq 1/2$, in place of those given in (3.1). We note that each F_Q has exactly one infinite boundary singularity (namely, e_1) and $F_Q(\mathbb{B})$ contains a real line. In particular, $\{ite_1 : t \in \mathbb{R}\} \subseteq F_Q(\mathbb{B})$. We call such a convex mapping *half-plane-like.*

We note that $F_Q(\mathbb{B})$ may possibly contain lines in other directions. For instance, if

$$Q(w) = -\frac{1}{2}\sum_{k=1}^{n-1} w_k^2, \qquad w \in \mathbb{C}^{n-1},$$

then $\|Q\| = 1/2$ and $F_Q(\mathbb{B})$ contains the real n-dimensional space spanned by $\{ie_1, e_2, \dots, e_n\}$. We note [17] that this is the maximum dimension of such a subspace:

THEOREM 3.2. *Let $f \in \mathcal{K}(\mathbb{B})$. If X is a real subspace of $\mathbb{C}^n$ such that $X \subseteq f(\mathbb{B})$, then $\dim X \leq n$.*

To see this, observe that $X \cap iX \neq \{0\}$ if $\dim X > n$. For $u \in X \cap iX \setminus \{0\}$, we have $u, iu \in X$. It follows that $\zeta \mapsto f^{-1}(\zeta u)$ is a bounded entire function of $\zeta \in \mathbb{C}$, which must be constant.

It certainly seems that a mapping F_Q for Q chosen so that $F_Q(\mathbb{B})$ contains a real n-dimensional space is "extremal" in some sense. This is the foundation of our work moving forward.

4. **HALF-PLANE-LIKE MAPPINGS**

Assume $n \geq 2$ throughout this section. Above, we noticed that F_Q given in (3.4) with $Q \in \mathcal{P}_2(n-1)$ and $\|Q\| \leq 1/2$ is what we called "half-plane-like" because it is a convex mapping with exactly one infinite boundary singularity that contains a line in its range. Here, we ponder the nature of such mappings to see that, under the right conditions, they must be of the form F_Q.

Let us begin by observing that if $f \in \mathcal{K}(\mathbb{B})$ and $u \in \partial\mathbb{B}$ are such that $\{tu : t \in \mathbb{R}\} \subseteq f(\mathbb{B})$, then $w + tu \in f(\mathbb{B})$ for all $w \in f(\mathbb{B})$ and $t \in \mathbb{R}$. This is geometrically intuitive, but see [3, 18] for arguments. For $t \in \mathbb{R}$, let

$$\psi_t(z) = f^{-1}(f(z) + tu), \qquad z \in \mathbb{B}.$$

It follows that $\{\psi_t : t \in \mathbb{R}\}$ is a continuous one-parameter subgroup of $\operatorname{Aut}\mathbb{B}$; i.e., $t \mapsto \psi_t$ is a continuous group-homomorphism of $\mathbb{R}$ into $\operatorname{Aut}\mathbb{B}$. Because ψ_t has no fixed points in $\mathbb{B}$ for $t \neq 0$, an argument based on the Denjoy–Wolff theorem in $\mathbb{B}$ (see [6], for instance) gives that there exist $a, b \in \partial\mathbb{B}$ such that

$$\lim_{t\to\infty} \psi_t(z) = a, \quad \lim_{t\to-\infty} \psi_t(z) = b, \qquad z \in \mathbb{B},$$

the limits being uniform on compact subsets of $\mathbb{B}$. Both a and b are infinite boundary singularities of f. Since half-plane-like mappings have only one infinite boundary singularity, we will consider the case $a = b$ here. The reader may consult [18] for a consideration of the case where $a \neq b$.

For the remainder of the section, we summarize the steps taken to analyze these half-plane-type mappings in [15, 17]. There is no loss of generality in assuming that $a = b = e_1$. Because the automorphisms ψ_t, $t \in \mathbb{R} \setminus \{0\}$, have e_1 as their only fixed point in $\overline{\mathbb{B}}$, their general form is known (see [1]). A detailed analysis shows that u (the direction vector of the line assumed to lie in $f(\mathbb{B})$) must be such that $u_1 \in i\mathbb{R}$. A complicated situation results when $\hat{u} \neq 0$, but the mappings F_Q considered above are such that u may be chosen to be ie_1, the only possibility (save for the redundant $u = -ie_1$) when $\hat{u} = 0$. We will therefore assume this latter condition moving forward. It can then be shown that there is an Hermitian operator $A\colon \mathbb{C}^{n-1} \to \mathbb{C}^{n-1}$ such that

$$\psi_t(z) = \frac{(z_1 + it(1 - z_1), e^{-itA}\hat{z})}{1 + it(1 - z_1)}, \qquad z \in \mathbb{B},\ t \in \mathbb{R}.$$

