AN EXACT FORMULA FOR THE VALUE OF A PORTFOLIO AT A RANDOM FINAL TIME

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We consider the classic formula obtained by Merton for the value of a contingent claim that expires at a fixed time T. In our case, the final time is random and depends on both the time t and the value of the stock at t. The partial integro-differential equation (PIDE), subject to the appropriate boundary conditions, is solved explicitly for a certain jump size distribution. The PIDE is first transformed into an integro-differential equation, and this equation is then reduced to an ordinary differential equation.

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1. INTRODUCTION

Let $S(t)$ denote the value of a stock at time t. We assume that $S(t)$ satisfies the stochastic differential equation

$$\mathrm{d}S(t) = r\,S(t)\,\mathrm{d}t + \sigma\,S(t)\,\mathrm{d}W(t), \tag{1}$$

where r is the instantaneous expected return on the stock, σ^2 is the instantaneous variance of the return and $\{W(t), t \geq 0\}$ is a standard Brownian motion. That is, $\{S(t), t \geq 0\}$ is a geometric Brownian motion.

Next, suppose that in addition to the *normal* variations of the stock price, there are also jumps due to *exceptional* events. Then $\{S(t), t \geq 0\}$ becomes a jump-diffusion process defined by

$$\frac{\mathrm{d}S(t)}{S(t)} = (r - \lambda\kappa)\,\mathrm{d}t + \sigma\,\mathrm{d}W(t) + (Y-1)\mathrm{d}N(t), \tag{2}$$

in which $\{N(t), t \geq 0\}$ is a Poisson process with rate λ, Y is the random jump size and $\kappa := E[Y-1]$ is the expected relative jump of $S(t)$ if the Poisson event occurs. We assume that $\{W(t), t \geq 0\}$, Y and $\{N(t), t \geq 0\}$ are independent.

In the above model, $Y-1$ is an impulse function that produces a finite jump from s to sY. The jump magnitudes are independent and identically distributed (i.i.d.) random variables.

Let $V(s,t)$ be the value of a contingent claim that depends on the stock price s and time t. Merton [13] has shown that $V(s,t)$ satisfies the partial integro-differential equation (PIDE)

$$V_t + \frac{\sigma^2 s^2}{2} V_{ss} + (r - \lambda\kappa) s V_s - (r+\lambda) V + \lambda \int_0^\infty V(sy,t) f_Y(y)\,\mathrm{d}y = 0, \tag{3}$$

where $f_Y(y)$ is the probability density function of Y. This equation has been studied, among others, by Andersen and Andreasen [4], Carr and Mayo [5] and Mayo [12]. In general, the authors proposed numerical methods to solve Eq. (3), subject to the appropriate boundary conditions.

In the case of a European option with expiration time T and strike price K, Eq. (3) is subject to the boundary conditions

$$V(0,t) = 0 \quad \text{and} \quad V(s,T) = \max\{0, s-K\}. \tag{4}$$

We could also try to solve the PIDE for a barrier option.

Let

$$X(t) := \ln[S(t)] \quad \text{and} \quad V^*(x,t) := V(e^x, t). \tag{5}$$

In this paper, instead of an option, we consider the value of a portfolio made up of only one stock. We assume that the value of the portfolio satisfies Eq. (3) and that the investor has decided to sell the stock the first time when either

$$X(t) - ct \le k_1 \quad \text{or} \quad X(t) - ct \ge k_2, \tag{6}$$

where c is a constant and $k_1 < X(0) < k_2$. Moreover, we choose

$$k_i = \gamma_i + x_0 \quad \text{for } i = 1, 2, \tag{7}$$

where $x_0 := X(0)$, $0 < \ln(\gamma_1) < 1$ and $\ln(\gamma_2) > 1$. We also assume that we can write that

$$V^*(x,t) = x - ct \quad \text{if } x - ct \notin (k_1, k_2). \tag{8}$$

We will try to solve the PIDE satisfied by V^* under the above condition for particular probability density functions f_Y.

Remark 1. In terms of the stock price, the investor sells when

$$S(t) \le e^{\gamma_1 + ct} S(0) \quad \text{or} \quad S(t) \ge e^{\gamma_2 + ct} S(0). \tag{9}$$

If $c = 0$, then the investment strategy is to sell if the stock price drops (respectively rises) to at least e^{γ_1}% (resp. e^{γ_2}%) of its initial value.

If $c > 0$, we could say that the investor takes the inflation rate into account, whereas the case when $c < 0$ implies that the investor does not want to keep the stock indefinitely and is willing to sell at a lesser profit (or larger loss) than hoped for. When c is positive, it could also be the fixed dividend yield on the stock.

