A SPLITTING INDEX THEOREM ON MANIFOLDS WITH CORNERS

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We prove an index formula for the Fredholm index of a fully elliptic geometric Dirac operator subject to second order local (e.g. Dirichlet and Neumann) boundary conditions on a manifold with corners. To calculate the index, we introduce a glueing construction and a corresponding Lie groupoid. We describe the Dirac operator subject to the boundary conditions via an equivariant family of Dirac operators on the fibers of the Lie groupoid. We show that the index splits into homotopy invariant contributions of the strata.

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1. **INTRODUCTION**

We study the index for boundary value problems on so-called Lie manifolds with boundary. We refer to [26] for an excellent survey by V. Nistor about the current state of research concerning index problems on Lie manifolds. On manifolds with boundary the Atiyah-Patodi-Singer (APS) index theorem is an index formula for Fredholm operators that consists of a local contribution, depending on the Riemannian metric and a non-local contribution, the η-invariant. The boundary conditions considered in the APS-theory are global projection conditions. In the present work, our goal is to describe a generalization of the APS-theory which involves general and mixed boundary conditions. Historically, boundary value problems were studied on manifolds with boundary, where the boundary may consist of disjoint components, on which different types of boundary conditions could be imposed. In the classical setup the problems under consideration involve questions of well-posedness for solutions, Fredholm conditions for the operators involved, as well as the index theory of partial differential equations subject to mixed boundary conditions. A mixed boundary value problem by definition is a partial differential equation subject to different boundary conditions on the different pieces of the boundary. In this note our main focus lies on the index theory for mixed boundary value problems and for convenience we will restrict attention to Dirac type operators. The classical Dirichlet problem and Neumann problem in such a mixed setup was investigated by S. Zaremba in 1910, [35]. Index formulae have been obtained by M. Gromov and H. B. Lawson [15] (relative index theorem) and Freed [14] for Dirac type operators on manifolds of odd dimension, see also [5]. In these works, the boundary is assumed to be compact and to possibly consist of disjoint pieces. We think it is an interesting question to study more general domains, that may consist of piecewise smooth boundaries. Examples considered in the literature are Lipschitz domains [16, 23, 24] and the complex analytic analogues of pseudoconvex domains [13]. A convenient framework for the consideration of such singular domains consists of certain manifolds with corners and singular structures defined on manifolds with corners, cf. [2, 22]. In our setup we study compact manifolds consisting of a boundary stratified by immersed submanifolds. Therefore, in contrast to previous results, we in particular allow intersecting boundary components and examine Dirac type operators subject to mixed boundary conditions on boundary components. The main focus of this work, however, will be the index theory of such boundary value problems. We obtain index formulae in the spirit of the index theorem of Atiyah-Patodi-Singer. In the special case with no mixed boundary conditions, posed on so-called regular strata, our results recover the known Atiyah-Patodi Singer index formula on a manifold with boundary [22] and more generally on manifolds with corners [8, 9, 10].

1.1. **Overview**

Dirac operators

A Lie manifold is a triple $(M, \mathcal{A}, \mathcal{V})$, where M is a compact manifold with corners and $\mathcal{V} \subset \Gamma(TM)$ is a Lie algebra of smooth vector fields, cf. [2]. For instance, $\mathcal{V}$ is assumed to be a subalgebra of the Lie algebra $\mathcal{V}_b$ of all vector fields tangent to the boundary strata and a finitely generated projective $C^\infty(M)$-module. The compact manifold with corners M is thought of as a compactification of a non-compact manifold with a degenerate, singular metric. We denote by ∂M the (stratified) boundary of M and by $M_0 = M \setminus \partial M$ the interior. By the Serre-Swan theorem there exists a vector bundle $\mathcal{A} \to M$ such that $\Gamma(\mathcal{A}) \cong \mathcal{V}$. We make the standing assumption that $\mathcal{A}_{M_0} \cong TM_0$, the tangent bundle on the interior. The bundle $\mathcal{A}$ has the structure of a Lie algebroid and we fix throughout $\varrho \colon \mathcal{A} \to TM$ to denote the *anchor* of the Lie algebroid, see e.g. [20]. Moreover, we need a Lie groupoid $\mathcal{G} \rightrightarrows M$. It is known that for any Lie structure with our assumption there is an s-connected

Lie groupoid $\mathcal{G}$ such that $\mathcal{A}(\mathcal{G}) \cong \mathcal{A}$, cf. [12]. Since we rely on the heat kernel of an arbitrary integrating Lie groupoid, we make the standing assumption that all Lie manifolds considered in this work, have heat kernels which are (transversally) smooth. It is conjectured, but yet not resolved in full generality, whether every Lie manifold has an integrating Lie groupoid with a smooth heat kernel, cf. [10, 33]. An *$\mathcal{A}$-metric* is a Riemannian metric $g = g_{\mathcal{A}}$ on the interior M_0 which extends to a positive definite symmetric bilinear form on $\mathcal{A}$, i.e. a Euclidian structure on $\mathcal{A}$. We call ∇ an *$\mathcal{A}$-connection* if ∇ is the Levi Civita connection on the interior M_0 which extends to a connection defined on $\mathcal{A}$ as described in [3]. Likewise if W is a $\mathrm{Cl}(\mathcal{A})$-module we call ∇^W an *admissible $\mathcal{A}$-connection* if

$$\nabla_X^W(c(Y)\varphi) = c(\nabla_X Y)\varphi + c(Y)(\nabla_X^W \varphi), \ X, Y \in \Gamma(\mathcal{A}), \ \varphi \in \Gamma(W).$$

Where Clifford multiplication is given by $c\colon \mathrm{Cl}(\mathcal{A}) \to \mathrm{End}(W)$ and ∇ is the Levi-Civita $\mathcal{A}$-connection with respect to a given $\mathcal{A}$-metric g, see also [3, 19]. With an admissible $\mathcal{A}$-connection we can define a general Atiyah-Singer type geometric Dirac operator $D = D^W$, cf. [19]. The *vector representation* furnishes a corresponding $\mathcal{G}$-invariant geometric Dirac operator $\not{D}$ such that $\varrho(\not{D}) = D$, cf. [19]. Here the vector representation is characterized by the equality: $(\varrho(P)f) \circ r = P(f \circ r)$, where r is the range map of the groupoid (a surjective submersion), $P \in \mathrm{End}(C^\infty(\mathcal{G}))$ and $f \in C^\infty(M)$, see also [2, 28]. Intuitively, we can think of the boundary strata of M as being *pushed to infinity*. This is the correct setup for APS-index theory as considered in [22] for the special case of the maximal Lie structure $\mathcal{V} = \mathcal{V}_b$ and a manifold with boundary.

Local boundary conditions

On the other hand, we would like to take into account also *local boundary conditions*, e.g. Dirichlet and Neumann condition in the setting of Lie manifolds. To this end, we need to modify slightly the definition of a Lie manifold and consider so-called Lie manifolds with boundary as introduced in [1]. As before, a *Lie manifold with boundary* consists of a manifold M with corners, whose boundary hyperfaces are $F_1, \ldots, F_k$, i.e. $M = M_0 \cup F_1 \cup \cdots \cup F_k$. However, one boundary hyperface, called the *regular boundary*, say $Y := F_1$, has a special role, similar to a boundary component of a Riemannian manifold with boundary. If we glue two copies of M along Y, then we obtain the double $M \cup_Y M$. We do this doubling such that the boundary of the double at the interior of $Y \cap F_i$, $i > 1$ has no corner. To keep the distinction in mind we use throughout the notation $\mathcal{F}_1(M)$ for the *singular* boundary hyperfaces of M, as opposed to the regular boundary stratum Y. Roughly speaking, a Lie structure with boundary on M is defined as the restriction of a Lie structure on $M \cup_Y M$ to one of the copies of M, see [1] for details. If Y_0 is the interior of Y, then the map ϱ defines a bundle isomorphism from $\mathcal{A}|_{Y_0}$ to $TM|_{Y_0}$, and Y carries an induced Lie structure, denoted by $\mathcal{B}$. In the following $\mathcal{W} := \Gamma(\mathcal{B}) \subset \Gamma(TY)$. We want to consider geometric admissible Dirac operators with boundary conditions posed with regard to the hypersurface Y. Here we also need to modify the definition of admissible Dirac operators for local boundary conditions of second order, i.e. boundary conditions posed for the associated Laplacian. We do so in the main body of the paper. In order to study general and mixed boundary conditions we modify also the definition of the Lie manifold with boundary further. We introduce the concept of a *decomposed Lie manifold* which consists of two parts $M = M_1 \cup_Y M_2$, glued at the regular hypersurface Y.

1.2. **Organization of the paper**

In the second section we recall the definition of a Lie manifold with boundary and study the properties of first order differential operators on Lie manifolds with boundary. The third section is concerned with decomposed Lie manifolds. We show that to any decomposed Lie manifold there is a Lie groupoid integrating the Lie structure. In the fourth section we define a Lie semi-groupoid and a convolution C^*-algebra for boundary value problems. The main result is the continuity of the field of C^*-algebras associated to the adiabatic semi-groupoid. We show that there is a functional calculus that can be suitably defined over the integrating groupoid of a decomposed Lie manifold. In the fifth section we define the renormalized trace on Lie manifolds with boundary. We prove the index theorem on decomposed Lie manifolds with mixed boundary conditions. The argument relies on a rescaling technique and the previously introduced functional calculus restricted to the adiabatic semi-groupoid. We finish by recalling the Fredholm conditions for Dirac operators and criteria for the equality of the renormalized index with the Fredholm index.

2. **GEOMETRIC DIRAC OPERATORS**

First order differential operators on a manifold with a Lie structure at infinity (a Lie manifold) are operators which are contained in the enveloping algebra generated by the Lie structure. For a Lie manifold with boundary, we discuss differential operators of first order which are symmetric to the boundary, a notion we introduce in this section. These operators are studied in the setting of measured Lie manifolds, a slightly more general context than oriented Lie manifolds. An essential feature of Lie manifolds with boundary is the existence of a tubular neighborhood. Since the interior of any Lie manifold is in particular a manifold with bounded geometry by [3], we can contrast this to the case of a manifold with bounded geometry and boundary. In the latter case the existence of a tubular neighborhood is part of the definition of a manifold with bounded geometry and boundary. For the (slightly) more restrictive class of Lie manifolds with boundary the existence of a tubular neighborhood can be derived as a theorem, i.e. as a consequence of the definition. This is an advantage of the category of Lie manifolds over other, slightly more general categories of non-compact manifolds. Later in this section we introduce geometric Dirac operators on spin Lie manifolds.

Lie manifold with boundary

We recall the definition of a Lie manifold with boundary from [1].

Definition 2.1. A *Lie manifold with boundary* is a Lie manifold $(M, \mathcal{A}, \mathcal{V})$ together with a submanifold $(Y, \mathcal{B}, \mathcal{W})$ such that the following conditions hold:

1) $(Y, \mathcal{B}, \mathcal{W}) \hookrightarrow (M, \mathcal{A}, \mathcal{V})$ is a *Lie submanifold*, i.e. $Y \subset M$ is a submanifold with corners where $\mathcal{B} \to Y$ is a C^∞-vector bundle such that $\Gamma(\mathcal{B}) \cong \mathcal{W}$ and $\mathcal{B} \hookrightarrow \mathcal{A}_{|Y}$ is a Lie subalgebroid.

2) The submanifold Y is *transverse* in M, i.e. $T_pM = \operatorname{span}\{\varrho(\mathcal{A}_p), T_pY\}$ for each $p \in \partial Y = Y \cap \partial M$.

Remark 2.2. i) Condition 2) is equivalent to $T_xM = T_xY + T_xF$ for each $x \in F \cap Y$ and each closed codimension one face $F \in \mathcal{F}_1(M)$. Notice that the Lie structure of vector fields $\mathcal{W}$ is - by definition of a Lie submanifold - a subalgebra of $\mathcal{V}_{|Y}$, precisely $\mathcal{W} = \{V_{|Y} : V \in \mathcal{V},\ V_{|Y} \text{ tangent to } Y\}$.

ii) Given a Lie manifold with boundary as above. Fix an $\mathcal{A}$-metric g and consider the exponential map $\exp\colon \mathcal{A} \to M$ which is the natural extension from the interior $\exp\colon TM_0 \to M_0$ [3], [1, Section 1.2]. Setting $\mathcal{N} := \frac{\mathcal{A}_{|Y}}{\mathcal{B}}$ the $\mathcal{A}$-normal bundle, where $\Gamma(\mathcal{B}) \cong \mathcal{W}$ and $\mathcal{B} \hookrightarrow \mathcal{A}_{|Y}$ is in particular a sub vector bundle. We obtain the exact sequence of vector bundles

$$\mathcal{B} \rightarrowtail \mathcal{A}_{|Y} \xrightarrow{q} \mathcal{N}.$$

This sequence splits and denote by $\eta\colon \mathcal{N} \to \mathcal{A}_{|Y}$ the splitting. We can choose η as an isomorphism $\eta\colon \mathcal{N} \xrightarrow{\sim} \mathcal{B}^\perp, q \circ \eta = \mathrm{id}_{\mathcal{N}}$ [1, p.13]. This furnishes the decomposition $\mathcal{A}_{|Y} \cong \mathcal{B} \oplus \mathcal{N}$.

iii) An application of the anchor ϱ of the Lie algebroid $\mathcal{A}$ yields an isomorphism

$$\frac{\mathcal{A}_p}{\mathcal{B}_p} = \mathcal{N}_p \xrightarrow{\sim} \frac{T_pM}{T_pY} \cong N_pY,\ p \in Y.$$

Hence, in particular, $\mathcal{N}_{|Y_0} \cong NY_0$ is an isomorphism over Y_0.

