GAP OPENING IN THE SPECTRUM OF SOME DIRAC-LIKE PSEUDO-DIFFERENTIAL OPERATORS

J.-M. BARBAROUX, H.D. CORNEAN, and S. ZALCZER

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In this paper, we study the opening of a spectral gap for a class of 2-dimensional periodic Hamiltonians which include those modelling multilayer graphene. The kinetic part of the Hamiltonian is given by $\boldsymbol{\sigma} \cdot \boldsymbol{F}(-i\nabla)$, where $\boldsymbol{\sigma}$ denotes the Pauli matrices and $\boldsymbol{F}$ is a sufficiently regular vector-valued function which equals 0 at the origin and grows at infinity. Its spectrum is the whole real line. We prove that a gap appears for perturbations in a certain class of periodic matrix-valued potentials depending on $\boldsymbol{F}$, and we study how this gap depends on different parameters.

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1. INTRODUCTION, MODEL AND MAIN RESULT

Graphene is a two-dimensional material made of carbon atoms on a honeycomb lattice. Among its remarkable properties is its energy band structure, with two bands crossing at the Fermi level [4]. This particular structure has suggested to model the dynamics of one electron in a graphene sheet by the free massless two-dimensional Dirac operator.

An interesting problem is to study the electronic properties of a material which is not a single sheet of graphene but several stacked layers of graphene. In this case, the dynamics of the electron can be approximated by an effective Hamiltonian which typically is a Nth order Dirac-like operator, N being the number of layers (see [7] and references therein).

One of the major problems linked with graphene is to tune an energy bandgap at the Fermi level, making graphene a semiconductor. To realize this, one of the possibilities is to use the so-called graphene antidot lattices, which consist of a sheet of graphene periodically patterned with obstacles such as holes. In the case of single-layer graphene antidot lattices, the gap opening has been numerically achieved in [5] and was proved in [1] with a mathematical approach. See also [2, 3] and references therein for rigorous studies of spectral properties of Dirac operators modelling graphene antidot lattices.

The goal of this article is to generalize such gap opening results to higher-order Hamiltonians, including the ones for multilayer graphene.

Namely, we want to study gap opening in the spectrum under periodic pertubations of the Hamiltonian

$$H_0 = \boldsymbol{\sigma} \cdot \boldsymbol{F}(-i\nabla)$$

on $L^2(\mathbb{R}^2, \mathbb{C}^2)$, where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ denotes the usual Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^3$. More precisely, this means that for any $\boldsymbol{\psi} \in \mathrm{Dom}(H_0)$

$$H_0\boldsymbol{\psi}(\mathbf{x}) = \left(\mathcal{F}^{-1}\left[\boldsymbol{\sigma} \cdot \boldsymbol{F}(\cdot)\mathcal{F}\boldsymbol{\psi}\right]\right)(\mathbf{x}),$$

where $\mathcal{F}$ denotes the Fourier transform on $L^2(\mathbb{R}^2, \mathbb{C}^2)$

$$\mathcal{F}\boldsymbol{\psi}(\boldsymbol{p}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i\boldsymbol{p}\cdot\mathbf{x}}\boldsymbol{\psi}(\mathbf{x})\mathrm{d}\mathbf{x}.$$

We suppose that the function $\boldsymbol{F}$ fulfils the following assumptions.

Hypothesis 1. (i) $\boldsymbol{F}$ belongs to $\mathcal{C}^3(\mathbb{R}^2, \mathbb{R}^3)$.

(ii) There exist constants K_0', $K_i > 0$ such that for all $\boldsymbol{p} \in \mathbb{R}^2$,

$$\begin{aligned} K_0'|\boldsymbol{p}|^d \leqslant |\boldsymbol{F}(\boldsymbol{p})| \leqslant K_0|\boldsymbol{p}|^d\,, \\ |D^i\boldsymbol{F}(\boldsymbol{p})| \leqslant K_i < \boldsymbol{p} >^{d-|i|} \end{aligned} \tag{1}$$

for some $d > 0$ and any multi-index i such that $1 \leqslant |i| \leqslant 3$. Here $< \boldsymbol{p} >= \sqrt{1 + |\boldsymbol{p}|^2}$ and D^i denotes the multi-index partial derivative operator.

(iii) There exists a 2×3 rank 2 matrix A such that in a neighbourhood of 0,

$$\boldsymbol{F}(\boldsymbol{p}) = |\boldsymbol{p}|^{d-1}A\boldsymbol{p} + O(|\boldsymbol{p}|^{d+1}).$$

Proposition 1. *The operator H_0 is unitarily equivalent to a multiplication operator by $\sigma_3|\boldsymbol{F}|$. It is then self-adjoint on* $\mathrm{Dom}(H_0) = \mathcal{F}^{-1}(\mathrm{Dom}(|\boldsymbol{F}(\cdot)|))$ *and its spectrum is given by the essential range of $\pm|\boldsymbol{F}|$, which is $\mathbb{R}$ under Hypothesis 1 (i) and (ii).*

Proof. The proof, identical to the one for the free Dirac operator, comes directly from the definition of H_0 through the Fourier transform which is unitary (cf. [11]). □

In order to open a gap around the zero energy, we will perturb H_0 with a periodic potential defined as follows. Let

$$\boldsymbol{\chi} = (\chi_1, \chi_2, \chi_3) : \mathbb{R}^2 \to \mathbb{R}^3$$

where each χ_i is a bounded function with compact support included in the set $\Omega =]-\frac{1}{2}, \frac{1}{2}]^2$.

Let $\beta > 0$ and $\alpha \in]0, 1]$. The perturbed Hamiltonian is

$$H(\alpha, \beta) = H_0 + \beta \sum_{\gamma \in \mathbb{Z}^2} \boldsymbol{\chi}\left(\frac{\mathbf{x} - \gamma}{\alpha}\right) \cdot \boldsymbol{\sigma}.$$

The operator $H(\alpha, \beta)$ is $\mathbb{Z}^2$-periodic and self-adjoint on $\mathrm{Dom}(H_0)$.

Remark 1. In [1], the authors treated the particular case corresponding to the free massless Dirac operator where $\boldsymbol{F}(\boldsymbol{p}) = (p_1,\, p_2,\, 0)$ and $\chi_1 = \chi_2 = 0$.

Let us denote $\Phi_i = \int_\Omega \chi_i(\mathbf{x}) \mathrm{d}\mathbf{x}$, $1 \leqslant i \leqslant 3$, and let us introduce the three-dimensional vector

$$\boldsymbol{\Phi} = (\Phi_i)_{1 \leqslant i \leqslant 3} \in \mathbb{R}^3.$$

We denote by $\boldsymbol{\Phi}_{||}$ the projection of $\boldsymbol{\Phi}$ on $\mathrm{Ran}(A)$ and by $\Phi_\perp$ the projection on $\mathrm{Ran}(A)^\perp$.

Here is the main result of our paper.

THEOREM 2. *Suppose that* $\Phi_\perp \neq 0$. *Let* $d' = \min(d, 2)$. *There exist some constants* $\lambda_0, C > 0$ *and* $\delta \in]0, 1[$ *with* $C\delta < \frac{|\Phi_\perp|}{2}$ *such that for any* $\alpha \in]0, 1/2]$ *and* $\beta > 0$ *satisfying* $\alpha^2 \beta < \lambda_0$, $\alpha^{d'} \beta < \delta$ *we have that the interval*

$$\left[-\alpha^2 \beta \left(\frac{|\Phi_\perp|}{2} - C\alpha^{d'}\beta\right), \alpha^2 \beta \left(\frac{|\Phi_\perp|}{2} - C\alpha^{d'}\beta\right)\right]$$

belongs to the resolvent set $\rho(H(\alpha, \beta))$.

Remark 3. In [1], this condition is achieved since the kinetic part is in the subspace spanned by σ_1 and σ_2 and the potential is in $\mathrm{Span}(\sigma_3)$.

As in [1], we use the Floquet-Bloch transformation to come to a problem on the unit square, where the gradient has a well-known eigenbasis. Then, we use a Feshbach map argument, separating the problem between constant and non-constant modes. While the estimate on the constant subspace is direct, we need to use decay of the resolvent of the free operator and repeated applications of the resolvent equation to prove the invertibility on the orthogonal.

The paper is organized as follows. In Section 2 we perform a detailed analysis of the integral kernel of the free resolvent, including its local singularities and off-diagonal decay. In Section 3 we give the proof of the main theorem, while in the Appendix we summarize the results we need from the Bloch-Floquet transformation.

2. RESOLVENT DECAY AND INTEGRAL KERNEL

In this chapter, we study the behaviour of the integral kernel for the free resolvent. Due to the expression of the kinetic energy, we do not have an explicit formula for this integral kernel. Nevertheless, we can prove that the integral kernel exists, it has integrable local singularities and has a sufficiently fast off-diagonal polynomial decay.

The result states as follows.

PROPOSITION 2. *Let M_d be the function defined on $\mathbb{R}^2 \times \mathbb{R}^2$ by:*

$$M_d(\mathbf{x}, \mathbf{x}') = \begin{cases} \frac{1}{|\mathbf{x}-\mathbf{x}'|^{2-d}} + 1 & \text{if } |\mathbf{x} - \mathbf{x}'| \leqslant 1 \text{ and } d \neq 2; \\ -\log|\mathbf{x} - \mathbf{x}'| + 1 & \text{if } |\mathbf{x} - \mathbf{x}'| \leqslant 1 \text{ and } d = 2; \\ \frac{1}{|\mathbf{x}-\mathbf{x}'|^3} & \text{if } |\mathbf{x} - \mathbf{x}'| \geqslant 1. \end{cases}$$

The operator $(H_0 - i)^{-1}$ has an integral kernel, denoted by $(H_0 - i)^{-1}(\mathbf{x}, \mathbf{x}')$, such that

$$\left|(H_0 - i)^{-1}(\mathbf{x}, \mathbf{x}')\right| \leqslant M_d(\mathbf{x}, \mathbf{x}') \,. \tag{2}$$

The proof of this Proposition is based on the following Lemma, and is postponed to the end of this section.

