CONTINUALLY RING HOMEOMORPHISMS IN METRIC SPACES WITH MEASURES

ELENA AFANAS'EVA and ANATOLY GOLBERG

Communicated by Lucian Beznea

We study the boundary behavior of mappings of continual domains in metric spaces with measures. Some sufficient conditions for continually ring Q-homeomorphisms and their inverses to be extended to the boundary are presented. These results involve the p-modulus technique and special types of domains, like domains with continually weakly flat and continually strongly accessible boundaries, continual QED domains, continual NED sets.

AMS 2010 Subject Classification: Primary 31B25, 30L10; Secondary 46E27, 30C65.

Key words: continually ring Q-homeomorphism, weakly flat boundary, strongly accessible boundary, QED domain, NED set.

1. INTRODUCTION

The recent developments of modern mapping theory motivate studying various types of regular domains, like domains of quasiextremal length (QED domains), domains with weakly flat and strongly accessible boundaries, null sets for extremal distance (NED sets), etc. in Euclidean spaces, metric spaces and Riemannian manifolds; see e.g. [12], [16], [20] and the references therein. Due to the famous Liouville theorem, even in $\mathbb{R}^3$ the class of conformal mappings is exhausted by Möbius transformations only, and quasiconformal mappings and their generalizations provide natural extensions of geometric aspects of analytic functions to higher dimensions.

The continually ring Q-homeomorphisms, whose boundary behavior is studied in the paper, have been introduced by the first author in [1]. The idea was to extend the well-known ring definition of quasiconformality by Gehring [7]. In fact, we study the main properties of mappings, whose p-moduli of the families of continua joining the boundary components of ring domains are integrally restricted from above. Such relations and establishing the main differential features of mappings in $\mathbb{R}^n$, $n \geq 2$, have been initiated by the second author in [9].

More precisely, we continue to establish the boundary behavior of mappings with integrally restricted p-moduli between continual domains of quasiextremal length in metric measure spaces started in [1], and investigate the appropriate properties of NED sets, QED domains and spaces, extending the theory of nonconformal moduli; see e.g. [5], [10], [16].

Note that the investigation of mappings in metric spaces with measures leads to interesting and deep applications to the fractal theory in $\mathbb{R}^n$, $n \geq 2$, which are intensively used in various fields of Mathematics and other sciences.

2. CONTINUAL CONNECTEDNESS IN TOPOLOGICAL SPACES

We recall some necessary definitions. A topological space is connected if it cannot be split into two nonempty distinct open sets. A topological space T is called *locally connected*, if for any point x_0 and its arbitrary neighborhood U there is a connected neighborhood $V \subseteq U$. Compact connected spaces is said to be *continua.* For any sets A, B and C of a topological space T we denote by $\Gamma(A, B; C)$ a family of all continua γ, which join A and B in C, e.g. such that $\gamma \cap A \neq \varnothing$, $\gamma \cap B \neq \varnothing$ and $\gamma \setminus \{A \cup B\} \subseteq C$.

A topological space T is called *continually connected*, if any pair of its points can be imbedded into a continuum γ located in T. By a *continual domain* in a topological space T we mean an arbitrary open continually connected set D. Also a space T is called *locally continually connected* at a point x_0, if for any neighborhood U of x_0 there is a neighborhood $V \subseteq U$, which is a continual domain in T. A space T is said to be *continually connected* at a point x_0, if for any its neighborhood U there exists a neighborhood $V \subseteq U$, such that $V \setminus \{x_0\}$ is a continual domain; cf. [16, p. 274]. Finally, a continual domain D is called *continually connected at a point* $x_0 \in \partial D$, if for any neighborhood U of x_0 there is a neighborhood $V \subseteq U$ provided that $V \cap D$ is a continual domain.

Now we present a continual counterpart of Proposition 2.1 in [18] (cf. [16, Proposition 13.1]), whose proof will be given below.

PROPOSITION 2.1. *Let T be a topological space with a base of topology $\mathcal{B}$, consisting of continually connected sets. Then any arbitrary open set Ω in T is connected if and only if Ω is continually connected.*

Recall that by a *base of topology T* we mean an arbitrary collection of open sets in T, such that any open set of T can be presented as a union of sets from this collection. This implies the following

COROLLARY 2.1. *An open set Ω in $\mathbb{R}^n$, $n \geq 2$, or, generally, in an arbitrary manifold, is connected if and only if Ω is continually connected.*

Remark 2.1. If a domain D in $\mathbb{R}^n$, $n \geq 2$, or in a manifold $\mathbb{M}^n$, $n \geq 2$, is locally connected at $x_0 \in \partial D$, then it is continually connected at x_0. Based on Proposition 2.1, we show later that the connectedness and continual connectedness of open sets are equivalent in a quite wide class so-called continually weakly flat spaces.

Proof of Proposition 2.1. 1) Sufficiency. Let Ω be continually connected. If Ω is nonconnected then $\Omega = \Omega_1 \cup \Omega_2$, where Ω_1 and Ω_2 are two some nonempty and distinct open sets. Let $x_1 \in \Omega_1$ and $x_2 \in \Omega_2$. Then due to the continual connectedness of Ω there is a continuum K in Ω, containing both points. On the other hand, the sets $K \cap \Omega_1$ and $K \cap \Omega_2$ are open in the topology T, both are nonempty distinct and $K = (K \cap \Omega_1) \cup (K \cap \Omega_2)$, which contradicts to the connectedness of K. This contradiction disproves the assumption that Ω is nonconnected.

2) Necessity. Assume that Ω is connected. Fix an arbitrary point $x_0 \in \Omega$ and denote by Ω_0 a set of all point x in Ω, which can be connected to x_0 by a finite sequence of sets $B_k \in \mathcal{B}$, $k = 1, ..., m$, such that $x_0 \in B_1$, $x \in B_m$ and $B_k \cap B_{k+1} \neq \varnothing$, $k = 1, ..., m-1$. Obviously, Ω_0 is continually connected; see e.g. [15].

First, the set Ω_0 is open. Indeed, if $x_* \in \Omega_0$ then by the construction there exists its neighborhood $B_* \in \mathcal{B}$, which lies in Ω_0.

Secondly, Ω_0 is closed in Ω. Assume by contradiction that $\partial\Omega_0 \cap \Omega \neq \varnothing$. Note that for any point $z_0 \in \partial\Omega_0 \cap \Omega$ there exist its neighborhood $B_0 \in \mathcal{B}$ and a point $y_0 \in \Omega_0$, since $z_0 \in \partial\Omega_0$. Thus, $z_0 \in \Omega_0$ due to the notation of Ω_0. However, Ω_0 is open, and, therefore, $\Omega_0 \cap \partial\Omega_0 = \varnothing$. This contradiction disproves the above assumption.

