STEKLOV EIGENVALUE PROBLEMS WITH INDEFINITE WEIGHT FOR THE (p, q)-LAPLACIAN

A. BOUKHSAS, A. ZEROUALI, O. CHAKRONE, and B. KARIM

Communicated by Lucian Beznea

This paper provides existence and non-existence results on a positive solution for the problem $\Delta_r u + \mu\Delta_{r'} u = |u|^{r-2}u + \mu|u|^{r'-2}u$, with a nonlinear boundary condition given by $\langle|\nabla u|^{r-2}\nabla u + |\nabla u|^{r'-2}\nabla u, \nu\rangle = \lambda m_r(x)|u|^{r-2}u$ on the boundary of the domain, with $\mu > 0$ and $1 < r \neq r' < \infty$, where Ω is a bounded domain in $\mathbb{R}^N$, ν is the outward unit normal vector on $\partial\Omega$, $\langle .,. \rangle$ is the scalar product of $\mathbb{R}^N$ and m_r is a weight function admitting sign-change. We show that existence and non-existence of a positive solution depend only on the relation between λ and the first eigenvalue of r-Laplacian with weight function m_r, whence it is independent of the operator $\Delta_{r'}$ and the parameter $\mu > 0$.

AMS 2020 Subject Classification: 35D30, 35J66, 35P30.

Key words: (p, q)-Laplacian, Steklov eigenvalue problem, indefinite weights, mountain pass theorem, global minimizer.

1. INTRODUCTION

In this paper, we are interested in the existence and non existence results for the following quasilinear elliptic equation:

$$P_{(r,r',\lambda,\mu)} \quad \begin{cases} \operatorname{div}[A^{(\mu)}_{r,r'}(\nabla u)] = A^{(\mu)}_{r,r'}(u)u & \text{in } \Omega, \\ \langle A^{(\mu)}_{r,r'}(\nabla u), \nu\rangle = \lambda m_r(x)|u|^{r-2}u & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$, ν is the outward unit normal vector on $\partial\Omega$, $\langle .,. \rangle$ is the scalar product of $\mathbb{R}^N$, $\lambda \in \mathbb{R}$, $\mu \geq 0$ and $1 < r \neq r' < \infty$. Let $\frac{N-1}{r-1} < s_r < \infty$ if $r < N$ and $s_r \geq 1$ if $r \geq N$. $A^{(\mu)}_{r,r'}(s) = |s|^{r-2}s + \mu|s|^{r'-2}s$ and the function weight $m_r \in \mathbb{M}_r$ may be unbounded and change sign, where $\mathbb{M}_r := \{m_r \in L^{s_r}(\partial\Omega); m_r^+ \not\equiv 0\}$.

We treat our equation $P_{(r,r',\lambda,\mu)}$ for $r = p$ and $r' = q$, and for $r = q$ and $r' = p$.

Although our problem $P_{(p,q,\lambda,\mu)}$ coincides with the second problem $P_{(q,p,\lambda,\mu)}$ by formally replacing p with q, we treat them separately in some cases (e.g. the proof of Theorem 4.1). One of the reasons for different treatment is that both of our problems should be solved in the space $W^{1,p}(\Omega)$ since the r-Laplacian naturally acts in $W^{1,r}(\Omega)$ for $1 < r < \infty$ and we are assuming $p > q$, hence we cannot transfer from one problem to another by merely replacing p with q.

Thus, *throughout this paper, we set* $W^{1,p}(\Omega)$ *as the space to find a solution of our* problems $P_{(p,q,\lambda,\mu)}$ and $P_{(q,p,\lambda,\mu)}$. However, it is proved that any solutions of our problems are of class $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ (see Remark 4.2). Throughout the paper we set $p = \max\{r; r'\}$ and $q = \min\{r; r'\}$.

In this paper, we say that $u \in W^{1,p}(\Omega)$ is a solution of $P_{(r,r',\lambda,\mu)}$ if the following holds

$$\int_\Omega \left(|\nabla u|^{r-2}\nabla u \nabla\varphi + |u|^{r-2}u\varphi\right)\mathrm{d}x + \mu\int_\Omega \left(|\nabla u|^{r'-2}\nabla u \nabla\varphi + |u|^{r'-2}u\varphi\right)\mathrm{d}x$$
$$= \lambda \int_{\partial\Omega} m_r |u|^{r-2}u\varphi \mathrm{d}\sigma$$

for all $\varphi \in W^{1,p}(\Omega)$, where $d\sigma$ is the $N-1$ dimensional Hausdorff measure.

Letting $\mu \to +0$ our problem $P_{(r,r',\lambda,\mu)}$ turns into the following weighted eigenvalue problem for the r-Laplacian:

$$P_{(r,\lambda)} \qquad \begin{cases} \Delta_r u = |u|^{r-2}u & \text{in} \quad \Omega, \\ |\nabla u|^{r-2}\frac{\partial u}{\partial \nu} = \lambda m_r(x)|u|^{r-2}u & \text{on} \quad \partial\Omega. \end{cases}$$

It is said that λ is an eigenvalue of $-\Delta_r$ with weight function m_r if Problem $P_{(r,\lambda)}$ has a non-trivial solution which is called an eigenfunction corresponding to λ. We denote the set of all eigenvalues of $-\Delta_r$ with weight function m_r by $\sigma(-\Delta_r, m_r)$. In particular, in the case of $m_r \equiv 1$, we write $\sigma(-\Delta_r)$ instead of $\sigma(-\Delta_r, 1)$. Similarly, in the non-homogeneous case, we say that λ is a *generalized eigenvalue* of $\Delta_r + \mu\Delta_{r'}$ with weight function m_r if $P_{(r,r',\lambda,\mu)}$ has a non-trivial solution. Denote the set of those λ's by $\sigma_G(\Delta_r + \mu\Delta_{r'}, m_r)$. The main purpose of this paper is to study the generalized eigenvalues of $\Delta_r + \mu\Delta_{r'}$.

Recently, many authors have studied (p,q)-Laplace equations (cf. [13], [18], [23], [27], [28], [29]). However, there are few results on generalized eigenvalue problems of the (p,q)-Laplacian. In [7] and [8], Benouhiba and Belyacine considered the equation

$$-\Delta_p u - \Delta_q u = \lambda g(x)|u|^{p-2}u \quad \text{in} \quad \mathbb{R}^{\mathbb{N}}$$

under several assumptions on $g \geq 0$. They showed the existence of principal eigenvalue and a *continuous* family of generalized eigenvalues λ. In [12, Theorem 4.2], Cingolani and Degiovanni proved the existence of a *non-trivial* solution for

$$-\Delta_p u - \mu\Delta u = \lambda|u|^{p-2}u + g(u) \quad \text{in} \ \ \Omega, \quad u = 0 \quad \text{on} \ \ \partial\Omega$$

in the case of $r = p > 2 (= r'), g \in C^1$ and $\lambda \notin \sigma(-\Delta_p)$. However, their result does not cover the resonant case $\lambda \in \sigma(-\Delta_p)$.

Under the Neumann boundary condition, Milhăilescu [20] gave a set of all generalized eigenvalues λ for $-\Delta_p u - \Delta u = \lambda u$ in Ω, where $r' = p > 2 (= r)$ and the case $r' = p < 2 (= r)$ was studied by M. Fărcăşeanu *et al.* in [15]. Also, this results were generalized by M. Mihăilescu and G. Moroşanu in [21] to the case when the Laplacian is replaced by a general q-Laplace operator (i.e. $r' = p \neq r = q$). In [24] the author has completely described generalized eigenvalue λ for which the following problem

$$\begin{cases} -\Delta_r u - \mu \Delta_{r'} u = \lambda m_r(x) |u|^{r-2} u & \text{in} \quad \Omega, \\ \qquad u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

has a positive solution, where $\mu > 0$ and $1 < r \neq r' < \infty$. In [9] was completed by investigating the asymptotic case when $\min\{r, r'\} \to \infty$.

