VARIATIONAL INCLUSIONS FOR A CLASS OF FRACTIONAL DIFFERENTIAL INCLUSIONS

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Communicated by Gabriela Marinoschi

We establish several fractional variational inclusions for solutions of a nonconvex fractional differential inclusion involving Caputo-Katugampola fractional derivative.

AMS 2010 Subject Classification: 34A60, 26A33, 26A42.

Key words: fractional derivative, variational inclusion, tangent cone, set-valued derivative.

1. INTRODUCTION

In the last years one may see a strong development of the theory of differential equations and inclusions of fractional order ([3, 5, 15, 16, 17] etc.). The main reason is that fractional differential equations are very useful tools in order to model many physical phenomena.

In the fractional calculus there are several fractional derivatives. From them, the fractional derivative introduced by Caputo allows to use Cauchy conditions which have physical meanings. Recently, a generalized Caputo-Katugampola fractional derivative was proposed in [14] by Katugampola and afterwards he provided the existence of solutions for fractional differential equations defined by this derivative. This Caputo-Katugampola fractional derivative extends the well known Caputo and Caputo-Hadamard fractional derivatives into a single form. Even if Katugampola fractional integral operator is an Erdélyi-Kober type operator ([10]) it is argued ([14]) that is not possible to derive Hadamard equivalence operators from Erdélyi-Kober type operators. Also, in some recent papers [1, 14, 18], several qualitative properties of solutions of fractional differential equations defined by Caputo-Katugampola derivative were obtained.

The present paper is concerned with fractional differential inclusions of the form

$$D_c^{\alpha,\rho}x(t) \in F(t,x(t)) \quad a.e.\ ([0,T]), \quad x(0)=x_0, \quad x'(0)=x_1 \tag{1.1}$$

where $\alpha \in (1,2]$, $\rho \geq 1$, $D_c^{\alpha,\rho}$ is the Caputo-Katugampola fractional derivative, $F : [0,T] \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map and $x_0, x_1 \in \mathbf{R}$.

The aim of this paper is twofold. On one hand, we briefly present some existing results in the literature concerning problem (1.1). Namely, we recall an existence result of Filippov type, a continuous selection of the solution set and a sufficient condition for local controllability along a reference trajectory in terms of certain variational fractional differential inclusion associated to problem (1.1).

On the other hand, we extend the results concerning the differentiability of solutions of differential inclusions with respect to initial conditions to the solutions of problem (1.1). In Control Theory, mainly, if we want to obtain necessary optimality conditions, it is essential to have several "differentiability" properties of solutions with respect to initial conditions. One of the most powerful result in the theory of differential equations, the classical Bendixson-Picard-Lindelöf theorem states that the maximal flow of a differential equation is differentiable with respect to initial conditions and its derivatives verify the variational equation. This result has been generalized in various ways to differential inclusions by considering several variational inclusions and proving corresponding theorems that extend Bendixson-Picard-Lindelöf theorem. The results we extend known as the contingent, the intermediate (quasitangent) and the circatangent variational inclusion are obtained in the "classical case" of differential inclusions. For this results and for a complete discussion on this topic we refer to [2].

The proofs of our results follows by an approach similar to the classical case of differential inclusions ([2, 12]) and use a recent result ([6]) concerning the existence of solutions of problem (1.1).

The paper is organized as follows: in Section 2 we present preliminary results to be used in the next section, Section 4 is devoted to a short survey of the results existing for our problem and in Section 4 we prove our main results.

2. PRELIMINARIES

Let Y be a normed space, $X \subset Y$ and $x \in \overline{X}$ (the closure of X).