Thus, Ω_0 is both open and closed in Ω, and, therefore, it coincides with Ω because of connectedness. This yields that Ω is continually connected and completes the proof of Proposition 2.1. $\square$

We say that a family Γ_1 of continua in an arbitrary topological space T *is minorated* by a family Γ_2 of continua in T and write $\Gamma_1 > \Gamma_2$, if for each continuum $\gamma_1 \in \Gamma_1$ there is a continuum $\gamma_2 \in \Gamma_2$, such that γ_2 is a subcontinuum of γ_1, namely $\gamma_2 \subseteq \gamma_1$.

The following statement is applied to the proofs of our main results. Its proof can be found in [1, Proposition 1].

Proposition 2.2. *Let Ω be an open set in a metric space (X, d). Then*

$$\Gamma(\Omega, X \setminus \Omega; X) > \Gamma(\Omega, \partial\Omega; \Omega)\,. \tag{1}$$

Recall that H^k, $k \in [0, \infty)$, denotes the *k-dimensional Hausdorff measure* of a set A in the metric space (X,d). More precisely, for A in (X,d),

$$H^k(A) := \sup_{\varepsilon>0} H^k_\varepsilon(A)\,,$$

$$H^k_\varepsilon(A) := \inf \sum_{i=1}^{\infty} (\operatorname{diam} A_i)^k\,, \tag{2}$$

where the infimum in (2) is taken over all coverings of A by the sets A_i with $\operatorname{diam} A_i < \varepsilon$; see e.g. [13]. Recall that $\operatorname{diam} A_i = \sup\limits_{x,y\in A_i} d(x,y)$, and if for any set A and $k_1 \geq 0$ the condition $H^{k_1}(A) < \infty$ holds, then $H^{k_2}(A) = 0$ for arbitrary real number $k_2 > k_1$. The following quantity

$$\dim_H A := \sup_{H^k(A)>0} k,$$

stands for the *Hausdorff dimension* of the set A.

Later on, we say that a continuum in a metric space (X,d) is *k-rectifiable* if its Hausdorff measure H^k is finite. 1-rectifiable continua γ are called for simplicity *rectifiable continua* or continua of *finite length*, and $H^1(\gamma)$ is the *length* of γ. In [6] the systems of measures in an abstract set $\mathcal{X}$ with a fixed main measure have been considered.

In our paper we consider a Borel measure $m^{(k)}_\gamma$ associated with some continuum γ in the metric space (X,d). The measure $m^{(k)}_\gamma$ is defined for any Borel set B in (X,d) as the Hausdorff measure H^k of $B\cap\gamma$ for fixed $k>0$. In that follows, for any continuum $\gamma\in\Gamma$, the measure $m_\gamma := m^{(1)}_\gamma$.

We develop here a p-modulus technique applicable to the families of continua in metric spaces which do not need to be linearly connected (by continuous paths). The simplest example for such continua can presented by a pseudoarc, whose never two points can be connected by a path; in particular any pseudoarc does not contain Jourdan arcs.

The following statement has been also proven in [1, Proposition 2]. For convenience of the reader, we repeat it here.

Proposition 2.3. *Let γ be a rectifiable continuum in a metric space (X,d), which joins two points $x_1 \in \overline{B(x_0,r_1)}$ and $x_2 \in X\setminus B(x_0,r_2)$, where $x_0\in X$, $0<r_1<r_2<\infty$. Suppose also that $\eta : [0,\infty]\to[0,\infty]$ is a Borel function. Then*

$$\int\limits_\gamma \eta(d(x,x_0))\,\mathrm{d}m_\gamma \geq \int\limits_{r_1}^{r_2} \eta(r)\,\mathrm{d}r\ . \tag{3}$$

Remark 2.2. In particular, the inequality (3) implies that for any continuum γ,

$$H^1(\gamma) \geq \operatorname{diam}\gamma\,. \tag{4}$$

However, the inequality

$$H^k(\gamma) \geq [\operatorname{diam}\gamma]^{\alpha_k} \tag{5}$$

does not hold for any nondegenerated continua and any k, except for $k = 1$, and any $\alpha_k \in \mathbb{R}$. Indeed, under $k < 1$, [13, Theorem VII.2] yields $1 > \dim_H \gamma \geq \dim\gamma = 0$, where $\dim\gamma$ denotes the topological dimension of γ, e.g. γ is totally disconnected; cf. [13, II.4.D]. Although the latter contradicts the fact that γ is a nondegenerated continuum. If $k > 1$, the inequality (4) is not valid too. It can be illustrated by the following example. Let $I = [0, 1]$. Obviously, $H^1(I) = 1 < \infty$ and, therefore, $H^k(I) = 0$ for any $k > 1$, and $\operatorname{diam} I = 1$. Thus, (5) does not hold for this simplest continuum I.

A nonnegative μ-measurable function $\rho : X \to [0, \infty]$ is called *admissible* for a family of continua Γ in (X, d) (abbr. $\rho \in \operatorname{adm}\Gamma$) if

$$\int\limits_X \rho\, \mathrm{d}m_\gamma \geq 1 \qquad \forall\, \gamma \in \Gamma\,.$$

3. METRIC MEASURE SPACES

Let now (X, d, μ) be a metric space with a Borel measure μ. Recall that the space (X, d, μ) is called *α-regular by Ahlfors*, if there exists a constant $C \geq 1$, such that

$$C^{-1} r^\alpha \leq \mu(B_r) \leq C r^\alpha$$

for all balls B_r in X of radius $r < \operatorname{diam} X$. It is well known that α-regular spaces have Hausdorff dimension α; see e.g. [11, p. 61]. A space (X, d, μ) is called *regular by Ahlfors*, if it is α-regular for some $\alpha \in (1, \infty)$.

We say also that a space (X, d, μ) is *upper α-regular at $x_0 \in X$*, if there is a constant $C > 0$ such that

$$\mu(B(x_0, r)) \leq C r^\alpha, \tag{6}$$

for all balls $B(x_0, r)$ centered at $x_0 \in X$ of radius $r < r_0$. A metric space (X, d, μ) is *upper regular* if the relation (6) holds at each point x for some $\alpha \in (1, \infty)$.

The *p-modulus*, $1 < p < \infty$, of the family Γ of continua γ in (X, d, μ) is defined by

$$M_p(\Gamma) = \inf_{\rho \in \operatorname{adm} \Gamma} \int\limits_X \rho^p(x)\, \mathrm{d}\mu(x).$$

The p-modulus of $\Gamma = \varnothing$ is defined by $M_p(\Gamma) = +\infty$.