LEMMA 1. *There exists some $C > 0$ such that for all $\boldsymbol{f}$ and $\boldsymbol{g} \in L^2(\mathbb{R}^2, \mathbb{C}^2)$ we have*

$$|\langle \boldsymbol{f}, (H_0 \pm i)^{-1} \boldsymbol{g} \rangle| \leqslant C \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\boldsymbol{f}(\mathbf{x})| M_d(\mathbf{x}, \mathbf{x}') |\boldsymbol{g}(\mathbf{x}')| \mathrm{d}\mathbf{x} \mathrm{d}\mathbf{x}'. \tag{3}$$

Remark 4. If $d > 2$ then M_d is bounded while if $d < 2$, M_d has a local singularity of the type $\frac{1}{|\mathbf{x}-\mathbf{x}'|^{2-d}}$. This is why we denote $d' = \min(d, 2)$; we can then write if $d \neq 2$ $M_d \leqslant \frac{2}{|\mathbf{x}-\mathbf{x}'|^{2-d'}}$.

Proof of Lemma 1. For $\boldsymbol{p} \in \mathbb{R}^2$, we define the 2×2 matrix

$$G(\boldsymbol{p}) = (\boldsymbol{\sigma} \cdot \boldsymbol{F}(\boldsymbol{p}) - i)^{-1}.$$

The operator of multiplication by G is bounded on $L^2(\mathbb{R}^2, \mathbb{C}^2)$.

Remind that $< \boldsymbol{p} >= \sqrt{1 + |\boldsymbol{p}|^2}$. We define, for $\epsilon > 0$, the regularized kernel

$$K_\epsilon(\mathbf{x}, \mathbf{x}') = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i\boldsymbol{p}\cdot(\mathbf{x}-\mathbf{x}')} G(\boldsymbol{p}) e^{-\epsilon<\boldsymbol{p}>} \mathrm{d}\boldsymbol{p} = \mathcal{F}^{-1}(G e^{-\epsilon<\cdot>})(\mathbf{x} - \mathbf{x}').$$

The estimates (1) of Hypothesis 1(ii) implies that for any multi-index N such that $|N| \leqslant 3$ there exists $C_N > 0$ such that

$$|D^N G(p)| \leqslant \frac{C_N}{< \boldsymbol{p} >^{|N|+d}}. \tag{4}$$

For $\mathbf{x} = (x_1, x_2)$ and $\mathbf{x}' = (x_1', x_2')$, choose $l \in \{1, 2\}$ such that $|\mathbf{x} - \mathbf{x}'| \leqslant \sqrt{2}|x_l - x_l'|$. By repeated integrations by part, we find that for any integer $M \leqslant 3$ we have :

$$(x_l - x_l')^M K_\epsilon(\mathbf{x}, \mathbf{x}') = i^M \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i\boldsymbol{p}\cdot(\mathbf{x}-\mathbf{x}')} \frac{\partial^M}{\partial p_l^M} \left(G(\boldsymbol{p}) e^{-\epsilon<\boldsymbol{p}>}\right) \mathrm{d}\boldsymbol{p}. \tag{5}$$

Hence we have

$$|x_l - x_l'|^M |K_\epsilon(\mathbf{x}, \mathbf{x}')| \leqslant \frac{1}{2\pi} \int_{\mathbb{R}^2} \left| \frac{\partial^M}{\partial p_l^M} \left(G(\boldsymbol{p}) e^{-\epsilon<\boldsymbol{p}>}\right) \right| \mathrm{d}\boldsymbol{p}. \tag{6}$$

Pick $M = 3$. We have then, by product rule and denoting $E(\boldsymbol{p}) = e^{-\epsilon<\boldsymbol{p}>}$,

$$\begin{aligned} \frac{\partial^3}{\partial p_l^3}(G(\boldsymbol{p})E(\boldsymbol{p})) =& \frac{\partial^3 G(\boldsymbol{p})}{\partial p_l^3} E(\boldsymbol{p}) + 3 \frac{\partial^2 G(\boldsymbol{p})}{\partial p_l^2} \frac{\partial E(\boldsymbol{p})}{\partial p_l} \\ &+ 3 \frac{\partial G(\boldsymbol{p})}{\partial p_l} \frac{\partial E(\boldsymbol{p})^2}{\partial p_l^2} + G(\boldsymbol{p}) \frac{\partial E(\boldsymbol{p})^3}{\partial p_l^3}. \end{aligned}$$

Furthermore, we have:

$$\begin{aligned} \frac{\partial E(\boldsymbol{p})}{\partial p_l} =& - \epsilon p_l \frac{E(\boldsymbol{p})}{< \boldsymbol{p} >}, \\ \frac{\partial^2 E(\boldsymbol{p})}{\partial p_l^2} =& - \epsilon \frac{E(\boldsymbol{p})}{< \boldsymbol{p} >} + \epsilon p_l^2 \frac{E(\boldsymbol{p})}{< \boldsymbol{p} >^3} + \epsilon^2 p_l^2 \frac{E(\boldsymbol{p})}{< \boldsymbol{p} >^2}, \\ \frac{\partial^3 E(\boldsymbol{p})}{\partial p_l^3} =& 3\epsilon p_l \frac{E(\boldsymbol{p})}{< \boldsymbol{p} >^3} + 3\epsilon^2 p_l \frac{E(\boldsymbol{p})}{< \boldsymbol{p} >^2} - 3\epsilon p_l^3 \frac{E(\boldsymbol{p})}{< \boldsymbol{p} >^5} \\ & - 3\epsilon^2 p_l^3 \frac{E(\boldsymbol{p})}{< \boldsymbol{p} >^4} - \epsilon^3 p_l^3 \frac{E(\boldsymbol{p})}{< \boldsymbol{p} >^3}. \end{aligned} \tag{7}$$

Moreover, since $x^k e^{-x}$ is bounded on $\mathbb{R}^+$ for all k, there exist some constants c_k such that for all $\epsilon > 0$ and $\boldsymbol{p} \in \mathbb{R}^2$

$$\epsilon^k e^{-\epsilon<\boldsymbol{p}>} \leqslant c_k < \boldsymbol{p} >^{-k}. \tag{8}$$

In the sequel, denoting by C a generic constant *independent of* ϵ, we obtain from (7), the above bound (8), and the fact that $|E(\boldsymbol{p})| \leqslant 1$ and $\frac{|p_l|}{<\boldsymbol{p}>} \leqslant 1$, that for $j \in \{1, 2, 3\}$,

$$\left| \frac{\partial^j E(\boldsymbol{p})}{\partial p_l^j} \right| \leqslant \frac{C}{< \boldsymbol{p} >^j}. \tag{9}$$

From (4) and (9) we obtain for $j \in \{0, 1, 2, 3\}$,

$$\int_{\mathbb{R}^2} \left| \frac{\partial^j}{\partial {p_j}^j} G(\boldsymbol{p}) \right| \left| \frac{\partial^{3-j}}{\partial {p_j}^{3-j}} E(\boldsymbol{p}) \right| \mathrm{d}\boldsymbol{p} \leqslant C$$

Hence, according to (6), we have $|K_\epsilon(\mathbf{x},\mathbf{x}')| \leqslant \frac{C}{|x_l-x_l'|^3}$ and thus

$$|K_\epsilon(\mathbf{x},\mathbf{x}')| \leqslant \frac{C}{|\mathbf{x}-\mathbf{x}'|^3}. \tag{10}$$

This estimate is only useful if $|\mathbf{x}-\mathbf{x}'| \geqslant 1$.

Let us now study the case $|\mathbf{x}-\mathbf{x}'| \leqslant 1$. We write

$$\begin{aligned} 2\pi K_\epsilon(\mathbf{x},\mathbf{x}') = &\int_{|\boldsymbol{p}|\leqslant 1} e^{i\boldsymbol{p}\cdot(\mathbf{x}-\mathbf{x}')} G(\boldsymbol{p}) e^{-\epsilon<\boldsymbol{p}>} \mathrm{d}\boldsymbol{p} \\ &+ \int_{1\leqslant|\boldsymbol{p}|\leqslant|\mathbf{x}-\mathbf{x}'|^{-1}} e^{i\boldsymbol{p}\cdot(\mathbf{x}-\mathbf{x}')} G(\boldsymbol{p}) e^{-\epsilon<\boldsymbol{p}>} \mathrm{d}\boldsymbol{p} \\ &+ \int_{|\mathbf{x}-\mathbf{x}'|^{-1}\leqslant|\boldsymbol{p}|} e^{i\boldsymbol{p}\cdot(\mathbf{x}-\mathbf{x}')} G(\boldsymbol{p}) e^{-\epsilon<\boldsymbol{p}>} \mathrm{d}\boldsymbol{p}. \end{aligned} \tag{11}$$

We simply bound the first term in the right hand side of (11) by

$$\left| \int_{|\boldsymbol{p}|\leqslant 1} e^{i\boldsymbol{p}\cdot(\mathbf{x}-\mathbf{x}')} G(\boldsymbol{p}) e^{-\epsilon<\boldsymbol{p}>} \mathrm{d}\boldsymbol{p} \right| \leqslant \int_{|\boldsymbol{p}|\leqslant 1} |G(\boldsymbol{p})| \mathrm{d}\boldsymbol{p}\,, \tag{12}$$

which is finite and independent of ϵ.

To bound the second term in the right hand side of (11), we use from Hypothesis 1(ii) that $|G(\boldsymbol{p})| \leqslant \frac{C}{|\boldsymbol{p}|^d}$ which yields

$$\left| \int_{1\leqslant|\boldsymbol{p}|\leqslant|\mathbf{x}-\mathbf{x}'|^{-1}} e^{i\boldsymbol{p}\cdot(\mathbf{x}-\mathbf{x}')} G(\boldsymbol{p}) e^{-\epsilon<\boldsymbol{p}>} \mathrm{d}\boldsymbol{p} \right| \leqslant 2\pi \int_1^{|\mathbf{x}-\mathbf{x}'|^{-1}} \frac{C}{r^d} r \mathrm{d}r \tag{13}$$

which is bounded by $C\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|^{2-d}} - 1\right)$ for $d \neq 2$ and by $C\log\left(|\mathbf{x}-\mathbf{x}'|^{-1}\right)$ for $d=2$.