From the multitude of the tangent cones in the literature (e.g., [2]) we recall only the contingent, the quasitangent and Clarke's tangent cones, defined, respectively by

$$K_x X = \{v \in Y; \quad \exists s_m \to 0+,\ \exists v_m \to v :\ x + s_m v_m \in X\},$$

$$Q_x X = \{v \in Y; \quad \forall s_m \to 0+,\ \exists v_m \to v :\ x + s_m v_m \in X\},$$

$$C_x X = \{v \in Y; \forall (x_m, s_m) \to (x, 0+),\ x_m \in X,\ \exists y_m \in X :\ \frac{y_m - x_m}{s_m} \to v\}.$$

This cones are related as follows: $C_xX \subset Q_xX \subset K_xX$.

Corresponding to each type of tangent cone, say τ_xX, one may introduce (e.g., [2]) a *set-valued directional derivative* of a multifunction $G(\cdot) : X \subset Y \to \mathcal{P}(Y)$ (in particular of a single-valued mapping) at a point $(x, y) \in \text{Graph}(G)$ as follows

$$\tau_y G(x; v) = \{w \in Y; \quad (v, w) \in \tau_{(x,y)}\text{Graph}(G)\}, \quad v \in \tau_x X.$$

We recall that a set-valued map, $A(.) : \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^n)$ is said to be a *convex* (respectively, closed convex) *process* if $\text{Graph}(A(.)) \subset \mathbf{R}^n \times \mathbf{R}^n$ is a convex (respectively, closed convex) cone.

Let $I := [0, T]$, denote by $C(I, \mathbf{R})$ the Banach space of all continuous functions from I to $\mathbf{R}$ endowed with the norm $|x|_C = \sup_{t\in I}|x(t)|dt$ and by $L^1(I, \mathbf{R})$ we denote the Banach space of Lebegue integrable functions $u(.) : I \to \mathbf{R}$ endowed with the norm $|u|_1 = \int_0^1 |u(t)|dt$.

Definition 2.1 ([14]). a) *The generalized left-sided fractional integral of order* $\alpha > 0$ of a Lebesgue integrable function $f : [0, \infty) \to \mathbf{R}$ is defined by

$$I^{\alpha,\rho} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s)\mathrm{d}s,$$

provided the right-hand side is pointwise defined on $(0, \infty)$ and $\Gamma(.)$ is (Euler's) Gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\mathrm{d}t$.

b) *The generalized fractional derivative*, corresponding to the generalized left-sided fractional integral of a function $f : [0, \infty) \to \mathbf{R}$ is defined by

$$D^{\alpha,\rho} f(t) = (t^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}t})^n (I^{n-\alpha,\rho})(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} (t^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}t})^n \int_0^t \frac{s^{\rho-1} f(s)}{(t^\rho - s^\rho)^{\alpha-n+1}} \mathrm{d}s$$

if the integral exists and $n = [\alpha] + 1$.

c) *The Caputo-Katugampola* generalized fractional derivative is defined by

$$D_c^{\alpha,\rho} f(t) = (D^{\alpha,\rho}[f(s) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} s^k])(t)$$

We note that if $\rho = 1$, the Caputo-Katugampola fractional derivative becames the well known Caputo fractional derivative. On the other hand, passing to the limit with $\rho \to 0+$, the above definition yields the Hadamard fractional derivative.

Definition 2.2. A function $x(.) \in C(I, \mathbf{R})$ is called a solution of problem (1.1) if there exists a function $f(.) \in L^1(I, \mathbf{R})$ with $f(t) \in F(t, x(t))$, *a.e.* (I) such that $D_c^{\alpha,\rho} x(t) = f(t)$, *a.e.* (I) and $x(0) = x_0, x'(0) = x_1$.

