ABOUT SOME RELATIVES OF PALINDROMES

VIOREL NIŢICĂ and ANDREI TÖRÖK

Communicated by Alexandru Zaharescu

We introduce two new classes of integers. The first class consists of numbers N for which there exists at least one integer A, such that the sum of A and the sum of digits of N, added to the reversal of the sum, gives N. The second class consists of numbers N for which there exists at least one integer A, such that the sum of A and the sum of the digits of N, multiplied by the reversal of the sum, gives N. All palindromes that either have an even number of digits or an odd number of digits and the middle digit even belong to the first class, and all squares of palindromes with at least two digits belong to the second class. These classes contain and are strictly larger than the classes of b-ARHardy numbers, respectively b-MRH numbers introduced in Niţică [6].

AMS 2010 Subject Classification: Primary 11B83, Secondary 11B99.

Key words: base, b-Niven number, reversal, additive b-Ramanujan-Hardy number, multiplicative b-Ramanujan-Hardy number, high degree b-Niven number, palindrome.

1. INTRODUCTION

Let $b \geq 2$ be a numeration base. In Niţică [6], motivated by a property of the taxicab number, 1729, we introduce the classes of *b-additive Ramanujan-Hardy (or b-ARH) numbers* and *b-multiplicative Ramanujan-Hardy (or b-MRH) numbers*. The first class consists of numbers N for which there exists at least one integer M, called *additive multiplier*, such that the product of M and the sum of base b digits of N, added to the reversal of the product, gives N. The second class consists of numbers N for which there exists at least one integer M, called *multiplicative multiplier*, such that the product of M and the sum of base b digits of N, multiplied by the reversal of the product, gives N. We show in [6, 8] the existence of infinite sets of b-ARH and b-MRH numbers and infinite sets of multipliers. Nevertheless, several questions asked in [6, 8] remain open.

Concerned with the sequences A005349, A067030, A305130, A305131, A306830, A323190 in *The On-Line Encyclopedia of Integer Sequences*, `http://oeis.org`

Viorel Niţică passed away on June 20, 2021

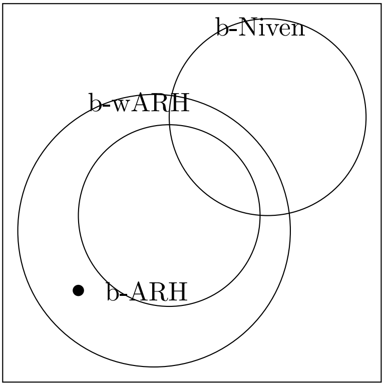
In particular we would like to find obstructions to the existence of multipliers and infinite sets of multipliers of fixed multiplicity.

In this paper we change the definitions above. We replace the product between the sum of digits and the multiplier by the sum of the sum of digits and a positive extra term. This gives two new classes of numbers, b-wARH and b-wMRH. These are strictly larger than those above. We believe that the study of these classes will bring some insight into the remaining open questions in [6, 8]. Another motivation for the study of these classes of numbers is the study of numerical palindromes. All palindromes that either have an even number of digits or an odd number of digits and the middle digit even belong to the first class, and all squares of palindromes with at least two digits belong to the second class. Some results in [6] bring new examples of b-Niven numbers. These are numbers divisible by the sum of their base b digits. See, for example, [1, 2, 3, 4, 7]). In particular, any b-MRH number is a b-Niven number. We expect the study here to shine new facets of this notion.

A computer search produced many $wARH$ numbers. There are 77 integers less than 10000 having this property; see sequence A305131 in the OEIS [5] and Table 1 in this paper. For example, 121212 has extra term 60597. The sum of the digits is 9, one has $9 + 60597 = 60606$, and $60606 + 60606 = 121212$.

A computer search produced also many $wMRH$ numbers. There are 365 integers less than 10000 having the property; see sequence A306830 in the OEIS [5] and Table 2 in this paper. For example, 2268 has extra term 18. The sum of the digits is 18, one has $18 + 18 = 36$, and $36 \times 63 = 2268$.

The paper is dedicated to the study of these classes of numbers. As a by-product we also clarify some relationships between the classes of numbers introduced here and in [6], and the well studied class of b-Niven numbers. The Venn diagrams in Figure 1, in which the universal set is the set of integers, record some relationships and lead to some open questions. The inclusion of the set of b-ARH numbers into the site of b-wARH numbers is proved in Proposition 7 and the inclusion of the set of b-MRH numbers into the set of b-wMRH numbers is proved in Proposition 16. We believe that each proper subset in the Venn diagrams contains an infinity of integers. Those subsets for which we already know this fact are marked by a full black dot. For the others, the question is open. See Corollary11 for an infinity of b-wARH numbers that are not b-Niven numbers. No large prime number can be either b-Niven or b-wMRH numbers. See the proof of Proposition 25 for an infinity of b-wMRH numbers that are not b-Niven numbers, and consequently neither b-MRH numbers.



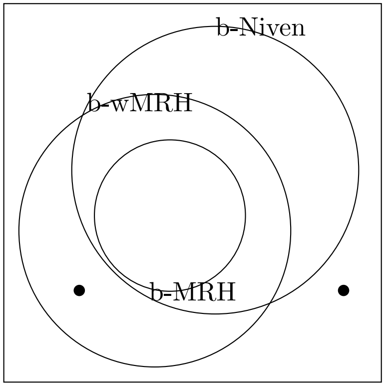


Figure 1

2. STATEMENTS OF THE MAIN RESULTS

In what follows let $b \geq 2$ be an arbitrary numeration base.

Definition 1. If N is a positive integer written in base b, we call *reversal* of N and let N^R denote the integer obtained from N by writing its digits in reverse order.

We observe that addition and multiplication are independent of the numeration base. The operation of taking the reversal is not.

Let $s_b(N)$ denote the sum of the base b digits of an integer N.

Definition 2. A positive integer N written in base b is called *weak b-additive Ramanujan-Hardy number,* or simply *b-wARH number*, if there exists an integer $A \geq 0$, called *additive extra term*, such that

$$N = A + s_b(N) + (A + s_b(N))^R, \tag{1}$$

where $(A + s_b(N))^R$ is the reversal of base b-representation of $A + s_b(N)$.