Remark 2. The random variable $\tau(x_0)$ defined by

$$\tau(x_0) = \inf\{t > 0 : X(t) - ct \notin (k_1, k_2) \mid X(0) = x_0\} \tag{10}$$

is known as a *first-passage time* in probability. Recently, the author published a number of papers on first-passage problems for jump-diffusion processes; see Lefebvre [9], [10] and [11].

The problem considered in this paper is a first-passage problem for a *function* of a jump-diffusion process. More precisely, it is a first-passage-*place* problem. There is a relatively large number of papers on first-passage-*time* problems for jump-diffusion processes; see, for instance, Abundo [1] and [2] and the references therein. In Peng and Liu [14], the authors calculated the moments of first-passage times for jump-diffusion processes with Markovian switching.

There are however few papers on first-passage-place problems. Kou and Wang [7] computed the joint moment-generating function of a first-passage time and place when the continuous part of the jump-diffusion process is a Wiener process (or Brownian motion) and the discrete part is a compound Poisson process.

In Abundo [1], the (positive and/or negative) jumps were assumed to be of constant size. In Kou and Wang [7], the jump size is an asymmetric double exponential distribution, whereas it is a hyper-Erlang distribution in Dong and Han [6]. Other possibilities include the log-normal (Merton [13]) and log-uniform (Ahlip and Prodan [3]) distributions.

In the next section, we will show, in a particular case, how to transform the PIDE satisfied by the function $V^*(x,t)$ into an ordinary differential equation (ODE). Then, we will consider various cases for the jumps.

2. TRANSFORMING THE PIDE INTO AN ODE

It is easy to show that the function $V^*(x,t) = V(e^x, t)$ satisfies the PIDE

$$\begin{aligned} 0 \;=\;& V_t^* + \frac{\sigma^2}{2} V_{xx}^* + \left(-\frac{\sigma^2}{2} + r - \lambda\kappa\right) V_x^* - (r+\lambda) V^* \\ &+ \lambda \int_0^\infty V^*(x + \ln y, t)\, f_Y(y)\, \mathrm{d}y. \end{aligned} \tag{11}$$

Thus, with $y^* := \ln y$, we obtain

$$\begin{aligned} 0 \;=\;& V_t^* + \frac{\sigma^2}{2} V_{xx}^* + \left(-\frac{\sigma^2}{2} + r - \lambda\kappa\right) V_x^* - (r+\lambda) V^* \\ &+ \lambda \int_{-\infty}^\infty V^*(x + y^*, t)\, f_Y\left(e^{y^*}\right) e^{y^*} \mathrm{d}y^*. \end{aligned} \tag{12}$$

Now, based on the condition (8), we assume that the function $V^*(x,t)$ can be expressed as follows:

$$V^*(x,t) = U(u), \quad \text{where } u := x - ct. \tag{13}$$

Remark 3. The technique used is a particular case of the method of similarity solutions and u is called the *similarity variable.*

Equation (12) becomes the integro-differential equation

$$\begin{aligned} 0 \;&=\; -cU' + \frac{\sigma^2}{2}U'' + \left(-\frac{\sigma^2}{2} + r - \lambda\kappa\right)U' - (r+\lambda)U \\ &\quad + \lambda \int_{-\infty}^{\infty} U(u+y^*)\, f_Y\left(e^{y^*}\right) e^{y^*} \mathrm{d}y^* \end{aligned} \tag{14}$$

and the boundary condition is

$$U(u) = u \quad \text{if } u \notin (k_1, k_2). \tag{15}$$

Next, setting $w := u + y^*$ in the integral, we find that

$$\begin{aligned} 0 \;&=\; \frac{\sigma^2}{2}U'' + \left(-c - \frac{\sigma^2}{2} + r - \lambda\kappa\right)U' - (r+\lambda)U \\ &\quad + \lambda \int_{-\infty}^{\infty} U(w)\, f_Y\left(e^{w-u}\right) e^{w-u} \mathrm{d}w. \end{aligned} \tag{16}$$

We will consider the case when $Y \sim \mathrm{U}(0,\beta)$. That is, Y is uniformly distributed on the interval $(0,\beta)$. It follows that

$$\kappa := E[Y-1] = \frac{\beta}{2} - 1 \tag{17}$$

and

$$\int_{-\infty}^{\infty} U(w)\, f_Y\left(e^{w-u}\right) e^{w-u} \mathrm{d}w = \frac{1}{\beta}\int_{-\infty}^{u+\ln\beta} U(w) e^{w-u} \mathrm{d}w, \tag{18}$$

so that

$$\begin{aligned} 0 \;&=\; \frac{\sigma^2}{2}U'' + \left(-c - \frac{\sigma^2}{2} + r - \lambda\kappa\right)U' - (r+\lambda)U \\ &\quad + \frac{\lambda}{\beta} \int_{-\infty}^{u+\ln\beta} U(w) e^{w-u} \mathrm{d}w. \end{aligned} \tag{19}$$

Hence, differentiating the above equation with respect to u, we obtain the following proposition.