We shall use the following notations for the solution sets of (1.1).

$$\mathcal{S}(x_0,x_1)=\{(x(.),f(.));\quad (x(.),f(.)) \text{ is a trajectory-selection pair of (1.1)}\}.$$

HYPOTHESIS 2.3. i) $F(.,.): I\times \mathbf{R}\to \mathcal{P}(\mathbf{R})$ *has nonempty closed values and for every* $x\in \mathbf{R}$, $F(.,x)$ *is measurable.*

ii) *There exist* $L(.)\in L^1(I,(0,\infty))$ *such that for almost all* $t\in I, F(t,.)$ *is* $L(t)$*-Lipschitz in the sense that*

$$d_H(F(t,x_1),F(t,x_2))\leq L(t)|x_1-x_2|\quad \forall\ x_1,x_2\in \mathbf{R},$$

where $d_H(A,C)$ *is Pompeiu-Hausdorff distance between closed sets* $A,C\subset \mathbf{R}$

$$d_H(A,C)=\max\{d^*(A,C),d^*(C,A)\},\quad d^*(A,C)=\sup\{d(a,C);a\in A\}.$$

On $C(I,\mathbf{R})\times L^1(I,\mathbf{R})$ we consider the following norm

$$|(x,f)|_{C\times L}=|x|_C+|f|_1\quad \forall\ (x,f)\in C(I,\mathbf{R})\times L^1(I,\mathbf{R}).$$

3. A SURVEY ON SOME RECENT RESULTS

The next result ([5]) is an extension of Filippov's theorem concerning the existence of solutions to a Lipschitzian differential inclusion ([11]) to fractional differential inclusions of the form (1.1). We recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov's theorem ([11]) consists in proving the existence of a solution starting from a given almost solution. Moreover, the result provides an estimate between the starting almost solution and the solution of the differential inclusion.

Consider $y_0,y_1\in \mathbf{R}$, $g(.)\in L^1(I,\mathbf{R})$ and $y(.)$ is a solution of the problem

$$D_c^{\alpha,\rho}y(t)=g(t)\quad y(0)=y_0,\quad y'(0)=y_1.$$

Denote

$$\eta=\frac{1}{1-I^{\alpha,\rho}L(T)}(|x_0-y(0)|+T|y'(0)-x_1|+I^{\alpha,\rho}q(T)).$$

THEOREM 3.1. *Assume that Hypothesis 2.3 is satisfied, assume that* $I^{\alpha,\rho}L(T)<1$ *and let* $y(.)\in C(I,\mathbf{R})$ *be such that there exists* $q(.)\in L^1(I,\mathbf{R})$ *with* $I^{\alpha,\rho}q(T)<+\infty$ *and* $d(D_c^{\alpha,\rho}y(t),F(t,y(t)))\leq q(t)$ *a.e.* (I).

Then there exists $x(.):I\to \mathbf{R}$ *a solution of problem (1.1) satisfying for all* $t\in I$

$$|x(t)-y(t)|\leq \eta\quad \forall t\in I,$$

$$|f(t)-g(t)|\leq L(t)\eta+q(t)\quad a.e.\ (I).$$

The proof of Theorem 3.1 may be found in [5].

At the same time, one may obtain the existence of solutions continuously depending on a parameter for problem (1.1). This result may be regarded also as a continuous version of Theorem 3.1. The proof is essentially based on the Bressan-Colombo selection theorem concerning the existence of continuous selections of lower semicontinuous multifunctions with decomposable values([4]).

HYPOTHESIS 3.2. (i) *S is a separable metric space, $a(.), b(.) : S \to \mathbf{R}$ and $c(.) : S \to (0, \infty)$ are continuous mappings.*

(ii) *There exists the continuous mappings $g(.), p(.) : S \to L^1(I, \mathbf{R})$, $y(.) : S \to C(I, \mathbf{R})$ such that*

$$(Dy(s))_c^{\alpha,\rho}(t) = g(s)(t) \quad a.e.\ t \in I, \quad \forall s \in S,$$

$$d(g(s)(t), F(t, y(s)(t)) \leq p(s)(t) \quad a.e.\ t \in I,\ \forall\ s \in S.$$

We use next the following notation

$$\xi(s) = \frac{1}{1 - |I^{\alpha,\rho}L|}(|a(s) - y(s)(0)| + T|b(s) - (y(s))'(0)| + c(s) + |I^{\alpha,\rho}p(s)|),$$

where $s \in S$, $|I^{\alpha,\rho}L| := \sup_{t \in I}|I^{\alpha,\rho}L(t)|$ and $|I^{\alpha,\rho}p(s)| := \sup_{t \in I}|I^{\alpha,\rho}p(s)(t)|$.