An analysis using this form of ψ_t allows us to conclude that there exists a holomorphic $G\colon \mathbb{C}^{n-1} \to \mathbb{C}^n$ with $G(0) = 0$ and $DG(0) = 0$ such that the mapping f with our assumptions has the form

$$(4.1)\quad f(z) = \left(\frac{z_1}{1-z_1}, \exp\left(\frac{z_1}{1-z_1}A\right)\frac{\hat{z}}{1-z_1}\right) + G\left(\exp\left(\frac{z_1}{1-z_1}A\right)\frac{\hat{z}}{1-z_1}\right).$$

The next step is to show that $A = 0$. To do this, we assume otherwise and choose a nonzero eigenvalue of A. Using this eigenvalue and an associated unit eigenvector, it is possible to construct two points of $f(\mathbb{B})$ whose midpoint fails to lie in $f(\mathbb{B})$, contradicting convexity. It therefore remains to address the characteristics of G.

The first significant property of G is that G must be a homogeneous polynomial of degree 2. To sketch the argument, let $G(w) = \sum_\alpha w^\alpha a_\alpha$, $w \in \mathbb{C}^{n-1}$, be the multi-index expansion of G. That is, each α is an $(n-1)$-tuple of nonnegative integers, $a_\alpha \in \mathbb{C}^n$ for all α, and $w^\alpha = \prod_{k=1}^{n-1} w_k^{\alpha_k}$. Fix α such that $|\alpha| = \sum_{k=1}^{n-1} \alpha_k \geq 3$ and $\rho \in (0,1)^{n-1}$ such that $\|\rho\| < 1/2$. Now for $r \in (0,1)$, let $z \in \mathbb{C}^n$ be such that $z_1 = r$ and $|z_k| = \rho_{k-1}\sqrt{1-r^2}$ for $2 \leq k \leq n$. With $w_k = z_{k+1}/(1-r)$ for $1 \leq k \leq n-1$, we use the assumption $|\alpha| \geq 3$ and some straightforward calculations involving (4.1) (with $A = 0$) to see that

$$w^\alpha a_\alpha = \int_{[0,2\pi]^{n-1}} e^{-i\langle\theta,\alpha\rangle} f(z_1, z_2 e^{i\theta_1}, \ldots, z_n e^{i\theta_{n-1}})\,\mathrm{d}m(\theta),$$

where $m \in P([0,2\pi]^{n-1})$ is normalized Lebesgue measure. The growth bound (2.1) then implies

$$\|w^\alpha a_\alpha\| \leq \int_{[0,2\pi]^{n-1}} \|f(z_1, z_2 e^{i\theta_1}, \ldots, z_n e^{i\theta_{n-1}})\|\,\mathrm{d}m(\theta) < \frac{1+r}{1-r}.$$

From the choice of w, we have

$$\|a_\alpha\|\rho^\alpha < \left(\frac{1-r}{1+r}\right)^{|\alpha|/2-1}.$$

Letting $r \to 1^-$ gives $a_\alpha = 0$.

The next notable property of G is that if we write $G(w) = (G_1(w), \hat{G}(w))$ for $w \in \mathbb{C}^{n-1}$, where $G_1\colon \mathbb{C}^{n-1} \to \mathbb{C}$ and $\hat{G}\colon \mathbb{C}^{n-1} \to \mathbb{C}^{n-1}$, then $\hat{G} = 0$. That is, $G(\mathbb{C}^{n-1}) \subseteq \operatorname{span}\{e_1\}$. To argue this, let $v \in \partial\mathbb{B}_{n-1}$, $r \in (0,1)$, $\zeta \in \mathbb{D}$, and λ be a square root of ζ. Convexity then gives that

$$\frac{r}{1-r}e_1 + \zeta\frac{1+r}{1-r}G(v) = \frac{f(r, \lambda\sqrt{1-r^2}\,v) + f(r, -\lambda\sqrt{1-r^2}\,v)}{2} \in f(\mathbb{B}).$$

We now define the analytic $g\colon \mathbb{D} \to \mathbb{B}$ by

$$g(\zeta) = f^{-1}\left(\frac{r}{1-r}e_1 + \zeta\frac{1+r}{1-r}G(v)\right).$$

We then have

$$g'(0) = Df(re_1)^{-1}\left(\frac{1+r}{1-r}G(v)\right) = ((1-r^2)G_1(v), (1+r)\hat{G}(v)).$$