PROPOSITION 2.1. *When the random variable Y is uniformly distributed over the interval $(0,1)$, the function U satisfies the ordinary differential equation*

$$\frac{\sigma^2}{2}U''' + (-c + r - \lambda\kappa)\,U'' - \left(c + \frac{\sigma^2}{2} + \lambda(\kappa+1)\right)U' - rU = 0 \tag{20}$$

for $k_1 < u < k_2$, where $\kappa = -1/2$.

In the case when β is such that $k_1 + \ln\beta \geq k_2$, we can write that

$$U(u + \ln\beta) = u + \ln\beta \quad \text{for } k_1 \leq u \leq k_2, \tag{21}$$

and we can state the following proposition.

PROPOSITION 2.2. *When $Y \sim U(0,\beta)$ and $k_1 + \ln\beta \geq k_2$, we can write that*

$$\begin{aligned} 0 \;=\; & \frac{\sigma^2}{2}U''' + (-c + r - \lambda\kappa)\,U'' - \left(c + \frac{\sigma^2}{2} + \lambda(\kappa+1)\right)U' \\ & - (r+\lambda)U + \lambda(u + \ln\beta) = 0 \end{aligned} \tag{22}$$

for $k_1 < u < k_2$, in which $\kappa = \frac{\beta}{2} - 1$.

In this paper, we assume that $c = 0$ and that $x_0 = 0$. Moreover, we choose the constants

$$r = 0.05, \quad \sigma = 0.25, \quad k_1 = \ln(0.9) \quad \text{and} \quad k_2 = \ln(1.1). \tag{23}$$

Hence, the investor sells the stock when its price drops or increases at least 10% compared to its initial value.

3. THE CASE WHEN THERE ARE NO JUMPS

Let us first consider the case when $\lambda = 0$, so that there are no jumps. Then, we must solve (see Eq. (16))

$$0.03125U'' + 0.01875U' - 0.05U = 0, \tag{24}$$

subject to

$$U[\ln(0.9)] = \ln(0.9) \quad \text{and} \quad U[\ln(1.1)] = \ln(1.1). \tag{25}$$

We find that

$$U(u) \simeq -0.3810e^{-1.6u} + 0.3841\,e^{u}. \tag{26}$$

Remark 4. When there are no jumps, we can solve the first-passage-place problem for the stock price by computing first the probability $p_0(u)$ that the process, starting from $u \in (k_1, k_2)$, will hit k_1 before k_2. This function satisfies the second-order ODE (see Lefebvre [8], for instance)

$$0.03125 p_0'' + 0.01875 p_0' = 0, \tag{27}$$

subject to

$$p_0(k_1) = 1 \quad \text{and} \quad p_0(k_2) = 0. \tag{28}$$

We find that

$$p_0(u) \simeq -7.8155 + 8.2755 e^{-0.6u}. \tag{29}$$

Then, we can write that the solution $U^*(u)$ to the first-passage problem is

$$\begin{aligned} U^*(u) &= \ln(0.9) p_0(u) + \ln(1.1) [1 - p_0(u)] \\ &\simeq 1.6637 - 1.6606 e^{-0.6u}. \end{aligned} \tag{30}$$

We can see in Figure 1 that the functions $U(u)$ and $U^*(u)$ almost coincide in the interval $[\ln(0.9), \ln(1.1)]$.

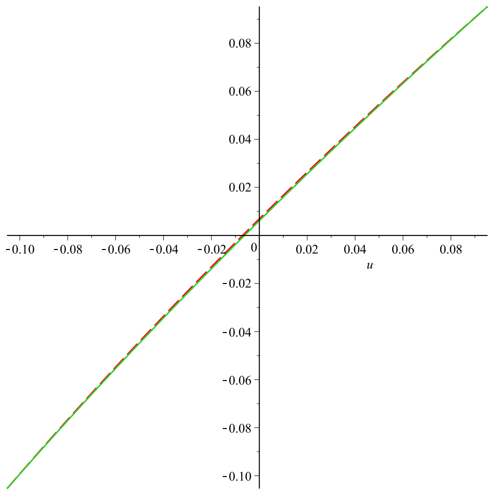

Figure 1 – Functions $U(u)$ and $U^*(u)$ when there are no jumps.

4. THE CASE WHEN Y IS UNIFORM OVER $(0, 1)$

Let us now assume that $\lambda > 0$ and that $Y \sim \mathrm{U}(0, 1)$. Hence, all the relative jumps are negative and $\kappa = -1/2$.