Under Steklov boundary condition, we have studied in [10] the problem $P_{(r,r',\lambda,\mu)}$ in the case where $r' = 2, r > 2$. In [1] J. Abreu and G. F. Madeira studied the following (p,2)-Laplacian Steklov problem

$$\begin{cases} -\Delta_p u - \Delta u &= \lambda a(x) u \quad \text{in } \Omega, \\ \langle |\nabla u|^{p-2} \nabla u + \nabla u, \nu \rangle &= \lambda b(x) u \quad \text{on } \partial\Omega. \end{cases}$$

for positive weight functions a and b satisfying appropriate integrability and boundedness assumptions. This result was generalised by L. Barbu and G. Moroşanu in [6] to the case where $p, q \in (1, \infty)$. For other results we cite [11, 14, 30].

Our purpose in this article is to extend these results obtained under Dirichlet boundary condition in [24] to nonlinear boundary condition, following the same approach, where Rayleigh quotient plays an important role (see Remark 2.2 for details).

The rest of this paper is organized as follows. In section 2, we give some preliminary results and lemmas which are needed in the proof of the main results. In section 3, we present and prove the non existence results. In section 4, we state and prove our existence result.

2. PRELIMINARY RESULTS

In this section we give some preliminary results needed for the proof of the main theorems. Throughout this paper, $\|u\|_{1,r} := \|u\|_{W^{1,r}(\Omega)}$ denotes the norm of Sobolev space $W^{1,r}(\Omega)$.

First, let us recall the first eigenvalue $\lambda_1(r, m_r)$ and the second eigenvalue $\lambda_2(r, m_r)$ of $-\Delta_r$ with weight function m_r.

It is well know that the first (smallest) positive eigenvalue $\lambda_1(r, m_r)$ is obtained by the following Rayleigh quotient

$$\lambda_1(r,m_r) := \inf\left\{\frac{\int_\Omega |\nabla u|^r \mathrm{d}x + \int_\Omega |u|^r \mathrm{d}x}{\int_{\partial\Omega} m_r|u|^r \mathrm{d}\sigma}; u \in W^{1,r}(\Omega), \int_{\partial\Omega} m_r|u|^r \mathrm{d}\sigma > 0\right\}. \tag{2.1}$$

Since there exist no *non-negative* eigenvalues provided $m_r \leq 0$, we set

$$\lambda_1(r,-m_r) = +\infty \quad \text{if} \;\; m_r \geq 0. \tag{2.2}$$

The principal eigenvalue problem $\lambda_1(r,m_r)$ play an important roles for existence and non-existence of a positive solution for our non-homogeneous problem $(P_{r,r',\lambda,\mu})$. The second (positive) eigenvalue $\lambda_2(r,m_r)$ of $-\Delta_r$ with weight function m_r it is defined by

$$\lambda_2(r,m_r) = \min\{\lambda > \lambda_1(r,m_r); \lambda \in \sigma(-\Delta_r,m_r)\}.$$

(Note that $\lambda_1(r,m_r)$ is *isolated* and $\sigma(-\Delta_r,m_r)$ is closed. It is also worth mentioning that $\lambda_1(r,m_r)$ has positive eigenfunctions $\varphi_1(r,m_r) \in C^{1,\alpha_r}(\overline{\Omega})$ with some $\alpha_r \in (0,1)$ (see [2]).

Next, we study Rayleigh quotient for our problems. For $r = p$ or q, we define the functional $\Phi_{(r,r',\mu)}(u)$ on $W^{1,p}(\Omega)$ as follows:

$$\Phi_{(r,r',\mu)}(u) := \|u\|_{1,r}^r + \frac{r\mu}{r'}\|u\|_{1,r'}^{r'} \tag{2.3}$$

for $u \in W^{1,p}(\Omega)$, where $r' = q$ if $r = p$ and $r' = p$ if $r = q$.

The following proposition is the result on Rayleigh quotient that is crucial to solve our problems.

PROPOSITION 2.1. *For $\mu > 0$ and $r = p$ or q we set*

$$\underline{\lambda}(r,r',\mu,m_r) := \inf\left\{\frac{\Phi_{(r,r',\mu)}(u)}{\Psi_r(u)}; u \in W^{1,p}(\Omega), \Psi_r(u) > 0\right\}, \tag{2.4}$$

where $\Psi_r(u) := \int_{\partial\Omega} m_r|u|^r \mathrm{d}\sigma$ and $\Phi_{(r,r',\mu)}(u)$ is the functional defined in (2.3) *Then,*

$$\underline{\lambda}(r,r',\mu,m_r) = \lambda_1(r,m_r)$$

holds for every $\mu > 0$. In addition, for every $\mu > 0$, the infimum in (2.4) *is not attained.*

Proof. Fix any $\mu > 0$. First, we treat the case of $r = p > q = r'$. From the definitions of $\lambda_1(p,m_p)$ and $\underline{\lambda}(p,q,\mu,m_p)$, it is obvious that $\underline{\lambda}(p,q,\mu,m_p) \geq \lambda_1(p,m_p)$. Let φ_1 be the positive eigenfunction corresponding to $\lambda_1(p,m_p)$ such that $\int_{\partial\Omega} m_p\varphi_1^p \mathrm{d}\sigma = 1$ (we may replace φ_1, if necessary, by $\varphi_1/(\int_{\partial\Omega} m_p\varphi_1^p \mathrm{d}\sigma)^{1/p}$ since $P_{(r,\lambda)}$ is a homogeneous problem). Thus, φ_1 satisfies $\|\varphi_1\|_{1,p}^p = \lambda_1(p,m_p)$.

Letting $t \to \infty$ in the following inequality:

$$\underline{\lambda}(p,q,\mu,m_p) \le \frac{\Phi_{(p,q,\mu)}(t\varphi_1)}{t^p} = \lambda_1(p,m_p) + \frac{p\mu\|\varphi_1\|_{1,q}^q}{qt^{p-q}}$$

(note that $\int_{\partial\Omega} m_p\varphi_1^p \mathrm{d}\sigma = 1$), we obtain $\underline{\lambda}(p,q,\mu,m_p) \le \lambda_1(p,m_p)$ because of $p > q$. Thus, in the case of $r = p > r' = q$, the first assertion is shown.