Definition 3. A positive integer N written in base b is called *weak b-multiplicative Ramanujan-Hardy number,* or simply *b-wMRH number*, if there exists an integer A, called *multiplicative extra term*, such that

$$N = (A + s_b(N)) \cdot (A + s_b(N))^R, \tag{2}$$

where $(A + s_b(N))^R$ is the reversal of base b-representation of $A + s_b(N)$.

To simplify the notation, let $s(N)$, wARH, wMRH denote $s_{10}(N)$, 10-wARH, 10-wMRH.

We observe that the notions of b-wARH and b-wMRH numbers are dependent on the base.

Example 4. The number $[121]_{10}$ is an wARH number with $A = 103$, but $[121]_9$ is not a 9-wARH number. The number $[121]_{10}$ is an wMRH number with $A = 7$, but $[121]_9$ is not a 9-wMRH number.

Once these notions are introduced and examples of such numbers found, several natural questions arise. Do there exist infinitely many b-wARH numbers? Do there exist infinitely many b-wMRH numbers? Do there exist infinitely many additive extra terms? Do there exist infinitely many multiplicative extra terms? All these questions are positively answered below.

In what follows, if x is a string of digits, we let $(x)^{\wedge k}$ denote the string obtained by repeating x k-times. We also let $[x]_b$ denote the value of the string x in base b.

The following proposition is of independent interest and it is also needed later.

PROPOSITION 5. *Let N be a base b integer. Then:*

a) $2s_b(N) \leq N$, *if N has at least two digits;*

b) $2s_b(N) + b - 1 \leq N \cdot b + \frac{b-1}{2}$, *if N has at least two digits;*

c) *If N has at least three digits, then*

$$s_b(N^2) \leq N. \tag{3}$$

The Proof of proposition 5 is done in Section 3

Remark 6. In Proposition 5, c), the condition that N has at least 3 digits is necessary, as shown by $N = [13]_{11}$.

The following proposition gives many examples of b-wARH numbers.

PROPOSITION 7. a) *Let N be a base b palindrome either with an even number of digits or with an odd number of digits and the digit in the middle position even. Then N is a b-wARH number.*

b) *Let N be a b-ARH number, Then N is a b-wARH number.*

COROLLARY 8. *For any string of digits I there exists an infinity of b-wARH numbers that contain I in their base b-representation.*

Proof. The string I is part of an infinity of base b palindromes with an even number of digits. □

COROLLARY 9. *For any integer N there exists an infinity of integers M such that $N \cdot M$ is a b-wARH number. Consequently, all integers are divisors of b-wARH numbers.*

Proof. It is proved in [8, Theorem 5] that for any integer N there exists an infinity of integers M such that $N \cdot M$ is a palindrome. If the palindrome has an even number of digits, we are done. Otherwise, if $P = N \cdot M$ is an arbitrary palindrome with k digits, consider the product $P \cdot [1(0)^{\wedge k-1}1]_b$, which is a palindrome with $2k$ digits. □

COROLLARY 10. *For any $b \geq 2$ there exist an infinity of arithmetic progressions of length b of b-wARH numbers.*

Proof. If I is a string of base b-digits of length at least 1, consider the following arithmetic progression of palindromes:

$$[I00I^R]_b, [I11I^R]_b, [I22I^R]_b, [I33I^R]_b, \ldots\ldots, [I(b{-}2)(b{-}2)I^R]_b, [I(b{-}1)(b{-}1)I]_b.$$

□

COROLLARY 11. *There exists an infinity of b-wARH numbers that are not b-Niven numbers.*

Proof. For any $k \geq 1$ define $N_k = [1(0)^{\wedge k}(b-1(b-1)(0)^{\wedge k}1]_b$. Then $s_b(N_k) = 2b$ and N_k is not divisible by b. But N_k are palindromes with even number of digits, so they are b-wARH numbers. □

We show in [6, Theorem 26] the existence of an infinity of integers that are not b-ARH. The following result has a similar proof.

PROPOSITION 12. *There exists an infinity of numbers that are not b-wARH numbers.*

The following result complements [6, Corollary 19], which applies only for b even.

PROPOSITION 13. *There exists an infinity of b-wARH numbers that are not b-MRH numbers.*

Proposition 13 is proved in Section 5.

Question 14. Does there exist an infinity of b-wARH numbers that are not b-ARH numbers?

PROPOSITION 15. *For any $b \geq 2$ there exists an infinity of b-wARH numbers and an infinity of extra terms.*

Proof. Consider the sequence $N_k = [1(0)^{\wedge k}(0)^{\wedge k}1]_b, k \geq 1$, with additive terms $A_k = b^{2k} - 2$. □

The following proposition gives many examples of b-wMRH numbers.

Proposition 16. a) *Let P be a a base b-palindrome with at least two digits and let $N = P^2$. Then N is a b-wMRH number.*

b) *Let N be a b-MRH number. Then N is a b-wMRH number.*

Corollary 17. *For any string of base b digits I there exists an infinity of b-wMRH numbers that contain I in their base b-representation.*

Proof. It is enough to show that the string I is part of an infinity of base b squares of base b palindromes. If $[I]_b$ is even, let $[J]_b$ be a k_0 digit string such that $2J = I$. Then I is part of the base b-representation of $\left([1(0)^{\wedge k}J(0)^{\wedge k}1]_b\right)^2$, for all $k \geq 3k_0$. If $[I]_b$ is odd, let $[J]_b$ be a k_0 digit string such that $2J + 1 = I$. Then I is part of the base b-representation of $\left(\left[J(0)^{\wedge k}1(0)^{\wedge k}1(0)^{\wedge k}J\right]_b\right)^2$ for all $k \geq 3k_0$. □

Corollary 18. *For any integer N there exists an infinity of integers M such that $N \cdot M$ is a b-wMRH number. Consequently, all integers are divisors of b-wMRH numbers.*

Proof. It is proved in [8, Theorem 5] that for any integer N there exists an infinity of integers M such that $N \cdot M$ is a palindrome. Then the product $N \cdot M \cdot (N \cdot M)$ is a b-MRH number. □

It is well known that there exists an infinity of numbers that are not b-Niven. As a b-MRH number is b-Niven, this gives an infinity of numbers that are not b-MRH numbers.

Proposition 19. *There exists an infinity of numbers that are not b-wMRH numbers.*

Proof. No prime number is b-wMRH number. □

Remark 20. The condition in Proposition 16 that P has at least 2 digits is necessary. Some squares of one digit numbers are b-wMRH number, for example 81, and some are not, for example 25.