To estimate the third term in the right hand side of (11), we need some more care. We choose $l \in \{1,2\}$ as before such that $|\mathbf{x}-\mathbf{x}'| \leqslant \sqrt{2}|x_l - x_l'|$.

Let us calculate

$$\int_{|\boldsymbol{p}|\geqslant|\mathbf{x}-\mathbf{x}'|^{-1}} -(x_l-x_l')^2 e^{i\boldsymbol{p}\cdot(\mathbf{x}-\mathbf{x}')} G(\boldsymbol{p}) e^{-\epsilon<\boldsymbol{p}>} \mathrm{d}\boldsymbol{p}\,,$$

which corresponds to the integral that we want to estimate multiplied by $-(x_l-x_l')^2$. For $\theta \in [0,2\pi)$ we define the vector $\boldsymbol{p}(\theta) = (|\mathbf{x}-\mathbf{x}'|^{-1}\cos\theta, |\mathbf{x}-\mathbf{x}'|^{-1}\sin\theta)$ and we denote by $p_l(\theta)$ its l-th component. Then, integrating by part with respect to the p_l variable and applying Gauss divergence theorem, we have

$$\begin{aligned}
&\int_{|\boldsymbol{p}|\geqslant|\mathbf{x}-\mathbf{x}'|^{-1}} -(x_l-x_l')^2 e^{i\boldsymbol{p}\cdot(\mathbf{x}-\mathbf{x}')}G(\boldsymbol{p})e^{-\epsilon<\boldsymbol{p}>}\mathrm{d}\boldsymbol{p}\\
(14)\qquad &= -\int_0^{2\pi} e^{-\epsilon<\boldsymbol{p}(\theta)>} i(x_l-x_l')e^{i\boldsymbol{p}(\theta)\cdot(\mathbf{x}-\mathbf{x}')}G(\boldsymbol{p}(\theta))p_l(\theta)\mathrm{d}\theta\\
&\quad -\int_{|\boldsymbol{p}|\geqslant|\mathbf{x}-\mathbf{x}'|^{-1}} i(x_l-x_l')e^{i\boldsymbol{p}\cdot(\mathbf{x}-\mathbf{x}')}\frac{\partial}{\partial p_l}\left(G(\boldsymbol{p})e^{-\epsilon<\boldsymbol{p}>}\right)\mathrm{d}\boldsymbol{p}.
\end{aligned}$$

Using the estimate (4), we get that the first term is bounded by

$$(15)\qquad 2\pi\frac{C_0}{(|\mathbf{x}-\mathbf{x}'|^{-1})^d}|\mathbf{x}-\mathbf{x}'|^{-1}|x_l-x_l'|\leqslant 2\pi C_0|\mathbf{x}-\mathbf{x}'|^d.$$

To estimate the second term in the right hand side of (14), we use a new integration by parts:

$$\begin{aligned}
&-\int_{|\boldsymbol{p}|\geqslant|\mathbf{x}-\mathbf{x}'|^{-1}} i(x_l-x_l')e^{i\boldsymbol{p}\cdot(\mathbf{x}-\mathbf{x}')}\frac{\partial}{\partial p_l}\left(G(\boldsymbol{p})e^{-\epsilon<\boldsymbol{p}>}\right)\mathrm{d}\boldsymbol{p}\\
(16)\qquad &= \int_0^{2\pi} e^{i\boldsymbol{p}(\theta)\cdot(\mathbf{x}-\mathbf{x}')}\frac{\partial}{\partial p_l}\left(Ge^{-\epsilon<\cdot>}\right)(\boldsymbol{p}(\theta))p_l(\theta)\mathrm{d}\theta\\
&\quad +\int_{|\boldsymbol{p}|\geqslant|\mathbf{x}-\mathbf{x}'|^{-1}} e^{i\boldsymbol{p}\cdot(\mathbf{x}-\mathbf{x}')}\frac{\partial^2}{\partial p_l^2}\left(G(\boldsymbol{p})e^{-\epsilon<\boldsymbol{p}>}\right)\mathrm{d}\boldsymbol{p}.
\end{aligned}$$

Using again that

$$\left|\frac{\partial}{\partial p_l}\left(G(\boldsymbol{p})e^{-\epsilon<\boldsymbol{p}>}\right)\right|\leqslant\frac{C}{|\boldsymbol{p}|^{d+1}}\ \text{and}\ \left|\frac{\partial^2}{\partial p_l^2}\left(G(\boldsymbol{p})e^{-\epsilon<\boldsymbol{p}>}\right)\right|\leqslant\frac{C}{|\boldsymbol{p}|^{d+2}},$$

we can bound the first term in the right hand side of (16) by

$$(17)\qquad 2\pi|\mathbf{x}-\mathbf{x}'|^{-1}\frac{C}{|\mathbf{x}-\mathbf{x}'|^{-(d+1)}}=2\pi C|\mathbf{x}-\mathbf{x}'|^d$$

and the second one by

$$(18)\qquad \int_{|\boldsymbol{p}|\geqslant|\mathbf{x}-\mathbf{x}'|^{-1}}\frac{C}{|\boldsymbol{p}|^{d+2}}\mathrm{d}\boldsymbol{p}=\frac{2\pi C}{d}|\mathbf{x}-\mathbf{x}'|^d$$

Putting together the estimates (15), (17) and (18), we find that

$$(19)\qquad \left|\int_{|\boldsymbol{p}|\geqslant|\mathbf{x}-\mathbf{x}'|^{-1}} e^{i\boldsymbol{p}\cdot(\mathbf{x}-\mathbf{x}')}G(\boldsymbol{p})e^{-\epsilon<\boldsymbol{p}>}\mathrm{d}\boldsymbol{p}\right|\leqslant C|\mathbf{x}-\mathbf{x}'|^{d-2}.$$

Adding the estimates (12), (13) and (19), we find that there exists a constant $C>0$ such that uniformly in ϵ

$$|K_\epsilon(\mathbf{x},\mathbf{x}')|\leqslant C\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|^{2-d}}+1\right),$$

if $|\mathbf{x}-\mathbf{x}'| \leqslant 1$, which together with the result (10) which holds for $|\mathbf{x}-\mathbf{x}'| \geqslant 1$ allows us to conclude that uniformly in ϵ we have

$$|K_\epsilon(\mathbf{x},\mathbf{x}')| \leqslant C M_d(\mathbf{x},\mathbf{x}')\,,$$

where M_d is the function defined in the statement of the Lemma.

We are now ready to prove the estimate (3) for $(H_0-i)^{-1}$. Let $\boldsymbol{f}$ and $\boldsymbol{g}$ be in the Schwartz space $\mathscr{S}(\mathbb{R}^2,\mathbb{C}^2)$. Then,

$$\langle \boldsymbol{f},(H_0-i)^{-1}\boldsymbol{g}\rangle = \int_{\mathbb{R}^2} \overline{\hat{\boldsymbol{f}}(\boldsymbol{p})} G(\boldsymbol{p})\hat{\boldsymbol{g}}(\boldsymbol{p})\mathrm{d}\boldsymbol{p}$$

by Parseval's identity. By dominated convergence, we have

$$\int_{\mathbb{R}^2} \overline{\hat{\boldsymbol{f}}(\boldsymbol{p})} G(\boldsymbol{p})\hat{\boldsymbol{g}}(\boldsymbol{p})\mathrm{d}\boldsymbol{p} = \lim_{\epsilon\to 0^+}\int_{\mathbb{R}^2} \overline{\hat{\boldsymbol{f}}(\boldsymbol{p})} G(\boldsymbol{p}) e^{-\epsilon<\boldsymbol{p}>}\hat{\boldsymbol{g}}(\boldsymbol{p})\mathrm{d}\boldsymbol{p} \tag{20}$$

and, by Parseval's identity again and denoting by $*$ the convolution product between $L^1(\mathbb{R}^2,\mathcal{M}_2(\mathbb{C}))$ and $L^2(\mathbb{R}^2,\mathbb{C}^2)$,

$$\begin{aligned}\int_{\mathbb{R}^2} \overline{\hat{\boldsymbol{f}}(\boldsymbol{p})} G(\boldsymbol{p}) e^{-\epsilon<\boldsymbol{p}>}\hat{\boldsymbol{g}}(\boldsymbol{p})\mathrm{d}\boldsymbol{p} &= \int_{\mathbb{R}^2} \overline{\boldsymbol{f}(\mathbf{x})}(K_\epsilon(\cdot,0)*\boldsymbol{g})(\mathbf{x})\mathrm{d}\mathbf{x}\\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\boldsymbol{f}(\mathbf{x})} K_\epsilon(\mathbf{x},\mathbf{x}')\boldsymbol{g}(\mathbf{x}')\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{x}'.\end{aligned}$$

Knowing that

$$\forall \epsilon>0,\ |K_\epsilon(\mathbf{x},\mathbf{x}')| \leqslant M_d(\mathbf{x},\mathbf{x}')\,,$$

we get, using (20), that

$$|\langle \boldsymbol{f},(H_0-i)^{-1}\boldsymbol{g}\rangle| \leqslant \int_{\mathbb{R}^2} |\boldsymbol{f}(\mathbf{x})| M_d(\mathbf{x},\mathbf{x}')|\boldsymbol{g}(\mathbf{x}')|\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{x}'\,.$$

This concludes the proof of Lemma 1. □

We are now ready to give the proof of Proposition 2.

Proof of Proposition 2. By equation (5), we know that for $\mathbf{x}\neq\mathbf{x}'$,

$$K_\epsilon(\mathbf{x},\mathbf{x}') = \frac{i}{(x_l-x_l')^3}\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i\boldsymbol{p}\cdot(\mathbf{x}-\mathbf{x}')}\frac{\partial^3}{\partial p_l^3}\left(G(\boldsymbol{p})e^{-\epsilon<\boldsymbol{p}>}\right)\mathrm{d}\boldsymbol{p}.$$

The integrand converges pointwise to $e^{i\boldsymbol{p}\cdot(\mathbf{x}-\mathbf{x}')}\frac{\partial^3 G(\boldsymbol{p})}{\partial p_l^3}$. Moreover, using (4), it is dominated by some integrable function independent of ϵ. Then, by dominated convergence, K_ϵ converges pointwise to some function of $\mathbf{x}$ and $\mathbf{x}'$ which will be denoted by $(H_0-i)^{-1}(\cdot,\cdot)$ and which trivially satisfies Inequality (2).