THEOREM 3.3. *Assume that Hypotheses 2.3 and 3.2 are satisfied.*

If $|I^{\alpha,\rho}L| < 1$, then there exist a continuous mapping $x(.) : S \to C(I, \mathbf{R})$ such that for any $s \in S$, $x(s)(.)$ is a solution of problem

$$D_c^{\alpha,\rho} z(t) \in F(t, z(t)), \quad z(0) = a(s), \quad z'(0) = b(s)$$

such that

$$|x(s)(t) - y(s)(t)| \leq \xi(s) \quad \forall (t, s) \in I \times S.$$

As a consequence of this result we obtained a continuous selection of the solution set of problem (1.1).

HYPOTHESIS 3.4. *Hypothesis 2.3 is satisfied, $|I^{\alpha,\rho}L| < 1$, there exists $p_0(.) \in L^1(I, \mathbf{R}_+)$ with $d(0, F(t, 0) \leq p_0(t)$ a.e. (I).*

COROLLARY 3.5. *Assume that Hypothesis 3.4 is satisfied. Then there exists a function $s(.,.) : I \times \mathbf{R}^2 \to \mathbf{R}$ with the following properties*

a) $s(., (\xi, \eta)) \in \mathcal{S}(\xi, \eta)$, $\forall (\xi, \eta) \in \mathbf{R}^2$.

b) $(\xi, \eta) \to s(., (\xi, \eta))$ *from $\mathbf{R}^2$ into $C(I, \mathbf{R})$ is continous.*

The proof of Theorem 3.3 and Corollary 3.4 may be found in [6].

In order to obtain a sufficient condition for local controllability along a reference trajectory of differential inclusion (1.1) we use the notion of derived cone to an arbitrary subset of a normed space introduced by M. Hestenes in [13].

A subset $D \subset \mathbf{R}^n$ is said to be a *derived set to* $X \subset \mathbf{R}^n$ at $x \in X$ if for any finite subset $\{w_1, ..., w_k\} \subset D$, there exist $s_0 > 0$ and a continuous mapping $\alpha(.) : [0, s_0]^k \to X$ such that $\alpha(0) = x$ and $\alpha(.)$ is (conically) differentiable at $s = 0$ with the derivative $\mathrm{col}[w_1, ..., w_k]$ in the sense that

$$\lim_{\mathbf{R}_+^k \ni \theta \to 0} \frac{||\alpha(\theta) - \alpha(0) - \sum_{i=1}^k \theta_i w_i||}{||\theta||} = 0.$$

A subset $C \subset \mathbf{R}^n$ is said to be a *derived cone* of X at x if it is a derived set and also a convex cone.

Among other properties of derived cones we recall the one in the next lemma and proved in [13].

Lemma 3.6. *Let $X \subset \mathbf{R}^n$. Then $x \in int(X)$ if and only if $C = \mathbf{R}^n$ is a derived cone at $x \in X$ to X.*

We consider next the reachable set of (1.1) defined by

$$R_F(T, X_0, X_1) := \{x(T); \quad x(.) \text{ is a solution of } (1.1)\}.$$

We define a certain variational fractional differential inclusion and we shall prove that the reachable set of this variational inclusion from derived cones $C_0 \subset \mathbf{R}$ to X_0 and $C_1 \subset \mathbf{R}$ to X_1 at time T is a derived cone to the reachable set $R_F(T, X_0, X_1)$.