Proposition 21. *For any $b \geq 2$ there exists an infinity of b-wMRH numbers and an infinity of extra terms.*

Proof. Consider the sequence $N_k = \left([1(0)^{\wedge k}1]_b\right)^2, k \geq 1$, with additive terms $A_k = b^k - 4$, if $b \geq 3$ and $A_k = b^k - 3$, if $b = 2$. □

Combining Proposition 7, c) and [6, Theorems 13,15] one has the following result.

THEOREM 22. a) *Consider the numbers*

$$N_k = [(1)^{\wedge k}]_b, \tag{4}$$

where b is even, $k = [1(0)^{\wedge p}]_b, p \geq 1$, p an arbitrary natural number. All numbers N_k are b-wARH numbers. Each N_k has a subset of additive multipliers of cardinality $2^{\frac{k-2p}{2}}$ consisting of all integers $k \cdot ([(1)^{\wedge p} I]_b)$, where I is a sequence of 0 and 1 of length $k - 2p$ in which no two digits symmetric about the center of the sequence are identical.

b) *Consider the numbers*

$$N_k = [(1)^{\wedge p}(10)^{\wedge k-2p}0(1)^{\wedge p}]_b, \tag{5}$$

where b is even and $k = [1(0)^{\wedge p}]_b, p \geq 1$, p arbitrary natural number. All numbers N_k are b-wARH numbers. For each N_k the set of additive extra terms has cardinality $(b-1)^{\frac{k-2p}{2}}$ and consists of all integers $2 \cdot ([(1)^{\wedge p} I0]_b - 1)$, where I is a concatenation of $k - 2p$ two digits strings of type $0\alpha, \alpha \neq 0$, in which any pair of nonzero digits symmetric about the center of $I0$ have their sum equal to b.

COROLLARY 23. *If b is even, there exists infinitely many b-wARH numbers that have at least two extra terms.*

Question 24. Do there exist infinitely many b-wMRH numbers that have at least two extra terms?

PROPOSITION 25. *There exists an infinity of b-wMRH numbers that are not b-MRH numbers.*

Question 26. Does there exist an infinitely of b-wARH numbers that are not b-wMRH?

Motivated by the results in Theorem 22, we introduce the following notions.

Definition 27. If N is a b-wARH number, let the *multiplicity* of N be the cardinality of the corresponding set of additive extra terms.

Definition 28. If N is a b-wMRH number, let the *multiplicity* of N be the cardinality of the corresponding set of multiplicative extra terms.

Theorem 22 has the following corollary.

COROLLARY 29. *The multiplicity of b-wARH numbers is unbounded for any even base.*

Question 30. Is the multiplicity of b-wMRH numbers bounded?

We show in [6, Theorem 25] an infinity of b-Niven numbers that are not b-MRH numbers. The following question is open.

Question 31. Does there exist an infinity of b-Niven numbers that are not b-wMRH numbers?

We show in Section 13 that 2 is not a multiplicative extra term for base 10. We do not know how to answer the following questions for any base.

Question 32. Do there exist infinitely many integers that are not additive extra terms?

Question 33. Do there exist infinitely many integers that are not multiplicative extra terms?

In what follows let $\lfloor x \rfloor$ denote the integer part, let $\ln x$ denote the natural logarithm and let $\log_b x$ denote base b logarithm of the positive real number x.

The following theorems give bounds for the number of digits in a b-wARH number with fixed extra term. Due to independent interest and in order to simplify the statements of other results we consider first the case when the extra term is $A = 0$.

THEOREM 34. *Let N be a b-wARH number with k digits and additive extra term $A = 0$. Then $N = 0$, $N = [11]_2$, $N = [22]_3$, or $N = [1(b-2)]_b$.*

THEOREM 35. *Let N be a b-wARH number with k digits and additive extra term A. Then*

$$k \leq A + 4.$$

COROLLARY 36. *For fixed additive extra term A and base b, the set of b-wARH numbers with extra term A is finite.*

THEOREM 37. *Let N be a b-wARH number with k digits and additive extra term A. Under the assumption $A \geq b^3$ one has:*

$$k \leq 2\lfloor \log_b A \rfloor. \tag{6}$$

The following theorems give bounds for the number of digits in a b-wMRH number with fixed extra term. Due to independent interest, we leave as open problem finding all b-wMRH numbers with extra term $A = 0$.

THEOREM 38. *Let N be a b-wMRH number with k digits and multiplicative extra term $A \geq 1$. Then*

$$k \leq \begin{cases} A + 4, & \text{if } b \geq 6; \\ A + 5, & \text{if } 2 \leq b \leq 5. \end{cases}$$

COROLLARY 39. *For fixed multiplicative extra terms A and base b, the set of b-wMRH numbers with extra term A is finite.*

THEOREM 40. *Let N be a b-wMRH number with k digits and multiplicative extra term $A \geq 1$. Under any of the following assumptions:*

- $b \geq 3$ *and* $A \geq b^3$;
- $b = 2$ *and* $A \geq b^2$;

one has

$$k \leq 3\lfloor \log_b A \rfloor. \tag{7}$$

We summarize the rest of the paper. Proposition 5 is proved in Section 3, Proposition 7 is proved in Section 4, Proposition 13 is proved in Section 5, Proposition 16 is proved in Section 6, Proposition 25 is proved in Section 7, Proposition 34 is proved in Section 8, Theorem 35 is proved in Section 9, Theorem 37 is proved in Section 10, Theorem 38 is proved in Section 11, and Theorem 40 is proved in Section 12. In Section 13 we show examples of wARH numbers and ask additional questions and in Section 14 we show examples of wMRH numbers and ask additional questions.