Next, we shall give the proof in the case of $r = q < r' = p$. It is easy to see that $\underline{\lambda}(p,q,\mu,m_p) \ge \lambda_1(p,m_p)$ (note that $W^{1,p}(\Omega) \subset W^{1,q}(\Omega)$). Let ψ_1 be the positive eigenfunction corresponding to $\lambda_1(q,m_q)$ such that $\int_{\partial\Omega} m_q\psi_1^q \mathrm{d}\sigma = 1$, so ψ_1 satisfies $\|\psi_1\|_{1,q}^q = \lambda_1(q,m_q)$. Recall that ψ_1 belongs to $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. Thus, we obtain $\underline{\lambda}(q,p,\mu,m_q) \le \lambda_1(q,m_q)$ by letting $t \to +0$ (note that $p - q > 0$) in the equality

$$\underline{\lambda}(q,p,\mu,m_q) \le \frac{\Phi_{(q,p,\mu)}(t\varphi_1)}{t^q} = \lambda_1(q,m_q) + \frac{qt^{p-q}\mu\|\psi_1\|_{1,p}^p}{p}.$$

Finally, we shall prove that $\underline{\lambda}(r,r',\mu,m_r)$ is not attained for any $\mu > 0$. To do so, by way of contradiction, we assume that there exist $\mu > 0$ and a function $u \in W^{1,p}(\Omega)$ such that $\int_{\partial\Omega} m_r|u|^r \mathrm{d}\sigma > 0$ and $\underline{\lambda}(r,r',\mu,m_r) = \Phi_{(r,r',\mu)}(u)/\Psi_r(u)$. Then, it follows from $u \ne 0$, the definition of $\lambda_1(r,m_r)$ (note that $W^{1,p}(\Omega) \subset W^{1,q}(\Omega)$ if $r = q$) and the first assertion of the proposition that

$$\begin{aligned}\underline{\lambda}(r,r',\mu,m_r) &= \frac{\Phi_{(r,r',\mu)}(u)}{\Psi_r(u)} \ge \lambda_1(r,m_r) + \frac{r\mu\|u\|_{1,r'}^{r'}}{r'\Psi_r(u)} \\ &> \lambda_1(r,m_r) = \underline{\lambda}(r,r',\mu,m_r).\end{aligned}$$

This is a contradiction. $\square$

Remark 2.2. If $\underline{\lambda}(r,r',\mu,m_r)$ was attained, we see that $P_{(r,r',\lambda,\mu)}$ had a positive solution with $\lambda = \underline{\lambda}(r,r',\mu,m_r)$. However, according to Proposition 2.1, we see that there does not exist the first (positive) generalized eigenvalue of $\Delta_r + \mu\Delta_{r'}$ for any $\mu > 0$ (refer also to Remark 3.3).

Under Neumann boundary condition, Mihăilescu has shown that

$$\lambda_1(p) := \inf\left\{\frac{\|\nabla u\|_p^p/p + \|\nabla u\|_2^2/2}{\|u\|_2^2/2}; u \in W^{1,p}\backslash\{0\}, \int_\Omega u\mathrm{d}x = 0\right\} > 0$$

and $\lim_{s\to p-0}\lambda_1(s) \le \lambda_1(p) \le \lim_{s\to p+0}\lambda_1(s)$, where $p > 2$. This case corresponds to one of $r' = p > 2 = r$, $m_r \equiv 1$ and $\mu = 1$ in our problems. Concerning our Steklov problem, it is easy to see that Rayleigh quotient $\underline{\lambda}(r,r',\mu,m_r)$ is continuous with respect to r', μ and m_r because $\underline{\lambda}(r,r',\mu,m_r)$ is independent of r' and μ, and by the continuity of $\lambda_1(r,m_r)$ with respect to $m_r \in L^{s_r}(\partial\Omega)$ (cf.[5]).

In what follows we state and prove the lemmas that we need to prove the existence result.
For $r = p$ and $r' = q$ we define the functional $I_{(p,q,\lambda,\mu)}$ on $W^{1,p}(\Omega)$ by

$$I_{(p,q,\lambda,\mu)}(u) := \frac{1}{p}\|u\|_{1,p}^p + \frac{\mu}{q}\|u\|_{1,q}^q - \frac{\lambda}{p}\int_{\partial\Omega} m_p u_+^p \mathrm{d}\sigma, \tag{2.5}$$

for $u \in W^{1,p}(\Omega)$ and $\mu > 0$.

LEMMA 2.3. *Let $\mu > 0$. Assume $0 \leq \lambda \neq \lambda_1(p, m_p)$. Then $I_{(p,q,\lambda,\mu)}$ satisfies the Palais-Smale condition.*

Proof. Fix any $\mu > 0$. Let $\{u_n\}$ be a Palais-Smale sequence of $I_{(p,q,\lambda,\mu)}$, that is,

$$I_{(p,q,\lambda,\mu)}(u_n) \to c \qquad \text{and} \qquad I'_{(p,q,\lambda,\mu)}(u_n) \to 0 \quad \text{in} \quad W^{1,p}(\Omega)^*$$

as $n \to \infty$ for some $c \in \mathbb{R}$. Let us first show that the sequence $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$. It is sufficient only to prove the boundedness of $\|u_n\|_{L^{ps'_p}(\partial\Omega)}$ because

$$\|u_n\|_{1,p}^p \leq pc + \lambda\|m_p\|_{L^{s_p}(\partial\Omega)}\|u_n\|^p_{L^{ps'_p}(\partial\Omega)}, \tag{2.6}$$

where $s'_p = s_p/s_p - 1$.
Suppose by contradiction that $\|u_n\|_{L^{ps'_p}(\partial\Omega)} \to +\infty$ and let $v_n := \frac{u_n}{\|u_n\|_{L^{ps'_p}(\partial\Omega)}}$.
The sequence v_n bounded in $W^{1,p}(\Omega)$. Indeed, dividing (2.6) by $\|u_n\|^p_{L^{ps'_p}(\partial\Omega)}$ we have

$$\|v_n\|_{1,p}^p \leq \frac{pc}{\|u_n\|_{L^{ps'_p}(\partial\Omega)}} + \lambda\|m_p\|_{L^{s_p}(\partial\Omega)}. \tag{2.7}$$

The inequality (2.7) implies the boundedness of $\{v_n\}$ in $W^{1,p}(\Omega)$. Therefore, we may suppose, up to a subsequence, that $v_n \rightharpoonup v$ (weakly) in $W^{1,p}(\Omega)$. By the compact embedding $W^{1,r}(\Omega) \subset L^{rs'_r}(\partial\Omega), (r = p, q)$ we have $v_n \to v$ strongly in $L^{rs'_r}(\partial\Omega)$ $(r = p, q)$. First we observe that $v^- \equiv 0$ in Ω. In fact, acting with $-u_n^-$ as test function, we have

$$\begin{aligned} o(1)\|u_n^-\|_{L^{ps'_p}(\partial\Omega)} &= \langle I'_{(\lambda,m_p,m_q)}(u_n), -u_n^-\rangle \\ &= \|u_n^-\|_{1,p} + \mu\|u_n^-\|_{1,p} \\ &\geq \|u_n^-\|_{1,q} \end{aligned} \tag{2.8}$$

the inequality (2.8) guarantees the boundedness of $\|v_n^-\|_{1,p}$ and so $\|v_n^-\|_{1,p} = \frac{\|u_n^-\|_{1,p}}{\|u_n^-\|_{L^{ps'_p}(\partial\Omega)}} \to 0$, thus $v^- \equiv 0$, holds, hence $v \geq 0$ in Ω.