If $h = \varphi_r \circ g$ (see (3.3)), then $h(0) = 0$, and hence by Schwarz's lemma, we find

$$\left\| \left(G_1(v), \sqrt{\frac{1+r}{1-r}}\,\hat{G}(v) \right) \right\| = \|D\varphi_r(re_1)g'(0)\| = \|h'(0)\| \le 1.$$

Since r may be arbitrarily close to 1, it must be that $\hat{G}(v) = 0$. Any number of elementary arguments then give that $\hat{G}(w) = 0$ for all $w \in \mathbb{C}^{n-1}$.

Finally, we may now observe that if $f = \sum_{k=1}^{\infty} P_k$ is the homogeneous polynomial expansion of f, then

$$P_2(z) = (z_1^2 + G_1(\hat{z}), z_1\hat{z}), \qquad z \in \mathbb{C}^n.$$

As noted above, $\|P_2\| \le 1$, which can be used to verify that $\|G_1\| \le 1/2$ (where the norm is in $\mathcal{P}_2(n-1)$). The following theorem summarizes our discussion.

Theorem 4.1. *Let $n \ge 2$ and $f \in \mathcal{K}(\mathbb{B})$ such that $\{tu : t \in \mathbb{R}\} \subseteq f(\mathbb{B})$ for some $u \in \partial\mathbb{B}$. The limits*

$$\lim_{t\to\infty} f^{-1}(f(z) + tu), \quad \lim_{t\to-\infty} f^{-1}(f(z) + tu), \qquad z \in \mathbb{B},$$

each converge uniformly on compact subsets of $\mathbb{B}$ to constants on $\partial\mathbb{B}$. If they both converge to e_1, then $u_1 \in i\mathbb{R}$. In the case that $u = \pm ie_1$, there exists $Q \in \mathcal{P}_2(n-1)$ such that $\|Q\| \le 1/2$ and $f = F_Q$.

5. A SUFFICIENT CONDITION FOR EXTREME POINTS OF $\overline{\mathrm{co}}\,\mathcal{K}(\mathbb{B})$

We begin the section with a lemma from [12].

Lemma 5.1. *Let $n \ge 2$ and $\Theta\colon \mathcal{P}_2(n-1) \to H(\mathbb{B}, \mathbb{C}^n)$ be given by $\Theta(Q) = F_Q$. Then Θ is an affine homeomorphism onto its range when $\mathcal{P}_2(n-1)$ has the norm topology and $H(\mathbb{B}, \mathbb{C}^n)$ has the topology of uniform convergence on compact subsets of $\mathbb{B}$.*

We denote closed balls in $\mathcal{P}_2(n)$ centered at the origin by

$$\mathcal{B}_r(n) = \{Q \in \mathcal{P}_2(n) : \|Q\| \le r\}, \qquad r > 0.$$

From Lemma 5.1 and the development in earlier sections, we see that Θ affinely and topologically embeds $\mathcal{B}_{1/2}(n-1)$ into $\mathcal{K}(\mathbb{B})$ when $n \ge 2$. It is immediate that a mapping F_Q for $Q \in \mathcal{B}_{1/2}(n-1)$ is *not* an extreme point of $\mathcal{K}(\mathbb{B})$ (and hence of $\overline{\mathrm{co}}\,\mathcal{K}(\mathbb{B})$) if Q is *not* an extreme point of $\mathcal{B}_{1/2}(n-1)$. In the following theorem from [12], we now will see that, conversely, the extreme points of $\mathcal{B}_{1/2}(n-1)$ indeed produce extreme points of $\overline{\mathrm{co}}\,\mathcal{K}(\mathbb{B})$.

THEOREM 5.2. *Let $n \geq 2$. If Q is an extreme point of $\mathcal{B}_{1/2}(n-1)$, then F_Q is an extreme point of $\overline{\mathrm{co}}\,\mathcal{K}(\mathbb{B})$.*

We note that mappings F_Q as in the theorem were shown to be extreme points of $\mathcal{K}(\mathbb{B})$ in [17]. That conclusion is subsumed in the above, stronger result. We complete the section by outlining the proof of this fundamental theorem with greater detail than we have given for previous results.