Then, by dominated convergence, we have that for all $\boldsymbol{f}$ and $\boldsymbol{g} \in L^2(\mathbb{R}^2, \mathbb{C}^2)$

$$\langle \boldsymbol{f}, (H_0 - i)^{-1}\boldsymbol{g}\rangle = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \boldsymbol{f}(\mathbf{x})(H_0 - i)^{-1}(\mathbf{x}, \mathbf{x}')\boldsymbol{g}(\mathbf{x}')\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{x}',$$

and

$$|(H_0 - i)^{-1}(\mathbf{x}, \mathbf{x}')| \leqslant M_d(\mathbf{x}, \mathbf{x}').$$

□

3. PROOF OF THE MAIN THEOREM

Let $\mathscr{S}(\mathbb{R}^2, \mathbb{C})$ be the Schwartz space of test functions, and let us fix $\Omega = (-\frac{1}{2}, \frac{1}{2}]^2$. We define the Bloch-Floquet transformation by the map

$$\begin{aligned} \mathcal{U}:\quad & \mathscr{S}(\mathbb{R}^2, \mathbb{C}) \subset L^2(\mathbb{R}^2, \mathbb{C}) \to L^2(\Omega^2) \\ & (\mathcal{U}\psi)(\mathbf{x}, \mathbf{k}) = \sum_{\boldsymbol{\gamma}\in\mathbb{Z}^2} e^{2i\pi\mathbf{k}\cdot(\mathbf{x}+\boldsymbol{\gamma})}\psi(\mathbf{x}+\boldsymbol{\gamma}), \end{aligned} \tag{21}$$

extended by density to $L^2(\mathbb{R}^2, \mathbb{C})$. It is possible to show (cf. [8]) that $\mathcal{U}$ is a unitary operator and that for $f \in L^2(\Omega^2)$, $\mathbf{x} \in \Omega$ and $\boldsymbol{\gamma} \in \mathbb{Z}^2$,

$$(\mathcal{U}^* f)(\mathbf{x}+\boldsymbol{\gamma}) = \int_\Omega e^{-2i\pi\mathbf{k}\cdot(\mathbf{x}+\boldsymbol{\gamma})} f(\mathbf{x}, \mathbf{k})\mathrm{d}\mathbf{k}.$$

We then define the Bloch-Floquet transformation componentwise on $L^2(\mathbb{R}^2, \mathbb{C}^2)$ which will be abusively again denoted by $\mathcal{U}$.

Applying this tranformation (see Proposition 4 in the appendix), we find

$$\begin{aligned} & \mathcal{U}H(\alpha, \beta)\mathcal{U}^* = \int_\Omega^\oplus h_{\mathbf{k}}(\alpha, \beta)\mathrm{d}\mathbf{k}, \\ & h_{\mathbf{k}}(\alpha, \beta) = \boldsymbol{\sigma}\cdot\boldsymbol{F}(-i\nabla_{\mathrm{per}} - 2\pi\mathbf{k}) + \beta\boldsymbol{\chi}_\alpha(\mathbf{x})\cdot\boldsymbol{\sigma}, \end{aligned} \tag{22}$$

where for each $\mathbf{k}$ the fiber Hamiltonian $h_{\mathbf{k}}(\alpha, \beta)$ is an operator defined on $L^2(\Omega, \mathbb{C}^2)$. Similarly, we wil denote

$$h_{\mathbf{k}}^{(0)} = \boldsymbol{\sigma}\cdot\boldsymbol{F}(-i\nabla_{\mathrm{per}} - 2\pi\mathbf{k}).$$

The operator ∇_{per} means here the gradient on $L^2(\Omega, \mathbb{C}^2)$ with periodic boundary conditions and $\boldsymbol{\chi}_\alpha(\mathbf{x}) = (\chi_{1,\alpha}(\mathbf{x}), \chi_{2,\alpha}(\mathbf{x}), \chi_{3,\alpha}(\mathbf{x})) := \boldsymbol{\chi}(\mathbf{x}/\alpha)$.

The spectra of $H(\alpha, \beta)$ and $h_{\mathbf{k}}(\alpha, \beta)$ are related through (see [10, Theorem XIII.85])

$$\mathrm{Spec}(H(\alpha, \beta)) = \overline{\bigcup_{\mathbf{k}\in\Omega} \mathrm{Spec}(h_{\mathbf{k}}(\alpha, \beta))}. \tag{23}$$

Picking $\Psi_{\mathbf{m}} = e^{2i\pi\mathbf{m}\cdot\mathbf{x}}$ for $\mathbf{m} \in \mathbb{Z}^2$ and $\mathbf{x} \in \Omega$, we get that the family of vectors $\Psi_{\mathbf{m}}$ is a basis of eigenvectors of $\boldsymbol{F}(-i\nabla_{\text{per}})$ satisfying if we denote $\boldsymbol{F} = (F_1, F_2, F_3)$

$$F_i(-i\nabla_{\text{per}})\Psi_{\mathbf{m}} = F_i(2\pi\mathbf{m})\Psi_{\mathbf{m}} \text{ for } i = 1, 2, 3.$$

We then define, for $\mathbf{m} \in \mathbb{Z}^2$, the projections

$$P_{\mathbf{m}} = |\Psi_{\mathbf{m}}\rangle\langle\Psi_{\mathbf{m}}| \otimes 1_{\mathbb{C}^2} \quad \text{and} \quad Q_{\mathbf{0}} = \text{Id} - P_{\mathbf{0}}\,.$$

We will use the Feshbach map method (see for example Lemma 6.1 of [9]) to reduce the spectral problem to problems on $P_{\mathbf{0}}L^2(\Omega, \mathbb{C}^2)$ and $Q_{\mathbf{0}}L^2(\Omega, \mathbb{C}^2)$. This method claims that $z \in \rho(h_{\mathbf{k}}(\alpha, \beta))$ if $Q_{\mathbf{0}}(h_{\mathbf{k}}(\alpha, \beta) - z)Q_{\mathbf{0}}$ is invertible on $Q_{\mathbf{0}}L^2(\Omega, \mathbb{C}^2)$ and the operator $\mathcal{F}_{P_{\mathbf{0}}}(z)$ defined on $P_{\mathbf{0}}L^2(\Omega, \mathbb{C}^2)$ and given by (24)

$$\mathcal{F}_{P_{\mathbf{0}}}(z) = P_{\mathbf{0}}(h_{\mathbf{k}}(\alpha, \beta) - z)P_{\mathbf{0}} - \beta^2 P_{\mathbf{0}}\boldsymbol{\chi}_\alpha \cdot \boldsymbol{\sigma} Q_{\mathbf{0}}(Q_{\mathbf{0}}(h_{\mathbf{k}}(\alpha, \beta) - z)Q_{\mathbf{0}})^{-1}Q_{\mathbf{0}}\boldsymbol{\chi}_\alpha \cdot \boldsymbol{\sigma} P_{\mathbf{0}}$$

is also invertible.

We will first prove that $P_{\mathbf{0}}h_{\mathbf{k}}(\alpha, \beta)P_{\mathbf{0}}$ has a spectral gap of order $\alpha^2\beta$ near 0 and then that the second term in the right hand side of (24) is small enough not to close the gap provided that z is in the interval given in the theorem. To show the invertibility of $P_{\mathbf{0}}(h_{\mathbf{k}}(\alpha, \beta) - z)P_{\mathbf{0}}$, we have to bound from below $|\boldsymbol{F}(\boldsymbol{p}) + \lambda\boldsymbol{\Phi}|$, where we remind that we have denoted $\boldsymbol{\Phi} = (\Phi_i)_{1 \leqslant i \leqslant 3}$, $\boldsymbol{\Phi}_{||}$ the projection of $\boldsymbol{\Phi}$ on $\text{Ran}(A)$ and $\Phi_\perp$ its projection on $\text{Ran}(A)^\perp$.

LEMMA 2. *Let $\alpha \in]0, 1[$ and $\beta > 0$. Then, for every $\mathbf{k} \in \Omega$ and $\Psi \in P_{\mathbf{0}}L^2(\Omega, \mathbb{C}^2)$, we have for $\alpha^2\beta$ small enough:*

$$\|P_{\mathbf{0}}h_{\mathbf{k}}(\alpha, \beta)P_{\mathbf{0}}\Psi\| \geqslant \frac{|\Phi_\perp|}{2}\alpha^2\beta\|\Psi\|.$$

Proof. We have, for $\mathbf{k} \in \Omega$:

$$P_{\mathbf{0}}h_{\mathbf{k}}(\alpha, \beta)P_{\mathbf{0}} = (\boldsymbol{\sigma} \cdot \boldsymbol{F}(-2\pi\mathbf{k}) + \alpha^2\beta\boldsymbol{\Phi} \cdot \boldsymbol{\sigma})P_{\mathbf{0}}.$$

Let us denote $\lambda = \alpha^2\beta$. For $\Psi \in P_{\mathbf{0}}L^2(\Omega, \mathbb{C}^2)$,

$$\begin{aligned}\|P_{\mathbf{0}}h_{\mathbf{k}}(\alpha, \beta)P_{\mathbf{0}}\Psi\|^2 &= \|\boldsymbol{\sigma} \cdot (\boldsymbol{F}(-2\pi\mathbf{k}) + \lambda\boldsymbol{\Phi})P_{\mathbf{0}}\Psi\|^2 \\ &\geqslant \inf_{\boldsymbol{p} \in \mathbb{R}^2} |\boldsymbol{F}(\boldsymbol{p}) + \lambda\boldsymbol{\Phi}|^2\|\Psi\|^2.\end{aligned}$$

The lower bound in the lemma would follow if we can prove the following statement: there exists $\lambda_0 > 0$ such that

$$\inf_{\boldsymbol{p} \in \mathbb{R}^2} |\boldsymbol{F}(\boldsymbol{p}) + \lambda\boldsymbol{\Phi}| \geqslant \frac{|\Phi_\perp|}{2}\lambda\,,$$

for $0 \leqslant \lambda \leqslant \lambda_0$.