Now, by taking $(v_n - v)/\|u_n^-\|^{p-1}_{L^{ps'_p}(\partial\Omega)}$ as test function, we have

$$
\begin{aligned}
o(1) &= \Big\langle I'_{(\lambda,m_p,m_q)}(u_n), \frac{(v_n - v)}{\|u_n^-\|^{p-1}_{L^{ps'_p}(\partial\Omega)}}\Big\rangle \\
&= \int_\Omega |\nabla v_n|^{p-2}\nabla v_n \nabla(v_n - v)\mathrm{d}x \\
&\quad + \frac{\mu}{\|u_n\|^{p-q}_{L^{ps'_p}(\partial\Omega)}}\int_\Omega |\nabla v_n|^{q-2}\nabla v_n\nabla(v_n - v)\mathrm{d}x \\
&\quad + \int_\Omega |v_n|^{p-2}v_n(v_n - v)\mathrm{d}x \\
&\quad + \frac{\mu}{\|u_n\|^{p-q}_{L^{ps'_p}(\partial\Omega)}}\int_\Omega |v_n|^{q-2}v_n(v_n - v)\mathrm{d}x \\
&\quad - \lambda\int_{\partial\Omega} m_p(v_n^+)^{p-2}v_n^+(v_n - v)\mathrm{d}\sigma \\
&= \int_\Omega |\nabla v_n|^{p-2}\nabla v_n\nabla(v_n - v)\mathrm{d}x + \int_\Omega |v_n|^{p-2}v_n(v_n - v)\mathrm{d}x \\
&\quad - \lambda\int_{\partial\Omega} m_p(v_n^+)^{p-2}v_n^+(v_n - v)\mathrm{d}\sigma + o(1)
\end{aligned}
\tag{2.9}
$$

because $q < p$, $\|u_n^-\|_{L^{ps'_p}(\partial\Omega)} \to +\infty$, v_n is bounded in $W^{1,p}(\Omega)$ and converge to v strongly in $L^{ps'_p}(\partial\Omega)$. Thus by (2.9) and (S_+) property of $\Delta_p u + u^{p-2}u$ in $W^{1,p}(\Omega)$, we deduce that $v_n \to v$ strongly in $W^{1,p}(\Omega)$. For any $\varphi \in W^{1,p}(\Omega)$, by taking $\frac{\varphi}{\|u_n^-\|^{p-1}_{L^{ps'_p}(\partial\Omega)}}$ as test function, we obtain

$$
\begin{aligned}
o(1) &= \Big\langle I'_{(\lambda,m_p,m_q)}(u_n), \frac{\varphi}{\|u_n\|^{p-1}_{L^{ps'_p}(\partial\Omega)}}\Big\rangle \\
&= \int_\Omega |\nabla v_n|^{p-2}\nabla v_n\nabla\varphi\mathrm{d}x + \frac{\mu}{\|u_n\|^{p-q}_{L^{ps'_p}(\partial\Omega)}}\int_\Omega |\nabla v_n|^{q-2}\nabla v_n\nabla\varphi\mathrm{d}x \\
&\quad + \int_\Omega |v_n|^{p-2}v_n\varphi\mathrm{d}x + \frac{\mu}{\|u_n\|^{p-q}_{L^{ps'_p}(\partial\Omega)}}\int_\Omega |v_n|^{q-2}v_n\varphi\mathrm{d}x \\
&\quad - \lambda\int_{\partial\Omega} m_p(v_n^+)^{p-2}v_n^+(v_n - v)\mathrm{d}\sigma.
\end{aligned}
\tag{2.10}
$$

Passing to the limit in (2.10), we see that v is a non-negative and non-trivial solution of problem $P_{(p,\lambda)}$ (note $v \geq 0$ and $\|v\|_{1,p} = 1$ and the associated eigenfunction v is $\in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, see [2]). According to maximum principle of Vasquez, we have $v > 0$ in $\overline{\Omega}$. This implies that $\lambda = \lambda_1(p, m_p)$

because any positive eigenvalue other than $\lambda_1(p, m_p)$ has no positive eigenfunction. Therefore, we obtain a contradiction since we assumed $\lambda \neq \lambda_1(p, m_p)$. Hence u_n is bounded in $W^{1,p}(\Omega)$. For a subsequence, $u_n \rightharpoonup u$ (weakly) in $W^{1,p}(\Omega)$ and $u_n \to u$ (strongly) in $L^{ps'_p}(\partial\Omega)$. We claim now that $u_n \to u$ in $W^{1,p}(\Omega)$. It suffices to prove that $\|u_n\|_{1,p} \to \|u\|_{1,p}$. Because $W^{1,p}(\Omega)$ is reflexive and uniformly convex. It is clear that

$$
\begin{aligned}
o(1) &= \left\langle I'_{(\lambda,m_p,m_q)}(u_n), u_n - u \right\rangle \\
&= \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u)\mathrm{d}x \\
&\quad + \int_\Omega |u_n|^{p-2}u_n(u_n - u)\mathrm{d}x \\
&\quad + \mu\int_\Omega |\nabla u_n|^{q-2}\nabla u_n \nabla(u_n - u)\mathrm{d}x \\
&\quad + \mu\int_\Omega |u_n|^{q-2}u_n(u_n - u)\mathrm{d}x + o(1).
\end{aligned} \tag{2.11}
$$

Using Hölder inequality and for $(r = p, q)$, we have

$$
\begin{aligned}
&\int_\Omega |\nabla u_n|^{r-2}\nabla u_n \nabla(u_n - u)\mathrm{d}x + \int_\Omega |u_n|^{r-2}u_n(u_n - u)\mathrm{d}x \\
&= \int_\Omega |\nabla u_n|^r \mathrm{d}x + \int_\Omega |\nabla u|^r \mathrm{d}x \\
&\qquad - \int_\Omega |\nabla u_n|^{r-2}\nabla u_n u\mathrm{d}x - \int_\Omega |\nabla u|^{r-2}\nabla u \nabla u_n \mathrm{d}x \\
&= \int_\Omega |u_n|^r \mathrm{d}x + \int_\Omega |u|^r \mathrm{d}x \\
&\qquad - \int_\Omega |u_n|^{r-2}u_n u\mathrm{d}x - \int_\Omega |u|^{r-2}u u_n \mathrm{d}x \\
&\geq \int_\Omega |\nabla u_n|^r \mathrm{d}x + \int_\Omega |\nabla u|^r \mathrm{d}x \\
&\qquad - \left(\int_\Omega |\nabla u_n|^r \mathrm{d}x\right)^{(r-1)/r}\left(\int_\Omega |\nabla u|^r \mathrm{d}x\right)^{1/r} + \int_\Omega |u_n|^r \mathrm{d}x \\
&\qquad + \int_\Omega |u|^r \mathrm{d}x - \left(\int_\Omega |u_n|^r \mathrm{d}x\right)^{(r-1)/r}\left(\int_\Omega |u|^r \mathrm{d}x\right)^{1/r} \\
&= \left(\|u_n\|_{1,r}^{r-1} - \|u\|_{1,r}^{r-1}\right)\left(\|u_n\|_{1,r} - \|u\|_{1,r}\right) \\
&\geq 0.
\end{aligned} \tag{2.12}
$$

Moreover, (2.11) and (2.12) imply that $\|u_n\|_{1,p} \to \|u\|_{1,p}$. Thus $u_n \to u$ strongly in $W^{1,p}(\Omega)$. $\square$

LEMMA 2.4. *Set*

$$X(d) := \left\{ u \in W^{1,p}(\Omega); \qquad \|u\|_{1,p}^p \le d\|u\|^p_{L^{ps'_p}(\partial\Omega)} \right\} \tag{2.13}$$

for $d > 0$. Then there exists $C = C(d) > 0$ such that

$$\|u\|_{1,p} \le C\|u\|_{L^{qs'_q}}(\partial\Omega) \qquad \text{for all } \ u \in X(d).$$