We consider the induced exponential map, by observing that $\varrho_{\mathcal{B}^\perp}$ is injective onto its image, i.e. we have

$$\begin{array}{ccc} \varrho_{|\mathcal{B}^\perp}\colon \mathcal{B}^\perp & \rightarrowtail & \varrho(\mathcal{B}^\perp) \subset TM \\ \big\downarrow & & \big\downarrow \\ \mathcal{N} & \longrightarrow & NY. \end{array}$$

For each $p \in F$, for a hyperface F in M, we have $\varrho(\mathcal{A}_p) \subset T_pF$. We set $\exp^\nu := \exp_{|\mathcal{B}^\perp}\colon \mathcal{B}^\perp \to M$, for the normal exponential map, which is a well-defined local diffeomorphism by the previous discussion.

Measured manifolds

For a given $\mathcal{A}$-metric g on a Lie manifold $(M, \mathcal{A}, \mathcal{V})$ we can associate a volume form which we denote throughout by μ_g.

Definition 2.3. A Lie manifold $(M, \mathcal{A}, \mathcal{V})$ is called *measured* if there is a nowhere-vanishing smooth one-density μ on M such that there is an $\mathcal{A}$-metric g on M with $\mu = \mu_g$.

PROPOSITION 2.4. *Let $(M, \mathcal{A}, \mathcal{V})$ be a Lie manifold with boundary $(Y, \mathcal{B}, \mathcal{W})$ such that (M, μ) is measured. Then there is an induced one-density ν on Y obtained from an induced $\mathcal{B}$-metric g_∂ on Y which turns (Y, ν) into a measured Lie manifold.*

Proof. Let g be an $\mathcal{A}$-metric on M such that $\mu = \mu_g$. Denote by $\mathcal{B}^\perp$ the complement of $\mathcal{B}$ defined with the metric g. If g_∂ is a metric obtained by restriction to Y, then g_∂ is a $\mathcal{B}$-metric on Y and $\mu_{g_\partial} = \nu$ yields a one-density such that $\nu = \mu_{|Y}$. □

Remark 2.5. If $(M, \mathcal{A}, \mathcal{V})$ is orientable, then we can trivialize the top degree part of $\Lambda^\bullet \mathcal{A}^*$ in order to obtain a global density $\mu = \mu_g$. Then μ yields a measured manifold (M, μ). In particular any spin Lie manifold is orientable, therefore, measured.

Global tubular neighborhood

We introduce the notation $Y_I := I \times Y$ for an interval $I \subset \mathbb{R}$. Also write $Y_{(\epsilon)} := (-\epsilon, \epsilon) \times Y$. By a *tubular neighborhood* of Y in M we mean a local diffeomorphism $\Psi\colon M \to M$ with an open neighborhood $Y \subset \mathcal{U} \subset M$ such that for a suitable $\epsilon > 0$ the restriction of Ψ yields a diffeomorphism $\Psi\colon \mathcal{U} \xrightarrow{\sim} Y_{(\epsilon)}$.

Definition 2.6. Given a Lie manifold $(M, \mathcal{A}, \mathcal{V})$ with boundary $(Y, \mathcal{B}, \mathcal{W})$, a *boundary defining function* ρ_Y adapted to a tubular neighborhood $\Psi\colon \mathcal{U} \xrightarrow{\sim} Y_{(\epsilon)}$ is an element of $C^\infty(M, \mathbb{R})$ with $\{\rho_Y = 0\} = Y$ and non-vanishing $d\rho_Y$ at Y such that

$$(\rho_Y \circ \Phi^{-1})(x', x_n) = x_n, \ x' \in Y, x_n \in (-\epsilon, \epsilon).$$

THEOREM 2.7. *Let $(M, \mathcal{A}, \mathcal{V})$ be a measured Lie manifold with boundary $(Y, \mathcal{B}, \mathcal{W})$. There is an open neighborhood $Y \hookrightarrow \mathcal{U} \hookrightarrow M$ and a local diffeomorphism $\Phi\colon M \to M$ such that $\Phi\colon (u, \psi) \in \mathcal{U} \xrightarrow{\sim} Y_{[0,r)}$ for some $r > 0$. Additionally, Φ has the following properties: Denote by $\underline{n}$ a fixed normal vector field $\underline{n} \in \Gamma(\mathcal{A}_{|Y})$ and by τ the accompanying 1-form such that $\tau(\underline{n}) = 1$ and $\tau_{|Y} = 0$.*

i) *We have $Y = u^{-1}(0)$, where $u \equiv \rho_Y$ denotes the boundary defining function*
of Y.

ii) *$\Phi_{|Y} = \mathrm{id}_Y$ and the following diagram commutes*

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\Phi} & Y_{[0,r)} \\ \downarrow & & \downarrow \mathrm{pr}_1 \\ Y & \xrightarrow{\mathrm{id}=\Phi_{|Y}} & Y. \end{array}$$

iii) *$d\Phi(\underline{n}) = \partial_u$ along Y.*

iv) *$\tau(\underline{n}) = du$ along Y.*

v) *$\Phi_*(\mu) = |du| \otimes \nu$ on $Y_{[0,r)}$ where μ is the measure of M and ν the induced measure on Y.*

Proof. Combine Proposition 2.4 with the tubular neighborhood theorem for Lie manifolds with boundary in [1, Theorem 2.7]. □

Model operators

We consider first order differential operators on a Lie manifold with boundary which are symmetric to the boundary. In the following we fix a Lie manifold $(M, \mathcal{A}, \mathcal{V})$ with boundary $(Y, \mathcal{B}, \mathcal{W})$ which is measured and we fix the volume element μ. We also fix Hermitian smooth vector bundles $E, F \to M$ and the normal vector field $\underline{n}$ with 1-form τ as well as a tubular neighborhood $\Phi\colon \mathcal{U} \to Y_{[0,r)}$ for some $r > 0$ from Theorem 2.7.

Definition 2.8. Let $D \in \mathrm{Diff}^1_{\mathcal{V}}(M, E, F)$, then D is called *boundary symmetric* (with regard to $\underline{n}$) if D is elliptic and setting $\sigma_0 := \sigma_D(\tau)$ we have that

$\sigma_0(x)^{-1} \circ \sigma_D(\xi)\colon E_x \to E_x$, $\sigma_D(\xi) \circ \sigma_0(x)^{-1}\colon F_x \to F_x$ are skew Hermitian for each $x \in Y,\ \xi \in \mathcal{B}^*_x$.

PROPOSITION 2.9. i) *The operator D is boundary symmetric if and only if the formal adjoint D^* is boundary symmetric.*

ii) *Boundary symmetry is independent of the choice of volume form μ.*

As a preparation for the analysis of the index problem for Dirac operators we extend next a result of [5] to the context of Lie manifolds with boundary.

LEMMA 2.10. *Let $D \in \mathrm{Diff}^1_{\mathcal{V}}(M; E, F)$ be elliptic and boundary symmetric. Fix the notation $\sigma_{(u,x)}\colon E_x \to F_x,\ (u,x) \in Y_{[0,r)},\ \sigma := \sigma_D(du),\ \sigma_{(0,x)} = \sigma_D(\tau(x)),\ x \in Y$. Then there are elliptic $\mathcal{W}$-differential operators*

$$A\colon C^\infty_c(Y_0, E_{|Y}) \to C^\infty_c(Y_0, E_{|Y}),\ \tilde{A}\colon C^\infty_c(Y_0, F_{|Y}) \to C^\infty_c(Y_0, F_{|Y})$$

such that

$$D = \sigma_t(\partial_t + A + R_t), \tag{1}$$

$$D^* = -\sigma_t^*(\partial_t + \tilde{A}_t + \tilde{R}_t), \tag{2}$$

where

$$R_t\colon C^\infty_c(Y_0, E_{|Y}) \to C^\infty_c(Y_0, E_{|Y}),\ \tilde{R}_t\colon C^\infty_c(Y_0, F_{|Y}) \to C^\infty_c(Y_0, F_{|Y})$$

are operators contained in $\mathrm{Diff}^l_{\mathcal{W}}(Y; E)$ and $\mathrm{Diff}^l_{\mathcal{W}}(Y; F)$ respectively, where $l \leq 1,\ t \in [0, r)$. Additionally, $R_t,\ \tilde{R}_t$ fufill the following estimates for $f \in C^\infty_c(Y_0, E_{|Y}),\ g \in C^\infty_c(Y_0, F_{|Y})$

$$\|R_t f\|_{L^2_{\mathcal{W}}(Y)} \leq C(t\|Af\|_{L^2_{\mathcal{W}}(Y)} + \|f\|_{L^2_{\mathcal{W}}(Y)}), \tag{3}$$

$$\|\tilde{R}_t g\|_{L^2_{\mathcal{W}}(Y)} \leq C(t\|\tilde{A}g\|_{L^2_{\mathcal{W}}(Y)} + \|g\|_{L^2_{\mathcal{W}}(Y)}). \tag{4}$$

Proof. With $x \in Y$ we can write $\mathcal{B}^*_x = \{\xi \in \mathcal{A}^*_x : \xi(\underline{n}) = 0\}$. If there is an A such that (1), (2) hold then observe that $\sigma_A(\xi) = \sigma_0(x)^{-1} \circ \sigma_D(\xi)$ using that $\sigma_0(x)^{-1} \circ \sigma_D(\xi)$ is skew Hermitian for $x \in Y,\ \xi \in \mathcal{B}^*_x$. The task is to find A formally selfadjoint with principal symbol σ_A. By definition, σ_A is composed of invertible symbols, hence such an A is elliptic over $Y_{[0,r)}$. We set $D = \sigma_t(\partial_t + \mathcal{D}_t)$, where $\mathcal{D}_t \in \mathrm{Diff}^1_{\mathcal{W}}(Y; E, E)$ is a family of elliptic operators with smooth coefficients, $t \in [0, r)$. Set $R_t := \mathcal{D}_t - A$, note that $\sigma(\mathcal{D}_0) = \sigma(A)$ and R_0 is a zero order operator. Therefore, by a Taylor expansion, we have the estimate

$$\|R_t f\|_{L^2_{\mathcal{W}}(Y)} \leq C'(t\|f\|_{H^1_{\mathcal{W}}(Y)} + \|f\|_{L^2_{\mathcal{W}}(Y)}).$$

Since $R_t\colon \mathcal{H}^1_{\mathcal{W}}(Y) \to \mathcal{H}^0_{\mathcal{W}}(Y) = L^2_{\mathcal{W}}(Y)$ is bounded and A is elliptic of first order we obtain by [1]

$$\|f\|_1 \leq C(\|f\|_0 + \|Af\|_0)$$

which gives (3). Via $\sigma_{D^*}(\xi) = -\sigma_D(\xi)^*$ we obtain

$$\sigma_{\tilde{A}}(\xi) = \sigma_{D^*}(\tau(x))^{-1}\sigma_{D^*}(\xi) = (\sigma_D(\tau(x))^*)^{-1} \circ \sigma_D(\xi)^*$$
$$= (\sigma_D(x)^*)^{-1} \circ \sigma_D(\xi) = (\sigma_D(\xi) \circ \sigma_0(x)^{-1})^*.$$

□

Twisted Dirac operators

Fix a spin Lie manifold $(M, \mathcal{A}, \mathcal{V})$, an $\mathcal{A}$-metric g and an $\mathcal{A}$-connection ∇^W. We recall the definition of the geometric twisted Dirac operators on Lie manifolds, cf. [3].

Fix a Clifford module $W \in \mathrm{Cl}(\mathcal{A})$ such that $W = W^+ \oplus W^-$ is $\mathbb{Z}_2$-graded compatible with the Clifford action: $\mathbf{c}(\mathrm{Cl}(\mathcal{A})^+)W^\pm \subset W^\pm$, $\mathbf{c}(\mathrm{Cl}(\mathcal{A})^-)W^\pm \subseteq W^\mp$.

Definition 2.11. The geometric Dirac operator $D = D^W$ on the Lie manifold is defined as $D = c \circ (\mathrm{id} \otimes \sharp) \circ \nabla^W$, where $\sharp$ is the isomorphism $\mathcal{A} \cong \mathcal{A}^*$ induced by the fixed compatible metric g, ∇^W denotes the $\mathcal{A}$-connection and c the Clifford multiplication:

$$\Gamma(W) \xrightarrow{\nabla^W} \Gamma(W \otimes \mathcal{A}^*) \xrightarrow{\mathrm{id}\otimes\sharp} \Gamma(W \otimes \mathcal{A}) \xrightarrow{c} \Gamma(W).$$

Since c is a $\mathcal{V}$-operator of order 0 and ∇^W is a $\mathcal{V}$-operator of order 1 we see that D is in $\mathrm{Diff}_{\mathcal{V}}^1(M; W)$. Additionally, $\sigma_1(D)\xi = ic(\xi) \in \mathrm{End}(W)$, hence invertible for $\xi \neq 0$, and D is elliptic.

We check that a geometric Dirac operator on a Lie manifold with boundary is boundary symmetric.