Hypothesis 3.7. i) *Hypothesis 2.3 is satisfied, $|I^{\alpha,\rho}L| < 1$ and $X_0, X_1 \subset \mathbf{R}$ are closed sets.*

ii) *$(z(.), f(.)) \in C(I, \mathbf{R}) \times L^1(I, \mathbf{R})$ is a trajectory-selection pair of (1.1) and a family $A(t, .) : \mathbf{R} \to \mathcal{P}(\mathbf{R})$, $t \in I$ of convex processes satisfying the condition*

$$A(t, u) \subset Q_{f(t)}F(t, .)(z(t); u) \quad \forall\, u \in dom(A(t, .)),\ a.e.\ t \in I$$

is assumed to be given and defines the variational inclusion

$$D_c^{\alpha,\rho} w(t) \in A(t, w(t)). \tag{3.1}$$

We mention that for any set-valued map $F(., .)$, one may find an infinite number of families of convex process $A(t, .)$, $t \in I$, satisfying condition (3.1). For example, we may take an "intrinsic" family of such closed convex process; namely, Clarke's convex-valued directional derivatives $C_{f(t)}F(t, .)(z(t); .)$.

Theorem 3.8. *Assume that Hypothesis 3.7 is satisfied, let $C_0 \subset \mathbf{R}$ be a derived cone to X_0 at $z(0)$ and $C_1 \subset \mathbf{R}$ be a derived cone to X_1 at $z'(0)$. Then the reachable set $R_A(T, C_0, C_1)$ of (3.1) is a derived cone to $R_F(T, X_0, X_1)$ at $z(T)$.*

The proof of Theorem 3.8, which essentially uses Theorem 3.3, may be found in [7].

From Theorem 3.8 and Lemma 3.6 it follows a sufficient condition for local controllability of the fractional differential inclusion (1.1) along a reference trajectory, $z(.)$ at time T, in the sense that

$$z(T) \in \operatorname{int}(R_F(T, X_0, X_1)).$$

THEOREM 3.9. *Let $z(.), F(.,.)$ and $A(.,.)$ satisfy Hypothesis 3.7, let $C_0 \subset \mathbf{R}$ be a derived cone to X_0 at $z(0)$ and $C_1 \subset \mathbf{R}$ be a derived cone to X_1 at $z'(0)$. If, the variational fractional differential inclusion in (3.1) is controllable at T in the sense that $R_A(T, C_0, C_1) = \mathbf{R}$, then the differential inclusion (1.1) is locally controllable along $z(.)$ at time T.*

4. THE MAIN RESULTS

Let $(y(.), g(.))$ be a trajectory-selection pair of problem (1.1). We wish to "linearize" (1.1) along $(y(.), g(.))$ by replacing it by several fractional variational inclusions.

Consider, first, the quasitangent variational inclusion

$$\begin{cases} D_c^{\alpha,\rho} w(t) \in Q_{g(t)}(F(t,.))(y(t); w(t)) \quad a.e.\ (I) \\ w(0) = u, \quad w'(0) = v, \end{cases} \tag{4.1}$$

where $u, v \in \mathbf{R}$.

THEOREM 4.1. *Consider the solution map $\mathcal{S}(.,.)$ as a set valued map from $\mathbf{R} \times \mathbf{R}$ into $C(I, \mathbf{R}) \times L^1(I, \mathbf{R})$ and assume that Hypothesis 2.3 is satisfied.*

Then, for any $u, v \in \mathbf{R}$ and any trajectory-selection pair (w, π) of the linearized inclusion (4.1) one has

$$(w, \pi) \in Q_{(y,g)}\mathcal{S}((y(0), y'(0); (u, v)).$$

Proof. Let $u, v \in \mathbf{R}$ and let $(w, \pi) \in C(I, \mathbf{R}) \times L^1(I, \mathbf{R})$ be a trajectory-selection pair of (4.1). By the definition of the quasitangent derivative and from the Lipschitzianity of $F(t, .)$, for almost all $t \in I$, we have

$$\begin{aligned} &\lim_{h\to 0+} \mathrm{d}(D_c^{\alpha,\rho} w(t), \tfrac{F(t,y(t)+hw(t)) - D_c^{\alpha,\rho} y(t)}{h}) = \\ &\lim_{h\to 0+} \mathrm{d}(\pi(t), \tfrac{F(t,y(t)+hw(t)) - g(t)}{h}) = 0. \end{aligned} \tag{4.2}$$