By the relation (1) from [6], Proposition 2.2 yields

Corollary 3.1. *For any open set Ω in a metric measure space (X, d, μ) with Borel measure μ,*

$$M_p(\Gamma(\Omega, X \setminus \Omega; X)) \leq M_p(\Gamma(\Omega, \partial\Omega; \Omega)) \qquad \forall\, p \in (1, \infty).$$

Let D and D' be two continual domains in spaces (X, d, μ) and $(X', d', \mu',)$ respectively, $Q : X \to (0, \infty)$ be a μ-measurable function and $p \in (1, \infty)$. We say that a homeomorphism $f : D \to D'$ is called *continually ring Q-homeomorphism at a point $x_0 \in \overline{D}$* with respect to p-modulus if the inequality

$$M_p(\Gamma(f(C_0),\, f(C_1);\, D')) \leq \int\limits_{A \cap D} Q(x) \cdot \eta^p(d(x, x_0))\, \mathrm{d}\mu(x)$$

holds for any ring $A = A(x_0, r_1, r_2) := \{x_0 \in X : r_1 < d(x, x_0) < r_2\}$, $0 < r_1 < r_2 < \infty$, for any two continua $C_0 \subset \overline{B(x_0,\, r_1)} \cap D$ and $C_1 \subset D \setminus B(x_0, r_2)$, and any Borel function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\, \mathrm{d}r \geq 1.$$

Finally, we say that a homeomorphism $f : D \to D'$ is *continually ring Q-homeomorphism in D*, if f is a continually ring Q-homeomorphism at any $x_0 \in \overline{D}$ (see Fig. 1).

Following [18], we say that the boundary of a continual domain D is *continually weakly flat at a point $x_0 \in \partial D$* with respect to p-modulus, $p \in (1, \infty)$, if for arbitrary $N > 0$ and any neighborhood U of the point x_0 there exists its neighborhood $V \subset U$ such that

$$M_p(\Gamma(E, F; D)) \geq N$$

for any two continua E and F in D crossing ∂U and ∂V.

We also say that the boundary of a continual domain D is *continually strongly accessible at a point $x_0 \in \partial D$* with respect to p-modulus, $p \in (1, \infty)$, if for any neighborhood U of the point x_0 there are a compact set $E \subset D$, a neighborhood $V \subset U$ of x_0, and a real number $\delta > 0$ such that

$$M_p(\Gamma(E, F; D)) \geq \delta$$

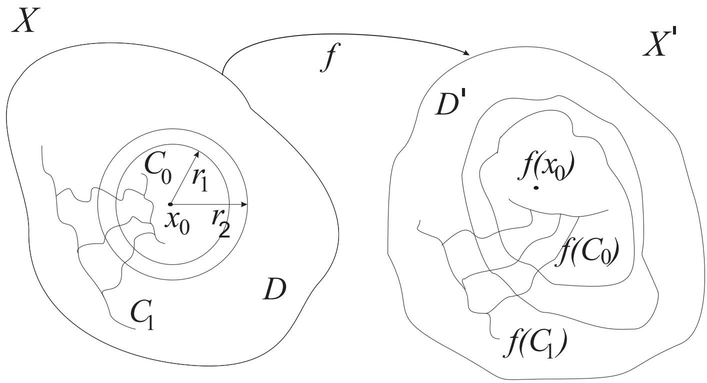

Figure 1

for any continuum F in D, crossing ∂U and ∂V.

Finally, the boundary of a continual domain D is called either *continually strongly accessible* with respect to p-modulus, $p \in (1, \infty)$, or *continually weakly flat* with respect to p-modulus, $p \in (1, \infty)$, if the corresponding property holds at each point of its boundary.

The notions of the weak flatness and strong accessibility were introduced for domains in $\mathbb{R}^n$ as generalizations of the corresponding P_1 and P_2-properties by Väisälä ([19]) and the quasiconformal accessibility and quasiconformal flatness by Näkki ([17]). In this paper we consider further extensions of the above notions since Γ stands for the family of continua.

Recall following [14] and [16] that $\varphi : X \to \mathbb{R}$ has *finite mean oscillation at a point* $x_0 \in X$, abbr. $\varphi \in FMO(x_0)$, if

$$\overline{\lim_{\varepsilon \to 0}} \ \frac{1}{\mu(B(x_0, \varepsilon))} \int_{B(x_0,\varepsilon)} |\varphi(x) - \widetilde{\varphi}_\varepsilon| \ \mathrm{d}\mu(x) < \infty \,, \tag{7}$$

where

$$\widetilde{\varphi}_\varepsilon = \frac{1}{\mu(B(x_0, \varepsilon))} \int\limits_{B(x_0,\varepsilon)} \varphi(x)\mathrm{d}\mu(x)$$

stands for the average of the function φ over the ball $B(x_0, \varepsilon) = \{x \in X : d(x, x_0) < \varepsilon\}$ with respect to the measure μ. Here the condition (7) assumes that φ is integrable with respect to the measure μ over some ball $B(x_0, \varepsilon)$, $\varepsilon > 0$. Any further details on the above class of functions can be found at [16].

The following result has been established in [18] for the case of families of curves. Here we extend it for the case of families of continua containing a priory fixed point.

Lemma 3.1. *Let the condition*

$$\int\limits_{A(x_0,\varepsilon,\varepsilon_0)} \psi^p(d(x,x_0))\,\mathrm{d}\mu(x) \;=\; o\left(\left[\int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\,\mathrm{d}t\right]^p\right) \tag{8}$$

hold as $\varepsilon \to 0$, *where* $\varepsilon_0 \in (0,\infty)$. *Suppose that* $\psi(t)$ *is a nonnegative function on* $(0,\infty)$, *satisfying* $0 < \int_\varepsilon^{\varepsilon_0} \psi(t)\,\mathrm{d}t < \infty$, $\forall\, \varepsilon \in (0,\varepsilon_0)$. *Then the p-modulus,* $p \in (1,\infty)$, *of all continua in* X, *containing* x_0, *vanishes.*

Remark 3.1. The condition (8) implies that under $\varepsilon \to 0$,

$$\int\limits_{A(x_0,\varepsilon,\varepsilon_1)} \psi^p(d(x,x_0))\,\mathrm{d}\mu(x) = o\left(\left[\int\limits_{\varepsilon}^{\varepsilon_1} \psi(t)\,\mathrm{d}t\right]^p\right) \qquad \forall\; \varepsilon_1 \in (0,\varepsilon_0). \tag{9}$$

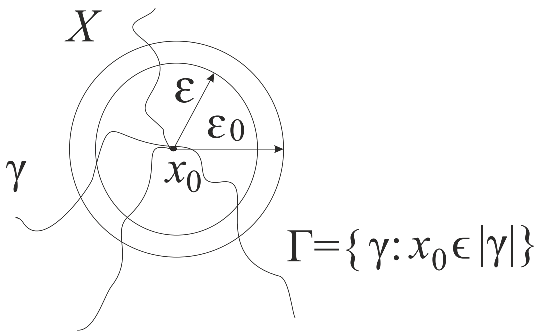

Figure 2

Proof of Lemma 3.1. Assume that Γ is a family of all continua in X, which contain the point x_0. For a sequence r_k such that $r_k \in (0,\varepsilon_0)$, $r_k \to 0$ as $k \to \infty$, denote by $\Gamma = \bigcup_{k=1}^{\infty} \Gamma_k$, where Γ_k is a family of all continua in X, containing the point x_0 and intersecting the spheres $S_k = S(x_0,r_k)$; see Fig. 2.

