Dedicated to the memory of Professor Mihnea Colţoiu

ON A NORMALITY CRITERION OF P. LAPPAN

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In this paper, as an application of Zalcman's lemma in $\mathbb{C}^n$, we give a sufficient condition for normality of a family of holomorphic functions of several complex variables, which generalizes previous known one-dimensional results of H. L. Royden, W. Schwick, and P. Lappan.

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1. INTRODUCTION

A family $\mathcal{F}$ of holomorphic functions on a domain $\Omega \subset \mathbb{C}^n$ is normal in Ω if every sequence of functions $\{f_j\} \subseteq \mathcal{F}$ contains either a subsequence which converges to a limit function $f \neq \infty$ uniformly on each compact subset of Ω, or a subsequence which converges uniformly to ∞ on each compact subset.

A family $\mathcal{F}$ is said to be normal at a point $z_0 \in \Omega$ if it is normal in some neighborhood of z_0. It is routine to confirm that a family of analytic functions $\mathcal{F}$ is normal in a domain Ω if and only if $\mathcal{F}$ is normal at each point of Ω.

For every function φ of class $\mathcal{C}^2(\Omega)$ we define at each point $z \in \Omega$ an Hermitian form

$$L_z(\varphi, v) := \sum_{k,l=1}^{n} \frac{\partial^2 \varphi}{\partial z_k \partial \overline{z}_l}(z) v_k \overline{v}_l \quad (v \in \mathbb{C}^n)$$

and call it the Levi form of the function φ at z.

For a holomorphic function f in Ω, we set

$$f^{\sharp}(z) := \max_{|v|=1} \sqrt{L_z(\log(1 + |f|^2), v)}. \tag{1}$$

This quantity $f^{\sharp}(z)$ is well defined since the Levi form $L_z(\log(1 + |f|^2), v)$ is nonnegative for all $z \in \Omega$. In particular, for $n = 1$ the formula (1) takes the form

$$f^{\sharp}(z) := \frac{|f'(z)|}{1 + |f(z)|^2}$$

and $z^\sharp$ coincides with the spherical metric on $\mathbb{C}$.

There are many criteria for $\mathcal{F}$ to be normal. A particularly useful one is Marty's criterion.

THEOREM 1 (Marty's Criterion, see [1]). *A family $\mathcal{F}$ of functions holomorphic on Ω is normal on $\Omega \subset \mathbb{C}^n$ if and only if for each compact subset $K \subset \Omega$ there exists a constant $M(K)$ such that at each point $z \in K$*

$$f^\sharp(z) \leq M(K)$$

for all $f \in \mathcal{F}$.

In [3], Lappan proved a "Five Point Theorem" for normal families of meromorphic functions in a domain $\Omega \subset \mathbb{C}$ using the generalization by Zalcman [8] to the result of Lohwater and Pommerenke.

THEOREM (Five Point Theorem, see Lappan [3]; cf. [9]). *A family $\mathcal{F}$ of meromorphic functions is normal on a region $\Omega \subset \mathbb{C}$ if and only if for each compact set $K \subset \Omega$ there exists a set $E = E(K) \subset \overline{\mathbb{C}}$ containing at least five points and a constant $M(K)$ such that*

$$f^\sharp(z) \leq M(K), z \in K, f(z) \in E, \tag{2}$$

for all $f \in \mathcal{F}$.

Moreover, a family $\mathcal{F}$ of holomorphic functions is normal in a region $\Omega \subset \mathbb{C}$ if (2) holds for all $f \in \mathcal{F}$ and the set E contains at least three points.

Marty's criterion is the principal ingredient of the proof of an extension of Zalcman's clever lemma characterizing non-normal families.

THEOREM 2 (Zalcman's Lemma, see [1]). *Let $\mathcal{F}$ be a family of functions holomorphic on $\Omega \subset \mathbb{C}^n$. Then $\mathcal{F}$ is not normal at some point $z_0 \in \Omega$ if and only if there exist sequences $f_j \in \mathcal{F}$, $z_j \to z_0$, $r_j \to 0$, such that the sequence*

$$g_j(z) := f_j(z_j + r_j z) \tag{3}$$

converges uniformly on compact subsets of $\mathbb{C}^n$ to a non-constant entire function g satisfying $g^\sharp(z) \leq g^\sharp(0) = 1$.

It is remarkable that non-normality can be described in terms of a convergent sequence. If $\{f_j\}$ were convergent, then the functions g_j given by (3) would converge to a constant on compact subsets of $\mathbb{C}^n$, since the radii r_j tend to 0. Zalcman's lemma says that arbitrarily small balls centered at z_j can be found where f_j is close to a non-trivial entire (holomorphic in $\mathbb{C}^n$) function, after rescaling.

2. THE RESULTS

Let us begin with a definition.

Definition 3. Let $g(\lambda)$ be a holomorphic function in $\mathbb{C}$, if the equation $g(\lambda) = a$, $a \in \mathbb{C}$, has no simple root then a is called a totally ramified value.

Note that an omitted value trivially satisfies this definition, but that it will be useful to distinguish between omitted values and non-omitted totally ramified values.

THEOREM 4 (R. Nevanlinna) [7, Theorem 17.3.10., p. 274]). *Let g be an entire (holomorphic in $\mathbb{C}$) function. Then g has at most two totally ramified (finite) values.*

In combination with Zalcman's Lemma in $\mathbb{C}^n$, one-dimensional Nevanlinna's Theorem immediately leads to the following theorem, which generalizes those obtained by Lappan [3] for normal families of holomorphic functions.