Let $\mathcal{E}$ be the set of extreme points of $\overline{\mathrm{co}}\,\mathcal{K}(\mathbb{B})$. By Montel's theorem (in conjunction with (2.1)), $\overline{\mathrm{co}}\,\mathcal{K}(\mathbb{B})$ is compact. Since the topology of uniform convergence on compact subsets of $\mathbb{B}$ is metrizable, we have that $\mathcal{E}$ is a Borel subset of $\mathcal{K}(\mathbb{B})$. (See [22].) By the Krein–Milman theorem, $\overline{\mathrm{co}}\,\mathcal{E} = \overline{\mathrm{co}}\,\mathcal{K}(\mathbb{B})$.

Fix an extreme point Q of $\mathcal{B}_{1/2}(n-1)$. By Choquet's theorem, there is a $\mu \in P(\mathcal{E})$ such that

$$\ell(F_Q) = \int_{\mathcal{E}} \ell \,\mathrm{d}\mu, \qquad \ell \in H(\mathbb{B}, \mathbb{C}^n)^*, \tag{5.1}$$

where $H(\mathbb{B}, \mathbb{C}^n)^*$ is the topological (i.e., continuous) dual space of $H(\mathbb{B}, \mathbb{C}^n)$.

For each $k \in \mathbb{N}$, the function $a_k \colon H(\mathbb{B}, \mathbb{C}^n) \to \mathbb{C}^n$ given by

$$a_k(f) = \frac{1}{2\pi i} \int_C \frac{f(\zeta e_1)}{\zeta^{k+1}} \,\mathrm{d}\zeta,$$

where $C \subseteq \mathbb{D}$ is a circle centered at 0, is linear and continuous. For $f \in H(\mathbb{B}, \mathbb{C}^n)$, we have

$$f(\zeta e_1) = f(0) + \sum_{k=1}^{\infty} \zeta^k a_k(f), \qquad \zeta \in \mathbb{D}.$$

By applying (5.1) in each coordinate, we obtain

$$e_1 = a_k(F_Q) = \int_{\mathcal{E}} a_k \,\mathrm{d}\mu, \qquad k \in \mathbb{N}.$$

Our work at the end of Section 2 makes clear that $\|a_k(f)\| \leq 1$ for all $k \in \mathbb{N}$ and $f \in \mathcal{E}$. Therefore the Borel measure ν_k defined by $\nu_k(A) = \mu(a_k^{-1}(A))$ for Borel sets $A \subseteq \mathbb{C}^n$ is seen to lie in $P(\overline{\mathbb{B}})$, and we have

$$e_1 = \int_{\overline{\mathbb{B}}} z \,\mathrm{d}\nu_k(z).$$

Since e_1 is an extreme point of the compact, convex set $\overline{\mathbb{B}}$, it must be that $\nu_k = \delta_{e_1}$ by Bauer's theorem (see [22]), meaning $a_k(f) = e_1$ for μ-almost every $f \in \mathcal{E}$. Thus if $\mathcal{E}_k = \{f \in \mathcal{E} : a_k(f) \neq e_1\}$, then

$$\tilde{\mathcal{E}} = \mathcal{E} \setminus \bigcup_{k=1}^{\infty} \mathcal{E}_k$$

is a Borel set with $\mu(\tilde{\mathcal{E}}) = 1$.

For $f \in \tilde{\mathcal{E}}$, we have

$$f(\zeta e_1) = \frac{\zeta}{1-\zeta} e_1, \qquad \zeta \in \mathbb{D}.$$

We therefore see that f satisfies the conditions of Theorem 4.1 with $u = ie_1$. Hence $f = F_{R_f}$ for some $R_f \in \mathcal{B}_{1/2}(n-1)$. Now the equation

$$F_Q(z) = \int_{\tilde{\mathcal{E}}} f(z)\,\mathrm{d}\mu(f), \qquad z \in \mathbb{B},$$

(using (5.1) coordinate-wise again) becomes

$$\frac{z}{1-z_1} + \frac{Q(\hat{z})}{(1-z_1)^2} e_1 = \frac{z}{1-z_1} + \frac{e_1}{(1-z_1)^2} \int_{\tilde{\mathcal{E}}} R_f(\hat{z})\,\mathrm{d}\mu(f), \qquad z \in \mathbb{B}.$$

If Θ is as in Lemma 5.1 and ν is the Borel measure defined by $\nu(\mathcal{A}) = \mu(\Theta(\mathcal{A}))$ for Borel sets $\mathcal{A} \subseteq \mathcal{P}_2(n-1)$, we have that $\nu \in P(\mathcal{B}_{1/2}(n-1))$ and

$$Q(w) = \int_{\mathcal{B}_{1/2}(n-1)} S(w)\,\mathrm{d}\nu(S), \qquad w \in \mathbb{C}^{n-1}.$$

Because Q is an extreme point of the compact, convex set $\mathcal{B}_{1/2}(n-1)$, an argument using Bauer's theorem once again gives $\nu = \delta_Q$. But then $\mu = \delta_{F_Q}$, meaning $F_Q \in \mathcal{E}$.