Moreover, since $g(t) \in F(t, y(t))$ a.e. (I), from Hypothesis 2.3, for all enough small $h > 0$ and for almost all $t \in I$, one has

$$\mathrm{d}(D_c^{\alpha,\rho}(y(t) + hw(t)), F(t, y(t) + hw(t))) = \mathrm{d}(g(t) + h\pi(t), F(t, y(t)+ hw(t))) \leq h(|\pi(t)| + L(t)|w(t)|)$$

By standard arguments (e.g., Lemmas 1.4 and 1.5 in [12]) the function $t \to \mathrm{d}(g(t)+h\pi(t), F(t,y(t)+hw(t)))$ is measurable. Therefore, using the Lebesgue dominated convergence theorem we infer

$$\int_0^T \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}(T^\rho - s^\rho)^{\alpha-1}s^{\rho-1}\mathrm{d}(D_c^{\alpha,\rho}(y(t)+hw(t)), F(t,y(t)+hw(t)))\mathrm{d}t \le \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}T^{\rho\alpha-1}\int_0^T \mathrm{d}(D_c^{\alpha,\rho}(y(t)+hw(t)), F(t,y(t)+hw(t)))\mathrm{d}t = o(h), \tag{4.3}$$

where $\lim_{h\to 0+}\frac{o(h)}{h} = 0$.

We apply Theorem 3.1 and by (4.3) we deduce the existence of $M \ge 0$ and of trajectory-selection pairs $(y_h(.), g_h(.))$ of the second-order differential inclusion in (1.1) satisfying

$$|y_h - y - hw|_C + |g_h - g - h\pi|_1 \le Mo(h),$$
$$y_h(0) = y(0) + hu, \quad y_h'(0) = y'(0) + hv,$$

which implies

$$\lim_{h\to 0+}\frac{y_h - y}{h} = w \quad \text{in} \quad C(I, \mathbf{R}),$$
$$\lim_{h\to 0+}\frac{g_h - g}{h_n} = \pi \quad \text{in} \quad L^1(I, \mathbf{R}).$$

Therefore

$$\lim_{h\to 0+}\mathrm{d}_{C\times L}((w,\pi), \frac{\mathcal{S}((y(0)+hu, y'(0)+hv)) - (y,g)}{h}) = 0$$

and the proof is complete. □

We consider next the variational inclusion defined by the Clarke directional derivative of the set-valued map $F(t,.)$, i.e., the so called circatangent variational inclusion

$$\begin{cases} D_c^{\alpha,\rho}w(t) \in C_{g(t)}(F(t,.))(y(t); w(t)) \quad a.e. \ (I) \\ w(0) = u, \quad w'(0) = v. \end{cases} \tag{4.4}$$

Theorem 4.2. *Consider the solution map $\mathcal{S}(.,.)$ as a set valued map from $\mathbf{R}\times\mathbf{R}$ into $C(I,\mathbf{R})\times L^1(I,\mathbf{R})$ and assume that Hypothesis 2.3 is satisfied.*