Proof. By way contradiction, we assume that $\forall n \in \mathbb{N}$, $\exists u_n \in X(d)$

$$\frac{1}{n}\|u_n\|_{1,p} > \|u_n\|_{L^{qs'_q}(\partial\Omega)}.$$

Set $v_n = \frac{u_n}{\|u_n\|_{1,p}}$, hence $\{v_n\}$ is bounded in $W^{1,p}(\Omega)$, then there exists $v \in W^{1,p}(\Omega)$ and a subsequence v_n, such that $v_n \rightharpoonup v$ (weakly) in $W^{1,p}(\Omega)$. By the compact embedding $W^{1,r}(\Omega) \subset L^{rs'_r}(r = p,q)$ we have $v_n \to v$ (strongly) in $L^{rs'_r}(\partial\Omega)(r = p,q)$. However, $\|v_n\|_{L^{qs'_q}(\partial\Omega)} < \frac{1}{n}$, $v_n \to 0$ in $L^{qs'_q}(\partial\Omega)$ by uniqueness of the limit we have $v = 0$, hence $v_n \to 0$ in $L^{ps'_p}(\partial\Omega)$, as $u_n \in X(d)$ we have

$$\|u_n\|_{1,p}^p \le d\|u_n\|^p_{L^{ps'_p}(\partial\Omega)}$$

$$\frac{1}{d} \le \frac{\|u_n\|^p_{L^{ps'_p}(\partial\Omega)}}{\|u_n\|^p_{1,p}} = \|v_n\|^p_{L^{ps'_p}(\partial\Omega)}$$

implies that $\frac{1}{d} \le 0 \Rightarrow d < 0$. This contradicts $d > 0$. $\square$

3. NON EXISTENCE RESULTS

In this section we prove the following non-existence results.

THEOREM 3.1. *Let $r = p$ or q. If $-\lambda_1(r,-m_r) \le \lambda \le \lambda_1(r,m_r)$ holds, then for any $\mu > 0$, $P_{(r,r',\lambda,\mu)}$ has no non-trivial solutions.*

THEOREM 3.2. *Let $r = p$ or q. If $\lambda_1(r,m_r) < \lambda \le \lambda_2(r,m_r)$ holds, then for any $\mu > 0$, $P_{(r,r',\lambda,\mu)}$ has no sign-changing solutions.*

Proof of Theorem 3.1. Fix any $\mu > 0$. Let u be a non-trivial solution of $P_{(r,r',\lambda,\mu)}$. By taking u as test function in $P_{(r,r',\lambda,\mu)}$, we have

$$0 < \|u\|^r_{1,r} < \|u\|^r_{1,r} + \mu\|u\|^{r'}_{1,r'} = \lambda \int_{\partial\Omega} m_r|u|^r \mathrm{d}\sigma \tag{3.1}$$

due to $\mu > 0$ and $\|u\|_{1,r'} > 0$. This yields that the following (a) or (b) occurs:
(a) $\lambda < 0$ and $\int_{\partial\Omega} m_r|u|^r \mathrm{d}\sigma < 0$;
(b) $\lambda > 0$ and $\int_{\partial\Omega} m_r|u|^r \mathrm{d}\sigma > 0$.

It is sufficient to consider only the case (b) because we may consider the pair of $-\lambda$ and $-m_r$ instead of λ and m_r provided m_r changes the sign. Thus, we may assume that $\int_{\partial\Omega} m_r|u|^r \mathrm{d}\sigma > 0$ and $\lambda > 0$, whence we have

$$\lambda_1(r, m_r) \leq \frac{\|u\|_{1,r}^r}{\int_{\partial\Omega} m_r|u|^r \mathrm{d}\sigma} < \frac{\|u\|_{1,r}^r + \mu\|u\|_{1,r'}^{r'}}{\int_{\partial\Omega} m_r|u|^r \mathrm{d}\sigma} = \lambda$$

by the definition of $\lambda_1(r, m_r)$. Our conclusion follows. $\square$

Remark 3.3. Here, we point out the relation between Proposition 2.1 and Theorem 3.1. Let u be a non-trivial solution of $P_{(r,r',\lambda,\mu)}$. Then, for each $s > 0$, multiplying problem $P_{(r,r',\lambda,\mu)}$ by s^{r-1}, we see that $v = su$ is a non-trivial solution of

$$\begin{cases} \Delta_r v + \mu s^{r-r'} \Delta_{r'} v = |v|^{r-2}v + \mu s^{r-r'}|v|^{r'-2}v & \text{in } \Omega, \\ |\nabla v|^{r-2}\frac{\partial v}{\partial \nu} + \mu s^{r-r'}|\nabla v|^{r'-2}\frac{\partial v}{\partial \nu} = \lambda m_r(x)|v|^{r-2}v & \text{on } \partial\Omega. \end{cases}$$

Choosing $s^{r-r'} = r/r'$ and taking $v = su$ as test function, we have

$$0 < \Phi_{(r,r',\mu)}(su) = \lambda \int_{\partial\Omega} m_r|su|^r \mathrm{d}\sigma,$$

where $\Phi_{(r,r',\mu)}$ is the functional defined by (2.3). Hence, if $\int_{\partial\Omega} m_r|su|^r \mathrm{d}\sigma > 0$ holds, then we obtain

$$\lambda_1(r, m_r) = \underline{\lambda}(r, r', \mu, m_r) < \frac{\Phi_{(r,r',\mu)}(su)}{\int_{\partial\Omega} m_r|su|^r \mathrm{d}\sigma} = \lambda$$

because $\underline{\lambda}(r, r', \mu, m_r)$ is not attained (see Proposition 2.1). This leads to the statement of Theorem 3.1.

Proof of Theorem 3.2. For the proof of Theorem 3.2, we recall a basic fact regarding the second (positive) eigenvalue $\lambda_2(r, m_r)$ of $-\Delta_r(1 < r < \infty)$ with weight function m_r (refer to [3], [4, Corollary 3.9]) Define

$$J_r(v) := \|v\|_{1,r}^r \quad \text{and} \quad \Psi_r(v) := \int_{\partial\Omega} m_r|v|^r \mathrm{d}\sigma \tag{3.2}$$

for $u \in W^{1,r}(\Omega)$. Then, it is known that $\lambda_2(r, m_r)$ is obtained by

$$\lambda_2(r, m_r) = \inf_{\gamma\in\Gamma(m_r)} \max_{t\in[0,1]} \widetilde{J}_r(\gamma(t)) \quad \text{with } \widetilde{J}_r := J_r |_{S(m_r)}, \tag{3.3}$$

$$S(m_r) := \big\{v \in W^{1,r}(\Omega); \Psi_r(v) = 1\big\}, \tag{3.4}$$

$$\Gamma(m_r) := \big\{\gamma \in C([0,1], S(m_r)); \gamma(0) = \varphi_1, \gamma(1) = -\varphi_1\big\}, \tag{3.5}$$

where $\varphi_1 = \varphi_1(r, m_r)$ is the positive eigenfunction corresponding to $\lambda_1(r, m_r)$ such that $\varphi_1 \in S(m_r)$.