6. THE EXTREME POINTS OF $\mathcal{B}_r(n)$.

Theorem 5.2 makes clear that an explicit identification of the extreme points of $\mathcal{B}_{1/2}(n-1)$ will produce concrete extreme points of $\overline{\mathrm{co}}\,\mathcal{K}(\mathbb{B})$ for $n \geq 2$. For generality, we will consider the extreme points of $\mathcal{B}_r(n)$ for $r > 0$ and $n \in \mathbb{N}$. To begin, consider the following subspace of $\mathbb{C}^n$ associated to $Q \in \mathcal{P}_2(n)$:

$$V(Q) = \operatorname{span}\{u \in \partial\mathbb{B} : |Q(u)| = \|Q\|\}.$$

Clearly, $1 \leq \dim V(Q) \leq n$. We now have the following from [17].

THEOREM 6.1. *Let $Q \in \mathcal{P}_2(n)$. There is an orthonormal basis $\{v_1, \dots, v_m\}$ of $V(Q)$ such that*

$$Q\left(\sum_{j=1}^m \lambda_j v_j + w\right) = \|Q\| \sum_{j=1}^m \lambda_j^2 + Q(w) \tag{6.1}$$

for all $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ and $w \in V(Q)^\perp$.

We briefly outline the main steps of the proof. There is a symmetric bilinear functional $L\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ such that $Q(z) = L(z,z)$ for all $z \in \mathbb{C}^n$. The proof begins by establishing the equality

$$L(u,v) = \|Q\| \operatorname{Re}\langle u, v\rangle \tag{6.2}$$

for all $u, v \in \partial\mathbb{B}$ such that $Q(u) = Q(v) = \|Q\|$. The desired basis for $V(Q)$ can then be constructed using the Gram–Schmidt process. Specifically, if we suppose $\{v_1, \ldots, v_k\}$ is an orthonormal set of vectors such that $Q(v_j) = \|Q\|$ for $j = 1, \ldots, k$ and there exists some $u \in \partial\mathbb{B} \setminus \operatorname{span}\{v_1, \ldots, v_k\}$ such that $Q(u) = \|Q\|$, then

$$v_{k+1} = \frac{u - \sum_{j=1}^{k} \langle u, v_j\rangle v_j}{\|u - \sum_{j=1}^{k} \langle u, v_j\rangle v_j\|}$$

is orthogonal to $v_1, \ldots, v_k$ and can be shown using (6.2) to satisfy $Q(v_{k+1}) = \|Q\|$.

To establish the expansion (6.1), we show that for $w \in V(Q)^\perp$, it must be that $L(v_j, w) = 0$ for all $j = 1, \ldots, m$. The result then follows by expanding the left-hand side of (6.1) using L.

The following theorem from [17] can then be used in conjunction with (6.1) to find the extreme points of $\mathcal{B}_r(n)$ explicitly.

Theorem 6.2. *Let $r > 0$. Then $Q \in \mathcal{P}_2(n)$ is an extreme point of $\mathcal{B}_r(n)$ if and only if $\|Q\| = r$ and $V(Q) = \mathbb{C}^n$.*

If $V(Q) \neq \mathbb{C}^n$, then we use compactness to let

$$\alpha = \sup_{u \in \partial\mathbb{B} \cap V(Q)^\perp} |Q(u)| < r.$$

If $\alpha > 0$, choose $\varepsilon > 0$ such that $0 < (1-\varepsilon)\alpha < (1+\varepsilon)\alpha < r$, and define $Q_1, Q_2 \in \mathcal{P}_2(n)$ by

$$Q_1\left(\sum_{j=1}^{m} \lambda_j v_j + w\right) = r \sum_{j=1}^{m} \lambda_j^2 + (1+\varepsilon) Q(w),$$
$$Q_2\left(\sum_{j=1}^{m} \lambda_j v_j + w\right) = r \sum_{j=1}^{m} \lambda_j^2 + (1-\varepsilon) Q(w)$$

for $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ and $w \in V(Q)^\perp$, where $\{v_1, \ldots, v_m\}$ is the orthonormal basis given by Theorem 6.1. Then $Q_1, Q_2 \in \mathcal{B}_r(n)$ and $Q = (Q_1 + Q_2)/2$. A similar approach addresses the case $\alpha = 0$.