Then, for any $u, v \in \mathbf{R}$ and any trajectory-selection pair (w,π) of the linearized inclusion (4.4) one has

$$(w,\pi) \in C_{(y,g)}\mathcal{S}((y(0), y'(0); (u,v)).$$

Proof. Let $u, v \in \mathbf{R}$, let $(w,\pi) \in C(I,\mathbf{R})\times L^1(I,\mathbf{R})$ be a trajectory-selection pair of (4.4), let (y_n, g_n) be a sequence of trajectory-selection pairs of (1.1) that converges to $(y,g) \in C(I,\mathbf{R})\times L^1(I,\mathbf{R})$ and let $h_n \to 0+$. Then there exists a subsequence $g_j(.) := g_{n_j}(.)$ such that

$$\lim_{j\to\infty} g_j(t) = g(t) \quad a.e. \ (I) \tag{4.5}$$

Denote $\lambda_j := h_{n_j}$. From (4.4) and from the definition of the Clarke directional derivative, for almost all $t \in I$ we have

$$\begin{aligned} &\lim_{h\to 0+} \mathrm{d}(D_c^{\alpha,\rho}w(t), \tfrac{F(t,y_j(t)+\lambda_j w(t))-D_c^{\alpha,\rho}y_j(t)}{\lambda_j}) = \\ &\lim_{h\to 0+} \mathrm{d}(\pi(t), \tfrac{F(t,y_j(t)+\lambda_j w(t))-g_j(t)}{\lambda_j}) = 0. \end{aligned} \tag{4.6}$$

Since $g_j(t) \in F(t, y_j(t))$ a.e. (I), for almost all $t \in I$, we get

$$\mathrm{d}(D_c^{\alpha,\rho}(y_j(t) + \lambda_j w(t)), F(t, y_j(t) + \lambda_j w(t))) = \mathrm{d}(g_j(t) + \lambda_j\pi(t), F(t, y_j(t) \\ +\lambda_j w(t))) \leq \lambda_j(|\pi(t)| + L(t)|w(t)|).$$

The last inequality together with Lebesgue's dominated convergence theorem implies

$$\begin{aligned} &\int_0^T \tfrac{\rho^{1-\alpha}}{\Gamma(\alpha)}(T^\rho - s^\rho)^{\alpha-1}s^{\rho-1}d(D_c^{\alpha,\rho}(y_j(t) + \lambda_j w(t)), F(t, y_j(t) + \lambda_j w(t)))\mathrm{d}t \leq \\ &\tfrac{\rho^{1-\alpha}}{\Gamma(\alpha)}T^{\rho\alpha-1}\int_0^T \mathrm{d}(D_c^{\alpha,\rho}(y_j(t) + \lambda_j w(t)), F(t, y_j(t) + \lambda_j w(t)))\mathrm{d}t = o(\lambda_j), \end{aligned} \tag{4.7}$$

where $\lim_{j\to\infty} \frac{o(\lambda_j)}{\lambda_j} = 0$.

We apply Theorem 3.1 and by (4.7) we deduce the existence of $M \geq 0$ and of trajectory-selections pairs $(\overline{y}_j(.), \overline{g}_j(.))$ of the fractional differential inclusion in (1.1) satisfying

$$|\overline{y}_j - y_j - \lambda_j w|_C + |\overline{g}_j - g_j - \lambda_j\pi|_1 \leq Mo(\lambda_j),$$

$$\overline{y}_j(0) = y(0) + \lambda_j u, \quad \overline{y}_j'(0) = y'(0) + \lambda_j v.$$

It follows that

$$\lim_{j\to\infty} \frac{\overline{y}_j - y}{\lambda_j} = w \quad \text{in} \quad C(I, \mathbf{R}),$$

$$\lim_{j\to\infty} \frac{\overline{g}_j - g}{\lambda_j} = \pi \quad \text{in} \quad L^1(I, \mathbf{R}),$$

which completes the proof. □

Finally, we consider the contingent variational inclusion

$$\begin{cases} D_c^{\alpha,\rho}w(t) \in \overline{co}K_{g(t)}(F(t,.))(y(t); w(t)) \quad a.e.\ (I) \\ w(0) = u, \quad w'(0) = v. \end{cases} \tag{4.8}$$

