Dedicated to the memory of Mihnea Colţoiu

DUALITY FOR GORENSTEIN MULTIPLE STRUCTURES ON SMOOTH ALGEBRAIC VARIETIES

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One gives a characterization of the Gorenstein nilpotent scheme structures on a smooth algebraic variety as support, in terms of a duality property of the graded objects associated to two canonical filtrations.

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1. **PRELIMINARIES**

Let X be a smooth connected algebraic variety over an algebraically closed field k. A Cohen-Macaulay scheme Y is called a *multiple structure on* X if the subjacent reduced scheme Y_{red} is X. In this case, all the local rings of Y have the same multiplicity (cf. [6], [7]), which is called *the multiplicity of* Y. To a given multiple structure Y on X one associates canonically three filtrations. To set the frame of the following considerations, let I be the (sheaf) ideal of X in Y and let m be the positive integer such that $I^m \neq 0$, $I^{m+1} = 0$. The three filtrations are:

1. Let $I^{(\ell)}$ be the ideal obtained throwing away the embedded components of I^ℓ and let Z_ℓ be the corresponding scheme. This gives the *Bănică-Forster filtration* (cf. [2]):

$$\begin{array}{ccccccccccc} \mathcal{O}_Y = I^{(0)} & \supset & I = I^{(1)} & \supset & I^{(2)} & \supset & \dots & \supset & I^{(m)} & \supset & I^{(m+1)} = 0 \\ & & X = Z_1 & \subset & Z_2 & \subset & \dots & \subset & Z_m & \subset & Z_{m+1} = Y \end{array}$$

Z_ℓ are not, in general, Cohen-Macaulay. But this is true if $\dim(X) = 1$. The graded associated object $\mathcal{B}(Y) = \bigoplus_{\ell=0}^{m} I^{(\ell)}/I^{(\ell+1)}$ is naturally a graded $\mathcal{O}_X$-algebra. If the schemes Z_ℓ are Cohen-Macaulay, the graded components of $\mathcal{B}(Y)$ are locally free sheaves on X.

2. Let X_ℓ be defined by $I_\ell = 0 : I^{m+1-\ell}$. Again, if $\dim(X) = 1$, X_ℓ are Cohen-Macaulay. This is also true if Y is locally complete intersection of

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multiplicity at most 6 (cf. [7]). In general, this is not always the case. When X_ℓ are Cohen-Macaulay, the quotients $I_\ell/I_{\ell+1}$ are locally free sheaves on X. This filtration was considered in [6].

3. Let Y_ℓ be the scheme given by $J_\ell = 0 : I_{m+1-\ell} = 0 : (0 : I^\ell)$. When X_ℓ is Cohen-Macaulay, Y_ℓ has the same property if Y is Gorenstein. The graded object $\mathcal{A}(Y) = \bigoplus_{\ell=0}^m J_\ell/J_{\ell+1}$ is a graded $\mathcal{O}_X$-algebra and $\mathcal{M}(Y) = \bigoplus_{\ell=0}^m I_\ell/I_{\ell+1}$ is a graded $\mathcal{A}(Y)$-module. This filtration was considered in [7]. Observe that $\mathcal{M}_m(Y) = \mathcal{A}_m(Y) = I_m = J_m$.
The system of the graded components $(\mathcal{A}_0(Y), \dots \mathcal{A}_m(Y); \mathcal{M}_0(Y), \dots \mathcal{M}_m(Y))$ is called *the type of* Y. Y is called *of free type* when all the graded pieces are locally free. As already remarked, in dimension 1, or if Y is lci of multiplicity up to 6, this is the case.

Recall some properties:

1) In general, the above filtrations are different. Take for instance

$$X = Spec(k),\ Y = Spec(k[x,y]/(x^3, xy, y^4)).$$

2) $Z_\ell \subset Y_\ell \subset X_\ell$.

2') there are canonical morphisms: $\mathcal{B}(Y) \to \mathcal{A}(Y) \to \mathcal{M}(Y)$.