Proposition 2.12. *Let $(M, \mathcal{A}, \mathcal{V})$ be a spin Lie manifold with boundary $(Y, \mathcal{B}, \mathcal{W})$. Then $(M, \mathcal{A}, \mathcal{V})$ is measured. A geometric Dirac operator $D = D^W$ acting on the C^∞ vector bundles $E, F \to M$ for a given Clifford module W is boundary symmetric with regard to Y for a fixed choice of normal vector field $\underline{n} \in \Gamma(\mathcal{A}_{|Y})$.*

Proof. Let $g = g_{\mathcal{A}}$ be an $\mathcal{A}$-metric. The Clifford relations are

$$2g_{\mathcal{A}}(\xi, \eta)\mathrm{id}_{E_x} = \sigma_D(\xi)^*\sigma_D(\eta) + \sigma_D(\eta)^*\sigma_D(\xi),\ \ \xi, \eta \in \mathcal{A}_x^*,$$
$$2g_{\mathcal{A}}(\xi, \eta)\mathrm{id}_{F_x} = \sigma_D(\xi)\sigma_D(\eta)^* + \sigma_D(\xi)^*\sigma_D(\eta),\ \ \xi, \eta \in \mathcal{A}_x^*.$$

Set $\sigma_0(x) = \sigma_D(\tau(x))$ where τ denotes the 1-form associated to the normal vector $\underline{n}$, i.e. $g_{\mathcal{A}}(\tau(x), \xi) = 0$. Extend ξ to $\mathcal{A}_x$ by setting $\xi(\underline{n}) = 0$. Then the Clifford relations yield $\sigma_0(x)^{-1} \circ \sigma_D(\xi) \colon E_x \to E_x$, $\sigma_D(\xi) \circ \sigma_0(x)^{-1} \colon F_x \to F_x$ are skew-Hermitian for each $x \in Y$, $\xi \in \mathcal{B}_x^*$. □

3. A GLUEING CONSTRUCTION

We introduce so-called decomposed Lie manifolds. These Lie manifolds consist of two parts, one part is a so-called Lie manifold of cylinder type and the complementary part is a standard Lie manifold in its own right. Furthermore, we show that for the corresponding Lie algebroid on a decomposed Lie manifold, we can obtain an integrating groupoid via a glueing of two groupoids, one on the cylinder part of the manifold and the other on the complement. We show that the resulting glued groupoid can be endowed with a smooth structure in a natural way.

Definition 3.1. A *decomposed Lie manifold* $(M, \mathcal{A}, \mathcal{V})$ with hypersurface $(Y, \mathcal{B}, \mathcal{W})$ is a Lie manifold such that $M = M_1 \cup M_2$ where $M_1 \cap M_2 = Y$. Additionally, M_2 is a Lie manifold of *cylinder type*, i.e. M_2 is diffeomorphic to a global tubular neighborhood of Y in M.

Any Lie manifold with boundary can be glued to a decomposed Lie manifold, up to a choice of tubular neighborhood, which follows by Theorem 2.7 and the following discussion. Let $(M, \mathcal{A}, \mathcal{V})$ be a decomposed Lie manifold such that $M_1 \cup M_2 = M$, $M_1 \cap M_2 = Y$. In the following we consider two groupoids: $\mathcal{G}_1 \rightrightarrows \mathring{M}_1$ where $\mathring{M}_1 := M_1 \setminus Y$ as well as $\mathcal{G}_2 \rightrightarrows Y \times [-1, 0]$. Here $M_2 \cong Y \times [-1, 0]$ is the cylinder-type part of M. Let $\mathcal{U}$ be a global tubular neighborhood of Y in M such that $\mathcal{U} = \mathcal{U}_+ \cup_Y \mathcal{U}_-$ is decomposed in the following sense. Fix a boundary defining function, i.e. smooth function $\rho \colon M \to \mathbb{R}$ such that $Y = \{\rho = 0\}$ and $d\rho$ is non-vanishing on Y. We also consider the strata $Y_+ := \{\rho = 1\}$ and $Y_- := \{\rho = -1\}$. Let $\mathcal{U}$ be decomposed into the collars $\mathcal{U}_+ \cong (0, 1]_u \times Y_+$ and $\mathcal{U}_- \cong [-1, 0)_u \times Y_-$. The groupoids $\mathcal{G}_1$, $\mathcal{G}_2$ are adapted to $(M, \mathcal{A}, \mathcal{V})$ and $(Y, \mathcal{B}, \mathcal{W})$. Let $\mathcal{V}_{\mathring{M}_1}$ be the Lie structure of $\mathring{M}_1$. Denote by $\mathcal{G}_1 = \mathcal{G}(\mathring{M}_1) \rightrightarrows \mathring{M}_1$ the Lie groupoid integrating the Lie structure $\mathcal{V}_{\mathring{M}_1}$. Let $\mathcal{H} \rightrightarrows Y$ be the Lie groupoid integrating the Lie structure $\mathcal{W}$. Define $\mathcal{G}_2 := \mathcal{H} \times ([-1, 0)^2 \cup \{0\} \times \mathbb{R}_+) \rightrightarrows Y \times [-1, 0]$. The groupoid structure of $\mathcal{G}_2$ is given by the pair groupoid structure on $[-1, 0)^2$ and $(\mathbb{R}_+, \cdot)$ viewed as a multiplicative group.

We are now in a position to state the following result.

THEOREM 3.2. *The groupoid $\mathcal{G} := \mathcal{G}_1 \cup \mathcal{G}_2 \rightrightarrows M$ has the C^∞-structure of a Lie groupoid such that $\mathcal{A}(\mathcal{G}) \cong \mathcal{A}$, i.e. $\mathcal{G}$ integrates the Lie structure of the decomposed Lie manifold $(M, \mathcal{A}, \mathcal{V})$.*

Proof. Define the auxiliary groupoid $\tilde{\mathcal{H}} := \mathcal{H} \times \mathbb{R} \rtimes \mathbb{R}_+ \rightrightarrows Y \times \mathbb{R}$, where $\mathbb{R} \rtimes \mathbb{R}_+ \rightrightarrows \mathbb{R}$ is the semi-direct product groupoid given by the multiplicative action of $(\mathbb{R}_+, \cdot)$ on $(\mathbb{R}, \cdot)$. It is immediate that $\tilde{\mathcal{H}}$ is a Lie groupoid. Set

$\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2 \rightrightarrows M$, then we exhibit a Lie groupoid structure on $\mathcal{G}$ with the help of $\tilde{\mathcal{H}}$, using a transport of structure argument. The definition of $\mathcal{G}$, together with the tubular neighborhood theorem for Lie manifolds then implies that $\mathcal{A}(\mathcal{G}) \cong \mathcal{A}$. We fix the collar neighborhoods as previously defined. We have $\mathcal{G}_2 = \tilde{\mathcal{H}}_{Y\times[-1,0]}^{Y\times[-1,0]}$. By transversality of Y in M we have a canonical diffeomorphism $(\mathcal{G}_1)_{\mathcal{U}}^{\mathcal{U}} \cong \mathcal{H} \times (0,1)^2$. Denote this diffeomorphism by ψ. We construct the diffeomorphism

$$\varphi \colon \tilde{\mathcal{H}}_{Y\times(-1,1)}^{Y\times(-1,1)} \xrightarrow{\sim} \mathcal{G}.$$

Here we restrict $\mathcal{G}$ without loss of generality to $\mathcal{U}$ and denote the restriction by the same letter. This will give the C^∞-structure at 0. Note that

$$\tilde{\mathcal{H}}_{Y\times(-1,1)}^{Y\times(-1,1)} = \mathcal{H} \times [(-1,0)^2 \cup \{0\} \times \mathbb{R}_+ \cup (0,1)^2].$$

Hence φ is given by gluing the identity and the diffeomorphism ψ. A similar type of argument shows that the structural maps of $\mathcal{G}$ are C^∞ maps. Therefore, $\mathcal{G}$ is a Lie groupoid. $\square$

4. CONTINUOUS FIELD OF C^*-ALGEBRAS

In this section we answer the question of how to associate a C^*-algebra to a semi-groupoid in a functorial way. To begin with we introduce suitable categories of fields of smooth groupoids and semi-groupoids as well as the category of fields of C^*-algebras. Then we describe a contravariant functor from the category of fields of smooth groupoids to the category of fields of C^*-algebras, or more generally, $C_0(T)$-algebras, where T is a compact Hausdorff space. We contrast this functoriality with the case of the smaller Lie category of smooth semi-groupoids. The next question is under what conditions a given field of semi-groupoids furnishes a continuous field of C^*-algebras. Here we are interested in a particular type of deformation semi-groupoid over the cylinder part of a given decomposed Lie manifold. We show that this semi-groupoid fulfills the necessary condition for functoriality with regard to a natural class of representations on Hilbert space. Then, as a preparation for our study of the heat kernel in the final section, we introduce a functional calculus on the convolution C^*-algebra of the deformation semi-groupoid on a decomposed Lie manifold.

Functoriality

We recall the definition of $C_0(T)$-C^*-algebras where T is a Hausdorff topological space. Then we establish what criteria are needed for the continuity of fields of C^*-algebras as defined via Lie groupoids or Lie semi-groupoids.

Recall that for a C^*-algebra $M(A)$ denotes the maximal unital C^*-algebra which contains A as an essential ideal. Denote by $Z(B)$ the center of a given C^*-algebra B.

Definition 4.1. A $C_0(T)$-algebra is a tuple (A, θ) where

$$\theta \colon C_0(T) \longrightarrow ZM(A)$$

is a $*$-homomorphism such that $\theta(C_0(T))A = A$.

Note that $a \in A$ can be identified with a family $a = (a_x)_{x \in T}$. Here $a_x \in A_x := A/C_x A,\ C_x := \{f \in C_0(T) : f(x) = 0\}$. The action of functions on T is implemented by θ and we often abuse notation by writing $f \cdot a$ instead of $\theta(f) \cdot a$. We also write $A^T := \overline{\theta(C_0(T))}A$ and call A *non-degenerate* if $A = A^T$.

Example 4.2. Consider $A = C_0(T)$ with $M(C_0(T)) = C_b(T)$, $ZM(C_0(T)) = C_b(T)$. We will provide further examples induced by the vast plethora of C^*-algebras associated to Lie groupoids.

Given two $C_0(T)$-algebras A, B an arrow is given by a $*$-homomorphism $\psi \colon A \to B$ which is $C_0(T)$-linear, i.e. $\psi(f \cdot a) = f \cdot \psi(a)$ for each $f \in C_0(T),\ a \in A$. Denote by $\mathbf{C}^*(T)$ the category with objects the $C_0(T)$-algebras and $C_0(T)$-linear $*$-homomorphisms as arrows between objects.

A particular case of $C_0(T)$-algebra is that given by a field of groupoids.

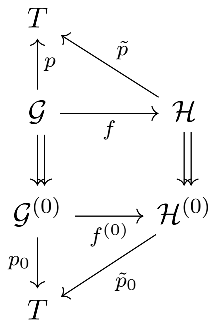
Definition 4.3. A *field* of Lie groupoids is a triple $(\mathcal{G}, T, p)$ where $\mathcal{G}$ is a Lie groupoid, T is a C^∞-manifold and $p \colon \mathcal{G} \to T$ is a submersion. We denote by p_0 the restriction of p to $\mathcal{G}^{(0)}$.

The *category of T-Lie groupoids* $\mathbf{LG}(T)$ consists of objects the fields of Lie groupoids. Let $(\mathcal{G}, T, p)$, $(\mathcal{H}, T, \tilde{p})$ be T-Lie groupoids. Recall that a strict Lie groupoid morphism is a tuple $(f, f^{(0)}) \colon \mathcal{G}(T) \to \mathcal{H}(T)$ such that the diagram

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow[f]{} & \mathcal{H} \\ \downdownarrows & & \downdownarrows \\ \mathcal{G}^{(0)} & \xrightarrow[f^{(0)}]{} & \mathcal{H}^{(0)}. \end{array}$$

commutes. An arrow in the category $\mathbf{LG}(T)$ is a tuple $(f, f^{(0)})$ with f and $f^{(0)}$

two C^∞-maps such that the following diagram commutes



The category $\mathbf{C}^*(T)$ has many useful stability properties, e.g. it is closed with regard to the formation of ideals, quotients, direct sums, suspensions and it is $C_0(T)$-stable. The correct tensor product inside the category is the completed maximal tensor product. Note that $\psi(A) \subset B^T$ for ψ a possibly degenerate arrow and $\mathrm{Mor}_{C_0(T)}(A, B) = \mathrm{Mor}_{C_0(T)}(A, B^T)$. We want to study semi Lie groupoids, i.e. Lie groupoids where inverses do not always exist. This can be viewed as merely the category of C^∞-manifolds with additional data and structural maps as given in a Lie groupoid. We are mainly interested in the question of how to assign a $C_0(T)$-algebra to a given Lie groupoid or semi Lie groupoid. To clarify these issues, we will first prove a functoriality result for the association

$$\mathbf{C}^* \colon \mathbf{LG}(T) \to \mathbf{C}^*(T).$$

In the functor $\mathbf{C}^* = \mathbf{C}^*_T$ we routinely suppress the dependency on T. For the following discussion, we also refer to [18, Section 5].

The map on objects. Let $(\mathcal{G}, T, p)$ be a T-Lie groupoid. We describe a left $C_0(T)$-module structure on $C_c^\infty(\mathcal{G})$. Define $(fa)(\gamma) = f(p(\gamma))a(\gamma),\ f \in C_0(T),\ a \in C_c^\infty(\mathcal{G})$. As a consequence $f(a * b) = (fa) * b = a * (fb)$ for each $f \in C_0(T),\ a, b \in C_c^\infty(\mathcal{G})$. Also $C_0(T)C_c^\infty(\mathcal{G}) = C_c^\infty(\mathcal{G})$ since if $f \in C_c(T)$ is chosen such that $f \equiv 1$ on $p(\mathrm{supp} a)$, then $a = fa \in C_0(T)C_c^\infty(\mathcal{G})$.