Pick the function

$$\rho(x) \;=\; \begin{cases} \psi(d(x,x_0)) \left(\int\limits_r^{r_k} \psi(t)\,\mathrm{d}t\right)^{-1}, & \text{if } x \in A(x_0,r,r_k), \\ 0, & \text{if } x \in X \backslash A(x_0,r,r_k). \end{cases}$$

For the family $\Gamma_k(r)$ intersecting the spheres S_k and $S(x_0,r)$, $r \in (0,r_k)$, this function is admissible in view of Proposition 2.3. Due to the fact that $\Gamma_k > \Gamma_k(r)$, we have

$$M_p(\Gamma_k) \;\le\; M_p(\Gamma_k(r)) \;\le\; \left(\int\limits_r^{r_k} \psi(t)\,\mathrm{d}t\right)^{-p} \int\limits_{A(x_0,r,r_k)} \psi^p(d(x,x_0))\,\mathrm{d}\mu(x)\,.$$

Hence, by the conditions (8) and (9), $M_p(\Gamma_k) = 0$, since $r \in (0, r_k)$ is arbitrary.

Now applying the subadditivity of the p-modulus, one gets

$$M_p(\Gamma) \leq \sum_{k=1}^{\infty} M_p(\Gamma_k) = 0\,,$$

which completes the proof of the lemma. □

Choosing in Lemma 3.1, $\psi(t) = 1/t$, we have the following conclusion.

COROLLARY 3.2. *Let for some $\varepsilon_0 \in (0, \infty)$, as $\varepsilon \to 0$, the condition*

$$\int\limits_{A(x_0,\varepsilon,\varepsilon_0)} \frac{\mathrm{d}\mu(x)}{d^p(x, x_0)} = o\left(\left[\log \frac{\varepsilon_0}{\varepsilon}\right]^p\right) \tag{10}$$

hold. Then the p-modulus, $p \in (1, \infty)$, of the family of all continua in X containing x_0, vanishes.

This result naturally extends Corollary 7.20 in [11] and Lemma 7.18 as well.

Remark 3.2. Note that for an upper α-regular metric space (X, d, μ), $\alpha > 1$, at a point x_0,

$$\int\limits_{r<d(x_0,x)<R_0} \frac{\mathrm{d}\mu(x)}{d^\alpha(x, x_0)} = O\left(\left[\log \frac{R_0}{r}\right]\right),$$

and, therefore, the condition (10) immediately holds; cf. [11, p. 54].

4. WEAKLY FLAT SPACES

Following [16, Chapter 13] and [18], we introduce some needed notions. A continually connected space (X, d, μ) is called *continually weakly flat at a point* $x_0 \in X$ with respect to p-modulus, $p \in (1, \infty)$, if for any neighborhood U of x_0 and arbitrary real $N > 0$ there exists a neighborhood $V \subseteq U$ of x_0 such that

$$M_p(\Gamma(E, F; X)) \geq N \tag{11}$$

for any pair of continua E and F in X intersecting both ∂V and ∂U. We say also that a continually connected space (X, d, μ) is *continually strongly accessible at a point* $x_0 \in X$ with respect to p-modulus, $p \in (1, \infty)$, if for any neighborhood U of x_0 there exist a neighborhood $V \subseteq U$ of x_0, a compact set E in X and a real number $\delta > 0$, such that $M_p(\Gamma(E, F; X)) \geq \delta$ for each continuum F in X intersecting ∂V and ∂U. Finally, a continually connected space (X, d, μ) is called *continually weakly flat* (*continually strongly accessible*), if the corresponding property holds at any point of this space.

Remark 4.1. In fact, in the definitions of continually weakly flat and continually strongly accessible spaces it is enough to restrict ourselves by neighborhoods of x_0 and, in particular, instead of U and V to choose sufficiently small balls (open and closed) centered at x_0. Moreover, by Proposition 2.2 one can consider continua E and F in $\overline{U}$. It is also clear that any continual domain in continually weakly flat space with respect to p-modulus, $p \in (1, \infty)$, is continually weakly flat space again due to the same statement (Proposition 2.2).

The proof of the following statement is similar to the corresponding proof of Proposition 3 in [1], therefore, we skip its proof.

PROPOSITION 4.1. *Let a space (X, d, μ) be continually weakly flat at a point $x_0 \in X$ with respect to p-modulus, $p \in (1, \infty)$. Then (X, d, μ) is continually strongly accessible at x_0 with respect to p-modulus.*

The next lemma provides a crucial tool for our research and goes back to the lines of Lemma 9.1 in [18]; cf. [16, Lemma 13.7].

LEMMA 4.1. *Let a space (X, d, μ) be continually weakly flat at a point $x_0 \in X$ with respect to p-modulus, $p \in (1, \infty)$. Then (X, d, μ) is locally continually connected at x_0.*

Proof. We start the proof by contradiction. Assume that the space X is not locally continually connected at the point x_0. Then there exists $r_0 \in (0, d_0)$, $d_0 = \sup_{x \in X} d(x, x_0)$, such that $\mu(U_{x_0}) := \mu(B(x_0, r_0)) < \infty$, and any neighborhood $V_{x_0} \subseteq U_{x_0}$ of x_0 has a continually connected component K_0, which contains x_0, and also continually connected components $K_1, \ldots, K_m, \ldots$, that are different from K_0, provided that $x_0 = \lim_{m \to \infty} x_m$ for some $x_m \in K_m$. The Janiszewski theorem and continual connectedness of X yield $\overline{K_m} \cap \partial V_{x_0} \neq \varnothing$ for all $m = 1, 2, \ldots$; see e.g. [15, §47, III, Thm 1].

The existence of such components of continual connectedness remains true for the case when the neighborhoods coincide, *i.e.* $V_{x_0} = U_{x_0} = B(x_0, r_0)$. Let $r_* \in (0, r_0)$, $K_i^* = K_i \cap \overline{B(x_0, r_*)}$ and $K_0^* = K_0 \cap \overline{B(x_0, r_*)}$. Then for any $i = 1, 2, \ldots$,

$$M_p(\Gamma(K_i^*, K_0^*; D)) \leq M_0 := \frac{\mu(U_{x_0})}{[r_0 - r_*]^p} < \infty. \tag{12}$$

Indeed, since the components K_i and K_0 cannot be connected by a continuum in $V_{x_0} = B(x_0, r_0)$ and any continuum connecting K_i^* and K_0^* should intersect the ring $B_0 \setminus \overline{B_*}$, due to the Darboux property on connected sets, the function

$$\rho(x) = \begin{cases} \frac{1}{r_0 - r_*}, & \text{if } x \in B(x_0, r_0) \setminus \overline{B(x_0, r_*)}\,, \\ 0, & \text{if } x \in X \setminus (B(x_0, r_0) \setminus \overline{B(x_0, r_*)}\,)\,, \end{cases}$$

is admissible for the family Γ_i of all rectifiable continua from $\Gamma(K_i^*, K_0^*; D)$; see [15, §46, I] and Proposition 2.3.