Let $r = p$ or $r = q$. We recall that $r' = q$ if $r = p$ and $r' = p$ if $r = q$. By way of contradiction, we assume that $P_{(r,\lambda,\mu)}$ has a sign-changing solution $u \in W^{1,p}(\Omega)$ for some $\mu > 0$ and λ satisfying $\lambda_1(r, m_r) < \lambda \le \lambda_2(r, m_r)$. Then, taking $\pm u_\pm$ as test function, we obtain

$$0 < \|u_+\|_{1,r}^r < \|u_+\|_{1,r}^r + \mu\|u_+\|_{1,r'}^{r'} = \lambda \int_{\partial\Omega} m_r u_+^r \mathrm{d}\sigma \quad \text{and} \tag{3.6}$$

$$0 < \|u_-\|_{1,r}^r < \|u_-\|_{1,r}^r + \mu\|u_-\|_{1,r'}^{r'} = \lambda \int_{\partial\Omega} m_r u_-^r \mathrm{d}\sigma \tag{3.7}$$

where $u_\pm := \max\{u_\pm, 0\}$. Note that $u_\pm \in W^{1,p}(\Omega) \subset W^{1,q}(\Omega)$ and $\Psi_r(u_\pm) = \int_{\partial\Omega} m_r u_\pm^r \mathrm{d}\sigma > 0$ because $\lambda > 0$. Combining these equalities and the argument in [22, Proposition 11], we can construct a continuous $\gamma_0 \in \Gamma(m_r)$ (see (3.5)) such that

$$\max_{t\in[0,1]} \widetilde{J}_r(\gamma_0(t)) < \lambda,$$

where $\widetilde{J}_r$ as in (3.3). This leads to $\lambda_2(r, m_r) < \lambda$ (see (3.3) for the characteristic of $\lambda_2(r, m_r)$, and hence it contradicts to $\lambda \le \lambda_2(r, m_r)$.

For readers' convenience, we sketch the existence of the path γ_0. Recalling that $\Psi_r(u_\pm) = \int_{\partial\Omega} m_r u_\pm^r \mathrm{d}\sigma > 0$, we define paths as follows:

$$\begin{aligned}
\gamma_1(t) &:= \frac{tu + (1-t)u_+}{\Psi_r(tu + (1-t)u_+)^{1/r}} = \frac{u_+ - tu_-}{\Psi_r(u_+ - tu_-)^{1/r}} \\
\gamma_2(t) &:= \frac{tu_+ + (1-t)u_-}{\Psi_r(tu_+ + (1-t)u_-)^{1/r}}, \\
\gamma_3(t) &:= \frac{(1-t)u - tu_-}{\Psi_r((1-t)u - tu_-)^{1/r}} = \frac{(1-t)u_+ - u_-}{\Psi_r((1-t)u_+ - u_-)^{1/r}}
\end{aligned}$$

for $t \in [0,1]$ (see (3.2) for the definition of Ψ_r. Then, by easy estimates, we have $\gamma_j(t) \in S(m_r)$ for $t \in [0,1]$ (see (3.4) for the definition of $S(m_r)$) and

$$\max_{t\in[0,1]} J_r(\gamma_j(t)) < \lambda, \tag{3.8}$$

for $j = 1, 2, 3$ by (3.6) and (3.7).

Now, we set $\mathcal{O}(c) := \{v \in S(m_r); J_r(v) < c\}$ for $c > 0$. Then, by the same argument as in [25, Lemma 31], we can show that any nonempty maximal open connected subset of $\mathcal{O}(c)$ contains at least one critical point of $\widetilde{J}_r$ with $J_r(w) = \beta$ corresponds to a non-trivial solution of $P_{(r,\beta)}$ (according to Lagrange multiplier rule) and belongs to $S(m_r) \cap C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. Thus an open interval $(\lambda_1(r, m_r), \lambda)$ contains no critical values of $\widetilde{J}_r$ (note $0 < \lambda_1(r, m_r) < \lambda \le \lambda_2(r, m_r)$). Hence, $\mathcal{O}(\lambda)$ contains exactly two critical points $\varphi_1 := \varphi_1(m_r)$ and $-\varphi_1$ because $\lambda_1(r, m_r)$ is simple.

Since any component of $\mathcal{O}(\lambda)$ is path-connected (cf. [4, Lemma 3.5]) and $\gamma_2(0) = -\gamma_3(1) = u_-/\Psi_r(u_-) \in \mathcal{O}(\lambda)$ (see (3.8)), there exists a continuous path γ_4 joining $u_-/\Psi_r(u_-)$ and φ_1 or $-\varphi_1$ in $\mathcal{O}(\lambda)$. Since J_r and Ψ_r are even, $-\gamma_i(t) \in S(m_r)$ and $J_r(\gamma_i(t)) = J_r(-\gamma_i(t))$ holds for every $t \in [0,1]$ and $1 \leq i \leq 4$.

Therefore, we can construct a path γ_0 such that $\max_t J_r(\gamma_0(t)) < \lambda$ by considering $\gamma_4^{-1}\cdot\gamma_4\cdot\gamma_2\cdot\gamma_1\cdot\gamma_3\cdot(-\gamma_4)$ or the inverse of it, where $\gamma_j^{-1}(t) := \gamma_j(1-t)$ and $\gamma_k \cdot \gamma_j$ denotes the path defined by $\gamma_k(2t)$ if $0 \leq t \leq 1/2$ and $\gamma_j(2t-1)$ if $1/2 < t \leq 1$. $\square$

4. EXISTENCE RESULT

In this section we prove the following existence result by dividing into cases $r = p > q = r'$ and $r = q < p = r'$.

THEOREM 4.1. *Let $r = p$ or q. If $\lambda > \lambda_1(r, m_r)$ or $\lambda < -\lambda_1(r, -m_r)$ holds, then for any $\mu > 0$, $P_{(r,r',\lambda,\mu)}$ has at least one positive solution.*

Due to Theorem 3.1, Theorem 4.1 and (2.2), we can completely describe the set of generalized eigenvalue as follows:

$$\sigma_G(\Delta_r + \mu\Delta_{r'}, m_r) = \begin{cases} (\lambda_1(r, m_r), \infty) & if \quad m_r \geq 0, \\ (-\infty, -\lambda_1(r, -m_r)) \cup (\lambda_1(r, m_r), \infty) & otherwise. \end{cases}$$

Hence, $\sigma_G(\Delta_r + \mu\Delta_{r'}, m_r)$ is an *open* unbounded set *independent* of the operator $\Delta_{r'}$ and the parameter $\mu > 0$. This is in contrast with the known fact that $\sigma(-\Delta_r, m_r)$ is *closed.* For $r = p$ or q, we define the functional $I_{(r,r',\lambda,\mu)}$ on $W^{1,p}(\Omega)$ as in (2.5), with $r = p$ and $r' = q$.

Remark 4.2. If $u \in W^{1,p}(\Omega)$ is a non-trivial critical point of $I_{(r,r',\lambda,\mu)}$ with $\mu > 0$, then u is a positive solution of $P_{(r,r',\lambda,\mu)}$. Indeed, by taking $-u_-$ as test function, we have

$$0 = \langle I'_{(r,r',\lambda,\mu)}(u), -u_- \rangle = \|u_-\|_{1,r}^r + \mu\|u_-\|_{1,r'}^{r'},$$

whence $u_- \equiv 0$. Thus, u satisfies

$$\int_\Omega \left(|\nabla u|^{r-2}\nabla u \nabla v + |u|^{r-2}uv\right)\mathrm{d}x + \mu\int_\Omega \left(|\nabla u|^{r'-2}\nabla u\nabla v + |u|^{r'-2}uv\right)\mathrm{d}x$$
$$= \lambda \int_{\partial\Omega} m_r u^{r-1} v \mathrm{d}\sigma$$

for any $v \in W^{1,p}(\Omega)$. This means that u is a non-negative solution of $P_{(r,r',\lambda,\mu)}$. The regularity result up to the boundary in [16] and [17] $u \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$. Moreover, then the maximum principle of Vasquez [26] can be applied to ensure positiveness of u.