3) The multiplications

$$\begin{aligned} \mathcal{A}_{\ell_1} \otimes \mathcal{A}_{\ell_2} &\to \mathcal{A}_{\ell_1+\ell_2} \\ \mathcal{A}_{\ell_1} \otimes \mathcal{M}_{\ell_2} &\to \mathcal{M}_{\ell_1+\ell_2} \end{aligned}$$

are never the zero maps for $\ell_1, \ell_1 \geq 0$, $\ell_1 + \ell_2 \leq m$ (cf. [7]).

4) One has the exact sequences:

$$0 \to \mathcal{M}_\ell(Y) \to \mathcal{O}_{X_{\ell+1}} \to \mathcal{O}_{X_\ell} \to 0$$
$$0 \to \mathcal{A}_\ell(Y) \to \mathcal{O}_{Y_{\ell+1}} \to \mathcal{O}_{Y_\ell} \to 0.$$

5) If Y is Gorenstein of free type, then X_ℓ and $Y_{m+1-\ell}$ are locally algebraically linked (cf. [6]). In particular, one has the exact sequences:

$$0 \to \omega_{X_{m+1-\ell}} \otimes \omega_Y^{-1} \to \mathcal{O}_Y \to \mathcal{O}_{Y_\ell} \to 0$$
$$0 \to \omega_{Y_{m+1-\ell}} \otimes \omega_Y^{-1} \to \mathcal{O}_Y \to \mathcal{O}_{X_\ell} \to 0.$$

6) If Y is Gorenstein of free type, then (cf. [7]):

(a) rank $\mathcal{A}_\ell(Y) =$ rank $\mathcal{M}_{m-\ell}(Y)$

(b) $\mathcal{A}_\ell(Y) = \mathcal{M}_\ell(Y)$ iff rank $\mathcal{A}_\ell(Y) =$ rank $\mathcal{A}_{m-\ell}(Y)$.

2. MAIN THEOREM

The aim of this theorem is to "explain" the equality 6)(a) from above. In fact, one gives a characterization of the Gorenstein multiple structures of free type on a smooth support which generalizes the result from [3].

THEOREM. *Let Y be a free type Cohen-Macaulay multiple structure on a smooth support X. Then Y is Gorenstein if and only if the following conditions are fulfilled:*

(a) $\mathcal{A}_m = \mathcal{M}_m$ *are line bundles.*

(b) *The canonical maps:*

$$\mathcal{A}_\ell \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}_{m-\ell}, \mathcal{M}_m) \cong \mathcal{M}_{m-\ell}^\vee \otimes \mathcal{M}_\ell$$

are isomorphisms.

Proof. Suppose Y is Gorenstein. Applying $\mathcal{H}om(?, \omega_Y))$ to the exact sequences:

$$0 \to \mathcal{A}_m(Y) \to \mathcal{O}_Y \to \mathcal{O}_{Y_m} \to 0$$

$$0 \to \mathcal{M}_m(Y) \to \mathcal{O}_Y \to \mathcal{O}_{X_m} \to 0$$

and taking the restrictions to X one gets:

$$\omega_Y|_X \cong \mathcal{A}_m(Y)^\vee \otimes \omega_X \cong \mathcal{M}_m(Y)^\vee \otimes \omega_X$$

and so (a) is fulfilled. Applying $\mathcal{H}om(?, \omega_Y))$ to the exact sequence:

$$0 \to \mathcal{M}_{m-\ell}(Y) \to \mathcal{O}_{X_{m-\ell+1}} \to \mathcal{O}_{X_{m-\ell}} \to 0$$

one gets the exact sequence:

$$\begin{array}{c} 0 \to \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_{X_{m-\ell}}, \omega_Y) \to \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_{X_{m-\ell+1}}, \omega_Y) \to \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M}_{m-\ell}(Y), \omega_Y) \to 0 \\ \| \\ \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}_{m-\ell}(Y), \omega_X) \end{array}$$

which tensored with ω_Y^{-1} gives the exact sequence:

$$\begin{array}{ccccccc} 0 \to & \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_{X_{m-\ell}}, \mathcal{O}_Y) & \to & \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_{X_{m-\ell+1}}, \mathcal{O}_Y) & \to & \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M}_{m-\ell}(Y), \omega_Y) \otimes \omega_Y^{-1} & \to 0 \\ & \| & & \| & & \| & \\ & 0 : (0 : I^{\ell+1}) & & 0 : (0 : I^\ell) & & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}_{m-\ell}(Y), \omega_X \otimes \omega_Y^{-1}) & \\ & & & & & \| & \\ & & & & & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}_{m-\ell}(Y), \mathcal{M}_m(Y)) & \end{array}$$

and so (b) is fulfilled.