Conversely, if $V(Q) = \mathbb{C}^n$, let $\{v_1, \ldots, v_n\}$ be the orthonormal basis given by Theorem 6.1 and suppose $Q_1, Q_2 \in \mathcal{B}_r(n)$ and $t \in (0,1)$ are such that $(1-t)Q_1 + tQ_2 = Q$. Evaluating both sides at each v_j shows that $Q_1(v_j) =$

$Q_2(v_j) = r$ for all $j = 1, \dots, n$. Following the proof of Theorem 6.1, we conclude that $Q_1 = Q_2 = Q$.

We summarize as follows.

Corollary 6.3. *Let $r > 0$ and $Q \in \mathcal{P}_2(n)$. Then Q is an extreme point of $\mathcal{B}_r(n)$ if and only if there is an orthonormal basis $\{v_1, \dots, v_n\}$ of $\mathbb{C}^n$ such that*

$$Q(z) = r \sum_{k=1}^{n} \langle z, v_k \rangle^2, \qquad z \in \mathbb{C}^n.$$

7. SOME, BUT ONLY SOME, EXTREME POINTS OF $\overline{\mathrm{co}}\,\mathcal{K}(\mathbb{B})$

Let $n \geq 2$ for this section. By $\mathcal{U}(n)$ we denote the group of unitary operators on $\mathbb{C}^n$. Evidently, $\mathcal{K}(\mathbb{B})$ and $\overline{\mathrm{co}}\,\mathcal{K}(\mathbb{B})$ are invariant under the transform $f \mapsto U^* \circ f \circ U|_{\mathbb{B}}$ for a given $U \in \mathcal{U}(n)$. Therefore if f is an extreme point of $\overline{\mathrm{co}}\,\mathcal{K}(\mathbb{B})$, then so is $U^* \circ f \circ U|_{\mathbb{B}}$ for each $U \in \mathcal{U}(n)$. Using this, we see from Theorem 5.2 and Corollary 6.3 that any mapping of the form

$$f(z) = \frac{1}{1 - \langle z, u \rangle} z + \frac{\sum_{k=1}^{n-1} \langle z, v_k \rangle^2}{2(1 - \langle z, u \rangle)^2} u, \qquad z \in \mathbb{B},$$

where $\{u, v_1, \dots, v_{n-1}\}$ is an orthonormal basis of $\mathbb{C}^n$, is an extreme point of $\overline{\mathrm{co}}\,\mathcal{K}(\mathbb{B})$. Let $\mathcal{F}$ denote the set of all mappings of this form. Then, of course, $\overline{\mathrm{co}}\,\mathcal{F} \subseteq \overline{\mathrm{co}}\,\mathcal{K}(\mathbb{B})$. Naturally, we wonder if equality holds. Unfortunately, it does not.

To see this, define $\ell \in H(\mathbb{B}, \mathbb{C}^n)^*$ by

$$\ell(f) = \frac{3}{\pi i} \int_C \frac{\langle f(\zeta e_2), e_1 \rangle}{\zeta^4} \, \mathrm{d}\zeta = \left\langle \frac{\partial^3 f}{\partial z_2^3}(0), e_1 \right\rangle,$$

where $C \subseteq \mathbb{D}$ is a circle centered at 0. For $f \in \mathcal{F}$, let $U \in \mathcal{U}(n)$ and $Q \in \mathcal{B}_{1/2}(n-1)$ such that $f = U^* \circ F_Q \circ U|_{\mathbb{B}}$. Write $Uz = (U_1 z, \hat{U} z)$ for $z \in \mathbb{C}^n$, where $U_1 \colon \mathbb{C}^n \to \mathbb{C}$ and $\hat{U} \colon \mathbb{C}^n \to \mathbb{C}^{n-1}$. For $u = U^* e_1$, we then have

$$\langle f(z), e_1 \rangle = \frac{z_1}{1 - \langle z, u \rangle} + \frac{u_1 Q(\hat{U} z)}{(1 - \langle z, u \rangle)^2}, \qquad z \in \mathbb{B}.$$

Then

$$|\ell(f)| = |u_1| |Q(\hat{U} e_2)| \left| \frac{\mathrm{d}^3}{\mathrm{d}\zeta^3} \frac{\zeta^2}{(1 - \overline{u}_2 \zeta)^2} \bigg|_{\zeta = 0} \right| = 12 |u_1| |u_2| |Q(\hat{U} e_2)|.$$

Note that $|Q(\hat{U} e_2)| \leq \|\hat{U} e_2\|^2 / 2 = (1 - |u_2|^2)/2$, and hence

$$|\ell(f)| \leq 6 |u_2| (1 - |u_2|^2)^{3/2} \leq \frac{9\sqrt{3}}{8},$$

the last inequality following from straightforward calculus.