3. PROOF OF PROPOSITION 5

Proof. a), b) Clearly b) implies a), so it is enough to prove b). Assume N has $n \geq 2$ digits. Then $N \geq b^{n-1}$ and $s_b(N) \leq n(b-1)$. To finish the proof, we show by induction on $n \geq 2$ that

$$2(b-1)n + (b-1) \leq b \cdot (b^{n-1}) + \frac{b-1}{2}. \tag{8}$$

Equation (8) is true if $n = 2$. Assume now that it is true for n and prove it for $n+1$. Induction hypothesis gives that:

$$\begin{aligned} 2(b-1)(n+1) + (b-1) &= 2(b-1)n + 2(b-1) + (b-1) \\ &\leq b \cdot (b^{n-1}) + \frac{b-1}{2} + 2(b-1). \end{aligned} \tag{9}$$

We still need to show that:

$$b \cdot (b^{n-1}) + \frac{b-1}{2} + 2(b-1) \leq b \cdot (b^n - 1) + \frac{b-1}{2}. \tag{10}$$

After some cancellation, equation (10) becomes $2 \leq b^n$, which is true for $n \geq 2, b \geq 2$.

c) Assume that N has $n \geq 3$ digits. Then $b^{n-1} \leq N \leq b^n - 1$. Hence

$$b^{2n-2} \leq N^2 \geq (b^n - 1)^2 = b^{2n} - 2b^n + 1. \tag{11}$$

So N has $2n-1$ digits, and$s_b(N^2) \leq (b-1)(2n-1)$. To finish the proof it is enough to show that

$$(b-1)(2n-1) \leq b^{n-1}. \tag{12}$$

Equation (12) is true for $n=3$ and $b \geq 3$. We assume $n \geq 4$ fixed and prove (12) by induction on $b \geq 3$. The induction hypothesis, $b \geq 3$, and the binomial expansion of $(1+b)^n$, imply that for all $b \geq 3$ one has that:

$$b(2n-1) = (b-1)(2n-1) + (2b-1) \leq b^n - 1 + (2n-1) \leq (b+1)^n - 1.$$

If $b=2$ equation (12) becomes $2n-1 \leq 2^{n-1}$, true for $n \geq 4$. There are only 4 integers with $b=2, n=3$, and for them (3) can be checked numerically. □

4. PROOF OF PROPOSITION 7

Proof. a) Assume first that $N = [a_1a_2 \dots a_na_n \dots a_2a_1]_b$.

Define $A = [a_1a_2 \dots a_n(0)^{\wedge n}]_b - s_b(N)$. Then $A \geq 0$ due to Lemma 5 a) applied to $[a_1a_2 \dots a_n(0]_b$. One has that:

$$\begin{aligned}(s_b(N)+A) + (s_b(N)+A)^R &= [a_1a_2 \dots a_n(0)^{\wedge n}]_b + ([a_1a_2 \dots a_n(0)^{\wedge n}]_b)^R \\ &= [a_1a_2 \dots a_n(0)^{\wedge n}]_b + [a_na_{n-1} \dots a_1]_b = N.\end{aligned}$$

Now assume that $N = [a_1a_2 \dots a_na_{n+1}a_n \dots a_2a_1]_b$, where a_{n+1} is even. Define $A = [a_1a_2 \dots a_n \left(\frac{a_{n+1}}{2}\right)(0)^{\wedge n}]_b - s_b(N)$. Then $A \geq 0$ due to Lemma 5 b) applied to $[a_1a_2 \dots a_n \left(\frac{a_{n+1}}{2}\right)]_b$. One has that:

$$\begin{aligned}(s_b(N)+A) + (s_b(N)+A)^R &= [a_1a_2 \dots a_n(0)^{\wedge n}]_b + ([a_1a_2 \dots a_n(0)^{\wedge n}]_b)^R \\ &= [a_1a_2 \dots a_n \left(\frac{a_{n+1}}{2}\right)(0)^{\wedge n}]_b + [\left(\frac{a_{n+1}}{2}\right) a_na_{n-1} \dots a_1]_b = N.\end{aligned}$$

b) Let N be a b-ARH number with additive multiplier $M \geq 1$. Then N is also a b-wARH number with extra term $A = s_b(N)(M-1)$. □

5. PROOF OF PROPOSITION 13

Proof. It is known that a base b number is divisible by $b-1$ only if and only if the sum of its digits is divisible by $b-1$. Consider the numbers

$$N_k = [(b-1)(0)^{\wedge k}(b-1)]_b, k \text{ even}.$$

It follows from Proposition 7, a), that the numbers N_k are $b-wARH$ numbers. If $b=2$, then $s_b(N_k)=2$, but N_k is odd, so N_k is not a b-MRH number. Assume $b \geq 4$. As $s_b(N) = 2(b-1)$ it follows that N_k is divisible by $b-1$, but not by $(b-1)^2$. Nevertheless, if N_k is b-MRH number then it must be divisible by $(b-1)^2$. If $b = 3$ consider the numbers $N_k = [2(0)^{\wedge k}2(0)^{\wedge k}2]_3$. It follows from Proposition 7, a), that the numbers N_k are $3 - wARH$ numbers. As N_k are divisible by 2, but not by 4, it follows that N_k are not $3 - MRH$ numbers. □

6. PROOF OF PROPOSITION 16

Proof. a) Let P base b palindrome and let $N = P^2$. Assume that P has at least three digits. It follows from Lemma 5 c), that $s_b(N) \leq P$. Let $A = P - s_b(N)$. Then N is a b-wMRH number with extra term A. Assume now that P has two digits. Then $P = [aa]_b$ for $1 \leq a \leq b-1$. We will show that formula (3) is still valid. Then the argument above can be applied again. We distinguish three cases.

Case 1. $2a^2 < b$ Then $P = a(b+1)$, $N = [a^2(2a^2)a^2]_b$, and $s_b(N) = 4a^2$. If $a > 1$ one has that:

$$s_b(N) = 4a^2 < 4 \cdot \frac{b}{2} = 2b < a(b+1) = P.$$

If $a = 1$ and $b \geq 3$ one has that:

$$s_b(N) = 4 \leq b + 1 = P.$$

If $a = 1$ and $b = 2$ then the condition $2a^2 < b$ is not satisfied.

Case 2. $a^2 < b \leq 2a^2$ We distinguish two subcases:
a) $a^2 + 1 < b$ and b) $a^2 + 1 = b$.

Subcase a). $s_b(N) = a^2 + 1 + 2a^2 - b + a^2 = 4a^2 + 1 - b < 3(b-1)$. If $a \geq 3$ then

$$s_b(N) < 3(b-1) < a(b+1) = P.$$

If $a = 1$, the condition $b \leq 2a^2$ implies that $b = 2$. In this case $P = [11]_2$ and

$$s_b(P^2) = s_b([10001]_2) = 2 \leq P = 3.$$

If $a = 2$, $b \in \{6, 7, 8\}$. So $P = [22]_6, P = [22]_7$ or $P = [22]_8$. These cases can be checked numerically.