Completion. Endow $C_c^\infty(\mathcal{G})$ with the inductive limit topology $\tau_\to$. We have the following condition

$$\begin{aligned} &\forall\, \pi \colon C_c^\infty(\mathcal{G}) \to \mathcal{L}(\mathcal{H}) \text{ continuous representation} \\ &\exists\, \varphi \colon C_0(T) \to \mathcal{L}(\mathcal{H}) \text{ unique representation s.t.} \\ (L) \qquad &\pi(fa) = \varphi(f)\pi(a). \end{aligned}$$

As shown in Lemma 1.13. of [29] this condition holds for Lie groupoids. By an application of (L) for given $f \in C_0(T),\ a \in C_c^\infty(\mathcal{G})$

$$\|\pi(fa)\| \leq \|\varphi(f)\|\|\pi(a)\| \leq \|f\|\|a\|$$

$$\Rightarrow \|fa\| \leq \|f\|\|a\|.$$

Therefore, for each $f \in C_0(T)$, the mapping $C_c^\infty(\mathcal{G}) \ni a \mapsto fa \in C_c^\infty(\mathcal{G})$ extends continuously to $C^*(\mathcal{G})$. Hence $C^*(\mathcal{G})$ has a canonical $C_0(T)$-Banach module structure. It needs to be checked that $C_0(T)C^*(\mathcal{G})$ is closed in $C^*(\mathcal{G})$ by a separate argument, cf. [18]. Then non-degeneracy is clear by $C_0(T)C^*(\mathcal{G}) \supset C_0(T)C_c^\infty(\mathcal{G}) = C_c^\infty(\mathcal{G})$. We have $f(a*b) = (fa)*b$ and $(fa)^* = f^*a^*$ for $a, b \in C^*(\mathcal{G})$, $f \in C_0(T)$, hence $C^*(\mathcal{G})$ is a $C_0(T)$-algebra.

The map on arrows. Let $(f, f^{(0)})\colon \mathcal{G}(T) \to \mathcal{H}(T)$ be a strict morphism of T-Lie groupoids. Define $\mathbf{C}^*(f, f^{(0)})$, in short $\mathbf{C}^*(f)\colon C^*(\mathcal{H}) \to C^*(\mathcal{G})$ via the assignment

$$\mathbf{C}^*(f)(a) = a \circ f,\ a \in C^*(\mathcal{H}).$$

Proposition 4.4. *We obtain a contravariant functor*

$$\mathbf{C}^*\colon \mathbf{LG}(T) \to \mathbf{C}^*(T).$$

Proof. Let $g \in C_0(T)$, then $\mathbf{C}^*(g{\cdot}a) = (g{\cdot}a)\circ f$ where g acts via multipliers on $C^*(\mathcal{H})$. Denote by $(C^*(\mathcal{G}), \theta)$ and $(C^*(\mathcal{H}), \tilde{\theta})$ the corresponding actions. By definition $\mathbf{C}^*(f)(g \cdot a) = \mathbf{C}^*(f)(\tilde{\theta}(g) \cdot a)$. It is then routine to verify that $\mathbf{C}^*(f)$ is $C_0(T)$-linear: Take $g \in C_0(T)$ fixed

$$\begin{aligned}
\mathbf{C}^*(f)(g \cdot a) &= \mathbf{C}^*(f)(\tilde{\theta}(g) \cdot a)(\gamma) \\
&= \mathbf{C}^*(f)((g \circ \tilde{p})a)(\gamma) = ((g \circ \tilde{p}) \cdot a)f(\gamma) \\
&= (g \circ \tilde{p})(f(\gamma)) \cdot a(f(\gamma)) = g(\tilde{p}(f(\gamma))) \cdot a(f(\gamma)) \\
&= g(p(\gamma))a(f(\gamma)) = \theta(g) \cdot \mathbf{C}^*(f)(a)(\gamma) = g \cdot \mathbf{C}^*(f)(a).
\end{aligned}$$

In the final line we used the commutativity

$$\begin{array}{ccc}
\mathcal{G} & \xrightarrow{f} & \mathcal{H} \\
& \searrow^{p} & \downarrow{\scriptstyle \tilde{p}} \\
& & T
\end{array}$$

□

What remains to be studied is the continuity of the resulting field of C^*-algebras.

Definition 4.5. A $C_0(T)$-algebra A is *continuous* if $x \mapsto \|a_x\| \in [0, \infty)$ is continuous for each $x \in T$.

Up until now the functoriality makes sense for $\mathbf{C}^*$ as well as $\mathbf{C}_r^*$. The second part, namely $\mathbf{C}^*$ mapping to *continuous* $C_0(T)$-algebras, requires that we restrict to *amenable* Lie groupoids. The reason is found in [18, Theorem

5.5] where upper semi-continuity of the field relies on the full C^*-algebra of the groupoid. We study next the general functoriality and continuity for Lie semi groupoids. There are two things to notice at the outset: *i)* For general functoriality of T semi-groupoids the condition (L) is no longer true in general, but becomes an axiom. *ii)* The proof of continuity of the field of C^*-algebras associated to a Lie groupoid crucially requires amenability. This is not really available anymore on Lie semi-groupoids. Let us address both of these problems in what follows. Denote by $\mathbf{LC}$ the *Lie category*, consisting of Lie semi-groupoids as objects and smooth functors as arrows between objects. In the same vein we also introduce the category of T Lie semi-groupoids $\mathbf{LC}(T)$. Denote by $\widetilde{\mathbf{LC}}(T)$ the subcategory of $\mathbf{LC}(T)$ consisting of objects the Lie semi-groupoids which fulfill condition (L). We have a functorial diagram

$$\begin{array}{ccc} \mathbf{LG}(T) & \xrightarrow{\mathbf{C}^*} & \mathbf{C}^*(T) \\ {\scriptstyle\iota}\Big\downarrow & \nearrow{\scriptstyle\widetilde{\mathbf{C}}^*} & \\ \widetilde{\mathbf{LC}}(T) & & \end{array}$$

where ι denotes the inclusion functor and $\widetilde{\mathbf{C}}^*$ is the functor constructed via representations given by (L).

Continuity for semi-groupoids

In the remainder of this work, we will be interested in a deformation semi-groupoid suitable for boundary value problems on Lie manifolds with boundary. We will define the reduced C^*-algebra associated to this semi-groupoid and show that this yields a continuous field of C^*-algebras. In order to give the reader a better appreciation of some of the difficulties involved in the study of the continuity of fields of C^*-algebras associated to semi-groupoids we have discussed above the case of Lie groupoids. We are given the following data: A Lie manifold $(M, \mathcal{A}, \mathcal{V})$ with boundary $(Y, \mathcal{B}, \mathcal{W})$. Let $\mathcal{G} \rightrightarrows M$ denote an integrating s-connected Lie groupoid, i.e. $\mathcal{A}(\mathcal{G}) \cong \mathcal{A}$. Fix the generalized exponential map (see also [10, 18]) $\mathrm{Exp}\colon \mathcal{A}(\mathcal{G}) \to \mathcal{G}$ as well as the exponential map $\exp\colon \mathcal{A}(\mathcal{G}) \to M$ induced by an invariant connection on $\mathcal{A}$. Consider the half space $\widetilde{\mathcal{A}} \subset \mathcal{A}$ which is defined as follows:

$$\widetilde{\mathcal{A}} := \{v \in \mathcal{A} : \exp(-tv) \in M, \ t > 0 \text{ small}\}.$$

This is the natural generalization of the half-space introduced in [4] for the case of a compact manifold with boundary and trivial Lie structures. We restrict the invariant connection ∇ of $\mathcal{A}$ to $\widetilde{\mathcal{A}}$ and also obtain the restriction of the generalized exponential which we denote by the same symbol $\mathrm{Exp}\colon \widetilde{\mathcal{A}} \to \mathcal{G}$. Then we define the deformation semi-groupoid $\widetilde{\mathcal{G}}^{ad} \rightrightarrows M \times I$ where $I = \mathbb{R}$ or $I = [0,1]$ and $I^* := I \setminus \{0\}$ as follows

$$\widetilde{\mathcal{G}}^{ad} := \mathcal{G} \times (0,1] \cup \widetilde{\mathcal{A}} \times \{0\}.$$

Note that a priori $\widetilde{\mathcal{G}}^{ad}$ has a natural semi-groupoid structure: the groupoid structure of $\mathcal{G}$, of $(0,1]$ viewed simply as a set as well as the semi-groupoid structure of $\widetilde{\mathcal{A}}$, viewed as a bundle of half-spaces. Note that $\widetilde{\mathcal{G}}^{ad} \subset \mathcal{G}^{ad}$ where $\mathcal{G}^{ad}$ is the adiabatic groupoid $\mathcal{G} \times (0,1] \cup \mathcal{A} \times \{0\}$. By the local diffeomorphism property of Exp we can describe a smooth structure on $\mathcal{G}^{ad}$. It is defined by glueing a neighborhood $\mathcal{O}$ of $\mathcal{A} \times \{0\}$ to $\mathcal{G} \times I^*$ via

$$\mathcal{O} \ni (v,t) \mapsto \begin{cases} v, \ t = 0 \\ (\mathrm{Exp}(-tv), t), \ t > 0. \end{cases}$$

Then the smooth structure of $\widetilde{\mathcal{G}}^{ad} \subset \mathcal{G}^{ad}$ is the one induced by $\mathcal{G}^{ad}$ with regard to the locally compact subspace topology, i.e. $C_c^\infty(\widetilde{\mathcal{G}}^{ad}) := C_c^\infty(\mathcal{G}^{ad})_{|\widetilde{\mathcal{G}}^{ad}}$. The next goal is to define a continuous field of C^*-algebras over the Lie semi-groupoid $\widetilde{\mathcal{G}}^{ad}$.

Definition 4.6. The C^*-algebra associated to $\widetilde{\mathcal{G}}^{ad}$ is defined as the completion $C_r^*(\widetilde{\mathcal{G}}^{ad}) := \overline{C_c^\infty(\widetilde{\mathcal{G}}^{ad})}^{\|\cdot\|}$. We define the norm $\|\cdot\|$ as the reduced norm with regard to the representation $\tilde{\pi} := (\pi, \pi^\partial)$ on the Hilbert space $\mathcal{H} := L^2(\mathcal{G}) \oplus L^2(\widetilde{\mathcal{A}}_{|Y})$. Here $\pi = (\pi_t)_{0<t\leq 1}$ where for $0 < t \leq 1$

$$\pi_t(f)\xi(\gamma) = \frac{1}{t^n}\int_{\mathcal{G}_{s(\gamma)}} f(\eta,t)\xi(\eta)\,d\mu_{s(\gamma)}(\eta).$$

Define the representation π_0^∂ on the Hilbert space $L^2(\widetilde{\mathcal{A}}_{|Y})$ by

$$\pi_0^\partial(f)\xi(v) = \int_{\widetilde{\mathcal{A}}_{\tilde{\pi}(v)}} f(v-w)\xi(w)\,\mathrm{d}w.$$

We also introduce a C^*-algebra associated to the half-space $\widetilde{\mathcal{A}}$.

Definition 4.7. Define the reduced C^*-algebra of $\widetilde{\mathcal{A}}$ in terms of the completion $C_r^*(\widetilde{\mathcal{A}}) := \overline{C_c^\infty(\widetilde{\mathcal{A}})}^{\|\cdot\|_{\tilde{\pi}_0}}$, where $\tilde{\pi}_0 = (\pi_0, \pi_0^\partial)$ is the representation of $C_c^\infty(\widetilde{\mathcal{A}})$ on the Hilbert space $\mathcal{H} := L^2(\mathcal{A}) \oplus L^2(\widetilde{\mathcal{A}}_{|Y})$. We define

$$\pi_0(f)\xi(v) = \int_{\mathcal{A}_{\pi(v)}} f(v-w)\xi(w)\,\mathrm{d}w$$

and

$$\pi_0^\partial(f)\xi(v) = \int_{\widetilde{\mathcal{A}}_{\tilde{\pi}(v)}} f(v-w)\xi(w)\,\mathrm{d}w.$$

Note that the above definition furnishes a field of C^*-algebras with $\varphi_t\colon C_r^\infty(\widetilde{\mathcal{G}}^{ad}) \to C_r^*(\widetilde{\mathcal{G}}^{ad})(t)$ where $C_r^*(\widetilde{\mathcal{G}}_t^{ad}) = C_r^*(\mathcal{G})$, $t \neq 0$ and $C_r^*(\widetilde{\mathcal{G}}_0^{ad}) = C_r^*(\widetilde{\mathcal{A}})$. This leads us to the following result.

THEOREM 4.8. *Let $(M, \mathcal{A}, \mathcal{V})$ be a decomposed Lie manifold with hypersurface $(Y, \mathcal{B}, \mathcal{W})$ and integrating Lie groupoid $\mathcal{G} \rightrightarrows M$. The field*

$$\left(C_r^*(\widetilde{\mathcal{G}}^{ad}), \{C_r^*(\widetilde{\mathcal{G}}_t^{ad}), \varphi_t\}_{t\in[0,1]}\right)$$

is a continuous field of C^*-algebras.

The continuity for the semi-groupoid $\widetilde{\mathcal{G}}^{ad}$ is a generalization of the result proven in [4], where the case of a compact manifold with boundary (and the trivial Lie structure of all vector fields) is studied. Since our case is vastly more general, we give more details below. We will describe the strategy of the argument, highlighting the main differences to the argument in loc. cit. for our case of general decomposed Lie manifolds with boundary.