In order to prove this, pick M such that $K_0' M^d - |\mathbf{\Phi}| = \frac{|\Phi_\perp|}{2}$ where K_0' is the constant appearing in the first inequality in (1) of Hypothesis 1(ii).

For $|\boldsymbol{p}| \geqslant M\lambda^{1/d}$ we have by the first inequality in (1):

$$|\boldsymbol{F}(\boldsymbol{p}) + \lambda\mathbf{\Phi}| \geqslant K_0'|\boldsymbol{p}|^d - \lambda|\mathbf{\Phi}| \geqslant (K_0'M^d - |\mathbf{\Phi}|)\lambda \geqslant \frac{|\Phi_\perp|}{2}\lambda. \tag{25}$$

For $|\boldsymbol{p}| \leqslant M\lambda^{1/d}$, by Hypothesis 1(iii), we have for some $K > 0$ and λ small enough:

$$\begin{aligned}|\boldsymbol{F}(\boldsymbol{p}) + \lambda\mathbf{\Phi}| = \left||\boldsymbol{p}|^{d-1}A\boldsymbol{p} + \lambda\mathbf{\Phi}_{||} + \lambda\Phi_\perp\right| + O(|\boldsymbol{p}|^{d+1}) \geqslant \lambda|\Phi_\perp| - K|\boldsymbol{p}|^{d+1} \\ \geqslant \lambda|\Phi_\perp| - KM^{d+1}\lambda^{(d+1)/d}.\end{aligned}$$

Define λ_0 such that

$$|\Phi_\perp| - KM^{d+1}\lambda_0^{1/d} = \frac{|\Phi_\perp|}{2}.$$

For $\lambda \leqslant \lambda_0$, the above estimate implies

$$|\boldsymbol{F}(\boldsymbol{p}) + \lambda\mathbf{\Phi}| \geqslant \frac{\lambda}{2}|\Phi_\perp|, \tag{26}$$

for all $\boldsymbol{p} \in \mathbb{R}^2$.

Equations (25) and (26) together conclude the proof. □

The invertibility of $Q_{\mathbf{0}}(h_{\mathbf{k}}(\alpha,\beta) - z)Q_{\mathbf{0}}$ on $Q_{\mathbf{0}}L^2(\Omega,\mathbb{C}^2)$ will require more technicality. We begin with the following estimates. Recall that $d' = \min(d,2)$.

Lemma 3. *There exists C such that for $|\alpha| \leqslant \frac{1}{2}$ and $\forall \mathbf{k} \in \Omega$ we have*

$$\|\sqrt{|\boldsymbol{\chi}_\alpha|}P_{\mathbf{0}}\| \leqslant \alpha; \tag{27}$$

$$\||\boldsymbol{\chi}_\alpha|^{1/2}(h_{\mathbf{k}}^{(0)} - i)^{-1}|\boldsymbol{\chi}_\alpha|^{1/2}\| \leqslant C\alpha^{d'}; \tag{28}$$

$$\||\boldsymbol{\chi}_\alpha|^{1/2}(h_{\mathbf{k}}^{(0)} - i)^{-1}\| \leqslant C\sqrt{\alpha^{d'}}, \tag{29}$$

where $h_{\mathbf{k}}^{(0)}$ has been defined in Equation (3).

Proof. As in [1], in order to show (27) we compute for $\boldsymbol{f}, \boldsymbol{g} \in L^2(\Omega,\mathbb{C}^2)$:

$$|\langle \boldsymbol{f}, \sqrt{|\boldsymbol{\chi}_\alpha|}P_{\mathbf{0}}\boldsymbol{g}\rangle| \leqslant |\langle \boldsymbol{f}, \sqrt{|\boldsymbol{\chi}_\alpha|}\Psi_0\rangle|\,|\langle \Psi_0, \boldsymbol{g}\rangle| \leqslant \|\boldsymbol{\chi}_\alpha\|_1^{1/2}\|\boldsymbol{f}\|_2\|\boldsymbol{g}\|_2 \leqslant \alpha\|\boldsymbol{f}\|_2\|\boldsymbol{g}\|_2.$$

For the next two inequalities, we need some notation: given an integral operator T, we denote its integral kernel by $T(\mathbf{x},\mathbf{x}')$. We now use the following identity proved in Proposition 3 in the appendix:

$$(h_{\mathbf{k}}^{(0)} - i)^{-1}(\mathbf{x},\mathbf{x}') = \sum_{\boldsymbol{\gamma}\in\mathbb{Z}^2} e^{2i\pi\mathbf{k}\cdot(\mathbf{x}+\boldsymbol{\gamma}-\mathbf{x}')}(H_0 - i)^{-1}(\mathbf{x}+\boldsymbol{\gamma},\mathbf{x}').$$

In the following, we will denote by C any constant independent of α and $\mathbf{k}$. Assume first that $d \neq 2$. Let $\boldsymbol{\Upsilon}$, $\boldsymbol{\Psi} \in L^2(\Omega, \mathbb{C}^2)$ with $\boldsymbol{\Upsilon}$ with support in $\Omega_\alpha = [-\frac{\alpha}{2}, \frac{\alpha}{2})^2$. According to Lemma 1 and Proposition 2, we have:

$$
\begin{aligned}
|\langle \boldsymbol{\Upsilon}, (h_{\mathbf{k}}^{(0)} - i)^{-1} \boldsymbol{\Psi} \rangle| &\leqslant \sum_{\gamma \in \mathbb{Z}^2} \iint_{\Omega_\alpha \times \Omega} |\boldsymbol{\Upsilon}(\mathbf{x})| |(H_0 - i)^{-1}|(\mathbf{x} + \boldsymbol{\gamma}, \mathbf{x}')| \boldsymbol{\Psi}(\mathbf{x}')| \mathrm{d}\mathbf{x} \mathrm{d}\mathbf{x}' \\
&\leqslant \sum_{\gamma \neq 0} \iint_{\Omega_\alpha \times \Omega} |\boldsymbol{\Upsilon}(\mathbf{x})| \frac{C}{|\mathbf{x} + \boldsymbol{\gamma} - \mathbf{x}'|^3} |\boldsymbol{\Psi}(\mathbf{x}')| \mathrm{d}\mathbf{x} \mathrm{d}\mathbf{x}' \\
&\quad + \iint_{\Omega_\alpha \times \Omega} |\boldsymbol{\Upsilon}(\mathbf{x})| \frac{C}{|\mathbf{x} - \mathbf{x}'|^{2-d'}} |\boldsymbol{\Psi}(\mathbf{x}')| \mathrm{d}\mathbf{x} \mathrm{d}\mathbf{x}'.
\end{aligned}
$$

In order to bound the first term, we see that there exists a constant C such that for all $|\alpha| \leqslant \frac{1}{2}$, $\mathbf{x} \in \Omega_\alpha$, $\mathbf{x}' \in \Omega$ and $\boldsymbol{\gamma} \in \mathbb{Z}^2 \setminus \{0\}$, we have

$$
\frac{1}{|\mathbf{x} + \boldsymbol{\gamma} - \mathbf{x}'|^3} \leqslant \frac{C}{|\boldsymbol{\gamma}|^3}.
$$

Thus the first term is bounded by

$$
C \|\boldsymbol{\Upsilon}\|_{L^1} \|\boldsymbol{\Psi}\|_{L^1}.
$$

For the second term, we have to bound

$$
\iint_{\Omega_\alpha \times \Omega} |\boldsymbol{\Upsilon}(\mathbf{x})| \frac{1}{|\mathbf{x} - \mathbf{x}'|^{2-d'}} |\boldsymbol{\Psi}(\mathbf{x}')| \mathrm{d}\mathbf{x} \mathrm{d}\mathbf{x}'.
$$

Hardy-Littlewood-Sobolev inequality (cf. [6, Theorem 4.3]) gives that there exists C such that:

$$
\iint_{\Omega_\alpha \times \Omega} |\boldsymbol{\Upsilon}(\mathbf{x})| \frac{1}{|\mathbf{x} - \mathbf{x}'|^{2-d'}} |\boldsymbol{\Psi}(\mathbf{x}')| \mathrm{d}\mathbf{x} \mathrm{d}\mathbf{x}' \leqslant C \|\boldsymbol{\Upsilon}\|_{\frac{4}{2+d'}} \|\boldsymbol{\Psi}\|_{\frac{4}{2+d'}}.
$$

By Hölder's inequality,

$$
\begin{aligned}
\| |\boldsymbol{\chi}_\alpha|^{1/2} \boldsymbol{f} \|_{\frac{4}{2+d'}} &= \left(\int_\Omega |\boldsymbol{\chi}_\alpha|^{\frac{2}{2+d'}} |\boldsymbol{f}|^{\frac{4}{2+d'}} \right)^{\frac{2+d'}{4}} \\
&\leqslant \left(\| |\boldsymbol{\chi}_\alpha|^{\frac{2}{2+d'}} \|_{\frac{2+d'}{d'}} \| |\boldsymbol{f}|^{\frac{4}{2+d'}} \|_{\frac{2+d'}{2}} \right)^{\frac{2+d'}{4}} \\
&= \| |\boldsymbol{\chi}_\alpha| \|_{\frac{2}{d'}}^{1/2} \|\boldsymbol{f}\|_2.
\end{aligned}
$$

A simple change of variable gives us that $\| |\boldsymbol{\chi}_\alpha| \|_{\frac{2}{d'}} = \alpha^{d'} \| |\boldsymbol{\chi}| \|_{\frac{2}{d'}} \leqslant \alpha^{d'} \| |\boldsymbol{\chi}| \|_\infty$.