Subcase b). $s_b(N) = 1+2a^2-b+a^2 = 3a^2-b-1 = 3(a^2+1)-b-1 = 2(b-1)$. If $a \geq 2$ then

$$s_b(N) = 2(b-1) \leq a(b+1) = P.$$

If $a = 1$ then $b = 2$ and $P = [11]_2$.

Case 3. $a^2 \geq b$ Note that each "carry over" in the computation of P^2 reduces $s_b(P^2)$ by b and also increases it by 1. We have at least 4 carry overs, so the largest value for $s_b(P^2)$ is $4a^2 - 4b + 4$. The inequality $s_b(P^2) \leq P$ becomes

$$4a^2 - 4b + 4 \leq a(b+1),$$

or equivalently

$$4a^2 - a(b+1) + 4(1-b) \leq 0, \text{ for } 1 \leq a \leq b-1. \tag{13}$$

If $b \geq 3$, the quadratic function in (13) has the vertex at $a = \frac{b+1}{2} \in (1, b-1)$, so its largest values in the interval [1,b-1] are reached in the endpoints. Since its value in $a = 1$ is $7 - 5b$ and its value in $a = b - 1$ is $6 - 7b$, it follows that (13) holds. If $b = 2$ the remaining case is $P = [11]_2$.

b) Let N be a b-MRH number with additive multiplier $M \geq 1$. Then N is a b-wMRH number with extra term $A = s_b(N)(M-1)$. □

7. PROOF OF PROPOSITION 25

Proof. It follows from Proposition 16 that it is enough to find an infinity of squares of palindromes that are not b-Niven numbers.

If $b = 2$ consider

$$N_k = \left([1(0)^{\wedge k}1(0)^{\wedge k}1]_2\right)^2 = [1(0)^{\wedge k-1}1(0)^{\wedge k-1}11(0)^{\wedge k-1}1(0)^{\wedge k+1}1]_2.$$

Then $s_b(N_k) = 6$ and N_k is not divisible by 2 because it is odd. If b is even, and $b \neq 2$, then consider $N_k = \left([1(0)^{\wedge k}1]_b\right)^2 = [1(0)^{\wedge k}2(0)^{\wedge k}1]_b$. Then $s_b(N_k) = 4$ and N_k is not divisible by 2 because it is odd.

If b is odd and b congruent to 0 or 2 modulo 3, consider the numbers

$$N_k = \left([1(0)^{\wedge k}1(0)^{\wedge k}1]_b\right)^2$$
$$= [1(0)^{\wedge k}2(0)^{\wedge k}3(0)^{\wedge k}2(0)^{\wedge k}1]_b.k+1 \text{ odd}.$$

Then $s_b(N_k) = 9$ and N_k is not divisible by 3 because $[1(0)^{\wedge k}1(0)^{\wedge k}1]_b$ is not divisible by 3. For the case, $b \geq 11$ congruent to 1 modulo 3, consider the numbers

$$N_k = \left([2(0)^{\wedge k}1(0)^{\wedge k}2]_b\right)^2$$
$$= [4(0)^{\wedge k}3(0)^{\wedge k}(10)(0)^{\wedge k}3(0)^{\wedge k}4]_b.k+1.$$

Then $s_b(N_k) = 24$ and N_k is not divisible by 3 because $[2(0)^{\wedge k}1(0)^{\wedge k}2]_b$ is not divisible by 3. If $b \leq 11$, then $b \in \{9, 7, 5, 3\}$ and these cases are covered above.

□

8. PROOF OF THEOREM 34

Let $N \geq 1$ be a b-wARH number with extra term $A = 0$ and k digits. Then N is also a b-ARH number with additive multiplier $M = 1$. It follows from [6, Theorem 35] that $k \leq 2$ if $b \geq 4$ and $k \leq 3$ if $b = 2$ or $b = 3$. If $k = 1$ and $N > 0$, then $s_b(N) + s_b(N)^R > N$, so we can assume $k \geq 2$. If $k = 2$, then $N = [\alpha\beta]_b$ with $1 \leq \alpha, \beta \leq b-1$. If $\alpha + \beta < b$, then the equation $s_b(N) + s_b(N)^R = N$ gives $\alpha(b-2) = \beta \leq b-1$, which implies $\alpha \leq 2$. If $\alpha = 0$, then $\beta = 0$, so $N = 0$. If $\alpha = 1$, then $\beta = b-2$ and $N = [1(b-2)]_2$. If $\alpha = 2$ then $b = 3$ and $\beta = 2$, so $N = [22]_3$. Assume now $\alpha + \beta \geq b$. Then $\alpha b + \beta = 2(1 + \alpha + \beta - b)$ which implies $2(b-2) \leq 2 + \beta - b \leq 1$. So $\alpha = 1$ and $b = 2$, which implies $\beta = 1$. So $N = [11]_2$. The remaining cases with $k = 3$ and $a = 2, a = 3$ are finite in number and do not give any other b-wARH number.

9. PROOF OF THEOREM 35

The case $A = 0$ is covered by Theorem 34. Assume that N is a b-wARH number with $k \geq 2$ digits and additive extra term $A \geq 1$. One has that:

$$b^{k-1} \leq N = (s_b(N) + A) + (s_b(N) + A)^R \leq (b+1)\left((b-1)k + A\right). \tag{14}$$

We show by induction on k that:

$$(b+1)\left((b-1)k + A\right) < b^{k-1}, \text{ for } k \geq A+5, b \geq 2, A \geq 1. \tag{15}$$

As (14) and (15) are contradictory, this finishes the proof of the theorem.

For $k = A + 5$, (15) gives that:

$$(b+1)\left((b-1)(A+5) + A\right) < b^{A+4}, b \geq 2, A \geq 1, \tag{16}$$

which we prove by induction on A.

If $A = 1$, (16) gives that $(b+1)\left(6(b-1)+1\right) < b^5$, which is true for $b \geq 2$.

We show the induction step in (16). From the induction hypothesis one has that:

$$b^{A+5} = b^{A+4}b \geq b(b+1)\left((b-1)(A+5) + A\right).$$

One still needs to show that

$$b(b+1)\left((b-1)(A+5) + A\right) \geq (b+1)\left((b-1)(A+6) + A + 1\right).$$

The last inequality follows from $b(A+5) \geq A+6$ and $bA \geq A+1$.