Proof. Note that the condition in the definition of the semi-algebroid $\widetilde{\mathcal{A}}$ only takes effect at the regular boundary stratum Y. The difficulty in the proof of continuity of the field is the upper and lower semi-continuity of the field at $t = 0$. If we restrict the groupoid outside any tubular neighborhood of Y in M then $\widetilde{\mathcal{A}}$ is identical to $\mathcal{A}$. For this restricted groupoid the argument for continuity of the field goes along the same lines as the proof given in [4]. Therefore, we can without loss of generality focus on the case where M is of cylinder type. Since M is assumed to be of cylinder type we identify $M \cong Y \times \overline{\mathbb{R}}_+$. Let $\mathcal{H} \rightrightarrows Y$ be a Lie groupoid such that $\mathcal{A}(\mathcal{H}) \cong \mathcal{B}$. The groupoid $\mathcal{G}$ takes the form $\mathcal{G} = \mathcal{H} \times (\mathbb{R}_+^2 \cup \{0\} \times \mathbb{R}_+) \rightrightarrows Y \times \overline{\mathbb{R}}_+$ with smooth structure as defined in Section 3. We first define the auxiliary algebra $C_{tc}^\infty(\widetilde{\mathcal{A}}) := C_c^\infty(\mathcal{A}) \oplus C_c^\infty(\mathcal{B} \times \mathbb{R}_+^2)$, which is a dense $*$-subalgebra of $C_r^*(\widetilde{\mathcal{A}})$. Choose a cutoff $\psi \in C_c^\infty(\mathcal{G})$ such that $0 \leq \psi \leq 1$ and $\psi_{|U} \equiv 1$ for a neighborhood $M \subset U \subset \mathcal{G}$ for which there is a corresponding neighborhood of the zero section, $M \subset \mathcal{O} \subset \mathcal{A}(\mathcal{G})$ for which $\mathrm{Exp}\colon \mathcal{O} \xrightarrow{\sim} \mathrm{supp}\psi \subset \mathcal{G}$ is a diffeomorphism. Throughout, we will make use of the lifting of an element $f \in C_c^\infty(\mathcal{A})$, or $f \in C_c^\infty(\widetilde{\mathcal{A}})$ to an element of $C_c^\infty(\widetilde{\mathcal{G}}^{ad})$, defined by

$$\tilde{f}(\gamma, t) := \psi(\gamma) f\left(-\frac{\mathrm{Exp}^{-1}(\gamma)}{t}\right). \tag{l}$$

We first show the lower semi-continuity of the field, i.e.

$$\liminf_{t\to 0} \|\varphi_t(a)\| \geq \max\{\|\pi_0(a)\|, \|\pi_0^\partial(a)\|\}. \tag{lsc}$$

The proof of lower semi-continuity is facilitated by a reduction of the proof of lower semi-continuity of representations of $C_c^\infty(\widetilde{\mathcal{G}}^{ad})$ and $C_c^\infty(\mathcal{H}^{ad} \times \overline{\mathbb{R}}_+)$.

Making use of the generalized exponential $\mathrm{Exp}_\partial\colon \mathcal{B} \to \mathcal{H}$, we can introduce a lifting (l) also for elements $f \in C_c^\infty(\mathcal{B} \times \mathbb{R}_+)$ to $C_c^\infty(\mathcal{H}^{ad} \times \mathbb{R}_+)$ and, abusing notation, we also denote by $\tilde{f}$. Fixing a Haar system $(\mu_{x,t})_{(x,t)\in M\times I}$, introduce the norms $\|\cdot\|_{\infty,t}$ on $\widetilde{\mathcal{G}}^{ad}$ by

$$\|g\|_{\infty,t}^2 := \sup_{x\in\mathcal{G}^{(0)}} \|g\|_{L^2(\widetilde{\mathcal{G}}^{ad}_{x,t},\mu_{x,t})}.$$

Abusing notation again, we use the same symbols $\|\cdot\|_{\infty,t}$ for the corresponding norms on $C_c^\infty(\mathcal{H}^{ad} \times \mathbb{R}_+)$, where we fix a Haar system $\mu_{x,t}^\partial$ on $\mathcal{H}^{ad}$ and the standard Lebesgue measure on $\mathbb{R}_+$. Define the representation $\rho_t\colon C_c^\infty(\widetilde{\mathcal{A}}) \to \mathcal{L}(L^2(\mathcal{G}))$ by

$$(\rho_t f)\xi(\gamma) = \int_{\mathcal{G}_{s(\gamma)}} \tilde{f}(\gamma\eta^{-1}, t)\xi(\eta)\,\mathrm{d}\mu_{s(\gamma)}(\eta).$$

A straightforward generalization of [4, Proposition 2.22], yields the density of $C_c^\infty(\widetilde{\mathcal{A}})$ in $C_c^\infty(\widetilde{\mathcal{G}}^{ad})$, which implies that it is sufficient to show the estimates

$$\liminf_{t\to 0} \|\rho_t(\tilde{f} + \tilde{K}\| \geq \|\pi_0(f)\|,\ f \oplus K \in C_{tc}^\infty(\widetilde{\mathcal{A}}), \tag{5}$$

$$\liminf_{t\to 0} \|\rho_t(\tilde{f} + \tilde{K}\| \geq \|\pi_0^\partial(f \oplus K)\|, f \in C_c^\infty(\mathcal{A}),\ K \in C_c^\infty(\mathcal{B} \times \mathbb{R}_+^2). \tag{6}$$

In order to show (5) we check that

$$\|\pi_t(\tilde{f} + \tilde{K})\|$$
$$= \sup\left\{\left\|\frac{1}{t^n}\int_{\mathcal{G}_\bullet} (\tilde{f}(\bullet\cdot\eta^{-1}, t) + \tilde{K}(\bullet\cdot\eta^{-1}, t)g(\bullet\cdot\eta^{-1}, t)\,\mathrm{d}\mu_\bullet(\eta)\right\|_{\infty,t} : \|g\|_\infty \leq 1\right\},$$

$$\|\pi_0(f)\| = \sup\left\{\left\|\int f(v,0)g(\bullet - v, 0)\,\mathrm{d}v\right\| : \|g\|_\infty \leq 1\right\}.$$

At this point we recall the structure of the groupoid $\widetilde{\mathcal{G}}^{ad}$ as $\mathcal{H} \times (\mathbb{R}_+^2 \cup \{0\} \times \mathbb{R}_+)$ and note that the set of g for which $g(\gamma, t) = 0$ with $s(\gamma) = (x', 0) \in Y \times \overline{\mathbb{R}}_+ \cong M$ is dense in $\{g \in C_c^\infty(\widetilde{\mathcal{G}}^{ad}) : \|g\|_\infty \leq 1\}$. The weak convergence of $\tilde{K}$ to zero yields for $g \in C_c^\infty(\widetilde{\mathcal{G}}^{ad})$

$$\lim_{t\to 0}\left\|\frac{1}{t^n}\int (\tilde{f}(\bullet\cdot\eta^{-1}, t) + \tilde{K}(\bullet\cdot\eta^{-1}, t))g(\bullet\cdot\eta^{-1}, t)\,\mathrm{d}\mu_\bullet(\eta)\right\|_{\infty,t}$$
$$= \left\|\int f(v,0)g(\bullet - v)\,\mathrm{d}v\right\|_{\infty,0}.$$

The equality (5) follows. We prove (6) by fixing $(a_t)_{t\in I}$ such that $a_t \to 0$ for $t \to 0$ and $\frac{a_t}{t} \to \infty$ for $t \to 0$. Define representations of $f \in C_c^\infty(\widetilde{\mathcal{G}}^{ad})$, $K \in C_c^\infty(\mathcal{B} \times \mathbb{R}_+^2)$ on $C_c^\infty(\mathcal{H}^{ad} \times \mathbb{R}_+^2)$ via

$$\eta_t(f)g(\gamma, t, b) = t^{n+1}\int_{[0,\frac{a_t}{t}]}\int_{\mathcal{G}_{s(\gamma)}} f(\gamma\eta^{-1}, tb, ta, t)g(\eta, t, a)\,\mathrm{d}\mu_{s(\gamma)}(\eta)\,\mathrm{d}a$$

for $t \neq 0$ and $b \in \left[0, \frac{a_t}{t}\right]$. As well as

$$\eta_0(f)g(v,0,b) = \int_{\mathcal{B}_x \times \mathbb{R}_+} f(v-w, b-a)g(w,0,a)\,\mathrm{d}w\,da,$$

$$\eta_t(K)g(\gamma,t,b) = t^{-n+1}\int_{[0,\frac{a_t}{t}]}\int_{\mathcal{G}_{s(\gamma)}} K(\mathrm{Exp}_\partial^{-1}(\gamma\eta^{-1}t^{-1}, b, a)g(\gamma\eta^{-1}, t, a)\,\mathrm{d}\mu_{s(\gamma)}(\eta)\,\mathrm{d}a.$$

$$\eta_0(K)g(v,0,b) = \int_{\mathcal{B}_x \times \mathbb{R}_+} K(v-w, b, a)g(w,0,a)\,\mathrm{d}w\,\mathrm{d}a.$$

Denote by P_t the operator given by multiplication with the characteristic function $\mathcal{H} \times [0,a_t]^2$, where $[0,a_t]^2$ is the pair groupoid. Also denote by D_t the dilation by t operator. Then we have $\|P_t\pi_t(f)P_t\| = \|D_tP_t\pi_t(f)P_tD_{t^{-1}}\| = \sup\{\|\eta_t(f)g\|_{\infty,t} : \|g\|_\infty \leq 1\}$. Hence $\|\pi_0^\partial(f \oplus K)\| = \sup\{\|\eta_0(f \oplus K)g\|_{\infty,0} : \|g\|_\infty \leq 1\}$. Let $g \in C_c^\infty(\widetilde{\mathcal{G}}^{ad})$ and note that for t small we can without loss of generality assume that g takes the form $g_0\left(-\frac{\mathrm{Exp}_\partial(\gamma\eta^{-1})}{t}, t, a\right)$ with $g_0 \in C_c^\infty(\mathcal{B} \times [0,1] \times \overline{\mathbb{R}}_+)$. Thence

$$\lim_{t\to 0}\|(\eta_t(\tilde{f} + \tilde{K})g\|_{\infty,t} = \|(\eta_0(f \oplus K)g\|_{\infty,0}$$

which implies (6). From (5) an (6) we obtain the lower semi-continuity (l). The upper semi-continuity is the inequality

$$(usc) \qquad \limsup_{t\to 0}\|\varphi_t(a)\| \leq \max\{\|\pi_0(a)\|, \|\pi_0^\partial(a)\|\}.$$

This follows by the density result of [4, Proposition 2.22]. The remainder of the argument is analogous to loc. cit. and we omit the details. The estimates (lsc) and (usc) together imply the continuity of the field of C^*-algebras. □

Functional calculus

On a given decomposed Lie manifold we define a functional calculus taking values in the reduced C^*-algebra of the deformation semi-groupoid considered in the previous section. Denote by $\mathcal{P}$ the set of functions in the Schwartz class $\mathbf{S}(\mathbb{R})$ which have compactly supported Fourier transform.

Theorem 4.9. *Let $(M, \mathcal{A}, \mathcal{V})$ be a decomposed Lie manifold with hypersurface $(Y, \mathcal{B}, \mathcal{W})$ with corresponding integrating groupoid $\mathcal{G} \rightrightarrows M$. Denote by $\mathbb{D} := (\not{D}_{x,t})_{(x,t)\in M\times I}$ an equivariant family of geometric Dirac operators associated to $\not{\nabla}^W$ on $\mathcal{G}^{ad}$. Then there exists a ring homomorphism*

$$\Psi_{\mathbb{D}} \colon C_0(\mathbb{R}) \to C_r^*(\mathcal{G}^{ad})$$

which is compatible with the representations (π, π^∂), *i.e. given* $\pi = (\pi_t)_{t \in I}$ *and* π^∂ *we have*

$$\pi_{x,t}(\Psi_{\mathbb{D}}(f)) = f(\not{D}_{x,t}),\ f \in \mathcal{P}, \tag{7}$$

as well as for $\pi_0^\partial \colon C_c^\infty(\widetilde{\mathcal{A}}) \to \mathcal{L}(L^2(\mathcal{G}_{|Y}))$

$$\pi_0^\partial(f)\xi(\gamma) = \int_{\widetilde{\mathcal{A}}_{r(\gamma)}} f(w)\psi(w)\xi(\mathrm{Exp}^{-1}(\gamma) - w)\,\mathrm{d}w \tag{8}$$

with

$$\pi_0^\partial(\Psi_{\mathbb{D}}(f)) = f(\not{D}_0),\ f \in \mathcal{P}.$$

Proof. Denote by $e^{i\tau \not{D}_{x,t}}$ the solution operator to the wave equation for $\not{D}_{x,t}$. For $f \in \mathcal{P}$ define via the functional calculus (cf. [11, Section 3.C])

$$f(\not{D}_{x,t}) = \frac{1}{2\pi}\int \hat{f}(\tau)e^{i\tau\not{D}_{x,t}}\,\mathrm{d}\tau.$$

By the estimates in the proof of Proposition 7.20 in [31], we obtain that $f(\not{D}_{x,t})$ is a smoothing operator of finite propagation speed. The family $f(\not{D}_{x,t})$ is $\mathcal{G}^{ad}$-equivariant. Take the reduced kernel k^f over $\mathcal{G}^{ad}$. By finite propagation speed and the equivariance it follows that k^f is compactly supported, see also [30]. Define $\Psi_{\mathbb{D}}(f)$ via the assignment $\gamma \mapsto k^f_{s(\gamma)}$. The latter assignment furnishes a ring homomorphism, cf. the proof of [31, Prop. 9.20]. The compatibility (7) follows since k^f is a reduced convolution kernel and (8) follows by the definition of $\not{D}_0$ on $\widetilde{\mathcal{A}}$. We obtain the L^2-action of $\Psi_{\mathbb{D}}$:

$$\begin{aligned} f(\not{D}_{x,t})g(\gamma) &= \pi_{x,t}(\Psi_{\mathbb{D}}(f))g(\gamma) \\ &= (\Psi_{\mathbb{D}}(f) * g)(\gamma),\ t > 0 \end{aligned}$$

as well as

$$\begin{aligned} f(\not{D}_0)g(v) &= \pi_0^\partial(\Psi_{\mathbb{D}}(f))g(v) \\ &= (\Psi_{\mathbb{D}}(f) * g)(v),\ t = 0. \end{aligned}$$

Altogether we have shown that $\Psi_{\mathbb{D}} \colon \mathcal{P} \to C_c^\infty(\mathcal{G}^{ad})$ is a ring homomorphism that is compatible with (π, π_0^∂). Since $\mathcal{P}$ is dense in $C_0(\mathbb{R})$ and $C_r^*(\widetilde{\mathcal{G}}^{ad})$ is defined as the completion with regard to the representations (π, π_0^∂), we obtain by the L^2-spectral theorem that the map $\mathcal{P} \to C_c^\infty(\mathcal{G}^{ad})$ is continuous with regard to the $C_0(\mathbb{R})$-norm. Hence $\Psi_{\mathbb{D}}$ extends continuously to a ring homomorphism $C_0(\mathbb{R}) \to C_r^*(\mathcal{G}^{ad})$. □

5. INDEX FORMULA

We give the proof of a generalized APS-type index formula on a decomposed spin Lie manifold for a geometric Dirac operator subject to local and non-local boundary conditions. The local boundary conditions are posed on a stratum of the cylinder part of the decomposed Lie manifold. On the other hand, we have non-local APS boundary conditions which are implicit on the complementary part of the manifold, where the boundary is pushed to infinity. The technique we employ to prove the index formula makes use of the theory outlined in the previous sections: The functional calculus from section 4 is used, combined with the rescaling bundle technique from [10], in order to derive an index formula on standard spin Lie manifolds.