However, the upper bound (12) for the p-modulus contradicts the condition of continual weak flatness of X at x_0. Indeed, in view of (11), there is $r \in (0, r_*)$ such that

$$M_p(\Gamma(K_{i_0}^*, K_0^*; D)) \geq M_0 + 1$$

for sufficiently large $i_0 = 1, 2, \ldots$, because in the corresponding sets $K_{i_0}^*$ with $d(x_0, x_{i_0}) < r$ and K_0^*, there are continua intersecting both $\partial B(x_0, r_*)$ and $\partial B(x_0, r)$; cf. Proposition 2.2.

Thus, the assumption that X is not locally continually connected fails. This completes the proof. □

Combining Lemma 4.1 and Corollary 2.1 with Proposition 2.1 yields the following conclusions.

Corollary 4.1. *Continually weakly flat spaces (X, d, μ) are locally connected.*

Corollary 4.2. *An open set Ω of any continually weakly flat space (X, d, μ) becomes continual domain if and only if Ω is connected.*

Corollary 4.3. *Any continual domain D in a continually weakly flat space (X, d, μ) is continually connected at a point $x_0 \in \partial D$ if and only if D is locally connected at x_0.*

Now combining Lemmas 3.1 and 4.1 with Corollary 2.1, we obtain

Corollary 4.4. *Let (X, d, μ) be a continually weakly flat space at a point $x_0 \in X$ with respect to p-modulus, $p \in (1, \infty)$, and the condition (8) (or, in particular, (10)) holds. Then (X, d, μ) is continually connected at x_0.*

Finally, Remark 13.8 in [16] allows us to conclude

Corollary 4.5. *If a space X is continually weakly flat with respect to p-modulus, $p \in (1, \infty)$, and upper p-regular at a point $x_0 \in X$, then X is continually connected at x_0.*

5. CONTINUAL DOMAINS OF QUASIEXTREMAL LENGTH

In this section we investigate a subclass of domains with continually weakly flat boundaries. We say that a continual domain D in (X, d, μ) is *continual domain of quasiextremal length* with respect to p-modulus, $p \in (1, \infty)$, abbr. *continual QED domain*, if

$$M_p(\Gamma(E, F; X)) \le K\, M_p(\Gamma(E, F; D))$$

for some finite number $K \ge 1$ and any continua E and F in D; see Fig. 3.

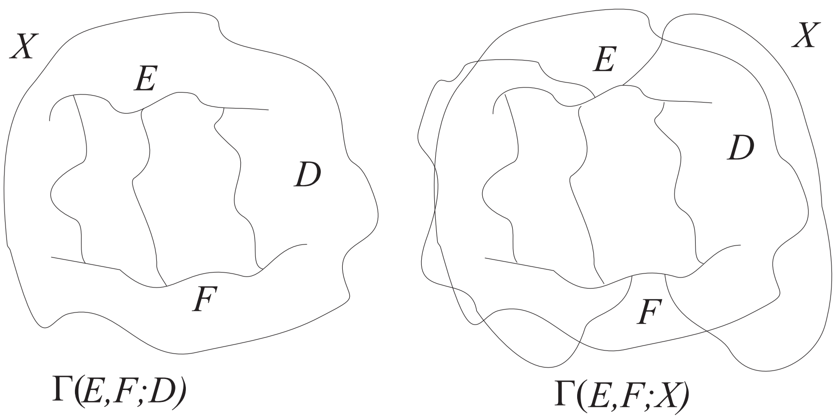

Figure 3

The class QED was introduced by Gehring and Martio in [8] under studying the boundary extension of quasiconformal mappings in higher dimensions. Note that continual QED domains provide a wider class of domains and have a special interest in view of various applications. Bounded convex domains, domains with smooth boundaries illustrate the simplest examples of the continual QED domains and the domains with continually weakly flat boundaries as well.

Obviously, by the definition, the continual QED domains in continually weakly flat spaces with respect to p-modulus, $p \in (1, \infty)$, have continually weakly flat boundaries. This fact allows us to formulate some results and continue our research regarding the boundary behavior of continually ring Q-homeomorphisms, started in [1].

Lemma 5.1. *Let f be a continually ring Q-homeomorphism with respect to p-modulus, $p \in (1, \infty)$, between two continual QED domains D and D' in continually weakly flat spaces X and X', respectively. Suppose that $\overline{D'}$ is* *compact and at a point $x_0 \in \partial D$ the following condition*

$$\int\limits_{A(x_0,\varepsilon,\varepsilon_0)} Q(x)\,\psi^p(d(x,x_0))\,\mathrm{d}\mu(x) = o\left(\left[\int\limits_{\varepsilon}^{\varepsilon_0}\psi(t)\,\mathrm{d}t\right]^p\right) \tag{13}$$

holds as $\varepsilon \to 0$. Here $\psi(t)$ is a nonnegative measurable function defined on $(0,\infty)$ such that

$$0 < \int\limits_{\varepsilon}^{\varepsilon_0}\psi(t)\,\mathrm{d}t < \infty \qquad \forall\,\varepsilon \in (0,\varepsilon_0).$$

Then $f : D \to D'$ extends to x_0 in (X', d', μ') by continuity.

Taking into account the appropriate properties of continual QED domains, Lemma 5.1 naturally follows from Lemma 4 in [1].

The following result is a conclusion from Lemma 5.1 with Corollary 4 and Remark 4 in [1].

Corollary 5.1. *In particular, the limit of $f(x)$ under $x \to x_0$ exists, if*

$$\int\limits_{A(x_0,\varepsilon,\varepsilon_0)} Q(x)\,\psi^p(d(x,x_0))\,\mathrm{d}\mu(x) < \infty \tag{14}$$

and

$$\lim_{\varepsilon\to 0}\int\limits_{\varepsilon}^{\varepsilon_0}\psi(t)\,\mathrm{d}t = \infty\,. \tag{15}$$

Theorem 2 in [1] and the corresponding properties of continual QED domains yield:

Theorem 5.1. *Let f be a continually ring Q-homeomorphism with respect to p-modulus, $p \in (1,\infty)$, between two continual QED domains D and D' in continually weakly flat spaces X and X', respectively. Suppose that $\overline{D'}$ is compact and at a point $x_0 \in \partial D$,*

$$\int\limits_{A(x_0,\varepsilon,\varepsilon_0)} \frac{Q(x)\,\mathrm{d}\mu(x)}{d(x,x_0)^p} = o\left(\left[\log\frac{\varepsilon_0}{\varepsilon}\right]^p\right), \qquad \varepsilon \to 0\,. \tag{16}$$

Then f admits a continuous extension to x_0 in (X', d').

Corollary 5.2. *In particular, the assertion of Theorem 5.1 holds if the singular integral*

$$\int \frac{Q(x)\,\mathrm{d}\mu(x)}{d(x,x_0)^p} \tag{17}$$

converges in a neighborhood of x_0.