Assume now that (a) and (b) are fulfilled. Let $x \in X$ be a fixed point and put $A := \mathcal{O}_{X,x}$, $B := \mathcal{O}_{Y,x}$. Let us denote (abusively) the sheaf stalks I_x, $(I_\ell)_x$ and $(J_\ell)_x$ by I, I_ℓ and J_ℓ, respectively. In particular, $A = B/I$.

To any finitely generated B-module N one associates the B-modules $N^\vee := \mathrm{Hom}_B(N, B)$ and $N' := \mathrm{Hom}_B(N, \omega_B)$. If J is an ideal of B then $(B/J)^\vee$ (resp., $(B/J)'$) can be identified to the set of elements of B (resp., ω_B) annihilated by J. In particular, $(B/I_{m+1-\ell})^\vee$ can be identified to J_ℓ. If B/J is Cohen-Macaulay of dimension $\dim B$ then $(B/J)' \simeq \omega_{B/J}$.

By hypothesis (a), $(B/I)^\vee = J_m \simeq A$. Since $A = B/I$ is Gorenstein (even regular) one has $(B/I)' \simeq \omega_A \simeq A$. Fix an isomorphism $(B/I)^\vee \xrightarrow{\sim} (B/I)'$. Since the map $\mathrm{Hom}_B(B, \omega_B) \to \mathrm{Hom}_B(J_m, \omega_B)$ is surjective (B/J_m is Cohen-Macaulay of dimension $\dim B$, hence $\mathrm{Ext}^i_B(B/J_m, \omega_B) = 0$ for $i > 0$) the above isomorphism can be extended to a morphism of B-modules $u\colon B \to \omega_B$. u induces, for any B-module N, a morphism $N^\vee \to N'$, namely $\mathrm{Hom}_B(\mathrm{id}_N, u)$. If N is annihilated by I, the image of any morphism of B-modules $N \to B$ (resp., $N \to \omega_B$) is contained in $(B/I)^\vee$ (resp., $(B/I)'$), hence $N^\vee \simeq \mathrm{Hom}_A(N, (B/I)^\vee)$ (resp., $N' \simeq \mathrm{Hom}_A(N, (B/I)'))$. Since u maps $(B/I)^\vee$ isomorphically onto $(B/I)'$ one deduces that, when N is annihilated by I, the morphism $N^\vee \to N'$ is an isomorphism.

Consider now, for $0 \le \ell \le m$, the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (B/I_{m-\ell})^\vee & \longrightarrow & (B/I_{m+1-\ell})^\vee & \longrightarrow & (I_{m-\ell}/I_{m+1-\ell})^\vee & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (B/I_{m-\ell})' & \longrightarrow & (B/I_{m+1-\ell})' & \longrightarrow & (I_{m-\ell}/I_{m+1-\ell})' & \longrightarrow & 0
\end{array}$$

whose vertical maps are induced by u.

The upper row is exact to the right by hypothesis (b) ($(B/I_{m+1-\ell})^\vee = J_\ell$ and $(I_{m-\ell}/I_{m+1-\ell})^\vee \simeq \mathrm{Hom}_A(I_{m-\ell}/I_{m+1-\ell}, I_m)$), while the lower row is exact to the right because $B/I_{m-\ell}$ is Cohen-Macaulay of dimension $\dim B$, hence $\mathrm{Ext}^i_B(B/I_{m-\ell}, \omega_B) = 0$, for $i > 0$. The right vertical map is an isomorphism because $I_{m-\ell}/I_{m+1-\ell}$ is annihilated by I. One deduces, by descending induction on $m \ge \ell \ge 0$, that $(B/I_{m+1-\ell})^\vee \to (B/I_{m+1-\ell})'$ is an isomorphism. In particular, for $\ell = 0$, one gets that $u\colon B \to \omega_B$ is an isomorphism hence B is Gorenstein. □