Consider a decomposed Lie manifold $(M, \mathcal{A}, \mathcal{V})$ with hypersurface $(Y, \mathcal{B}, \mathcal{W})$, i.e. $M = M_1 \cup M_2$ where M_2 is of cylinder type. We study a geometric admissible Dirac operator defined on M. The operator D is assumed to be decomposed into two geometric admissible Dirac operators, i.e. $D_{|M_1} = D_1$, $D_{|M_2} = D_2$. The operator D_2 is of model type and admits second order local boundary conditions, while D_1 is a graded geometric admissible Dirac operator on the complement M_1. Fix the notation $\widetilde{\mathcal{A}} \to M$ for the semi-Lie algebroid as introduced in the previous section. With the help of the functional calculus introduced in the previous section and the rescaling argument from [10], we can separate the index calculation for the two cases of D_1 and D_2.

Renormalizable Lie manifolds

Similarly, as in [10] we define a renormalized trace for renormalizable Lie manifolds. Denote by $\Omega^1(\mathcal{A})$ the 1-forms on $\mathcal{A}$ and by $\dot{C}^\infty(M, \Omega^1(\mathcal{A}))$ the smooth sections vanishing to all orders at the boundary strata of M.

Definition 5.1. A Lie manifold $(M, \mathcal{V}, \mathcal{A})$ is called *renormalizable* if there is a functional ${}^{\mathcal{V}}\mathrm{Tr}\colon C^\infty(M, \Omega^1(\mathcal{A})) \to \mathbb{C}$, $f \mapsto {}^{\mathcal{V}}\!\!\fint f$ with the following properties:

1) The integral $\int_M f$ exists for $f \in \dot{C}^\infty(M, \Omega^1(\mathcal{A}))$, and the functional ${}^{\mathcal{V}}\mathrm{Tr}$ is a linear extension.

2) There is a minimal $k \in \mathbb{R}$ such that $G(f)(z) = \int_M \rho^z f$ defines a function $G(f)$ holomorphic on $\Re(z) > k - 1$, which extends meromorphically to $\mathbb{C}$.

Then we can define

$$
{}^{\mathcal{V}}\!\!\fint_M f : \text{ regularized value (zero order Taylor coefficient) at } z = 0 \text{ of } G(f).
$$

Lemma 5.2. *Let $(M, \mathcal{A}, \mathcal{V})$ be a renormalizable Lie manifold with boundary $(Y, \mathcal{B}, \mathcal{W})$. Then $(Y, \mathcal{B}, \mathcal{W})$ is renormalizable as well.*

Proof. We construct the functional $H\colon C^\infty(Y, \Omega^1(\mathcal{B})) \to \mathbb{C},\ f \mapsto {}^{\mathcal{W}}\!\!\fint_Y f$ as a linear extension. Let ρ_Y denote the boundary defining function of Y, i.e.

$$\rho_Y := \prod_{F \in \mathcal{F}_1(Y)} \rho_F.$$

By Theorem 2.7 there is a global tubular neighborhood $Y \hookrightarrow \mathcal{U} \hookrightarrow M$ such that the boundary defining function ρ_Y of Y is expressed in the coordinates of $\mathcal{U}$, i.e.

$$(\rho_Y \circ \nu)(x_1, x') = x_1,\ (x_1, x') \in Y_{(\epsilon)}$$

where $\nu\colon Y_{(\epsilon)} = Y \times (-\epsilon, \epsilon) \xrightarrow{\sim} \mathcal{U}$. Fix the local coordinates over $\mathcal{U}$ by $x' = (x_2, \cdots, x_{n-1})$, then $\rho = x_2 x_3 \cdots x_{n-1}$. Since the degeneracy index k of M is finite, there is $l \geq k$ such that

$$H(f)(z) = \int_Y \rho^z f$$

is holomorphic in $\{\Re(z) \geq l - 1\}$ and extends meromorphically to $\mathbb{C}$. Then ${}^{\mathcal{W}}\!\!\fint_Y f = \mathrm{Reg}_{z=0} H(f)(z)$ is defined and finite. Hence $(Y, \mathcal{B}, \mathcal{W})$ is renormalizable. □

Example 5.3. We refer to [10, Section 5] for a large class of examples of renormalizable Lie manifolds where it is shown that so-called *exact* Lie manifolds are renormalizable. Here a Lie structure $\mathcal{V}$ of a Lie manifold $(M, \mathcal{V}, \mathcal{A})$ is called *exact* if, near each face with boundary defining function x_1, the Lie structure is generated by vector fields of the form $\left\{x_1^{k_1}\partial_{x_1}, \ldots, x_1^{k_n}\partial_{x_n}\right\}$ for arbitrary $\{k_l :\ 1 \leq l \leq n\}$.

Cylinder type index formula

Fix the geometric admissible Dirac operator D on a Lie manifold $(M, \mathcal{A}, \mathcal{V})$ of cylinder type with boundary $(Y, \mathcal{B}, \mathcal{W})$. The Dirac operator $D = D^W$ is defined via a Clifford module $W \in \mathrm{Cl}(\widetilde{\mathcal{A}}) - \mathrm{mod}$. After an application of Lemma 2.10 without loss of generality D is assumed to be of model type, i.e.

$$D = \begin{pmatrix} i\partial_u & iD_\partial^- \\ -iD_\partial^+ & -i\partial_u \end{pmatrix} \tag{9}$$

where D_∂ is an admissible geometric Dirac operator on $(Y, \mathcal{B}, \mathcal{W})$, i.e.

$$D_\partial = \begin{pmatrix} 0 & iD_\partial^- \\ -iD_\partial^+ & 0 \end{pmatrix}$$

for a fixed admissible connection ∇^{W_∂} and $W_\partial \in \mathrm{Cl}(\mathcal{B})$ – mod. Note that $D^*D = -\partial_u^2 + D_\partial^2$. Let $\varphi \in \Gamma(W)$ be a smooth section such that $\varphi = \varphi^+ \oplus \varphi^-$ corresponding to the grading $W = W^+ \oplus W^-$. Denote by $B^\pm$ the local boundary condition $(\varphi_{|Y})^\pm = 0$. Note that $(D, B^+)^* = (D, B^-)$. We study the *induced boundary conditions* for *(a)* the case D^*D, where $\varphi^+(0, y) = 0$ and $(\partial_u \varphi^- + D_\partial^+ \varphi^+)_{u=0} = 0$. Since D_∂^+ is a tangential operator it follows that $D_\partial^+ \varphi^+ = 0$. Therefore, in case *(a)* the local boundary conditions are subdivided into Dirichlet and Neumann condition

$$(Da) \qquad \varphi^+(0, y) = 0,$$

$$(Na) \qquad \partial_u \varphi^-(0, y) = 0.$$

The induced boundary conditions in the case *(b)* of DD^* are for $\psi \in \Gamma^\infty(W)$ given by

$$(Nb) \qquad \partial_u \psi^+(0, y) = 0,$$

$$(Db) \qquad \psi^-(0, y) = 0.$$

Definition 5.4. A *model type* Dirac operator is an operator of the form (9) with second order boundary conditions $B^\pm$ as specified above, such that the generalized ellipticity (Fredholm) conditions for boundary value problems on Lie manifolds with boundary, as stated in [7, Def. 9.4], are fulfilled.

Define the *renormalized index* as defined via the renormalized super trace

$$^{\mathcal{V}}\mathrm{ind}(D) = \lim_{t \to \infty} {}^{\mathcal{V}}\mathrm{Tr}_s(e^{-tD^*D}) - {}^{\mathcal{V}}\mathrm{Tr}_s(e^{-tDD^*}).$$

We have the following result concerning the renormalized index of a model form Dirac operator.

THEOREM 5.5. *Let* $(M, \mathcal{A}, \mathcal{V})$ *be a renormalizable, spin Lie manifold of cylinder type with boundary* $(Y, \mathcal{B}, \mathcal{W})$ *and let* D *be an admissible graded Dirac operator of model type* (9). *Then the renormalized index of* D*, subject to boundary condition* $B^\pm$*, is*

$$^{\mathcal{V}}\mathrm{ind}(D) = \lim_{t \to \infty} {}^{\mathcal{V}}\mathrm{Tr}_s(e^{-tD^2}) = \mp \frac{1}{2} {}^{\mathcal{W}}\mathrm{ind}(D_\partial).$$

Proof. By full ellipticity, i.e. Theorem 5.8, the renormalized indices agree with the Fredholm index. Set $K_\pm(t, u) := {}^{\mathcal{V}}\mathrm{Tr}_s(e^{-tD^*D} - e^{-tDD^*})$ for the density with regard to the boundary condition $B^\pm$. Fix the volume forms μ on M and the induced volume form μ_∂ on Y, by Proposition 2.4. Write $K_+(t, u) = K_1(t, u) - K_2(t, u)$, then we have

$$K_\pm(t, u) := {}^{\mathcal{V}}\mathrm{Tr}(e^{-tD^*D} - e^{-tDD^*}) = K_1^\pm(t, u) - K_2^\pm(t, u)$$

$$= {}^{\mathcal{W}}\!\!\!\!\!\!\!\int_Y \int_{\overline{\mathbb{R}}_+} \operatorname{tr} \tilde{K}_1(t,v,v,y,y)\,\mathrm{d}v\,\mathrm{d}\mu_\partial(y) - {}^{\mathcal{W}}\!\!\!\!\!\!\!\int_Y \int_{\overline{\mathbb{R}}_+} \operatorname{tr} \tilde{K}_2(t,v,v,y,y)\,\mathrm{d}v\,\mathrm{d}\mu_\partial(y).$$

We next calculate the kernels with regard to (Da), (Na) separately. We make use of the calculations in [21], see also [34] to obtain the following formulae. Case (Da) and (Db):

$$\tilde{K}_1(t,v,w,y,z) = \frac{1}{\sqrt{4\pi t}} \left\{ \exp\left(\frac{(v-w)^2}{4t} - \exp\left(-\frac{(v+w)^2}{4t} \right) \right) \right\} \times e^{-tD_\partial^+ D_\partial^-}(t,y,z).$$

$$\tilde{K}_2(t,v,w,y,z) = \frac{1}{\sqrt{4\pi t}} \left\{ \exp\left(-\frac{(v-w)^2}{4t} \right) - \exp\left(-\frac{(v+w)^2}{4t} \right) \right\} \times e^{-tD_\partial^- D_\partial^+}(t,y,z).$$

Case (Na) and (Nb):

$$\tilde{K}_1(t,v,w,y,z) = \frac{1}{\sqrt{4\pi t}} \left\{ \exp\left(-\frac{(v-w)^2}{4t} \right) + \exp\left(-\frac{(v+w)^2}{4t} \right) \right\} \times e^{-tD_\partial^+ D_\partial^-}(t,y,z),$$

$$\tilde{K}_2(t,v,w,y,z) = \frac{1}{\sqrt{4\pi t}} \left\{ \exp\left(-\frac{(v-w)^2}{4t} \right) + \exp\left(-\frac{(v+w)^2}{4t} \right) \right\} \times e^{-tD_\partial^- D_\partial^+}(t,y,z).$$

With the help of these explicit formulae we obtain the trace density $K_1(t,u)$ of e^{-tD^*D} subject to B^+ equals

$${}^{\mathcal{W}}\!\!\!\!\!\!\!\int_Y \int_{\overline{\mathbb{R}}_+} \operatorname{tr} K_1(t,v,w,y,y)\,\mathrm{d}v\,\mathrm{d}\mu_\partial(y) = \frac{1}{\sqrt{4\pi t}} {}^{\mathcal{W}}\mathrm{Tr}(e^{-tD_\partial^- D_\partial^+}) \left\{ 1 - \exp\left(-\frac{u^2}{t} \right) \right\} + \frac{1}{\sqrt{4\pi t}} {}^{\mathcal{W}}\mathrm{Tr}(e^{-tD_\partial^+ D_\partial^-}) \left\{ 1 + \exp\left(-\frac{u^2}{t} \right) \right\}.$$

Similarly, for $K_2(t,u)$, the density of e^{-tDD^*}. Altogether we obtain

$$K_+(t,u) = \frac{e^{-\frac{u^2}{t}}}{\sqrt{\pi t}} \left({}^{\mathcal{W}}\mathrm{Tr}(e^{-tD_\partial^+ D_\partial^-}) - {}^{\mathcal{W}}\mathrm{Tr}(e^{-tD_\partial^- D_\partial^+}) \right).$$

By an application of the McKean-Singer formula (see also [21]) we have

$$\int_0^\infty K_+(t,u)\,\mathrm{d}u = -\frac{1}{2} {}^{\mathcal{W}}\mathrm{ind}(D_\partial).$$

For the case B^- we obtained

$$\int_0^\infty K_-(t,u)\,\mathrm{d}u = \frac{1}{2} {}^{\mathcal{W}}\mathrm{ind}(D_\partial)$$

by an analogous calculation. □

Rescaling for semi-groupoids

For any decomposed Lie manifold our aim is to find a topological interpretation of the renormalized index. We prove a generalized APS-type index formula. The formula for the class of non-compact manifolds involves a local contribution, depending on the metric and a non-local contribution which also depends on restrictions of the Dirac operator to the boundary (the so-called indicial symbol). The formula without the generalized boundary conditions holds a priori for any Dirac operator which is not required to be Fredholm and the renormalized index may take non-integer values. In the case with boundary conditions, as specified above, we make use of the assumption that the boundary value problem is generalized elliptic, i.e. Atiyah-Bott-Shapiro-Lopatanskii elliptic in the sense of [7]. We also state criteria for the equality of the Fredholm index with the renormalized index as well as the conditions for the Dirac operator to be Fredholm. The main new feature of our formula over the special case investigated already in [10] is that it extends to the case of local boundary conditions in a singular setting. For a given renormalizable Lie manifold $(M, \mathcal{A}, \mathcal{V})$ and graded Dirac operator D we fix the definition of the renormalized η-invariant ${}^{\mathcal{V}}\eta(D) := \frac{1}{2}\int_0^\infty {}^{\mathcal{V}}\mathrm{Tr}_s([D, De^{-tD^2}])\,dt$.