We show the induction step in (15). From the induction hypothesis one has that:

$$b^k = b^{k-1}b \geq b(b+1)\left((b-1)k + A\right).$$

One still needs to show that

$$b(b+1)\left((b-1)k+A\right) \geq (b+1)\left((b-1)(k+1)+A\right).$$

Last inequality is equivalent to

$$b(b-1)k+bA \geq (b-1)(k+1)+A,$$

which follows due to $bk \geq k+1$ and $b \geq 1$.

10. **PROOF OF THEOREM 37**

Proof. Assume that N is a b-wARH number with $k \geq 2$ digits and additive extra term $A \geq 1$. One has (14). We show by induction on k that

(17) $$b^{k-1} > (b+1)\left((b-1)k+A\right), A \geq b^3, k \geq 2\lfloor \log_b A \rfloor, b \geq 2,$$

which is in contradiction to (14) and finishes the proof of the theorem.

In order to prove (17) for $k = 2\lfloor \log_b A \rfloor$ it is enough to show that

(18) $$b^{2\log_b A} > (b^2-1)(2\log_b A+1)+(b-1)A, b \geq 2, A \geq b^3,$$

which we will prove by induction on A. If $A = b^3$, then (18) becomes $b^6 > (b^2-1)\cdot 7+(b-1)b^3$, which is true for $b \geq 2$. we how the induction step in (18). From induction hypothesis follows that

$$(A+1)^2 = a^2+2A+1 > (b^2-1)(\log_b A^2+1)+(b-1)A+2A+1.$$

One still needs to check that:

$$(b^2-1)(\log_b A^2+1)+(b-1)A+2A+1 \geq (b^2-1)(\log_b (A+1)^2+1)+(b-1)(A+1).$$

Last equation is equivalent to $(b^2-1)\log_b\left(\frac{A}{A+1}\right)+2A+1 > b-1$, which is clearly true if $A \geq b^3$.

It remains to show the induction step in (17). From induction hypothesis follows that

$$b^k = b\cdot b^{k-1} > (b+1)((b-1)k+A).$$

One still needs to show

$$(b+1)((b-1)k+A) \geq (b+1)((b-1)(k+1)+A.$$

Last equation is equivalent to $(b-1)^2k+(b-1)A \geq b-1$, which is clearly true for $A \geq 1, b \geq 2$. $\square$

11. PROOF OF THEOREM 38

Proof. Assume that N is a b-wMRH number with $k \geq 2$ digits and additive extra term $A \geq 1$. One has that:

$$b^{k-1} \leq N = (s_b(N) + A) \cdot (s_b(N) + A)^R \leq b\left((b-1)k + A\right)^2. \tag{19}$$

In order to prove the theorem for $b \geq 6$, one shows by induction on k that:

$$b\left((b-1)k + A\right)^2 < b^{k-1}, \text{ if } k \geq A + 5, A \geq 1, b \geq 6. \tag{20}$$

If $k = A + 5$ (20) becomes

$$b\left((b-1)(A+5) + A\right)^2 < b^{A+4}. \tag{21}$$

We prove (21) by induction on $A \geq 1$.

If $A = 1$, (21) becomes $b\left((b-1)6 + 1\right)^2 < b^5$, which is true for $b \geq 6$. We show the induction step in (21). It follows from the induction hypothesis that

$$b^{A+5} = b \cdot b^{A+4} > b^2\left((b-1)(A+5) + A\right)^2.$$

One still needs to check that

$$b^2\left((b-1)(A+5) + A\right)^2 \geq b(b-1)(A+6) + A + 1)^2.$$

Last equation is equivalent to

$$\sqrt{b}(b-1)(A+5) + \sqrt{b}A \geq (b-1)(A+6) + A + 1$$

which is clearly true if $b \geq 6$. We show the induction step in (20). It follows from the induction hypothesis that

$$b^k = b \cdot b^{k-1} > b^2\left((b-1)k + A\right)^2.$$

One still needs to check that

$$b^2\left((b-1)k + A\right)^2 \geq b((b-1)(k+1) + A)^2.$$

Last equation is equivalent to

$$\sqrt{b}(b-1)k + \sqrt{b}A \geq (b-1)(k+1) + A,$$

which is clearly true if $b \geq 6$.

Assume now $2 \leq b \leq 5$. One shows by induction on k that:

$$b\left((b-1)k + A\right)^2 < b^{k-1}, \text{ if } k \geq A + 6, A \geq 1. \tag{22}$$

This finishes the proof of the theorem if $2 \leq b \leq 5$.

If $k = A+6$ then (22) becomes the following equation which is proved by induction on $A \geq 1$.

$$b\left((b-1)(A+6) + A\right) < 5^{A+5}, 2 \leq b \leq 5. \tag{23}$$

□

12. PROOF OF THEOREM 40

Proof. Assume that N is a b-wMRH number with $k \geq 2$ digits and additive extra term $A \geq 1$. One has (19). In order to finish the proof of the theorem in the case $b \geq 3$ one shows by induction on k that

$$b^{k-1} > b(b-1)\left((b-1)k + A\right) \text{ for } k \geq 3\lfloor \log_b A \rfloor + 1, b \geq 3, A \geq b^3. \tag{24}$$

To prove (24) for $k = 3\lfloor \log_b A \rfloor + 1$ it is enough to show by induction on A that:

$$b^{3\log_b A - 3} > (b-1)\left((b-1)(3\log_b A + 1) + A\right), b \geq 3, A \geq b^2. \tag{25}$$

If $A = b^3$, (24) becomes $b^6 > (b-1)\left((b-1)\cdot 10 + b^3\right)$, which is true for $b \geq 3$.