Remark 1. The case of embedded multiple structures, (as in [2], [6], [7]) *i.e.*, $X \subset Y \subset P$, with X, P smooth connected algebraic varieties and Y a multiple structure on X, leads to similar filtrations. Denote now by I the ideal of X in P, by J the ideal of Y in P and suppose $I^{m+1} \subset J$, $I^m \not\subset J$. Then the filtrations are (we consider now only the ideals):

1. Let $I^{(k)}$ be the ideal obtained throwing away the embedded components of $I^{(k)} + J$.

2. Let $I_\ell = J : I^{m+1-\ell}$.

3. Let $J_\ell = J : (J : I^\ell)$.

All the above considerations apply also in this case.

Remark 2. In the case of quasiprimitive structures, *i.e.*, when $\operatorname{rank}\mathcal{A}_\ell = \operatorname{rank}\mathcal{M}_\ell = 1$ for all ℓ, all the above filtrations are equal, whence $\mathcal{B} = \mathcal{A} = \mathcal{M}$, so one can express the conditions in the theorem only in terms of Bănică-Forster filtration. This was done in [3] for the case of quasiprimitive multiple structures on smooth curves in a threefold.

Remark 3. The above duality gives a direct explanation of the identities in the Chern classes of the bundles which appear in various constructions in [7] (*e.g.*, 4.14, 4.16. *loc.cit.*).

3. EXAMPLES

A class of nilpotent structures Y on a smooth connected algebraic variety X which are of free type is of those which are locally monomial, *i.e.*, the local completed rings of Y are of the shape $B = k[[\mathbf{x}, \mathbf{u}]]/J$, where $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{u} = (u_1, \ldots u_m)$ are indeterminants and J is an ideal generated by monomials in $x_1, \ldots, x_n$ such that $B_{\text{red}} = A = k[[u_1, \ldots, u_m]]$.

We exemplify the use of the above duality for the case when the local structure is given by ideals of the shape $J = (x_1^3, x_2^3, \mathbf{x}, u_1, \ldots u_m)$, where $\mathbf{x}$ is a finite set of variables, namely we consider X a closed connected smooth algebraic variety in a projective space $\mathbb{P}$ and consider a locally Cohen-Macaulay "nilpotent" (*i.e.*, not reduced) algebraic scheme Y in $\mathbb{P}$ with $Y_{\text{red}} = X$. If $\mathcal{I}$, $\mathcal{J}$ are the ideals of X, respectively of Y, in $\mathbb{P}$, suppose that they are defined locally by ideals of the shape $I = (x, y, \mathbf{z})$, $J = (x^3, y^3, \mathbf{z})$ in $k[[[x, y, \mathbf{z}, \mathbf{u}]]]$, where $\mathbf{z}$ and $\mathbf{u}$ are finite sets of variables. A direct computation shows that the three filtrations of this nilpotent structure coincide and, locally, they are defined by

$$\begin{aligned} J_5 = \ & (x^3, y^3, \mathbf{z}) \subset J_4 = (x^3, x^2y^2, y^3, \mathbf{z}) \subset J_3 = (x^3, x^2y, xy^2, y^3, \mathbf{z}) \\ & \subset J_2 = (x^2, xy, y^2, \mathbf{z}) \subset J_1 = I = (x, y, \mathbf{z}). \end{aligned}$$

Then the three graded objects associated to Y coincide and they have the shape:

$$\mathcal{B}(Y) = \mathcal{A}(Y) = \mathcal{M}(Y) = \mathcal{O}_X \oplus E \oplus F \oplus G \oplus L$$

where E, F, G, L are vector bundles of rank, respectively, 2, 3, 2, 1. The local description implies that F is the second symmetric power of E, and the above duality Theorem gives:

$$G \cong E^\vee \otimes L\ ,\ S^2(E) \cong (S^2(E))^\vee \otimes L.$$

From here it follows the equality of Chern classes $3c_1(L) = 6c_1(E)$.

Let X be a line in $\mathbb{P}^n, n \geq 3$. Take $(x : y : \mathbf{z} : u : v)$ homogeneous coordinates in $\mathbb{P}^n$, such that $(x, y, \mathbf{z})$ defines the line X and (u, v) are homogeneous coordinates on X.