The index of a geometric Dirac operator on a decomposed Lie manifold is determined by two principal symbols and the corresponding boundary restrictions of the singular strata.

Lemma 5.6. *Given a boundary value problem D of the type $B^{\pm}$ with conditions as stated in the next Theorem. Then the Fredholm index splits:*

$$\mathrm{ind}(D) = f(D_1) + g(D_\partial), \tag{10}$$

where f and g are homotopy invariant functionals of the full symbol of D_1 and D_∂ respectively.

Proof. The first part is the boundary symbol of the model operator. The latter operator by assumption fulfills a generalized Shapiro-Lopatinskii-Atiyah-Bott condition. Secondly, the operator on the cylinder part of the decomposed Lie manifold has a boundary symmetry property. The proof goes along the same lines as the proof of [25, Theorem 2.3], except that we make use of the deformation groupoid $\mathcal{G}$ as constructed in section 3. Set $\mathcal{G}_1 = \mathcal{G}(\mathring{M}_1)$ and $\mathcal{G}_2 = \mathcal{H} \times (]-1,0)^2 \cup \{0\} \times \mathbb{R}_+)$. Denote by $e_0 \colon C^*(\mathcal{G}) \to C^*(\mathcal{H} \times \mathbb{R}_+)$, induced by restriction of $\mathcal{G}$ to the saturated subgroupoid $\mathcal{H} \times \mathbb{R}_+$. Note that the kernel of e_0 is contractible and thereby trivial in K-theory. Denote by e_1 the restriction homomorphism $C^*(\mathcal{G}) \to C^*(\mathcal{G}_1)$. We obtain the pushforward:

$$\pi_! := (e_1)_* \circ (e_0)_*^{-1} \colon K_0(C^*(\mathcal{H} \times \mathbb{R}_+)) \to K_0(C^*(\mathcal{G}_1)).$$

Consider two boundary value problems $(D, B^{\pm})$, $(\widetilde{D}, \widetilde{B}^{\pm})$ of the type under consideration such that the principal symbols are homotopic and the boundary symbols are equal. Setting

$$\operatorname{ind}(A) = \operatorname{ind}(\widetilde{D}) - \operatorname{ind}(D)$$

where $(A, C^{\pm})$ is a boundary value problem contained in the Boutet de Monvel calculus of [7]. Denote by σ the principal symbol and by σ_∂ the boundary symbol as specified in [7]. In particular, the principal symbol fulfills $\sigma(A) = \widetilde{\sigma} \cdot \sigma^{-1}$ and the boundary symbol is unitary. Using surgery, similar to the constructions in Section 3, we obtain that $\operatorname{ind}(A)$ is equal to the index of a fully elliptic pseudodifferential operator with principal symbol $\sigma_{\partial t} \cdot \sigma_{\partial 0}^{-1} = \sigma$ on $Y \times S^1$. By [9, Theorem 5.1], the latter equals $\pi_![\sigma_\partial(A)]$. This term is the Atiyah-Patodi-Shapiro-Lopatinskii obstruction. Since the generalized ellipticity condition holds by assumption, the obstruction vanishes. This concludes the proof. □

THEOREM 5.7. *Let $(M, \mathcal{A}, \mathcal{V})$ be a decomposed renormalizable spin Lie manifold with hypersurface $(Y, \mathcal{B}, \mathcal{W})$ such that $M = M_1 \cup_Y M_2$. Denote by $D = D^W$ a geometric Dirac operator for an admissible $\widetilde{\mathcal{A}}$-connection $\tilde{\nabla}^W$ with $D_{|M_1} = D_1$, $D_{|M_2} = D_2$, where D_2 is of model type on the cylinder M_2 and D_1 is a geometric Dirac operator over M_1. Then subject to the boundary condition $B^{\pm}$ fulfilling the generalized Shapiro-Lopatinskii-Atiyah-Bott condition we have*

$$\begin{aligned} {}^{\mathcal{V}}\operatorname{ind}(D) &= {}^{\mathcal{V}_1}\!\!\!\!\!\!\!\int_{M_1} \hat{A}(\nabla_1) \wedge \exp F^{W_2/S}\, \mathrm{d}\mu + {}^{\mathcal{V}_1}\eta(D_1) \\ &\mp \frac{1}{2}\, {}^{\mathcal{W}}\!\!\!\!\!\!\!\int_{Y} \hat{A}(\nabla_\partial) \wedge \exp F^{W_\partial/S}\, \mathrm{d}\nu_\partial + {}^{\mathcal{W}}\eta(D_\partial). \end{aligned} \tag{11}$$

Proof. According to Lemma 5.6 we can calculate the indices of D_1 and D_∂ separately. Let $W \to M$ be the $\mathrm{Cl}(\widetilde{\mathcal{A}})$ module compatible with the Clifford action. Denote by $\hom(W) \to M$ the bundle with fibers $\hom(W)_x \cong \hom(W_x, W_x) \cong \mathrm{Cl}(\widetilde{\mathcal{A}}_x \otimes \mathbb{C}) \otimes \mathrm{End}_{\mathrm{Cl}}(W_x)$ and by $\mathrm{Hom}(W) \to M$ the homomorphism bundle. We have $\mathrm{Hom}(W)_{|\widetilde{\mathcal{A}}} \cong \mathrm{Cl}(\widetilde{\mathcal{A}} \otimes \mathbb{C}) \otimes \mathrm{End}_{Cl}(W)$. Fix the Lie groupoid $\mathcal{G} \rightrightarrows M$ with $\mathcal{A}(\mathcal{G}) \cong \mathcal{A}$ as given in Theorem 3.2. We can lift $\mathrm{Hom}(W)$ to a bundle over $\widetilde{\mathcal{G}}^{ad}$ using the range and source map, i.e. by forming the pullback bundle $r^*(\mathrm{Hom}(W)) \otimes s^*(\mathrm{Hom}(W)^*)$. Since no confusion will arise we denote this lifted bundle by the same symbol $\mathrm{Hom}(W) \to \widetilde{\mathcal{G}}^{ad}$. Consider the geometric Dirac operator $\not{\mathbb{D}}$ on $\mathcal{G}$ such that $\varrho(\not{\mathbb{D}}) = D$ (cf. [19]). Then the reduced heat kernel k_t on $\widetilde{\mathcal{G}}^{ad}$ is well-defined and we have the asymptotic expansion of the pointwise trace of k_t at $x \in M$ (cf. [33]):

$$\mathrm{tr}_x e^{-t\not{D}^2} \sim (4\pi)^{-\frac{n}{2}} \sum_{i=0}^{\infty} a_i(x) t^{\frac{n}{2}-i}, \ t \to 0^+, \ x \in M \subset \mathcal{G}$$

for $a_i \in \Gamma^\infty(\mathrm{Cl}(\widetilde{\mathcal{A}}^*) \otimes \mathrm{End}_{\mathrm{Cl}(\widetilde{\mathcal{A}}^*)}(W))$. We have the filtration by Clifford degree $\mathrm{Cl}_0 \subseteq \mathrm{Cl}_1 \subseteq \cdots \subseteq \mathrm{Cl}_n(\widetilde{\mathcal{A}} \otimes \mathbb{C})$. Topologically $\widetilde{\mathcal{G}}^{ad}$ is a manifold with corners in its own right and we can view $\widetilde{\mathcal{A}}$ as a particular boundary stratum. By restricting the generalized exponential mapping $\mathrm{Exp}\colon \mathcal{A} \to \mathcal{G}$ to the half space $\widetilde{\mathcal{A}}$ we obtain a tubular neighborhood and a normal direction (cf. [18]). Denote by $N = \partial_t$ the corresponding normal vector field in $\widetilde{\mathcal{G}}^{ad}$. Define

$$^{\mathcal{V}}\mathcal{D} := \{u \in C_c^\infty(\widetilde{\mathcal{G}}^{ad}, \mathrm{Hom}(W)) : \nabla_N^p u_{|\widetilde{\mathcal{A}}} \in C^\infty(\widetilde{\mathcal{A}}, \mathrm{Cl}_{n-p} \otimes \mathrm{End}_{\mathrm{Cl}}(W)), 0 \le p \le n\}.$$

By a proof analogous to [10, Proposition 6.4], we obtain that the filtration $\{Cl_j\}$ can be extended by parallel transport along ∇_N to a neighborhood of $\widetilde{\mathcal{A}}$ inside $\widetilde{\mathcal{G}}^{ad}$. Denote by $\{\tilde{\mathrm{Cl}}_j\}$ the extended filtration. Then we have the alternative description

$$^{\mathcal{V}}\mathcal{D} = \{u \in C_c^\infty(\widetilde{\mathcal{G}}^{ad}, \mathrm{Hom}(W)) : u = \sum_{j=0}^{n} t^{n-j} u_j + t^{n+1} u', \ \text{near } \widetilde{\mathcal{A}}\}$$

where $u_j \in C_c^\infty(\widetilde{\mathcal{G}}^{ad}, \tilde{\mathrm{Cl}}_{n-j} \otimes \mathrm{End}_{\mathrm{Cl}}(W))$ and $u' \in C_c^\infty(\widetilde{\mathcal{G}}^{ad}, \mathrm{Hom}(W))$. By the Serre-Swan theorem there is a *rescaling bundle* $\mathbb{E} \to \widetilde{\mathcal{G}}^{ad}$ such that $C_c^\infty(\widetilde{\mathcal{G}}^{ad}, \mathbb{E}) = i_{\mathrm{Cl}}^* \mathcal{D}$. Here $i_{\mathrm{Cl}}\colon \mathbb{E} \to \mathrm{Hom}(W)$ is a bundle map which is an isomorphism over the interior $\mathcal{G}_{(0,1]}^{ad}$. It is not hard to check that we obtain a canonical isomorphism of Clifford algebras $\mathbb{E}_{\widetilde{\mathcal{A}}} \cong \Lambda\widetilde{\mathcal{A}}^* \otimes \mathrm{End}_{\mathrm{Cl}}(W)$. The structure of $\mathbb{E}$ will make sure that we extract the correct coefficient in the formal heat kernel expansion. Set $\mathbb{D} := (t\not{D}_x)_{(x,t)\in M\times I}$ for the family of Dirac operators over $\widetilde{\mathcal{G}}^{ad}$. Consider $f(x) = e^{-x^2}$ and assume for technical reasons that f is convolved with a function λ which has Fourier transform with large compact support. Denote this convolution by $\tilde{f} := f * \lambda$. By the definition of the functional calculus in Theorem 4.9 we obtain that $\Psi_{\mathbb{D}}(\tilde{f}) = t^n k_{t^2}$ as an element of $C_r^*(\widetilde{\mathcal{G}}^{ad})$. This follows from the action of the functional calculus $f(\not{D}_{x,t})g(\gamma) = \pi_{x,t}(\Psi_{\mathbb{D}}(f) * g)(\gamma)$ for $t > 0$ which yields

$$f(t\not{D})g(\gamma) = \int_{\mathcal{G}_{s(\gamma)}} \Psi_{\mathbb{D}}(f)(\gamma\eta^{-1}) g(\eta) t^{-n} \, \mathrm{d}\mu_{s(\gamma)}(\eta).$$

The scaling factor t^{-n} enters by a choice of Haar system as in [18, (6.8)]. Set $l_t := \Psi_{\mathbb{D}}(f)_{|\mathcal{G}_\Delta}$, then $l_t(\gamma) = t^n k_{t^2}(\gamma)$ for $t \neq 0$. Define the diagonal