Theorem 4.3. *Consider the solution map $\mathcal{S}(.,.)$ as a set valued map from $\mathbf{R} \times \mathbf{R}$ into $C(I, \mathbf{R}) \times L^\infty(I, \mathbf{R})$, with $L^\infty(I, \mathbf{R})$ supplied with the weak-* topology and assume that Hypothesis 2.3 is satisfied.*

Then for any $u, v \in \mathbf{R}$ one has

$$K_{(y,g)}\mathcal{S}((y(0), y'(0); (u, v)) \subset \\ \{(w, \pi); (w, \pi) \text{ is a trajectory-selection pair of } (4.8)\}.$$

Proof. Let $u, v \in \mathbf{R}$ and let $(w, \pi) \in K_{(y,g)}\mathcal{S}((y(0), y'(0); (u, v))$. According to the definition of the contingent derivative there exist $h_n \to 0+, u_n \to u, v_n \to v$, $w_n(.) \to w(.)$ in $C(I, \mathbf{R})$, $\pi_n(.) \to \pi(.)$ in weak-* topology of $L^\infty(I, \mathbf{R})$ and $c > 0$ such that

$$\begin{aligned} &|\pi_n(t)| \le c \quad a.e. \ (I), \\ &g(t) + h_n\pi_n(t) \in F(t, y(t) + h_n w_n(t)) \quad a.e. \ (I), \\ &w_n(0) = u_n, w_n'(0) = v_n. \end{aligned} \tag{4.9}$$

Therefore,

$$\begin{aligned} &w_n(.) \quad \text{converges pointwise to} \quad w(.) \\ &\pi_n(.) \quad \text{converges weak in } L^1(I, \mathbf{R}) \text{ to} \quad \pi(.) \end{aligned} \tag{4.10}$$

We apply Mazur's theorem (e.g., [9]) and we find that there exists

$$v_m(t) = \sum_{p=m}^{\infty} a_m^p \pi_p(t)$$

$v_m(.) \to \pi(.)$ (strong) in $L^1(I, \mathbf{R})$, where $a_m^p \ge 0$, $\sum_{p=m}^\infty a_m^p = 1$ and for any m, $a_m^p \neq 0$ for a finite number of p.

Therefore, a subsequence (again denoted) $v_m(.)$ converges la $\pi(.)$ a.e.. From (4.9) for any p and for almost all $t \in I$

$$w_p'(t) \in \frac{1}{h_p}(F(t, y(t) + h_p w_p(t)) - g(t)) \cap cB$$

Let $t \in I$ be such that $v_m(t) \to \pi(t)$ and $g(t) \in F(t, y(t))$. Fix $n \ge 1$ and $\epsilon > 0$. From (3.9) there exists m such that $h_p \le 1/n$ and $|w_p(t) - w(t)| \le 1/n$ for any $p \ge m$.

If, we denote

$$\phi(z, h) := \frac{1}{h}(F(t, y(t) + hz) - g(t)) \cap cB$$

then

$$v_m(t) \in \text{co}(\cup_{h \in (0, \frac{1}{n}], z \in B(w(t), \frac{1}{n})} \phi(z, h))$$

and if $m \to \infty$, we get

$$\pi(t) \in \overline{\text{co}}(\cup_{h \in (0, \frac{1}{n}], z \in B(w(t), \frac{1}{n})} \phi(z, h)).$$

Since, $\phi(z, h) \subset cB$, we infer that

$$\pi(t) \in \overline{\text{co}} \cap_{\epsilon > 0, n \ge 1} (\cup_{h \in (0, \frac{1}{n}], z \in B(w(t), \frac{1}{n})} \phi(z, h) + \epsilon B).$$

On the other hand,

$$\cap_{\epsilon > 0, n \ge 1} (\cup_{h \in (0, \frac{1}{n}], z \in B(w(t), \frac{1}{n})} \phi(z, h) + \epsilon B) \subset K_{g(t)} F(t, .)(y(t); w(t))$$

and the proof is complete. $\square$

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