Hence, picking $\boldsymbol{\Upsilon} = \boldsymbol{\Psi} = |\boldsymbol{\chi}_\alpha|^{1/2} \boldsymbol{f}$ in the above estimates yields

$$
\left| \langle \boldsymbol{f}, |\boldsymbol{\chi}_\alpha|^{1/2} (h_{\mathbf{k}}^{(0)} \pm i)^{-1} |\boldsymbol{\chi}_\alpha|^{1/2} \boldsymbol{f} \rangle \right| \leqslant C \|\boldsymbol{\chi}_\alpha \boldsymbol{f}\|_1^2 + C \alpha^{d'} \|\boldsymbol{f}\|_2^2.
$$

An application of Cauchy-Schwarz inequality gives that

$$\| |\boldsymbol{\chi}_\alpha| \boldsymbol{f}\|_1 \leqslant \| |\boldsymbol{\chi}_\alpha| \|_{L^2} \|\boldsymbol{f}\|_2 \leqslant \sqrt{3}\alpha \|\boldsymbol{f}\|_2.$$

This concludes the proof of (28).

The proof of (29) is similar: we have to take $\boldsymbol{\Upsilon} = |\boldsymbol{\chi}_\alpha|\boldsymbol{f}$ and $\boldsymbol{\Psi} = \boldsymbol{f}$. We do not give further details.

If $d = 2$, we can prove these inequalities for any $\tilde{d} < 2$ and then take the supremum. □

LEMMA 4. *For $\boldsymbol{f} \in Q_{\mathbf{0}}\mathrm{Dom}(h_{\mathbf{k}}^{(0)})$, we have $\|h_{\mathbf{k}}^{(0)} Q_{\mathbf{0}} \boldsymbol{f}\| \geqslant \pi^d K_0' \|\boldsymbol{f}\|$.*

Proof. As we show in Proposition 4 in the appendix, we have, for $m \in \mathbb{Z}^2$ and $k \in \Omega$,

$$P_{\mathbf{m}} h_{\mathbf{k}}^{(0)} P_{\mathbf{m}} = \boldsymbol{\sigma} \cdot \boldsymbol{F}(2\pi(\mathbf{m} - \mathbf{k})) P_{\mathbf{m}}.$$

Hence, according to inequality (1) of Hypothesis 1(ii), we have for $\boldsymbol{f} \in Q_{\mathbf{0}}\mathrm{Dom}(h_{\mathbf{k}}^{(0)})$:

$$\begin{aligned}\|h_{\mathbf{k}}^{(0)} Q_{\mathbf{0}} \boldsymbol{f}\|^2 = \sum_{\mathbf{m}\neq 0} \|\sigma \cdot \boldsymbol{F}(2\pi(\mathbf{m}-\mathbf{k}))P_{\mathbf{m}}\boldsymbol{f}\|^2 \geqslant K_0'^2 \sum_{m\neq 0} (2\pi|\mathbf{m}-\mathbf{k}|)^{2d} \|P_{\mathbf{m}}\boldsymbol{f}\|^2 \\ \geqslant K_0'^2 \pi^{2d} \sum_{\mathbf{m}\neq 0} \|P_{\mathbf{m}}\boldsymbol{f}\|^2.\end{aligned}$$

□

For a self-adjoint operator T and an orthogonal projection Q, we define the resolvent set

$$\rho_Q(T) := \{z \in \mathbb{C} \text{ such that } Q(T-z)Q : \mathrm{Ran}Q \to \mathrm{Ran}Q \quad \text{is invertible}\}.$$

We set

$$\begin{aligned} R_0(z) &:= \left(Q_{\mathbf{0}}(h_{\mathbf{k}}^{(0)} - z)Q_{\mathbf{0}}\restriction_{\mathrm{Ran}Q_{\mathbf{0}}}\right)^{-1}, \quad z \in \rho_{Q_{\mathbf{0}}}(h_{\mathbf{k}}^{(0)}), \\ R(z) &:= \left(Q_{\mathbf{0}}(h_{\mathbf{k}}(\alpha,\beta) - z)Q_{\mathbf{0}}\restriction_{\mathrm{Ran}Q_{\mathbf{0}}}\right)^{-1}, \quad z \in \rho_{Q_{\mathbf{0}}}(h_{\mathbf{k}}(\alpha,\beta)). \end{aligned}$$

For $i \in \{1,2,3\}$, we define the operators $U_i : L^2(\Omega, \mathbb{C}^2) \to \mathrm{Ran}Q_{\mathbf{0}}$ and $W_i : \mathrm{Ran}Q_{\mathbf{0}} \to L^2(\Omega, \mathbb{C}^2)$ by:

$$W_i = \sqrt{\beta}(\sqrt{|\chi_{i,\alpha}|}\sigma_i)Q_{\mathbf{0}} \text{ and } U_i = \sqrt{\beta}Q_{\mathbf{0}}\,\mathrm{sgn}(\chi_{i,\alpha})\sqrt{|\chi_{i,\alpha}|}.$$

LEMMA 5. *There exists $C > 0$ independent of $\alpha \in]0, 1/2]$ and $\beta > 0$ such that for any $|z| \leqslant K_0'\pi^d/2$ and any j, l*

$$\|W_j R_0(z) U_l\| \leqslant C\alpha^{d'}\beta.$$

Proof. Due to Lemma 4, we have for $|z| \leqslant K_0'\pi^d/2$,

$$\|R_0(z)\| \leqslant \frac{2}{K_0'\pi^d}\,. \tag{30}$$

Using the first resolvent identity, we get for $j, l = 1, 2, 3$:

$$\begin{aligned} W_j R_0(z) U_l &= W_j R_0(i) U_l + (z-i) W_j R_0(i)^2 U_l \\ &\quad + (z-i)^2 W_j R_0(i) R_0(z) R_0(i) U_l. \end{aligned} \tag{31}$$

We shall separately estimate each term on the right hand side of (31).

Since $h_{\mathbf{k}}^{(0)}$ commutes with the projections $P_{\mathbf{m}}$ we have

$$R_0(i) = (h_{\mathbf{k}}^{(0)} - i)^{-1} - (P_{\mathbf{0}}(h_{\mathbf{k}}^{(0)} - i)P_{\mathbf{0}}\restriction_{\mathrm{Ran}P_{\mathbf{0}}})^{-1}. \tag{32}$$

Note that, due to the definition of U_l and W_j, for any z such that $|z| \leqslant K_0'\pi^d/2$

$$W_j(P_{\mathbf{0}}(h_{\mathbf{k}}^{(0)} - z)P_{\mathbf{0}}\restriction_{\mathrm{Ran}P_{\mathbf{0}}})^{-1} U_l = 0. \tag{33}$$

The identity (32) together with inequalities (28) and (33) imply that there exists $c > 0$, such that for $|\alpha| < 1/2$ and all $k \in \Omega$

$$\begin{aligned} \|W_j R_0(i) U_l\| &= \|W_j (h_{\mathbf{k}}^{(0)} - i)^{-1} U_l\| \\ &\leqslant c\beta\alpha^{d'}. \end{aligned} \tag{34}$$

This bounds the first term on the right hand side of (31).

To estimate the second one we first notice that

$$(h_{\mathbf{k}}^{(0)} - i)^{-2} = (Q_{\mathbf{0}}(h_{\mathbf{k}}^{(0)} - i)Q_{\mathbf{0}}\restriction_{\mathrm{Ran}Q_{\mathbf{0}}})^{-2} + (P_{\mathbf{0}}(h_{\mathbf{k}}^{(0)} - i)P_{\mathbf{0}}\restriction_{\mathrm{Ran}P_{\mathbf{0}}})^{-2}.$$

It is easy to see that the equation (33) remains true with a power -2. Then,

$$\begin{aligned} \|(z-i) W_j R_0(i)^2 U_l\| &= \|(z-i) W_j (h_{\mathbf{k}}^{(0)} - i)^{-2} U_l\| \\ &\leqslant \sqrt{1 + \frac{K_0'^2 \pi^{2d}}{4}}\, \beta \|\sqrt{|\boldsymbol{\chi}_\alpha|}(h_{\mathbf{k}}^{(0)} - i)^{-1}\| \, \|(h_{\mathbf{k}}^{(0)} - i)^{-1}\sqrt{|\boldsymbol{\chi}_\alpha|}\| \\ &\leqslant C\beta\alpha^{d'}, \end{aligned} \tag{35}$$

for some $C > 0$ independent of α and β, where we used (29) in the last inequality.

Finally, we bound the last term on the right hand side of (31). Observe that from inequalities (27) and (29) we obtain that there exists $c, C > 0$ such that for all $\alpha \leqslant 1/2$

$$\begin{aligned} \|\sqrt{|\boldsymbol{\chi}_\alpha|} R_0(i)\| &\leqslant \|\sqrt{|\boldsymbol{\chi}_\alpha|}(h_{\mathbf{k}}^{(0)} - i)^{-1}\| + \|\sqrt{|\boldsymbol{\chi}_\alpha|} P_{\mathbf{0}}(P_{\mathbf{0}}(h_{\mathbf{k}} - i)P_{\mathbf{0}})^{-1}\| \\ &\leqslant c\sqrt{\alpha^{d'}} + c\alpha \leqslant C\sqrt{\alpha^{d'}}. \end{aligned}$$

Therefore, using (30) and (29)

$$\begin{aligned}&\|(z-i)^2 W_j R_0(i) R_0(z) R_0(i) U_l\|\\&\leqslant \beta |z-i|^2 \|\sqrt{|\boldsymbol{\chi}_\alpha|} R_0(i)\| \, \|R_0(z)\| \, \|R_0(i)\sqrt{|\boldsymbol{\chi}_\alpha|}\|\\&\leqslant C\beta|z-i|^2\alpha^{d'}.\end{aligned}$$

Summing the latter bound together with (34) and (35) (in view of (31)) concludes the proof. □

LEMMA 6. *Let*

$$\mathcal{S} = \{z \in \rho_{Q_\mathbf{0}}(h_\mathbf{k}^{(0)}) : \sup_{j,l} \|W_j R_0(z) U_l\| < 1/3\}.$$

Then, for every $z \in \mathcal{S}$, we have $z \in \rho_{Q_\mathbf{0}}(h_\mathbf{k}(\alpha,\beta))$.