Here and later on we assume $Q(x) = 0$ outside of D. Similarly to Theorem 3 from [2] together with Theorem 1.1 in [3] we obtain the following statement.

Theorem 5.2. *Let* f *be a continually ring* Q*-homeomorphism with respect to* p*-modulus,* $p \in (1, \infty)$*, between two continual QED domains* D *and* D' *in continually weakly flat spaces* X *and* X'*, respectively, and let* $\overline{D'}$ *be compact. If* $Q \in L^1_\mu(D)$*, then the inverse homeomorphism* $g = f^{-1}$ *admits a continuous extension* $\overline{g} : \overline{D'} \to \overline{D}$.

Combining Theorem 4 and Corollary 7 in [1] with the appropriate properties of continual QED domains, one gets

Theorem 5.3. *Let* f *be a continually ring* Q*-homeomorphism with respect to* p*-modulus,* $p \in (1, \infty)$*, between two continual QED domains* D *and* D' *in continually weakly flat spaces* X *and* X'*, respectively, and let* $\overline{D}$ *and* $\overline{D'}$ *be compact. If* $Q \in L^1_\mu(D)$ *satisfies either (16) or (17) at each point* $x_0 \in \partial D$*, then* f *admits a homeomorphic extension* $\overline{f} : \overline{D} \to \overline{D'}$.

Finally, combining Theorems 3 and 5 from [1] with the corresponding properties of QED domains yields:

Theorem 5.4. *Let* f *be a continually ring* Q*-homeomorphism with respect to* p*-modulus,* $p \in [2, \infty)$*, between two continual QED domains* D *and* D' *in continually weakly flat spaces* X *and* X'*, respectively, and let* $\overline{D}$ *and* $\overline{D'}$ *be compact. If at some point* $x_0 \in \partial D$*, the function* $Q : X \to (0, \infty)$ *has a finite mean oscillation at* $x_0 \in \partial D$*,*

$$\mu(B(x_0, 2r)) \le \gamma \cdot \log^{p-2} \frac{1}{r} \cdot \mu(B(x_0, r)) \qquad \forall\, r \in (0, r_0)\,, \tag{18}$$

and the space (X, d, μ) *is upper* p*-regular at* x_0*, then* f *continuously extends to* x_0*. If the last two conditions hold at each point* $x_0 \in \partial D$*, then* f *admits a homeomorphic extension to* $\overline{f} : \overline{D} \to \overline{D'}$.

Remark 5.1. As it was mentioned in [16, Remark 13.11], in the case of regular by Ahlfors spaces the doubling measure condition holds. This condition is much stronger than (18). Due to the compactness of $\overline{D}$, the function Q is integrable in a neighborhood of ∂D, which can be derived from the finite mean oscillation at points of ∂D. Recall that to ensure $Q \in FMO(x_0)$ at $x_0 \in \partial D$, it suffices to assume

$$\overline{\lim_{\varepsilon\to 0}}\ \frac{1}{\mu(B(x_0,\varepsilon))}\int_{B(x_0,\varepsilon)} Q(x)\ \mathrm{d}\mu(x) < \infty\,.$$

6. CONTINUAL NULL SETS FOR EXTREMAL DISTANCE

Recall that similarly to $\mathbb{R}^n$, $n \geq 2$, sets A of continual null sets for extremal distance in continually weakly flat spaces with respect to p-modulus, $p \in (1, \infty)$, cannot contain inner points. Moreover, they cannot split the space X even locally. This means that $D \setminus A$ has only one component of continual connectedness for any continual domain D in X. Thus, the complement to such sets A in X provides a quite partial case of continually QED domains. Therefore, the continual null sets for extremal distance in continually weakly flat spaces with respect to p-modulus, $p \in (1, \infty)$, play the same role in various removable problems for singular sets under quasiconformal mappings and their generalizations as in $\mathbb{R}^n$, $n \geq 2$. We start with the definition for such sets.

A closed set A in a space (X, d, μ) is called *continual null set for extremal distance* with respect to p-modulus, $p \in (1, \infty)$, abbr. continual NED set, if

$$M_p(\Gamma(E, F; D)) = M_p(\Gamma(E, F; D \setminus A)) \tag{19}$$

for any continual domain D in X and any continua E and F in D; see Fig. 4.

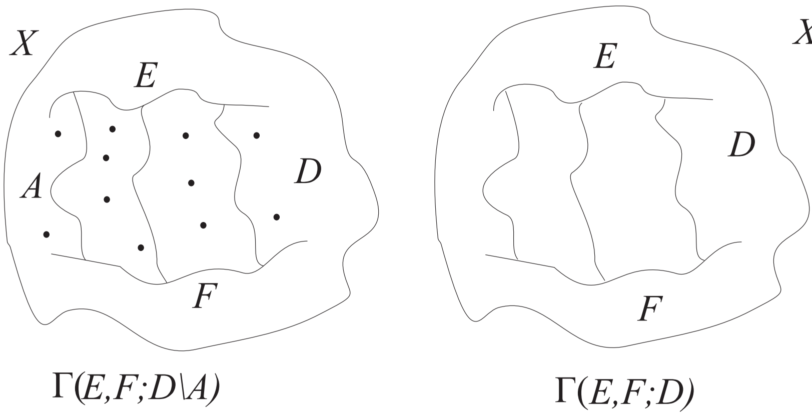

Figure 4

The following proposition is a continual counterpart of Proposition 9.2 in [18] with respect to p-modulus.

Proposition 6.1. *Let A be a continual NED set in a continually weakly flat space (X, d, μ) with respect to p-modulus, $p \in (1, \infty)$, which does not degenerate into one element set. Then*

1) A has no inner points;

2) $D \setminus A$ is a continual domain for any continual domain D in X.

Proof. 1) We show that A does not contain any inner points by contradiction. Assume that there exists a point $x_0 \in A$, which belongs to A with a ball $B(x_0, r_0)$ for some $r_0 > 0$, *i.e.* $B(x_0, r_0) \subseteq A$. Let also $x_* \in X$, $x_* \neq x_0$, and γ be a continuum containing both x_0 and x_* in X. Since always there exists a smaller ball $B^* = B(x_0, r^*) \subseteq A$, which does not contain x_*, one can find its subcontinuum $\gamma^* \subseteq \overline{B^*}$, such that x_0 and $x_1 \in \partial B^*$ belong to γ^*. For details, see Proposition 1 in [1]. Hence, letting $E = F = \gamma^*$, we have $M_p(\Gamma(E, F, X)) = \infty$, because the space X is continually weakly flat with respect p-modulus, $p \in (1, \infty)$. On the other hand, γ^* does not intersect with $X \setminus A$, then $\Gamma(E, F; X \setminus A) = \varnothing$, and, therefore, $M_p(\Gamma(E, F; X \setminus A)) = 0$. This contradicts the condition (19). Thus, the continual NED set A has no inner points.