We show the induction step in (25). It follows from the induction hypothesis that

$$b^{3\log_b(A+1)-3} = b^{3\log_b A - 3} \cdot \left(\frac{A+1}{A}\right)^3$$
$$> \left(\frac{A+1}{A}\right)^3 \cdot (b-1)\left((b-1)(3\log_b A + 1) + A\right).$$

One still needs to show

$$\left(\frac{A+1}{A}\right)^3 \cdot (b-1)((b-1)(3\log_b A + 1) + A)$$
$$\geq (b-1)\left((b-1)(3\log_b (A+1) + 1) + (A+1)\right).$$

The last inequality follows due to the following inequalities which are true for $A \geq b^2, b \geq 3$:

$$\left(\frac{A+1}{A}\right)^3 \cdot (b-1)((b-1)(3\log_b A + 1) > (b-1)^2(3\log_b(A+1)+1),$$
$$\left(\frac{A+1}{A}\right)^3 \cdot A > A + 1.$$

We show the induction step in (24). It follows from the induction hypothesis that

$$b^k = b \cdot b^{k-1} > b(b-1)\left((b-1)k + A\right).$$

One still needs to show

$$b(b-1)\left((b-1)k + A\right) \geq (b-1)\left((b-1)(k+1) + A\right).$$

Last inequality follows from the following inequalities which are obvious for $b \geq 2$:

$$b(b-1)k \geq (b-1)(k+1, \qquad bA \geq A.$$

If $b=2$ one shows by induction on k that:

$$2^{k-1} > 2(k+A), \text{ for } k \geq 3\lfloor \log_2 A \rfloor, A \geq 4, \tag{26}$$

which is contradictory to (19) and ends the proof of the theorem.

In order to prove (26) for $k = 3\lfloor \log_2 A \rfloor$, it is enough to show by induction on A that:

$$2^{3\log_2 A - 1} \geq 2\left(3\log_2 A + 4\right), A \geq 4. \tag{27}$$

If $A=4$, (27) becomes $2^5 \geq 12$, which is true. We show the induction step in (27). It follows from the induction hypothesis that:

$$2^{3\log_2(A+1)-1} = \left(\frac{A+1}{A}\right)^3 \cdot 2^{3\log_2 A - 1} \geq \left(\frac{A+1}{A}\right)^3 \cdot 2\left(3\log_2 A + 4\right).$$

One still needs to show that

$$\left(\frac{A+1}{A}\right)^3 \cdot 2\left(3\log_2 A + 4\right) \geq 2\left(3\log_2(A+1) + 4\right).$$

The last inequality is true for $A \geq 4$ due to $A^A \geq A+1$. □

13. EXAMPLES OF wARH NUMBERS

We list in Table 1 small wARH numbers N and one of their extra terms A. We did not find any number that is not an additive extra term. This suggests that the answer to Question 32 is negative. We conjecture that all integers are additive extra terms. We observe from Table 1 that certain extra terms, for example 2, have associated several wARH numbers, respectively $210, 55$. The last observation motivates the following definition and questions.

Definition 41. If A is an additive extra term in a base b, let the *multiplicity* of A be the cardinality of the corresponding set of bw-ARH numbers.

Question 42. If we fix the multiplicity and the base, is the set of additive extra terms infinite?

Question 43. If we fix the base, is the multiplicity of additive extra terms bounded?

14. EXAMPLES OF wMRH NUMBERS

We list in Table 2 small wMRH numbers N and all their extra terms A. Theorem 38 shows that a wMRH number with multiplier 2 has at most 7 digits. A computer search through all integers with at most 6 digits shows that 2 is not a multiplicative extra term. This motivates Question 33.

Table 1 – All 365 wARH numbers less than 10000 and one of their extra term

N	A	N	A	N	A	N	A	N	A	N	A	N	A	N	A
0	0	362	170	827	149	1251	270	1656	711	2662	1045	5005	994	7546	1573
10	4	363	120	828	99	1252	319	1661	1046	2761	1774	5104	1183	7557	1032
11	8	382	178	847	157	1271	278	1675	670	2772	1053	5115	1002	7656	1671
12	3	383	128	848	107	1272	327	1676	719	2871	1872	5214	1281	7766	1769
14	2	403	145	867	165	1291	286	1695	678	2882	1061	5225	1010	7777	1048
16	1	404	95	868	115	1292	335	1696	727	2981	1970	5324	1379	7876	1867
18	0	423	153	887	173	1312	352	1716	744	2992	1069	5335	1018	7887	1056
22	7	424	103	888	123	1313	401	1717	793	3002	996	5434	1477	7986	1965
33	6	443	161	908	140	1331	1022	1736	752	3102	1185	5445	1026	7997	1064
44	5	444	111	909	90	1332	360	1737	801	3113	1004	5544	1575	8008	991
55	4	463	169	928	148	1333	409	1756	160	3212	1283	5555	1034	8107	1180
66	3	464	119	929	98	1352	368	1771	1054	3223	1012	5654	1673	8118	999
77	2	483	177	948	156	1353	417	1776	768	3322	1381	5665	1042	8217	1278
88	1	484	127	949	106	1372	376	1777	817	3333	1020	5764	1771	8228	1007
99	0	504	144	968	164	1373	1425	1796	776	3432	1479	5775	1050	8327	1376
101	98	505	94	969	114	1392	384	1797	825	3443	1028	5874	1869	8338	1015
110	17	524	152	988	172	1393	433	1877	842	3542	1577	5885	1058	8437	1474
121	25	525	102	989	122	1413	450	1818	891	3553	1036	5984	1967	8448	1023
132	33	544	160	1001	998	1414	499	1837	850	3652	1675	5995	1066	8547	1572
141	114	545	110	1009	148	1433	458	1838	899	1663	1044	6006	993	8558	1031
143	41	584	176	1010	107	1434	507	1854	907	3762	1773	6105	1182	8657	1670
154	49	585	126	1029	156	1441	1030	1858	907	1773	1052	6215	1280	8668	1039
161	22	605	143	1030	115	1453	466	1877	866	3872	1871	6226	1009	8767	1768
165	57	606	101	1049	164	1454	515	1878	915	3883	1060	6325	1378	8778	1047
176	65	625	151	1050	123	1473	474	1881	1062	3982	1969	6336	1017	8877	1866
181	130	626	101	1069	172	1474	523	1897	874	3993	1068	6435	1476	8888	1055
187	73	645	159	1070	131	1493	482	1898	923	4004	1184	6446	1025	8987	1964
198	81	646	109	1089	180	1494	531	1918	940	4103	1184	6545	1574	8988	1063
201	147	665	167	1090	139	1514	548	1938	948	4114	1003	6556	1033	9009	990
202	97	666	117	1110	156	1515	567	1958	956	4213	1282	6666	1041	9108	1179
221	155	685	175	1111	205	1534	556	1978	964	4224	1011	6765	1770	9119	998
222	105	686	125	1130	164	1535	605	1991	1070	4323	1380	6875	1868	9218	1277
241	163	706	142	1131	213	1551	1038	1998	972	4334	1478	6886	1057	9229	1006
242	113	707	92	1150	172	1554	564	2002	997	4444	1027	6985	1966	9328	1375
261	171	726	150	1151	221	1555	613	2101	1186	4543	1576	6996	1065	9339	1014
262	121	727	100	1170	180	1574	572	2112	1005	4554	1035	7007	92	9438	1473
281	179	746	158	1171	229	1575	621	2211	1284	4654	1674	7106	1181	9449	1022
282	129	747	108	1190	188	1594	580	2222	1013	4664	1043	7117	1000	9548	1571
302	146	766	166	1191	237	1595	629	2332	1021	4763	1772	7216	1279	9559	1030
303	96	767	116	1211	254	1615	646	2431	1480	4774	1051	7227	1008	9658	669
322	154	786	174	1212	303	1616	695	2442	1029	4873	1870	7326	1377	9669	1038
323	104	787	124	1221	1014	1635	654	2541	578	4884	1059	7337	1016	9768	1767
342	162	807	141	1231	262	1636	703	2552	1037	4983	1968	7436	1475	9779	1046
343	112	808	91	1232	311	1655	662	2651	1676	4994	1067	7447	1024	9878	1865
9889	1054	9988	1963	9999	1062										

