TWO RESULTS ABOUT THE RATIONAL NUMBERS

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Communicated by Lucian Beznea

In the first part we show that the exponential diophantine problem for $\mathbb{Q}$ is algorithmically undecidable. The second part studies the question about a continuous approximation function for the square root in the rationals, in the larger context of continuous choice functions for generic predicates.

AMS 2010 Subject Classification: 11D61, 14G05, 03C07, 03C35, 03E25.

Key words: exponential diophantine equations, rational numbers, decision problems, undecidability, choice, densely ordered sets, order topology, continuous functions, roots, $\aleph_0$-categorical theories.

1. INTRODUCTION

My talk in the Number Theory section of the Congress of the Romanian Mathematicians (Galati, 2019) contained two separated results. Both of them have consequences in the arithmetic of the field $\mathbb{Q}$ of rational numbers. The first one is related with Hilbert's Tenth Problem for $\mathbb{Q}$, which is a notoriously open problem. I prove that the decision problem for exponential diophantine systems of equations over the rationals is not algorithmically solvable. The other one concerns the possibility to continuously compute some approximations in rationals, as for example a continuous approximation function for the square root. The method presented works in the very general setting of countable densely ordered sets without endpoints and translates the problem in the problem to find a continuous choice function with values in some generic predicate. Because of this very general setting, the result has a plethora of possible applications. This note is a short survey presentation of those results, without proofs.

2. THE EXPONENTIAL DIOPHANTINE PROBLEM FOR $\mathbb{Q}$

There were two steps leading to the negative solution of Hilbert's Tenth Problem. First Martin Davis, Hilary Putnam and Julia Robinson proved in [3] that it is undecidable if exponential diophantine equations have solutions in $\mathbb{N}$. Later Yuri Matiyasevich shown that the exponential relation (x, y, x^y) is diophantine in $\mathbb{N}$ and concluded that it is undecidable whether diophantine equations have solutions in $\mathbb{N}$, see [4].

In the present note is shown that the Davis–Putnam–Robinson step for $\mathbb{Q}$ reduces easily to Hilbert's Tenth Problem for $\mathbb{N}$. This implies the fact that the exponential diophantine problem for $\mathbb{Q}$ is undecidable. With other words, the Davis-Putnam-Robinson Theorem works for the field of rational numbers.

The following result has been already known to Euler, see Dickson's History [2]:

LEMMA. *The system $x^y = y^x \ \wedge\ 0 < x < y$ has no other solutions in $\mathbb{Q}$ as all the pairs:*

$$x_n = \left(1 + \frac{1}{n}\right)^n \quad , \quad y_n = \left(1 + \frac{1}{n}\right)^{n+1} \quad ,$$

where $n \in \mathbb{N} \setminus \{0\}$.

$$\mathbb{Q} \models \quad \alpha > 0 \ \leftrightarrow\ \exists\, x_1, \ldots, x_5 \ \ \alpha = x_1^2 + \cdots + x_4^2 \ \wedge\ \alpha x_5 = 1.$$

It follows that in $\mathbb{Q}$ the set $\mathbb{N}$ of natural numbers is exponential diophantine as one has:

$$\mathbb{Q} \models \quad w \in \mathbb{N} \ \leftrightarrow\ w = 0 \vee \exists x, y \Big(x > 0 \wedge y - x > 0 \wedge x^y = y^x \wedge wy = (w+1)x. \Big)$$

In conclusion, for every diophantine equation we can effectively construct an exponential diophantine system of equations which has solutions in rational numbers if and only if the diophantine equation has solutions in natural numbers. This implies that the decision problem for exponential diophantine systems of equations over the rationals is not algorithmically solvable. For a more detailed report, see author's paper [6].

3. ON CONTINUOUS APPROXIMATIONS OF THE SQUARE ROOT, AND RELATED PROBLEMS

The field of rational numbers $\mathbb{Q}$, like the field of real numbers $\mathbb{R}$, satisfy the following condition:

$$\forall\, x, y\, \exists\, z \quad 0 < x < y \to x < z^2 < y.$$

A Skolem function for this formula would be a function $f(x, y)$ such that:

$$\forall\, x, y \quad 0 < x < y \to x < f(x, y)^2 < y.$$

In $\mathbb{R}$ there are plenty of such continuous Skolem functions, as:

$$z(x,y) = \sqrt{\frac{x+y}{2}},\ z(x,y) = \frac{\sqrt{x}+\sqrt{y}}{2},\ z(x,y) = \sqrt[4]{xy},$$

to mention just three of them.

The primary goal of this note is to show that there are such continuous Skolem functions in the rationals.

It turns out that the best setting for this problem is to study the countable structures $(A, P, <)$ such that $(A, <)$ is a densely ordered set without endpoints and P denotes a generic predicate in this set. This means that the subset defined by P is infinite, co-infinite, dense and co-dense in the set A. It is known that densely ordered sets without endpoints are $\aleph_0$-categorical, see for example [5]. By a similar proof, based on the back-and-forth principle, the densely ordered sets without endpoints endowed with a generic predicate are $\aleph_0$-categorical as well.

The problem about the continuous Skolem function corresponding to the approximation of the square root can be seen as the problem to find a continuous function $f : \mathbb{Q}_+ \to P$ such that:

$$\forall\, x, y \quad 0 < x < y \to x < f(x,y) < y.$$

Here P denotes the set of those positive rationals which are squares of rational numbers. We observe that the structure $(\mathbb{Q}_+, P, <)$ is an example of countable densely ordered set without end-point, endowed with a generic predicate. According to the $\aleph_0$-categoricity, this structure is isomorphic with any other structure of this type.

So the problem reformulates to the question if there is some continuous choice function for the countable densely ordered set without end-point, endowed with a generic predicate, which must take values only in the predicate. Because of the categoricity result, it is sufficient to construct such a structure together with the corresponding function.

Lemma 3.1. *There is a subset $D \subset \mathbb{R}$ with the following properties:*

1. *D is countable.*
2. *D is dense in $\mathbb{R}$.*
3. *D is linearly independent over $\mathbb{Q}$.*

Definition 3.2. Let $C \subset \mathbb{R}$ be the subset of $\mathbb{R}$ generated by D under the operation:

$$z(x,y) = \frac{x+y}{2},$$

which is also the intersection of all subsets E of $\mathbb{R}$ which contain D and are closed under $z(x,y)$. Let $P = C \setminus D$. The order $<$ on C is the order induced from $\mathbb{R}$.

LEMMA 3.3. *The structure $(C, P, <)$ is a countable densely ordered set without end-point, endowed with a generic predicate.*

THEOREM 3.4. *All countable densely ordered set without end-point, endowed with a generic predicate, admit continuous choice.*

Moreover, one can prove the following:

THEOREM 3.5. *Let $(A, P, <)$ be a countable model of T_g such that there is computable bijective enumeration of A and the predicate P, as like the order relation $<$ is effectively decidable. Then the structure A has an effectively computable continuous choice $z : H_A \to A$.*

In conclusion, (apply for example [1]) in many applications, and specially in the case of the continuous approximation of the square root, the continuous choice function is effectively computable. For a more detailed report, see author's paper [7].

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