Proof of the case $r = p > q = r'$ in Theorem 4.1. Fix any $\mu > 0$. We prove the existence of a non-trivial critical point of $I_{(p,q,\lambda,\mu)}$ only in the case of $\lambda > \lambda_1(p, m_p)$ because the other case can be treated by replacing λ and $\lambda_1(p, m_p)$ with $-\lambda$ and $\lambda_1(p, -m_p)$ (note $\lambda m_p = (-\lambda(-m_p))$.

First, we claim that there exist $\delta > 0$ and $\rho > 0$ such that

$$I_{(p,q,\lambda,\mu)}(u) \ge \delta \qquad \text{whenever} \qquad \|u\|_{L^{qs'_q}(\partial\Omega)} = \rho. \tag{4.1}$$

Indeed, for $u \in W^{1,p}(\Omega)$ such that $\int_{\partial\Omega} m_p(u_+)^p d\sigma \le 0$, we have

$$I_{(p,q,\lambda,\mu)}(u) \ge \frac{1}{p}\|u\|_{1,p}^p + \frac{\mu}{q}\|u\|_{1,q}^q \ge \frac{\mu\lambda_1(q,1)}{q}\|u\|_{L^{qs'_q}(\partial\Omega)}^q \tag{4.2}$$

(note that $\mu, \lambda > 0$), where $\lambda_1(q, 1)$ is the first eigenvalue of Δ_q with weight function $m_q \equiv 1$. For $d = \lambda\|m_p\|_{L^{ps'_p}(\partial\Omega)}$, we consider $X(d)$ by (2.13).

For any $u \notin X(d)$ (that is, $\|u\|_{1,p}^p > d\|u\|_{L^{ps'_p}(\partial\Omega)}^p$), we have

$$\begin{aligned} I_{(p,q,\lambda,\mu)}(u) &\ge \frac{1}{p}\|u\|_{1,p}^p + \frac{\mu}{q}\|u\|_{1,q}^q - \frac{\lambda\|m_p\|_{L^{ps'_p}(\partial\Omega)}}{p}\|u\|_{L^{ps'_p}(\partial\Omega)}^p \\ &\ge \frac{\mu\lambda_1(q,1)}{q}\|u\|_{L^{qs'_q}(\partial\Omega)}^q. \end{aligned} \tag{4.3}$$

Concerning the last case, that is, $u \in X(d)$ and $\int_{\partial\Omega} m_p u_+^p d\sigma > 0$, it follows from the definition of $\lambda_1(p, m_p)$ that such u satisfies

$$\|u\|_{1,p}^p \ge \|u_+\|_{1,p}^p \ge \lambda_1(p, m_p)\int_{\partial\Omega} m_p u_+^p d\sigma.$$

Hence, for such u, we obtain

$$\begin{aligned} I_{(p,q,\lambda,\mu)}(u) &\ge \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1(p,m_p)}\right)\|u\|_{1,p}^p + \frac{\mu}{q}\|u\|_{1,q}^q \\ &\ge \left(1 - \frac{\lambda}{\lambda_1(p,m_p)}\right)C^p\|u\|_{L^{qs'_q}(\partial\Omega)}^p + \frac{\mu\lambda_1(q,1)}{q}\|u\|_{L^{qs'_q}(\partial\Omega)}^q \end{aligned} \tag{4.4}$$

due to Lemma 2.4 and the definition of $\lambda_1(p, m_p)$ (not $\lambda/\lambda_1(p, m_p) > 1$), where $C = C(d)$ is the constant in Lemma 2.4.

Thus, noting that $p > q$, our claim (4.1) is shown by taking a sufficiently small $\|u\|_{L^{qs'_q}(\partial\Omega)}$ in (4.4) and by (4.2) and (4.3).

Now, we can choose a sufficiently large $R > 0$ such that

$$\|R\varphi_1\|_{L^{qs'_q}(\partial\Omega)} > \rho \qquad \text{and} \qquad I_{(p,q,\lambda,\mu)}(R\varphi_1) < 0, \tag{4.5}$$

where $\rho > 0$ is the constant in (4.1) and φ_1 is the positive eigenfunction corresponding to $\lambda_1(p, m_p)$ satisfying $\int_{\partial\Omega} m_p\varphi_1^p d\sigma = 1$. In fact, for a sufficiently large $R > 0$ we have

$$\frac{I_{(p,q,\lambda,\mu)}(R\varphi_1)}{R^p} = \frac{\lambda_1(p,m_p)-\lambda}{p} + \frac{\mu\|\varphi_1\|_{1,q}^q}{qR^{p-q}} < 0$$

because of $\lambda > \lambda_1(p,m_p)$ and $p > q$. Recalling that $I_{(p,q\lambda,\mu)}$ satisfies the Palais-Smale condition by of Lemma 2.3, the properties pointed out in (4.1) and (4.5) allow us to apply the mountain pass theorem, which guarantees the existence of a positive critical value $c \geq \delta$ of $I_{(p,q,\lambda,\mu)}$, with $\delta > 0$ in (4.1), namely

$$c := \inf_{\gamma\in\Sigma} \max_{t\in[0,1]} I_{(p,q,\lambda,\mu)}(\gamma(t)),$$
$$\Sigma := \big\{\gamma \in C([0,1]), W^{1,p}(\Omega)); \gamma(0) = 0, \gamma(1) = R\varphi_1\big\}. \qquad \square$$

Proof of the case $r = q < p = r'$ in Theorem 4.1. In this case, our functional in (2.5) is written as follows:

$$I_{(q,p,\lambda,\mu)}(u) = \frac{\mu}{p}\|u\|_{1,p}^p + \frac{1}{q}\|u\|_{1,q}^q - \frac{\lambda}{q}\int_{\partial\Omega} m_q u_+^q \mathrm{d}\sigma.$$

Fix any $\mu > 0$. By Remark 4.2, it is sufficient to show the existence of a non-trivial critical point of $I_{(q,p,\lambda,\mu)}$ only in the case $\lambda > \lambda_1(q,m_q)$ because when $\lambda < 0$ we can argue with $-\lambda$ and $-m_q$.

First, we note that $I_{(q,p,\lambda,\mu)}$ is weakly lower semi-continuous on $W^{1,p}(\Omega)$ since $m_q \in L^s(\partial\Omega)$ and the embedding of $W^{1,p}(\Omega)$ into $L^{qs'}(\partial\Omega)$ is compact. For every $u \in W^{1,p}(\Omega)$ using Hölder inequality, we obtain

$$\begin{aligned} I_{(q,p,\lambda,\mu)}(u) &\geq \frac{\mu}{p}\|u\|_{1,p}^p + \frac{1}{q}\|u\|_{1,q}^q - \frac{\lambda}{q}\|m_q\|_{L^{s_q}(\partial\Omega)}\|u\|^q_{L^{qs'_q}(\partial\Omega)} \\ &\geq \frac{\mu}{p}\|u\|_{1,p}^p + \frac{1}{q}\|u\|_{1,q}^q - \frac{C\lambda}{q}\|m_q\|_{L^{s_q}(\partial\Omega)}\|u\|_{1,q}^q \\ &\geq \frac{\mu}{p}\|u\|_{1,p}^p - \frac{C\lambda}{\lambda_1(p,1)^{q/p}}\|m_q\|_{L^{s_q}(\partial\Omega)}\|u\|_{1,p}^q. \end{aligned}$$

This implies that $I_{(q,p,\lambda,\mu)}$ is coercive and bounded from below on $W^{1,p}(\Omega)$ because $\mu > 0$ and $p > q$. Consequently, by the standard argument [19, Theorem 1.1], we can obtain a global minimizer u_0 of $I_{(q,p,\lambda,\mu)}$.