$\mathcal{G}_\Delta := \{\gamma \in \widetilde{\mathcal{G}}^{ad} : s(\gamma) = r(\gamma)\} \subset \widetilde{\mathcal{G}}^{ad}$. For $t \neq 0$ the supertrace functional maps $\mathrm{tr}_s \colon C_c^\infty(\mathcal{G}_\Delta, \mathbb{E}_\Delta) \to t^n C_c^\infty(\mathcal{G})$, cf. [31, Proposition 11.4]. Therefore, $t^{-n}\mathrm{tr}_s(l_t)$ extends smoothly to $t = 0$ and hence $t^{-n}\mathrm{tr}_s(l_t) = \mathrm{tr}_s(l_0) + o(t)$. We are therefore, reduced to calculate $\mathrm{tr}_s(l_0)$. Note that since l_0 lives on the semi-algebroid $\widetilde{\mathcal{A}}$ we have to be careful to incorporate the boundary conditions posed on Y. First note that $\widetilde{\mathcal{A}}(\mathcal{G})_{|M_2} \cong \widetilde{\mathcal{A}}(\mathcal{G}_2)$ is the semi-algebroid corresponding to the cylinder type manifold M_2. On the other hand, $\widetilde{\mathcal{A}}(\mathcal{G})_{|M_1} \cong \mathcal{A}(\mathcal{G}_1)$ is the Lie algebroid corresponding to the Lie manifold (without boundary) M_1. In the second case the calculation goes analogous to the proof given in [10]. We repeat the key steps of the argument for completeness before dealing with the first case. Denote by $\Phi \colon U^{ad} \times V \xrightarrow{\sim} \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$ a diffeomorphism where we write $U^{ad} = U \times \mathbb{R}$. The coordinates induced by Φ should form a *parametrization* of the adiabatic groupoid of $\mathcal{G}_1$, cf. [18] and [28, p.145]. Let $\alpha_x = \alpha_{\mathcal{A}_x(\mathcal{G}_2)\cap V}$ denote the restriction of the generalized exponential map Exp_x on the fiber $\mathcal{G}_x$. If V is chosen sufficiently small we can fix a local geodesic coordinate system $\alpha_x(\gamma) = (a_1, \cdots, a_m) =: a$. Let $\Phi_{x,t}$ be the restriction of Φ to $V \times \{x\} \times \{t\}$. An elementary calculation yields $\Phi_{x,t}(\eta) = \frac{1}{t}(\alpha_x(\eta) - a)$. Applying the Lichnerowicz formula on the complete Riemannian manifold $(\mathcal{G}_x, g_x)$ and taking the limit as $t \to 0$ we obtain (cf. [10])

$$\not{D}_{x,0}^2 = -\sum_i \left(\partial_i^x + \frac{1}{4}\sum_j R_{ij}^x a_j\right)^2 + \sum_{i<j} F^{W_x/S}(e_i, e_j)(a_j)(a_j).$$

The differential equation of the heat kernel of $\not{D}_{x,0}^2$ is a harmonic oscillator. Applying [6] we have the solution in terms of a formal power series in the scalar curvature R_{ij}^x and the exponential of the twisting bundle $\exp F^{W_x/S}$. By the $\mathcal{G}$-invariance of the curvature tensor as well as the twisting curvature and the Lichnerowicz theorem for Lie manifolds given in [10, Theorem 2.4], it follows from [6, p. 164] and [31, Proposition 12.25, 12.26] the integrand $\hat{A}(\nabla) \wedge \exp F^{S/W}$ in the trace formula. Thus, we have shown that

$$\lim_{t\to 0^+} {}^{\mathcal{V}}\mathrm{Tr}_\mathrm{s}(e^{-tD_1^2}) = {}^{\mathcal{V}_1}\!\!\int_{M_1} \hat{A}(\nabla_1) \wedge \exp F^{W/S}\,\mathrm{d}\mu.$$

To obtain the limit $t \to \infty$ we notice that

$$\lim_{t\to\infty} {}^{\mathcal{V}}\mathrm{Tr}_\mathrm{s}(e^{-tD_1^2}) - \lim_{t\to 0^+} {}^{\mathcal{V}}\mathrm{Tr}_\mathrm{s}(e^{-tD_1^2}) = \int_0^\infty \partial_t {}^{\mathcal{V}}\mathrm{Tr}_\mathrm{s}(e^{-tD_1^2})\,\mathrm{d}t.$$

Observe that $\partial_t {}^{\mathcal{V}}\mathrm{Tr}_\mathrm{s}(e^{-tD_1^2}) = {}^{\mathcal{V}}\mathrm{Tr}_\mathrm{s}(\partial_t e^{-tD_1^2})$. Setting

$${}^{\mathcal{V}_1}\eta(D_1) := \frac{1}{2}\int_0^\infty {}^{\mathcal{V}}\mathrm{Tr}_\mathrm{s}(D_1^2 e^{-tD_1^2})\,\mathrm{d}t$$

this completes the proof of the first part of the index formula. Now consider the problem on the cylinder type Lie manifold $(M_2, \mathcal{A}_2, \mathcal{V}_2)$. Here we apply the reduction given in Theorem 5.5. This effectively reduces the problem to the calculation of ${}^{\mathcal{W}}\mathrm{ind}(D_\partial)$. The operator D_∂ can be viewed as an odd graded geometric Dirac operator on the Lie manifold $(Y, \mathcal{B}, \mathcal{W})$. Hence we can use the above rescaling approach for the integrating groupoid $\mathcal{H} \rightrightarrows Y$. With the same analysis as above, applied to the groupoid $\mathcal{H}$, we obtain the index formula also on the cylinder part of the decomposed Lie manifold. □

If we impose additional conditions on the integrating groupoid, we can show that the renormalized index of Fredholm operators equals the Fredholm index. In such a case the previous index theorem yields an actual generalization of Atiyah-Singer index theory. For a given geometric admissible Dirac operator $D = D^W$ on a Lie manifold $(M, \mathcal{A}, \mathcal{V})$ we denote by $\mathcal{R}(D) := \oplus_F \mathcal{R}_F(D)$ where the direct sum ranges over all codimension one singular hyperfaces of M and $\mathcal{R}_F(D)$ denotes the restriction of D to the stratum F. The *indicial symbol* is more easily understood in the context of groupoids. If $\not{D}$ is the corresponding Dirac operator on an integrating Lie groupoid $\mathcal{G} \rightrightarrows M$, then $\mathcal{R}_F(\not{D}) := (\not{D}_x)_{x \in F}$. The groupoid $\mathcal{G} \rightrightarrows M$ is called *strongly amenable* if the natural action $C_r^*(\mathcal{G}) \hookrightarrow \mathcal{L}(\mathcal{H})$ on the Hilbert space $\mathcal{H} := L^2(M_0)$ is injective.

THEOREM 5.8. *Let $(M, \mathcal{A}, \mathcal{V})$ be a renormalizable Lie manifold for which there is an integrating Lie groupoid $\mathcal{G} \rightrightarrows M$ such that $\mathcal{G}$ is of polynomial growth, Hausdorff and strongly amenable. Then for any geometric admissible Dirac operator $D = D^W$ with pointwise invertible indicial symbol $\mathcal{R}(D)$ we have ${}^{\mathcal{V}}\mathrm{ind}(D) = \mathrm{ind}(D)$.*

Proof. From [27] we have that D is Fredholm if and only if $\mathcal{R}(D)$ is pointwise invertible by the conditions imposed on the groupoid $\mathcal{G}$. Assume that D is Fredholm. Then ${}^{\mathcal{V}}\mathrm{ind}(D) = \mathrm{ind}(D)$ follows by the argument in [17, Section 2.2], see also [10, Theorem 1.2]. □

Example 5.9. The conditions on the Lie groupoid as stated in the Theorem hold in numerous special cases of Lie manifolds. We refer to [27] for an overview.

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REFERENCES

[1] B. Ammann, A. Ionescu, and V. Nistor, *Sobolev spaces on Lie manifolds and regularity for polyhedral domains.* Doc. Math. **11** (2006), 161-206.

[2] B. Ammann, R. Lauter, and V. Nistor, *Pseudo-differential operators on manifolds with a Lie structure at infinity.* Ann. Math. (2) **165** (2007), *3*, 717–747.

[3] B. Ammann, R. Lauter, and V. Nistor, *On the geometry of Riemannian manifolds with a Lie structure at infinity.* Int. J. Math. Math. Sci. **2004** (2004), *1-4*, 161–193.

[4] J. Aastrup, R. Nest, and E. Schrohe, *A continuous field of C^*-algebras and the tangent groupoid for manifolds with boundary.* J. Funct. Anal. **237** (2006), *2*, 482–506.

[5] C. Bär and W. Ballmann, *Boundary value problems for elliptic differential operators of first order.* In: H.-D. Cao et al. (Eds.), *Lectures given at the JDG symposium on geometry and topology.* Surv. Differ. Geom. **17** (2012), 1–78.

[6] N. Berline, E. Getzler, and M. Vergne, *Heat Kernels and Dirac Operators.* Grundlehren Math. Wiss. **298**, Springer Science & Business Media, Berlin, 1992.

[7] K. Bohlen, *Boutet de Monvel operators on Lie manifolds with boundary.* Adv. Math. **312** (2017), 234-285.

[8] K. Bohlen, *Positive scalar curvature metrics on manifolds with controlled geometry at infinity.* Oberwolfach Rep. **1732** (2017), 2275–2278.

[9] K. Bohlen and J.M. Lescure, *A geometric approach to K-homology for Lie manifolds.* arXiv preprint arXiv:1904.04069 (2019). To appear in Ann. Sci. Éc. Norm. Supér. (4).

[10] K. Bohlen and E. Schrohe, *Getzler rescaling via adiabatic deformation and a renormalized local index formula.* J. Math. Pures Appl. **120** (2018), *9*, 220–252.

[11] P.R. Chernoff, *Essential self-adjointness of powers of generators of hyperbolic equations.* J. Funct. Anal. **12** (1973), 401–414.

[12] C. Debord, *Holonomy groupoids of singular Foliations.* J. Differential Geom. **58** (2001), *3*, 467-500.

[13] C. Fefferman, *On Kohn's microlocalization of $\overline{\partial}$ problems.* In: T. Bloom et al. (Eds.), *Modern methods in complex analysis.* Ann. of Math. Stud. **137**, Princeton Univ. Press, Princeton, NJ, 1995.

[14] D.S. Freed, *Two index theorems in odd dimensions.* Comm. Anal. Geom. **6** (1998), *2*, 317–329.

[15] M. Gromov and H.B. Lawson, *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds.* Publ. Math. Inst. Hautes Études Sci. **58** (1983), 83–196.

[16] D. Jerison and C.E. Kenig, *The inhomogeneous Dirichlet problem in Lipschitz domains.* J. Funct. Anal. **130** (1995), *1*, 161–219.

[17] M. Lesch, H. Moscovici, and M.J. Pflaum, *Connes-Chern character for manifolds with boundary and eta cochains.* Mem. Amer. Math. Soc. **220** (2012), *1036*.

[18] N.P. Landsman and B. Ramazan, *Quantization of Poisson algebras associated to Lie algebroids.* Contemp. Math. **282** (2001), 159–192.

[19] R. Lauter and V. Nistor, *Analysis of geometric operators on open manifolds: A groupoid approach.* Progr. Math. **198** (2001), 181–229.

[20] K.C.H. Mackenzie, *General Theory of Lie Groupoids and Lie Algebroids.* London Math. Soc. Lecture Note Ser. **213**, Cambridge Univ. Press, 2005.

[21] H.P. McKean and I.M. Singer, *Curvature and the eigenvalues of the Laplacian.* J. Differential Geom. **1** (1967), *1*, 43–69.

[22] R.B. Melrose, *The Atiyah-Patodi-Singer Index Theorem.* Research Notes in Mathematics **4**, A. K. Peters, Wellesley, 1993.

[23] D. Mitrea, M. Mitrea, and M. Taylor, *Layer Potential, the Hodge Laplacian and Global Boundary Problems in Nonsmooth Riemannian Manifolds.* Mem. Am. Math. Soc. **713**, Providence, RI, 2001.

[24] M. Mitrea and M. Taylor, *Boundary layer methods for Lipschitz domains in Riemannian manifolds.* J. Funct. Anal. **163** (1999), 181-251.

[25] V. Nazaikinskii, A. Savin, B.W. Schulze, and B. Sternin, *Elliptic theory on manifolds with nonisolated singularities: V. Index formulas for elliptic problems on manifolds with edges.* Univ. Potsdam, Institut für Mathematik, Potsdam, 2003.

[26] V. Nistor, *Analysis on singular spaces: Lie manifolds and operator algebras.* J. Geom. Phys. **105** (2016), 75–101.

[27] V. Nistor, *Pseudodifferential operators on non-compact manifolds and analysis on polyhedral domains.* Contemp. Math. **366** (2005), 307–328.

[28] V. Nistor, A. Weinstein, and P. Xu, *Pseudodifferential operators on differential groupoids.* Pacific J. Math. **189** (1999), 117–152.

[29] J. Renault, *A groupoid Approach to C^*-algebra.* Lecture Notes in Math. **793**, Springer, Berlin, 1980.

[30] J. Roe, *Finite propagation speed and Connes' foliation algebra.* Math. Proc. Cambridge Philos. Soc. **102** (1987), 459–466.

[31] J. Roe, *Elliptic Operators, Topology and Asymptotic Methods.* Second Edition, Pitman Research Notes in Mathematics Series, **395**, Longman, Harlow, New York 1998.

[32] J. Roe, *Index Theory, Coarse Geometry, and Topology of Manifolds.* CBMS Reg. Conf. Ser. Math. **90**, Providence, RI: American Mathematical Society (AMS), 1996.

[33] B.K. So, *Exponential coordinates and regularity of groupoid heat kernels.* Cent. Eur. J. Math. **12** (2014), *2*, 284-297.

[34] M.E. Zadeh, *Heat equation approach to index theorems on odd dimensional manifolds.* Bull. Belg. Math. Soc. Simon Stevin **16** (2009), *4*, 647–664.

[35] S. Zaremba, *Sur un probléme mixte relatif á l'équation de Laplace.* Bulletin international de l'Académie de Sciences de Cracovie, Classe des Sciences Mathématiques et Naturelles, Serie A: Sciences mathématiques (1910), 313-344.

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