Now let $g \in \overline{\mathrm{co}}\,\mathcal{F}$. By Choquet's theorem, there is $\mu \in P(\mathcal{F})$ such that

$$|\ell(g)| = \left| \int_{\mathcal{F}} \ell \,\mathrm{d}\mu \right| \leq \frac{9\sqrt{3}}{8}.$$

However, Theorem 3.1 gives that the mapping $h(z) = z + z_2^3 e_1/3$, $z \in \mathbb{B}$, is such that $h \in \mathcal{K}(\mathbb{B})$. But $\ell(h) = 2$ then implies $h \notin \overline{\mathrm{co}}\,\mathcal{F}$.

8. EPILOGUE

As noted in Section 1, $\overline{\mathrm{co}}\,\mathcal{K} = \mathcal{R}$, and we conclude this note with a short consideration of the relationship between $\overline{\mathrm{co}}\,\mathcal{K}(\mathbb{B})$ and the generalization of $\mathcal{R}$ to $\mathbb{B}$ given by

$$\mathcal{R}(\mathbb{B}) = \Big\{ f \in H(\mathbb{B}, \mathbb{C}^n) : f(0) = 0,\ Df(0) = I, \\ \text{and } \mathrm{Re}\langle f(z), z\rangle > \frac{\|z\|^2}{2} \text{ for } z \in \mathbb{B} \setminus \{0\} \Big\}.$$

Here, we will see that $\overline{\mathrm{co}}\,\mathcal{K}(\mathbb{B}) \neq \mathcal{R}(\mathbb{B})$ when $n \geq 2$, providing a negative answer to a question posed in [13].

Recall that the family

$$\mathcal{M} = \Big\{ f \in H(\mathbb{B}, \mathbb{C}^n) : f(0) = 0,\ Df(0) = I, \\ \text{and } \mathrm{Re}\langle f(z), z\rangle > 0 \text{ for } z \in \mathbb{B} \setminus \{0\} \Big\}$$

provides a generalization of the Carathéodory class of functions $p \in H(\mathbb{D})$ satisfying $p(0) = 1$ and $\mathrm{Re}\, p(z) > 0$ for $z \in \mathbb{D}$. It is well known that $\mathcal{M}$ is compact [8], and $\mathcal{R}(\mathbb{B})$ is the image of $\mathcal{M}$ under the continuous transformation

$$f \mapsto \frac{f + I|_{\mathbb{B}}}{2}, \tag{8.1}$$

and hence $\mathcal{R}(\mathbb{B})$ is compact. In addition, $\mathcal{R}(\mathbb{B})$ is clearly convex.

In [13], we proved that $\mathcal{K}(\mathbb{B}) \subseteq \mathcal{R}(\mathbb{B})$, which immediately gives $\overline{\mathrm{co}}\,\mathcal{K}(\mathbb{B}) \subseteq \mathcal{R}(\mathbb{B})$. This result was actually known to T.J. Suffridge before its publication. Of note, we actually showed that $\mathcal{K}(\mathbb{B}) \subseteq \mathcal{G}(\mathbb{B}) \subseteq \mathcal{R}(\mathbb{B})$, where $\mathcal{G}(\mathbb{B})$ is a family of *quasi-convex mappings* introduced by Roper and Suffridge [25], who proved the first inclusion. While it is clear that $\overline{\mathrm{co}}\,\mathcal{G}(\mathbb{B}) \subseteq \mathcal{R}(\mathbb{B})$, whether $\overline{\mathrm{co}}\,\mathcal{G}(\mathbb{B})$ and $\mathcal{R}(\mathbb{B})$ are equal remains unknown.

That $\overline{\mathrm{co}}\,\mathcal{K}(\mathbb{B}) \neq \mathcal{R}(\mathbb{B})$ follows immediately from the following theorem [12].