Proof. Let $z \in \mathcal{S} \cap \mathbb{R}$. Put $z_\epsilon = z + i\epsilon$. The set $\mathcal{S}$ being open, for $\epsilon > 0$ small enough $z_\epsilon \in \mathcal{S}$. We denote $\boldsymbol{U} = (U_1, U_2, U_3)$ and $\boldsymbol{W} = (W_1, W_2, W_3)$. Applying the second resolvent identity several times, we find for any $N > 0$:

$$R(z_\epsilon) = R_0(z_\epsilon)\left(\sum_{n=0}^{N}(-1)^n\left(\boldsymbol{U}\boldsymbol{W}^T R_0(z_\epsilon)\right)^n + T_{N+1}\right)$$

with

$$T_{N+1} = (-1)^{N+1}\left(\boldsymbol{U}\boldsymbol{W}^T R_0(z_\epsilon)\right)^N \boldsymbol{U}\boldsymbol{W}^T R(z_\epsilon).$$

But we have that

$$\left(\boldsymbol{U}\boldsymbol{W}^T R_0(z_\epsilon)\right)^N = \sum_{i_1,\dots,i_N} U_{i_1} W_{i_1} R_0(z_\epsilon) U_{i_2} \dots W_{i_{N-1}} R_0(z_\epsilon) U_{i_N} W_{i_N} R_0(z_\epsilon)$$

and then

$$\left\|\left(\boldsymbol{U}\boldsymbol{W}^T R_0(z_\epsilon)\right)^N\right\| \leqslant C3^N\left(\sup_{j,l}\|W_j R_0(z_\epsilon) U_l\|\right)^{N-1}$$

which tends to 0 as $N \to \infty$ since $z_\epsilon \in \mathcal{S}$.

Then, at fixed ϵ, we have

$$R(z_\epsilon) = R_0(z_\epsilon)\sum_{n=0}^{\infty}(-1)^n\left(\boldsymbol{U}\boldsymbol{W}^T R_0(z_\epsilon)\right)^n. \tag{36}$$

Using the definition of $\mathcal{S}$ and equation (30), we obtain that this resolvent is bounded uniformly in ϵ, so we can take the limit $\epsilon \to 0$ and thus

$$\forall z \in \mathcal{S}; z \in \rho_{Q_\mathbf{0}}(h_\mathbf{k}(\alpha,\beta)).$$

□

We are now ready to study the invertibility of Feshbach's operator $\mathcal{F}_{P_{\mathbf{0}}}(z)$ for $\alpha^{d'}\beta$ small enough. To this purpose, we use the two following lemmas (similar to Lemmas 2.2 and 2.3 of [1]):

Lemma 7. *There exists a constant $\delta \in]0,1[$ such that, for all $\alpha \in]0,1/2[$ and $\beta > 0$ satisfying $\alpha^{d'}\beta < \delta$, $Q_{\mathbf{0}}(h_{\mathbf{k}}(\alpha,\beta) - z)Q_{\mathbf{0}}$ is invertible on the range of $Q_{\mathbf{0}}$ for all $z \in [-K_0'\pi^d/2, K_0'\pi^d/2]$ and $\mathbf{k} \in \Omega$.*

Proof. Notice that the proof of this lemma follows from Lemma 6 since $z \in \mathcal{S}$ provided $\alpha^{d'}\beta$ is small enough, according to Lemma 5. □

Put

$$\mathcal{B}_{P_{\mathbf{0}}}(z) = \beta^2 P_{\mathbf{0}}\boldsymbol{\chi}_\alpha \cdot \boldsymbol{\sigma} Q_{\mathbf{0}}(Q_{\mathbf{0}}(h_{\mathbf{k}}(\alpha,\beta) - z)Q_{\mathbf{0}})^{-1}Q_{\mathbf{0}}\boldsymbol{\chi}_\alpha \cdot \boldsymbol{\sigma} P_{\mathbf{0}}.$$

Lemma 8. *There exist two constants $\delta \in]0,1[$ and $C > 0$ such that, for all $\alpha \in]0,1/2[$ and $\beta > 0$ satisfying $\alpha^{d'}\beta < \delta$, we have*

$$\|\mathcal{B}_{P_{\mathbf{0}}}(z)\boldsymbol{\psi}\| \leqslant C\beta^2\alpha^{2+d'}\|\boldsymbol{\psi}\|$$

for all $z \in [-K_0'\pi^d/2, K_0'\pi^d/2]$, $k \in \Omega$ and $\boldsymbol{\psi} \in L^2(\Omega, \mathbb{C}^2)$.

Proof. According to equation (36), we have that

$$\begin{aligned} R(z) &= R_0(z) + R_0(z)\sum_{n\geqslant 1}(-1)^n \left(\boldsymbol{U}\boldsymbol{W}^T R_0(z)\right)^n \\ &= R_0(z) - R_0(z)\sum_{n\geqslant 0}(-1)^n\boldsymbol{U}\left(\boldsymbol{W}^T R_0(z)\boldsymbol{U}\right)^n \boldsymbol{W}^T R_0(z) \\ &= R_0(z) - R_0(z)\boldsymbol{U}(I_3 + \boldsymbol{W}^T R_0(z)\boldsymbol{U})^{-1}\boldsymbol{W}^T R_0(z). \end{aligned}$$

We remark that $\boldsymbol{W}^T R_0(z)\boldsymbol{U}$ is an operator acting on $\left(L^2(\Omega,\mathbb{C}^2)\right)^3$.

The definition of $\mathcal{B}_{P_{\mathbf{0}}}$ and these equalities give us 2 terms to estimate. On the one hand,

$$\begin{aligned} \beta^2\|P_{\mathbf{0}}\boldsymbol{\chi}_\alpha \cdot \boldsymbol{\sigma} R_0(z)\boldsymbol{\chi}_\alpha \cdot \boldsymbol{\sigma} P_{\mathbf{0}}\| &\leqslant \beta\|P_{\mathbf{0}}|\boldsymbol{\chi}_\alpha|^{1/2}\|\sum_{i,j}\|W_i R_0(z)U_j\|\||\boldsymbol{\chi}_\alpha|^{1/2}P_{\mathbf{0}}\| \\ &\leqslant c\beta^2\alpha^{2+d'}, \end{aligned}$$

where we used Lemma 5 and (27).

On the other hand, assuming that $\alpha^{d'}\beta$ is so small that $\|\boldsymbol{W}^T R_0(z)\boldsymbol{U}\| < 1/2$ we have

$$\begin{aligned} &\beta^2\|P_{\mathbf{0}}\boldsymbol{\chi}_\alpha \cdot \boldsymbol{\sigma} R_0(z)\boldsymbol{U}(I_3 + \boldsymbol{W}^T R_0(z)\boldsymbol{U})^{-1}\boldsymbol{W}^T R_0(z)\boldsymbol{\chi}_\alpha \cdot \boldsymbol{\sigma} P_{\mathbf{0}}\| \\ &\leqslant \beta\|P_{\mathbf{0}}|\boldsymbol{\chi}_\alpha|^{1/2}\|\sum_{i,j}\|W_i R_0(z)U_j\|\|(I_3 + \boldsymbol{W}^T R_0(z)\boldsymbol{U})^{-1}\| \end{aligned}$$

$$\times \sum_{i,j} \|W_i R_0(z) U_j\| \| |\boldsymbol{\chi}_\alpha|^{1/2} P_{\mathbf{0}}\| \leqslant c\beta^3 \alpha^{2+2d'}.$$

The latter inequality together with (3) finishes the proof of the lemma. □

Proof of Theorem 2. In view of (23) it is enough to show the invertibility of the Feshbach operator uniformly in $\mathbf{k} \in \Omega$. Using Lemmas 2 and 8 we get that for any $\boldsymbol{\psi} \in P_{\mathbf{0}} L^2(\Omega, \mathbb{C}^2)$

$$\begin{aligned} \|\mathcal{F}_{P_{\mathbf{0}}}(z)\boldsymbol{\psi}\| &\geqslant \|(P_{\mathbf{0}}(h_{\mathbf{k}}(\alpha,\beta) - z)P_{\mathbf{0}})\boldsymbol{\psi}\| - \|\mathcal{B}_{P_{\mathbf{0}}}(z)\boldsymbol{\psi}\| \\ &\geqslant (\beta\alpha^2 \frac{|\boldsymbol{\Phi}|}{2} - |z| - c\alpha^{2+d'}\beta^2)\, \|\boldsymbol{\psi}\|\,. \end{aligned}$$

This concludes the proof by picking $\alpha^{d'}\beta$ so small that $\frac{|\boldsymbol{\Phi}|}{2} > c\alpha^{d'}\beta$. The theorem is then proven with $C = \frac{c}{2}$. □

APPENDIX

BLOCH-FLOQUET TRANSFORMATION

In this appendix, we study the Bloch-Floquet transformation applied to our operator $H(\alpha, \beta)$. Because the potential is bounded and $\mathbb{Z}^2$-periodic, it is enough to study this transformation applied to H_0. Because H_0 is unbounded, we prefer to work with its resolvent and we start with the following proposition. Recall that we have denoted by $(H_0 - i)^{-1}(\mathbf{x}, \mathbf{x}')$ the integral kernel of $(H_0 - i)^{-1}$.