Finally, to see that $u_0 \neq 0$, we shall show that $\min_{W^{1,p}(\Omega)} I_{(q,p,\lambda,\mu)} < 0$. Recall that the eigenfunction ψ_1 corresponding to $\lambda_1(q,m_q)$ is positive and belongs to $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$.

Because we are considering the case $\lambda > \lambda_1(q,m_q)$, our claim is proved by the following inequality

$$I_{(q,p,\lambda,\mu)}(t\psi_1) = t^q\left(\frac{t^{p-q}\mu}{p}\|\psi_1\|_{1,p}^p + \frac{\lambda_1(q,m_q)-\lambda}{q}\right) < 0$$

for sufficiently small $t > 0$, where we take ψ_1 such that $\int_{\partial\Omega} m_q \psi_1^q \mathrm{d}\sigma = 1$. Hence, the proof is complete. $\square$

Acknowledgments. The authors would like to thank the anonymous reviewers for their helpful and constructive comments that greatly contributed to improving the final version of the paper.

REFERENCES

[1] J. Abreu and G. Madeira, *Generalized eigenvalues of the $(p,2)$-Laplacian under a parametric boundary condition.* Proc. Edinb. Math. Soc. **63** (2020), *1*, 287-303.

[2] A. Anane, O. Chakrone and N. Moradi, *Regularity of the solutions to a nonlinear boundary problem with indefinite weight.* Bol. Soc. Parana. Mat. **29** (2011), *1*, 17-23.

[3] A. Anane, *Étude des valeurs propres et de la résonance pour l'opérateur p-Laplacian.* Thèse de Doctorat, Université Libre de Bruxelles, 1987.

[4] A. Anane, O. Chakrone, B. Karim and A. Zerouali, *The first nonprincipal eigenvalue for a Steklov problem.* Glob. J. Pure Appl. Math. **5** (2009), *3*, 217-228.

[5] A. Anane, O. Chakrone, B. Karim and A. Zerouali, *Nonresonance between the first two eigenvalues for a Steklov problem.* Nonlinear Anal. **72** (2010), 2974-2981.

[6] L. Barbu and G. Moroşanu, *Full description of the eigenvalue set of the Steklov (p,q)-Laplacian.* J. Differential Equations **290** (2021), 1-16.

[7] N. Benouhiba and Z. Belyacine, *A class of eigenvalue problems for the (p,q)-Laplacian in* $\mathbb{R}$. Int. Pure Appl. Math. **50** (2012), 727-737.

[8] N. Benouhiba and Z. Belyacine, *On the solutions of (p,q)-Laplacian problem at resonance.* Nonlinear Anal. **77** (2013), 74-81.

[9] M. Bocea and M. Mihăilescu, *Existence of nonnegative viscosity solutions for a class of problems involving the ∞-Laplacian.* NoDEA Nonlinear Differential Equations Appl. **23** (2016), 1-21.

[10] A. Boukhsas, A. Zerouali, O. Chakrone and B. Karim, *Positive solutions for a $(p,2)$-Laplacian Steklov problem.* Mathematica **64** (2022), *2*, 201–210.

[11] A. Boukhsas and B. Ouhamou, *Steklov eigenvalues problems for generalized (p,q)-Laplacian type operators.* Mem. Differ. Equ. Math. Phys. **85** (2022), 35-51.

[12] S. Cingolani and M. Degiovanni, *Nontrivial solutions for p-Laplace equations with right-hand side having p-linear growth at infinity.* Comm. Partial Differential Equations **30** (2005), 1191-1203.

[13] L. Faria, O. Miyagaki and D. Motreanu, *Comparison and positive solutions for problems with (p,q)-Laplacian and convection term.* Proc. Edinb. Math. Soc. (2) **57** (2014) *3*, 687-698.

[14] M. Fărcăşeanu, A. Grecu, M. Mihăilescu and D. Stancu-Dumitru, *Perturbed eigenvalue problems: an overview.* Studia Univ. Babeş-Bolyai Math. **66** (2021), 55-73.

[15] M. Fărcăşeanu, M. Mihăilescu and D. Stancu-Dumitru, *On the set of eigenvalues of some PDEs with homogeneous Neumann boundary condition.* Nonlinear Anal. **116** (2015), 19-25.

[16] G.M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations.* Nonlinear Anal. **12** (1988), 1203-1219.

[17] G.M. Lieberman, *The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations.* Comm. Partial Differential Equations **16** (1991), 311-361.

[18] S.A. Marano and N.S. Papageorgiou, *Constant sign and nodal solutions of coercive (p, q)-Laplacian problems.* Nonlinear Anal. **77** (2013), 118-129.

[19] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian System.* Appl. Math. Sci. **74**, Springer-Verlag, New York, 1989.

[20] M. Mihailescu, *An eigenvalue problem possessing a continuous family of eigenvalues plus an isolated eigenvalue.* Commun. Appl. Anal. **10** (2011), 701-708.

[21] M. Mihăilescu and G. Moroşanu, *Eigenvalues of $-\Delta_p - \Delta_q$ under Neumann boundary condition.* Canad. Math. Bull. **59** (2016), 606-616.

[22] D. Motreanu and M. Tanaka, *Generalized eigenvalue problems of nonhomogeneous elliptic operators and their application.* Pacific J. Math. **265** (2013), 151-184.

[23] N.E. Sidiropoulos, *Existence of solutions to indefinite quasilinear elliptic problems of (p, q)-Laplacian type.* Elect. J. Differential Equations **162** (2010), 1-23.

[24] M. Tanaka, *Generalized eigenvalue problems for (p, q)-Laplace equation with indefinite weight.* J. Math. Anal. Appl. **419** (2014), *2*, 1181-1192.

[25] M. Tanaka, *Existence of the Fucik type spectrums for the generalized p-Laplace operators.* Nonlinear Anal. **75** (2012), 3407-3435.

[26] J.L. Vazquez, *A strong maximum principle for some quasi-linear elliptic equations*. Appl. Math. Optim. **12** (1984), *1*, 191-202.

[27] M. Wu and Z. Yang, *A class of (p, q)-Laplacian type equation with potentials eigenvalue problem in $\mathbb{R}^N$.* Bound. Value Probl. **2009** (2009), ID 185319, 19 p.

[28] H. Yin and Z. Yang, *A class of (p, q)-Laplacian type equation with noncaveconvex nonlinearities in bounded domain.* J. Math. Anal. Appl. **382** (2011), 843-855.

[29] H. Yin and Z. Yang, *Multiplicity of positive solutions to a (p, q)-Laplacian equations involving critical nonlinearity.* Nonlinear Anal. **75** (2012), 3021-3035.

[30] A. Zerouali, A. Boukhsas, O. Chakrone and B. Karim. *Resonant Steklov eigenvalue problem involving the (p, q)-Laplacian.* Afr. Mat. **30** (2019), *1*, 171-179.

Received November 24, 2021

Moulay Ismail University of Meknes
FST Errachidia, LMIMA Laboratory, Rolali Group, Morocco
abdelmajidboukhsas@gmail.com

Regional Centre of Trades Education and Training
Department of Mathematics, Oujda, Morocco
abdellahzerouali@yahoo.fr

Moulay Ismail University of Meknes
FST Errachidia, LMIMA Laboratory, Rolali Group, Morocco
karembelf@gmail.com

Mohammed First University of Oujda
Faculty of Sciences, Department of Mathematics
Oujda, Morocco
chakrone@yahoo.fr