2) Due to Corollary 4.2, it is enough to establish that $D \setminus A$ is connected. Denote by Ω_* one of connected components of the open set $D \setminus A$. Note that Ω_* is open in X, since by Corollary 4.1, X is locally connected; see [4, I.11.11]. Similarly, Ω as the union of all other connected components of the set $D \setminus A$ is also open. Assume that $\Omega \neq \varnothing$.

Put $\overline{\Omega}^0 := \overline{\Omega} \setminus \partial\overline{\Omega}$, $\overline{\Omega}_*^0 := \overline{\Omega}_* \setminus \partial\overline{\Omega}_*$, *i.e.* $\overline{\Omega}^0$ and $\overline{\Omega}_*^0$ are the interiors of closures of Ω and Ω_*, respectively. Then $\overline{\Omega}^0 \neq \varnothing$, since $\Omega \subseteq \overline{\Omega}^0$, $B := \partial\overline{\Omega} \cap D = \partial\overline{\Omega}_* \cap D = A \setminus \{(A \cap \overline{\Omega}^0) \cup (A \cap \overline{\Omega}_*^0)\}$, and due to the first part of the proof, $A^0 := A \setminus \partial A = \varnothing$, *i.e.* $A = \partial A$. Note also that $B \neq \varnothing$ by connectedness of D, since otherwise $D = \overline{\Omega}^0 \cup \overline{\Omega}_*^0$.

Considering the continual domain D as a metric space and $\overline{\Omega}^0$ as its subset, by Proposition 2.2, there exists a continuum in $\overline{\overline{\Omega}^0} \cap D = \overline{\Omega} \cap D$, such that $\gamma \cap \overline{\Omega}^0 \neq \varnothing$ and $\gamma \cap \partial\Omega \cap \partial\Omega_* \cap D \neq \varnothing$, because $\partial\overline{\Omega} \subseteq \partial\Omega$ and $\partial\overline{\Omega}_* \subseteq \partial\Omega_*$. Let $x_0 \in \gamma \cap \partial\Omega \cap \partial\Omega_* \cap D$, $x_* \in \Omega_*$ $x_n \in \Omega_*$, $x_n \to x_0$ under $n \to \infty$. Note that Ω_* is continually connected by Corollary 4.2, and, therefore, there is a sequence of continua γ_n in Ω_*, connecting the points x_* and x_n. Then $M_p(\Gamma(\gamma, \gamma_n; D)) \to \infty$ as $n \to \infty$, since D is a continually weakly flat space; cf. Remark 4.1. On the other hand, $\Gamma(\gamma, \gamma_n; D \setminus A) = \varnothing$, and, hence, $M_p(\Gamma(\gamma, \gamma_n; D \setminus A)) = 0$. This contradicts the assumption that $D \setminus A$ is not connected. Thus, $D \setminus A$ is a continual domain, which completes the proof. □

The proof of the following lemma follows the lines of the proof of Lemma 9.4 in [18], although it requires some modifications and efforts needed for continually weakly flat spaces.

LEMMA 6.1. *Let X and X' be continually weakly flat spaces with respect to p-modulus, $p \in (1,\infty)$, D be a continual domain in X, $A \subset D$ be a continual NED set in X, and f be a continually ring Q-homeomorphism from $G = D \setminus A$ to X'. If the limit set*

$$A' := C(A, f) = \{x' \in X' : x' = \lim_{k\to\infty} f(x_k), x_k \in G, \lim_{k\to\infty} x_k \in A\}$$

is a continual NED set in X' and $G' = f(G)$, then $D' = G' \cup A'$ is a continual set in X'. Moreover, there are continual domains D^* in X, satisfying $A \subset D^*$, $\overline{D^*} \subset D$ and $A' \cap A^* = \varnothing$, where $A^* := f(\partial D^*)$.

Proof. First, we note that the continual NED set A is closed in the compact space X and, therefore, $\varepsilon_0 = \operatorname{dist}(A, \partial D) > 0$. Thus, A lies in the open set

$$\Omega = \{x \in X : \operatorname{dist}(x, A) < \varepsilon\}$$

for any (fixed) $\varepsilon \in (0, \varepsilon_0)$, which is also located in D. Since A is compact, A is contained in a finite number of components of (continual) connectedness $\Omega_1, \ldots, \Omega_m$ of the set Ω. Let x_0 be an arbitrary point of the continual domain D and let $x_k \in \Omega_k$, $k = 1, \ldots, m$. Then there exist continua γ_k in D, containing x_0 and x_k, $k = 1, \ldots, m$. Note that the set $C = \bigcup\limits_{k=1}^{m} \gamma_k$ is also a continuum in D and $\delta_0 = \operatorname{dist}(C, \partial D) > 0$; see e.g. [15, Thm 1, Ch. 5, § 47].

Consider open sets $D_\delta = \{x \in D : \operatorname{dist}(x, \partial D) > \delta\}$. The triangle inequality implies that the set

$$C_0 = C \bigcup \left(\bigcup_{k=1}^{m} \Omega_k \right)$$

lies in D_δ for any $\delta \in (0, d_0)$, where $d_0 = \min(\varepsilon_0 - \varepsilon, \delta_0)$. Moreover, C_0 is contained in one component of (continual) connectedness D^*_δ of the set D_δ, since the set C_0 is (continually) connected; cf. [4, Proposition I.11.11] and Corollary 4.1.

By the construction, $\overline{D^*_\delta} \subset D$, and D^*_δ are continual domains in X, and, hence, continually weakly flat spaces with respect to p-modulus, $p \in (1,\infty)$. By Proposition 6.1, the sets $G_\delta = D^*_\delta \setminus A$ are continual domains with continually weakly flat boundaries A in the spaces D^*_δ, $\delta \in (0, d_0)$.

Let $f_\delta = f|_{G_\delta}$ and $g_\delta = (f_\delta)^{-1} : G'_\delta \to G_\delta$, where $G'_\delta = f_\delta(G_\delta)$. Then, the symmetry

$$A' = C(A, f_\delta), \quad A = C(A', g_\delta), \qquad \forall\, \delta \in (0, d_0)\,,$$

has place; cf. [16, Proposition 13.5]. Note that $\partial D^*_\delta, \delta \in (0, d_0)$ are compact subsets of the continual domain G, and, therefore, $f(\partial D^*_\delta)$ are compact subsets of the continual domain $G' = f(G)$, which do not intersect with A' (see again [16, Proposition 13.5]). Thus, $d_\delta = \mathrm{dist}\,(A', f(\partial D^*_\delta)) > 0$ for any $\delta \in (0, d_0)$. By Lemma 4.1, the space X' is continually connected, and, therefore, for any point $x_0 \in A'$ there exists a continual domain $U \subset B(x_0, d_\delta)$, which is a neighborhood of x_0, and, by Proposition 6.1, $V = U \setminus A'$ is also a continual domain which is a continual subdomain of G' by construction. Thus, $D' = G' \cup A'$ is a continual domain in X'. The proof is complete. □

Applying Lemma 5.1, Proposition 6.1 and Lemma 6.1, we obtain the following statement for the continual NED sets; cf. Remarks 13.5 and 13.6 in [16].