PROPOSITION 3. *Let $\mathcal{U}$ be the Bloch-Floquet transformation as defined in (21).*

$$\mathcal{U}(\boldsymbol{\sigma} \cdot \boldsymbol{F}(-i\nabla) - i)^{-1}\mathcal{U}^* = \int_\Omega^\oplus g_{\mathbf{k}} \mathrm{d}\mathbf{k}\,,$$

where, for $\mathbf{k} \in \Omega$, the operator $g_{\mathbf{k}} : L^2(\Omega, \mathbb{C}^2) \to L^2(\Omega, \mathbb{C}^2)$ has an integral kernel given by

$$g_{\mathbf{k}}(\mathbf{x}, \mathbf{x}') = \sum_{\boldsymbol{\gamma} \in \mathbb{Z}^2} e^{2i\pi\boldsymbol{\gamma}\cdot\mathbf{k}} e^{2i\pi\mathbf{x}\cdot\mathbf{k}} (H_0 - i)^{-1}(\mathbf{x} + \boldsymbol{\gamma}, \mathbf{x}') e^{-2i\pi\mathbf{x}'\cdot\mathbf{k}}.$$

Proof. Let $\boldsymbol{f} \in \mathcal{C}^\infty(\Omega \times \mathbb{R}^2)^2$ such that $\boldsymbol{f}$ is $\mathbb{Z}^2$-periodic with respect to the second variable and $\mathbf{x} \in \mathbb{R}^2$. To avoid heavy notation, we will denote $K_0(\mathbf{x}, \mathbf{x}') = (H_0 - i)^{-1}(\mathbf{x}, \mathbf{x}')$. We can then write

$$((\boldsymbol{\sigma} \cdot \boldsymbol{F}(-i\nabla) - i)^{-1}\mathcal{U}^*\boldsymbol{f})(\mathbf{x}) = \int_{\mathbb{R}^2} K_0(\mathbf{x}, \mathbf{x}')(\mathcal{U}^*\boldsymbol{f})(\mathbf{x}')\mathrm{d}\mathbf{x}'$$

$$= \sum_{\gamma' \in \mathbb{Z}^2} \int_\Omega K_0(\mathbf{x}, \mathbf{x}' + \gamma') \int_\Omega e^{-2i\pi(\mathbf{x}'+\gamma')\cdot \mathbf{k}'} \boldsymbol{f}(\mathbf{x}', \mathbf{k}')\mathrm{d}\mathbf{k}'\mathrm{d}\mathbf{x}'.$$

The Fourier coefficients $\hat{\boldsymbol{f}}(\mathbf{x}', \gamma') = \int_\Omega e^{-2i\pi(\mathbf{x}'+\gamma')\cdot \mathbf{k}'} \boldsymbol{f}(\mathbf{x}', \mathbf{k}')\mathrm{d}\mathbf{k}'$ decay faster than any polynomial in γ' uniformly in $\mathbf{x}'$. The integral kernel $K_0(\mathbf{x} + \gamma, \mathbf{x}' + \gamma')$ has a decay like $|\gamma - \gamma'|^{-3}$ when $|\gamma - \gamma'|$ is larger than 3 uniformly in $\mathbf{x}$ and $\mathbf{x}'$. Moreover, $K_0(\mathbf{x} + \gamma, \mathbf{x}' + \gamma')$ is absolutely integrable with respect to $\mathbf{x}'$ and

$$\int_\Omega |K_0(\mathbf{x} + \gamma, \mathbf{x}' + \gamma')|\mathrm{d}\mathbf{x}' \leqslant C$$

uniformly in $\mathbf{x} \in \Omega$ and $|\gamma - \gamma'| \leqslant 3$. These facts justify the interchange of the various series below:

$$\begin{aligned}
&(\mathcal{U}(\boldsymbol{\sigma} \cdot \boldsymbol{F}(-i\nabla) - i)^{-1}\mathcal{U}^* \boldsymbol{f})(\mathbf{x}, \mathbf{k}) \\
&= \sum_{\gamma \in \mathbb{Z}^2} e^{2i\pi \mathbf{k}\cdot(\mathbf{x}+\gamma)} \sum_{\gamma' \in \mathbb{Z}^2} \int_\Omega K_0(\mathbf{x} + \gamma, \mathbf{x}' + \gamma')\hat{\boldsymbol{f}}(\mathbf{x}', \gamma')\mathrm{d}\mathbf{x}' \\
&= \sum_{\gamma' \in \mathbb{Z}^2} \sum_{\gamma \in \mathbb{Z}^2} e^{2i\pi \mathbf{k}\cdot\mathbf{x}} e^{2i\pi \mathbf{k}\cdot(\gamma-\gamma')} \int_\Omega K_0(\mathbf{x} + (\gamma - \gamma'), \mathbf{x}')e^{2i\pi\gamma'\cdot\mathbf{k}}\hat{\boldsymbol{f}}(\mathbf{x}', \gamma')\mathrm{d}\mathbf{x}' \\
&= \sum_{\gamma' \in \mathbb{Z}^2} \sum_{\tilde{\gamma} \in \mathbb{Z}^2} e^{2i\pi \mathbf{k}\cdot\tilde{\gamma}} e^{2i\pi \mathbf{k}\cdot\mathbf{x}} \int_\Omega K_0(\mathbf{x} + \tilde{\gamma}, \mathbf{x}')e^{2i\pi\mathbf{k}\cdot\gamma'}\hat{\boldsymbol{f}}(\mathbf{x}', \gamma')\mathrm{d}\mathbf{x}' \\
&= \int_\Omega \sum_{\gamma \in \mathbb{Z}^2} e^{2i\pi \mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} e^{2i\pi \mathbf{k}\cdot\gamma} K_0(\mathbf{x} + \gamma, \mathbf{x}') \sum_{\gamma' \in \mathbb{Z}^2} e^{2i\pi \mathbf{k}\cdot(\gamma'+\mathbf{x}')}\hat{\boldsymbol{f}}(\mathbf{x}', \gamma')\mathrm{d}\mathbf{x}'.
\end{aligned}$$

In the last line, we identify the Fourier series representation of $\boldsymbol{f}(\mathbf{x}', \cdot)$ at the point $\mathbf{k}$. We finally obtain

$$\int_\Omega \sum_{\gamma \in \mathbb{Z}^2} e^{2i\pi \mathbf{k}\cdot(\mathbf{x}+\gamma-\mathbf{x}')} K_0(\mathbf{x} + \gamma, \mathbf{x}')\boldsymbol{f}(\mathbf{x}', \mathbf{k})\mathrm{d}\mathbf{x}' = \int_\Omega g_k(\mathbf{x}, \mathbf{x}')\boldsymbol{f}(\mathbf{x}', \mathbf{k})\mathrm{d}\mathbf{x}',$$

which concludes the proof of Proposition 3. □

PROPOSITION 4. *We have*

$$\mathcal{U}H_0\mathcal{U}^* = \int_\Omega^\oplus h_{\mathbf{k}}^{(0)}\mathrm{d}\mathbf{k}\,,$$

with

$$h_{\mathbf{k}}^{(0)} = \boldsymbol{\sigma} \cdot \boldsymbol{F}(-i\nabla_{\mathrm{per}} - 2\pi\mathbf{k}).$$

Proof. According to Theorem XIII.85 of [10], to prove (22), we need to show that, for $\mathbf{k} \in \Omega$ $g_{\mathbf{k}} : L^2(\Omega, \mathbb{C}^2) \to L^2(\Omega, \mathbb{C}^2)$ satisfies

$$g_{\mathbf{k}} = (h_{\mathbf{k}}^{(0)} - i)^{-1}.$$

To this purpose, we will denote by $(e_j)_{j=1,2}$ the vectors of the standard basis in $\mathbb{C}^2$ and $\Psi_{\mathbf{m}} = e^{2i\pi\mathbf{m}\cdot\mathbf{x}}$. We will prove that for all $\mathbf{m} \in \mathbb{Z}^2$ and $j \in \{1,2\}$ we have

$$g_{\mathbf{k}}(\Psi_{\mathbf{m}} \otimes e_j) = (h_{\mathbf{k}}^{(0)} - i)^{-1}(\Psi_{\mathbf{m}} \otimes e_j) = (\boldsymbol{\sigma} \cdot \boldsymbol{F}(2\pi(\mathbf{m} - \mathbf{k})) - i)^{-1}(\Psi_{\mathbf{m}} \otimes e_j).$$

Recall the notation $G(\boldsymbol{p}) = (\boldsymbol{\sigma} \cdot \boldsymbol{F}(\boldsymbol{p}) - i)^{-1} \in \mathcal{B}(\mathbb{C}^2)$. We have that

(37)

$$\begin{aligned}
&\{g_k(\Psi_{\mathbf{m}} \otimes e_j)\}(\mathbf{x}) \\
&= \int_\Omega g_k(\mathbf{x}, \mathbf{x}')(\Psi_{\mathbf{m}} \otimes e_j)(\mathbf{x}')\mathrm{d}\mathbf{x}' \\
&= \Psi_{\mathbf{m}}(\mathbf{x}) \int_\Omega \mathrm{d}\mathbf{x}' \sum_{\boldsymbol{\gamma}\in\mathbb{Z}^2} e^{2i\pi\mathbf{k}\cdot(\mathbf{x}+\boldsymbol{\gamma}-\mathbf{x}')} e^{2i\pi\mathbf{m}\cdot(-\mathbf{x}+\mathbf{x}')} \frac{1}{2\pi}\mathcal{F}^{-1}(G)(\mathbf{x} - \mathbf{x}' + \boldsymbol{\gamma})e_j.
\end{aligned}$$

Because both $\mathbf{m}$ and $\boldsymbol{\gamma}$ are in $\mathbb{Z}^2$ we have $e^{2\pi i\mathbf{m}\cdot\boldsymbol{\gamma}} = 1$, hence in (37) we can replace $e^{2i\pi\mathbf{m}\cdot(-\mathbf{x}+\mathbf{x}')}$ with $e^{-2i\pi\mathbf{m}\cdot(\mathbf{x}+\boldsymbol{\gamma}-\mathbf{x}')}$. Thus, after a change of variables we obtain

$$\begin{aligned}
\{g_{\mathbf{k}}(\Psi_{\mathbf{m}} \otimes e_j)\}(\mathbf{x}) &= \Psi_{\mathbf{m}}(\mathbf{x})\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-2i\pi(\mathbf{m}-\mathbf{k})\cdot\mathbf{y}}\mathcal{F}^{-1}(G)(\mathbf{y})\mathrm{d}\mathbf{y}\, e_j \\
&= G(2\pi(\mathbf{m} - \mathbf{k}))\,(\Psi_{\mathbf{m}} \otimes e_j)(\mathbf{x}).
\end{aligned}$$

This ends the proof of the proposition. □

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Aix Marseille Univ, Université de Toulon, CNRS,
CPT, Marseille, France
`jean-marie.barbaroux@univ-tln.fr`

Department of Mathematical Sciences, Aalborg University
Skjernvej 4A, 9220 Aalborg Ø, Denmark
`cornean@math.aau.dk`

Aix Marseille Univ, Université de Toulon, CNRS,
CPT, Marseille, France
`sylvain.zalczer@univ-tln.fr`