THEOREM 8.1. *If $n \geq 2$ and $Q \in \mathcal{B}_{1/2}(n-1)$, then F_Q (as in (3.4)) is not an extreme point of $\mathcal{R}(\mathbb{B})$.*

The proof relies on showing that mappings of the form

$$f_\alpha(z) = F_Q(z) + \frac{\alpha Q(\hat{z})}{(1-z_1)^2}(z - z_1 e_1), \qquad z \in \mathbb{B},$$

lie in $\mathcal{R}(\mathbb{B})$ when $\alpha \in \mathbb{C}$ is such that $|\alpha| \leq 1/2$. For $Q \neq 0$, we then have $F_Q = (f_\alpha + f_{-\alpha})/2$ for $0 < |\alpha| \leq 1/2$. The case $Q = 0$ is simple.

To examine this situation a little more, we let $H_0(\mathbb{D}) = \{f \in H(\mathbb{D}) : f(0) = 0\}$, $n \geq 2$, and $Q \in \mathcal{P}_2(n-1)$ and define the operator $\Psi_Q \colon H_0(\mathbb{D}) \to H(\mathbb{B}, \mathbb{C}^n)$ by

$$\Psi_Q[f](z) = \left(f(z_1) + \frac{f(z_1)^2 Q(\hat{z})}{z_1^2}, \frac{f(z_1)}{z_1}\hat{z} \right), \qquad z \in \mathbb{B}.$$

This is a modification of an extension operator introduced in [9] for families of locally univalent mappings. We have the following from [12].

THEOREM 8.2. *Let $n \geq 2$ and $Q \in \mathcal{P}_2(n-1)$. Then $\Psi_Q[\mathcal{R}] \subseteq \mathcal{R}(\mathbb{B})$ if and only if $\|Q\| \leq 1/2$.*

To sketch the proof, let $f \in \mathcal{R}$ and $F = \Psi_Q[f]$. The only significant verification needed is that $\operatorname{Re}\langle F(z), z\rangle > \|z\|^2/2$ for $z \in \mathbb{B} \setminus \{0\}$ when $\|Q\| \leq 1/2$. A calculation shows that this is implied by the inequality

$$|z_1|\|\hat{z}\|^2 |\varphi(z_1)|^2 < \|z\|^2 (2 \operatorname{Re} \varphi(z_1) - 1), \qquad z \in \mathbb{B} \setminus \{0\},$$

where $\varphi(\zeta) = f(\zeta)/\zeta$ for $\zeta \in \mathbb{D}$. If $\mu \in P(\partial\mathbb{D})$ is such that (1.1) holds, we see that this inequality is implied by

$$|z_1|\|\hat{z}\|^2 \int_{\partial\mathbb{D}} \frac{\mathrm{d}\mu(u)}{|1 - uz_1|^2} < \|z\|^2 \int_{\partial\mathbb{D}} \frac{1 - |z_1|^2}{|1 - uz_1|^2} \,\mathrm{d}\mu(u), \qquad z \in \mathbb{B} \setminus \{0\}.$$

This, in turn, follows from establishing $|z_1|\|\hat{z}\|^2 < \|z\|^2(1 - |z_1|^2)$ for $z \in \mathbb{B} \setminus \{0\}$ by a straightforward calculation.

The converse follows by showing that $F_Q = \Psi_Q[f] \notin \mathcal{R}(\mathbb{B})$ when $\|Q\| > 1/2$, where $f(\zeta) = \zeta/(1-\zeta)$ for $\zeta \in \mathbb{D}$. This particular observation is notable because F_Q is an extreme point of $\overline{\text{co}}\,\mathcal{K}(\mathbb{B})$ when Q is an extreme point of $\mathcal{B}_{1/2}(n-1)$. Although this mapping is not an extreme point of $\mathcal{R}(\mathbb{B})$, we see that any increase in the norm of Q will cause F_Q to lie outside of $\mathcal{R}(\mathbb{B})$. This at least suggests that F_Q is a support point of $\mathcal{R}(\mathbb{B})$. This may, in fact, be true for any $Q \in \mathcal{P}_2(n-1)$ with $\|Q\| = 1/2$. Either result would imply the same property for the mapping

$$g(z) = \frac{1+z_1}{1-z_1} z + \frac{2Q(\hat{z})}{(1-z_1)^2} e_1, \qquad z \in \mathbb{B},$$

relative to the family $\mathcal{M}$, using the transform (8.1).

To our knowledge, the mappings F_Q, where Q is an extreme point of $\mathcal{B}_{1/2}(n-1)$, are the only known extreme points of $\mathcal{K}(\mathbb{B})$, much less of $\overline{\mathrm{co}}\,\mathcal{K}(\mathbb{B})$, when $n \geq 2$, and we have seen there are more to be found. We look forward to what future developments on this topic will bring.

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