Corollary 6.1. *Let X and X' be compact continually weakly flat spaces with respect to p-modulus, $p \in (1, \infty)$, D be a continual domain in X, $A \subset D$ be a continual NED set in X. Suppose that f is a continually ring Q-homeomorphism, acting from $G = D \setminus A$ to X', such that the limit set $C(A, f)$ is a continual NED set in X'. If at some point $x_0 \in A$ the condition (13) holds, then f admits a continue extension to the point x_0.*

Remark 6.1. In particular, f extends by continuity to $x_0 \in A$, if for this point one from the conditions (14)–(15), (16), (17) or (18) holds assuming $Q \in \mathrm{FMO}(x_0)$.

Based on [18, Lemma 9.6] and applying the properties of continually ring Q-homeomorphisms with respect to p-modulus, we get the following result.

Corollary 6.2. *Let X and X' be compact continually weakly flat spaces with respect to p-modulus, $p \in (1, \infty)$, D be a continual domain in X, $A \subset D$ be a continual NED set in D, and f be a continually ring Q-homeomorphism from $G = D \setminus A$ to X' with a continual NED set $A' = C(A, f)$. If $Q \in L^1_\mu(D)$, then the inverse homeomorphism $g = f^{-1} : G' \to G$, $G' = f(G)$, admits a continuous extension $\overline{g} : D' \to D$, where $D' = G' \cup A'$.*

Remark 6.2. Thus, if $Q \in L^1_\mu(G)$ satisfies either the condition (18) or one of conditions (14)–(17) with $Q \in \mathrm{FMO}(x_0)$, the inequality of doubling measure

$$\mu(B(x_0, 2r)) \leq \gamma \cdot \mu(B(x_0, r)) \qquad \forall\, r \in (0, r_0)$$

at each point $x_0 \in A$, then any continually ring Q-homeomorphism f acting from a continual domain $G = D \setminus A$ to X' with continual NED domains A and $A' = C(A, f)$, has a homeomorphic extension $\overline{f} : D \to D'$, where $D' = G' \cup A'$ and $G' = f(G)$.

By Theorem 13.12 in [16] and due to the corresponding properties of continually ring Q-homeomorphisms with respect to p-modulus, one yields

Corollary 6.3. *Let X and X' be compact continually weakly flat spaces with respect to p-modulus, $p \in [2, \infty)$, D be a continual domain in X, $A \subset D$ be a continual NED set in X, and f be a continually ring Q-homeomorphism from $G = D \setminus A$ to X' with the continual NED set $A' := C(A, f)$. If Q has a finite mean oscillation and X is a p-regular by Ahlfors at any point $x_0 \in A$, then f admits a homeomorphic extension $\overline{f} : D \to D'$, where $D' = G' \cup A'$, $G' = f(G)$.*

The above results can be transferred to the case of smooth Riemannian manifolds which are closely related to various problems of modern theoretical physics and to the Loewner spaces, Carnot and Heisenberg groups as well.

REFERENCES

[1] E. S. Afanas'eva, *On the boundary behavior of one class of mappings in metric spaces.* Ukrainian Math. J. **66** (2014), *1*, 16–29.

[2] E. S. Afanas'eva, *Generalized quasi-isometries on smooth Reimannian manifolds.* Math. Notes **102** (2017), *1*, 12–21.

[3] E. Afanas'eva and A. Golberg, *Finitely bi-Lipschitz homeomorphisms between Finsler manifolds.* Anal. Math. Phys. **10** (2020), *4*, 48, 16 pp.

[4] N. Bourbaki, *Elements of mathematics. General topology.* **Part 1**. Hermann, Paris; Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1966.

[5] M. Cristea, *Boundary behaviour of the mappings satisfying generalized inverse modular inequalities.* Complex Var. Elliptic Equ. **60** (2015), *4*, 437–469.

[6] B. Fuglede, *Extremal length and functional completion.* Acta Math. **98** (1957), 171–219.

[7] F. W. Gehring, *Rings and quasiconformal mappings in space.* Trans. Amer. Math. Soc. **103** (1962), 353–393.

[8] F. W. Gehring and O. Martio, *Quasiextremal distance domains and extension of quasiconformal mappings.* J. Analyse Math. **45** (1985), 181–206.

[9] A. Golberg, *Differential properties of (α, Q)-homeomorphisms.* Further progress in analysis, 218–228, World Sci. Publ., Hackensack, NJ, 2009.

[10] A. Golberg and R. Salimov, *Logarithmic Hölder continuity of ring homeomorphisms with controlled p-module.* Complex Var. Elliptic Equ. **59** (2014), *1*, 91–98.

[11] J. Heinonen, *Lectures on analysis on metric spaces.* Universitext. Springer-Verlag, New York, 2001.

[12] D. A. Herron and P. Koskela, *Quasiextremal distance domains and conformal mappings onto circle domains*. Complex Variables Theory Appl. **15** (1990), *3*, 167–179.

[13] W. Hurewicz and H. Wallman, *Dimension Theory*. Princeton Mathematical Series, Vol. 4. Princeton University Press, Princeton, N. J., 1941.

[14] A. A. Ignat'ev and V. I. Ryazanov, *Finite mean oscillation in mapping theory*. Ukr. Math. Bull. **2** (2005), *3*, 403–424.

[15] K. Kuratowski, *Topology. Vol. II*, New edition, revised and augmented. Academic Press, New York-London; Panstwowe Wydawnictwo Naukowe Polish Scientific Publishers, Warsaw, 1968.

[16] O. Martio, V. Ryazanov, U. Srebro, and E. Yakubov, *Moduli in modern mapping theory*. Springer Monographs in Mathematics. Springer, New York, 2009.

[17] R. Näkki, *Boundary behavior of quasiconformal mappings in n-space*. Ann. Acad. Sci. Fenn. Ser. A I **484**, 1970.

[18] V. I. Ryazanov and R. R. Salimov, *Weakly planar spaces and boundaries in the theory of mappings*. Ukr. Math. Bull. **4** (2007), *2*, 199–234.

[19] J. Väisälä, *Lectures on n-dimensional quasiconformal mappings*. Lecture Notes in Mathematics, Vol. 229. Springer-Verlag, Berlin-New York, 1971.

[20] Sh. Sh. Yang, *QED domains and NED sets in $\overline{\mathbb{R}}^n$*. Trans. Amer. Math. Soc. **334** (1992), *1*, 97–120.

Institute of Applied Mathematics and Mechanics of the NAS of Ukraine
1 Dobrovol'skogo St., Slavyansk 84100, Ukraine
es.afanasjeva@gmail.com

Holon Institute of Technology
Department of Mathematics
52 Golomb St., P.O.B. 305, Holon 5810201, Israel
golberga@hit.ac.il