THEOREM 5. *Let $\mathcal{F}$ be a family of holomorphic functions on a domain $\Omega \subset \mathbb{C}^n$ with the property that for each compact set $K \subset \Omega$ there is a function $h_K : [0, \infty) \to [0, \infty)$, which is finite for at least three points in $\mathbb{C}$, such that*

$$f^\sharp(z) \leq h_K(|f(z)|) \tag{4}$$

for all $f \in \mathcal{F}$ and $z \in K$. Then $\mathcal{F}$ is normal in Ω.

Proof of Theorem 5. We shall show that assumption that $\mathcal{F}$ is not normal at a point $z_0 \in \Omega$ leads to a contradiction. Since all features of the theorem are local, and translation and scale-change invariant, it suffices to consider the case when $z_0 = 0$ and $\Omega = B(0, 1)$. If $\mathcal{F}$ is not normal at 0, it follows from Zalcman's lemma [1, Theorem 3.1] that there exist $f_j \in \mathcal{F}$, $z_j \to 0$, $\rho_j \to 0$, such that the sequence

$$g_j(z) := f_j(z_j + \rho_j z)$$

converges uniformly on compact subsets of $\mathbb{C}^n$ to a non-constant entire function g satisfying $g^\sharp(z) \leq g^\sharp(0) = 1$.

Let $K \subset B(0, 1)$ be a closed ball in $\mathbb{C}^n$ about 0. Let $E = E(K) = \{a_1, a_2, a_3\}$ be the set of three distinct values in $\mathbb{C}$ such that the function h_K is finite on E and let $M = M(K)$ be the maximal value of h_K on E.

Define the value set by

$$A_{g_j}(a_l) := \{\xi \in \mathbb{C}^n : g_j(\xi) = a_l\} = g_j^{-1}[\{a_l\}].$$

Suppose that $\zeta^l \in A_g(a_l)$. Choose $R > 0$ such that $\zeta^l \in S_R = \{\xi \in \mathbb{C}^n : |\xi| < R\}$.

By Hurwitz' theorem ([4, Corollary p. 80])

$$A_{g_j}(a_l) \cap \{\xi \in \mathbb{C}^n : |\xi - \zeta^l| < 1/n\} \neq \emptyset$$

for j sufficiently large since g is not a constant function. It is routine to show that there exist a sequence $\{p_j^l\} \subset S_K$, such that $p_j^l \to \zeta^l$ and $g_j(p_j^l) = a_l$.

Since $\rho_j \to 0$ we see that $z_j + \rho_j p_j^l \in K$ for j sufficiently large. Now by (4), $f_j^\sharp(z_j + \rho_j p_j^l) \leq h_K(|f_j(z_j + \rho_j p_j^l)|) \leq M$ for j sufficiently large, so that

$$g^\sharp(\zeta^l) = \lim_{k\to\infty} g_j^\sharp(p_j^l) = \lim_{k\to\infty} \rho_j f_j^\sharp(z_j + \rho_j p_j^l) \leq \lim_{k\to\infty} \rho_j M = 0.$$

Thus $g^\sharp(\zeta^l) = 0$.

Since

$$g^\sharp(\zeta^l) = \max_{\{v\in\mathbb{C}^n:|v|=1\}} \frac{|\frac{d}{d\lambda} g(\zeta^l + \lambda \cdot v)|_{\lambda=0}/d\lambda|}{1 + |g(\zeta^l)|^2} = 0$$

it follows a_l is a finite totally ramifies value for $g(\zeta^l + \lambda \cdot v)$ (for every $v \in \mathbb{C}^n$ with $|v| = 1$).

If $\{\xi \in C^n : \xi = \zeta^l + \lambda \cdot v\} \cap A_g(a^k) \ni \zeta^k$ then $\zeta^k = \zeta^l + \lambda_k \cdot v$ for some $\lambda_k \in \mathbb{C}$. Arguing as above we have

$$g'(\zeta^l + \lambda_k \cdot v) = 0.$$

Hence a_k is a totally ramified value for $g(\zeta^l + \lambda \cdot v)$.

If $\{\xi \in \mathbb{C}^n : \xi = \zeta^l + \lambda \cdot v\} \cap A_g(a^k) = \emptyset$ then a^k is omitted value for $g(\zeta^l + \lambda \cdot v)$ and hence a totally ramified (finite) value of the function $g(\zeta^l + \lambda \cdot v)$.

Thus a_1, a_2, a_3 are three totally ramified (finite) values for the entire function $g\Big(\zeta^l + \lambda \cdot v\Big)$. By Nevanlinna's theorem $g\Big(\zeta^l + \lambda \cdot v\Big)$ is constant. Since v was arbitrary the function g is constant, a contradiction. Thereby, the theorem is proved. □

One almost immediate consequence of the (proof of the) above Theorem is the following extension and sharpening of Schwick's extension [6] of a theorem of Royden [5] (see also [2]).

THEOREM 6. *Let $\mathcal{F}$ be a family of holomorphic functions on a domain $\Omega \subset \mathbb{C}^n$ with the property that for each compact set $K \subset \Omega$ there is a function $h_K : [0, \infty) \to [0, \infty)$, which is finite somewhere on $(0, \infty)$, such that*

$$(1 + |f(z)|^2) f^\sharp(z) \leq h_K(|f(z)|)$$

for all $f \in \mathcal{F}$ and $z \in K$. Then $\mathcal{F}$ is normal in Ω.

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