Table 2 – All 77 wMRH numbers less than 10000 with all their multiplicative extra terms

N	A	N	A	N	A	N	A	N	A
0	0	574	25	1612	16, 52	3600	591	5929	52
1	0	640	70	1729	0, 63	3627	21, 75	6400	790
10	9	736	7, 16	1855	16, 34	3640	43, 52	6624	51, 78
40	16	765	33	1936	25	4000	1996	6786	51, 60
81	0	810	81	1944	9, 54	4030	123, 303	7360	214, 304
90	21	900	291	2268	18, 45	4032	39, 75	7650	132, 192
100	99	976	39	2296	9, 63	4275	39, 57	7663	57, 75
121	7	1000	999	2430	36, 45	4356	48	7744	66
160	33	1008	15, 33	2500	493	4606	23, 78	8100	891
250	43	1089	15	2520	11, 201	4840	204	8722	70, 79
252	3, 12	1207	7, 61	2668	7, 70	4900	687	9000	2991
360	51	1210	106	2701	27, 63	4930	42, 69	9760	138, 588
400	196	1300	21, 48	2944	27, 45	5092	51, 160	9801	81
403	6, 24	1458	0, 63	3025	45	5605	43, 79		
484	6	1462	21, 30	3154	25, 70	5740	124, 94		
490	57	1600	393	3478	25, 52	5848	43, 61		

We observe from Table 2 that certain wMRH numbers, for example, 252, 403, and 736, have several extra terms (respectively $\{3, 12\}$, $\{6, 24\}$, $\{7, 16\}$). This suggests a positive answer to Question 24. The table does not show any example of wMRH number with 3 multiplicative extra term. The smallest example we found is 63504 with extra terms 234, 423, 126.

We also observe from Table 1 that certain extra terms, for example 7, have associated several wMRH numbers, respectively $121, 736, 1207, 2668$. The last observation motivates the following definition and questions.

Definition 44. If A is a multiplicative extra term in base b, let the *multiplicity* of A be the cardinality of the corresponding set of b-wMRH numbers.

Question 45. If we fix the multiplicity and the base, is the set of multiplicative extra terms infinite?

Question 46. If we fix the base, is the multiplicity of multiplicative extra terms bounded?

15. **CONCLUSION**

In this paper we introduce two new classes of integers. The first class consists of all numbers N for which there exists at least one integer A, such that the sum of A and the sum of digits of N, added to the reversal of the sum, gives N. The second class consists of all numbers N for which there exists at least one integer A, such that the sum of A and the sum of the digits of N, multiplied by the reversal of the sum, gives N. All palindromes that either have an even number of digits or an odd number of digits and the middle digit even belong to the first class, and all squares of palindromes with at least two digits belong to the second class. These classes contain and are strictly larger than the classes of b-ARH numbers, respectively b-MRH numbers introduced in Niţică [6]. We show many examples of such numbers and ask several questions that may lead to future research. In particular, we try to clarify the relationships between these classes of numbers and the well studied class of b-Niven numbers. Most of our results are true in a general numeration base.

Acknowledgments. The authors would like to thank the editor and the referee for valuable comments that helped them write a better paper.

REFERENCES

[1] C. N. Cooper and R. E. Kennedy, *On consecutive Niven numbers.* Fibonacci Quart. **21** (1993), 146–151.

[2] J. M. De Koninck and N. Doyon, *Large and small gaps between consecutive Niven numbers.* J. Integer Seq. **6** (2003), Article 03.2.5.

[3] H. G. Grundman, *Sequences of consecutive Niven numbers.* Fibonacci Quart. **32** (1994), 174–175.

[4] H. Fredricksen, E. J. Ionaşcu, F. Luca, and P. Stănică, *Remarks on a sequence of minimal Niven numbers.* In: *Sequences, Subsequences, and Consequences*, Lec. Notes in Comp. Sci., Vol. 4893. Springer, 2007, pp. 162–168.

[5] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences.* http://oeis.org.

[6] V. Niţică, *About some relatives of the taxicab number.* J. of Int. Seq. **21** (2018), Article 18.9.4.

[7] V. Niţică, *High degree b-Niven numbers.* Integers **21** (2021), Paper No. A101.

[8] V. Niţică, *Infinite sets of b-additive and b-multiplicative Ramanujan-Hardy numbers.* J. of Int. Seq. **22** (2019), *4*, Article 19.4.3.

Received August 29, 2019

Viorel Niţică
West Chester University of Pennsylvania
Department of Mathematics
West Chester, PA 19383, USA
and
Institute of Mathematics of the Romanian Academy
P.O. Box 1–764, RO-70700 Bucharest, Romania

Andrei Török
University of Houston, Department of Mathematics
Houston, TX 77204, USA
and
Institute of Mathematics of the Romanian Academy
P.O. Box 1–764, RO-70700 Bucharest, Romania.
torok@math.uh.edu