First, we take $\lambda = 0.05$. Then, from Eq. (20), we must solve

$$0.03125 U'''(u) + 0.075 U''(u) - 0.05625 U'(u) - 0.05 U(u) = 0. \tag{31}$$

The general solution of this third-order linear and homogeneous ODE with constant coefficients is

$$(32)\quad U(u) = c_1 e^u + c_2 \exp\left\{\frac{(-17+\sqrt{129})}{10}\,u\right\} + c_3 \exp\left\{\frac{(-17-\sqrt{129})}{10}\,u\right\},$$

where c_1, c_2 and c_3 are arbitrary constants.

Making use of the boundary conditions (25), we obtain

$$(33)\quad \begin{aligned} U(u) \;\simeq\;& c_1 e^u + (-1.7023 c_1 + 0.4467)\, e^{-0.5642u} \\ &+ (0.6724 c_1 - 0.4297)\, e^{-2.8358u}. \end{aligned}$$

To determine the value of the constant c_1, we can substitute the above expression into the integro-differential equation (19) satisfied by the function U (with $\beta = 1$). We find that

$$(34)\quad c_1 \simeq 0.5988.$$

Hence, we have

$$(35)\quad U(u;\lambda = 0.05) \simeq 0.5988\, e^u - 0.5727\, e^{-0.5642u} - 0.02708\, e^{-2.8358u}$$

for $u \in [\ln(0.9), \ln(1.1)]$.

Proceeding as above with $\lambda = 1$, we obtain

$$(36)\quad U(u;\lambda = 1) \simeq 1.5477\, e^u - 1.6239\, e^{-0.8642u} + 0.1997\, e^{-18.51u}.$$

As can been seen in Figure 2, this function is sometimes smaller than $\ln(0.9)$ because of the jumps.

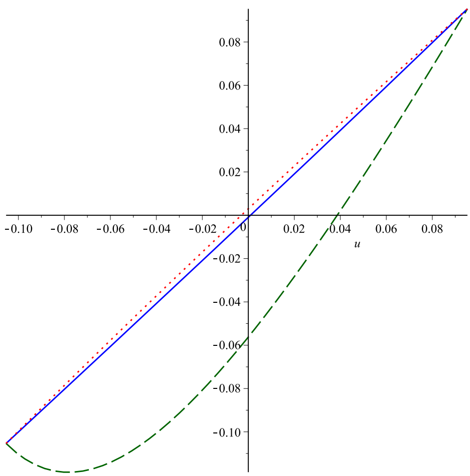

Figure 2 – Functions $U(u;\lambda = 0)$ (dotted line), $U(u;\lambda = 0.05)$ (solid line) and $U(u;\lambda = 1)$ (dashed line).

5. THE CASE WHEN Y IS UNIFORM OVER $(0, \beta)$

Finally, we assume that $Y \sim \mathrm{U}(0, \beta)$, with $\beta > 1$. Hence, the relative jumps can be positive or negative.

We take $\beta = 1.9$. It follows that $\kappa = (\beta/2) - 1 = -0.05$. With $\lambda = 5$, we find that

$$(37) \quad U(u) \simeq -0.3008 + 0.9901\, u - 0.01004\, e^{-u} + 0.05301\, e^{16.94u} + 0.1106\, e^{-9.540u}$$

for $u \in [\ln(0.9), \ln(1.1)]$. This function is shown in Figure 3.

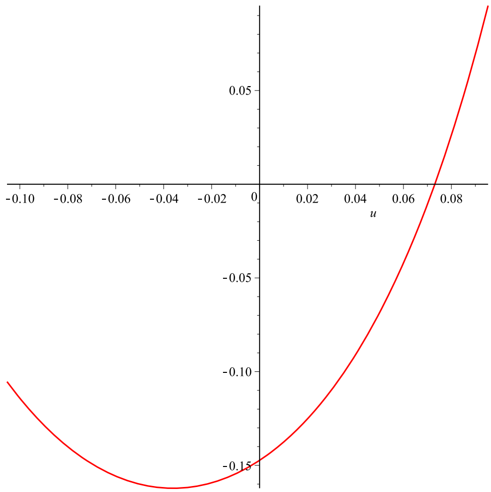

Figure 3 – Function $U(u)$ when $\beta = 1.9$ and $\lambda = 5$.

6. CONCLUDING REMARKS

In this paper, we found explicit and exact solutions to a classic partial integro-differential equation in financial mathematics under the assumption that the investor sells a stock when the stock price increases or decreases at least a certain percentage of its initial value.

The solutions were obtained in the case when the random jump size is uniformly distributed over the interval $(0, \beta)$. The parameter β was first chosen equal to 1, so that all the relative jumps are negative. Then, we chose $\beta = 1.9$, which implies that there can be both positive and negative relative jumps. Notice however that the expected value of the relative jumps is negative, which is realistic.

Next, we could try to solve our problem in the case when $c \neq 0$. Finally, we could consider optimal control problems for the model presented in this paper.

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