Example 1. When $n = 3$ the ideal $\mathcal{I}$ of X is of the shape (x, y) and we show that $\mathcal{J}$ is globally complete intersection of the form (x^3, y^3). Indeed,

1. $E \cong \mathcal{I}/\mathcal{I}^2 \cong 2\mathcal{O}_X(-1)$ is the conormal bundle of X in $\mathbb{P}^3$.
2. It follows $L \cong \mathcal{O}_X(-4)$, $F \cong 3\mathcal{O}_X(-2)$, $G \cong 2\mathcal{O}_X(-3)$.
3. We have globally $\mathcal{J}_2 = \mathcal{I}^2$, $\mathcal{J}_3 = \mathcal{I}^3$, so in the canonical filtration $Y_2 =$ first infinitesimal neighbourhood and $Y_3 =$ the second infinitesimal neighbourhood of X. Then Y_4 is given by an exact sequence:

$$0 \to \mathcal{J}_4/\mathcal{I}\mathcal{J}_3 \to \mathcal{J}_3/\mathcal{I}\mathcal{J}_3 (\cong \mathcal{I}^3/\mathcal{I}^4 \cong 4\mathcal{O}_X(-3)) \xrightarrow{p} G(\cong 2\mathcal{O}_X(-3)) \to 0\,.$$

Observe that p is not an arbitrary surjection; its kernel should contain, locally in each point, two perfect cubes in $S^3(\mathcal{I}/\mathcal{I}^2)$. After a change of coordinates in $\mathbb{P}^3$, if necessary, $\mathcal{J}_4$ is defined by the ideal (x^3, x^2y^2, y^3).

4. Finally, Y_5 is given by the exact sequence:

$$0 \to \mathcal{J}_5/\mathcal{I}\mathcal{J}_4 \to \mathcal{J}_4/\mathcal{I}\mathcal{J}_4 (\cong \mathcal{O}_X(-4) \oplus 2\mathcal{O}_X(-3)) \xrightarrow{q} L(\cong \mathcal{O}_X(-4)) \to 0.$$

where q is the canonical projection.

This proves the assertion.

Example 2. We take a line X in $\mathbb{P}^4$; the ideal of Xin $\mathbb{P}^4$ is (x, y, z) and the homogeneous coordinates on X are u, v. For simplicity, we take $E \cong \mathcal{O}_X(-1) \oplus \mathcal{O}_X$. Then $F \cong \mathcal{O}_X(-2) \oplus \mathcal{O}_X(-1) \oplus \mathcal{O}_X$, $L \cong \mathcal{O}_X(-2)$, $G \cong \mathcal{O}_X(-2) \oplus \mathcal{O}_X(-1)$.

1. $\mathcal{J}_2$ should be given by an exact sequence

$$0 \to \mathcal{J}_2/\mathcal{I}^2 \to \mathcal{I}/\mathcal{I}^2 (\cong 3\mathcal{O}_X(-1)) \xrightarrow{p_1} E(\cong \mathcal{O}_X(-1) \oplus \mathcal{O}_X) \to 0\,.$$

We take $p_1 = \begin{pmatrix} 0 & 0 & 1 \\ -v & u & 0 \end{pmatrix}$ and then $\mathcal{J}_2 =$ ideal defined by $(ux+vy)+(x, y, z)^2$ and one gets $\mathcal{J}_2/\mathcal{I}^2 \cong \mathcal{O}_X(-2)$ and $\mathcal{I}^2/\mathcal{I}\mathcal{I}_2 \cong \mathcal{O}_X(-2) \oplus \mathcal{O}_X(-1) \oplus \mathcal{O}_X$. From the exact sequence:

$$0 \to \mathcal{I}^2/\mathcal{I}\mathcal{J}_2 \to \mathcal{J}_2/\mathcal{I}\mathcal{J}_2 \to \mathcal{J}_2/\mathcal{I}^2 \to 0$$

one gets $\mathcal{J}_2/\mathcal{I}\mathcal{J}_2 \cong 2\mathcal{O}_X(-2) \oplus \mathcal{O}_X(-1) \oplus \mathcal{O}_X$.

2. $\mathcal{J}_3$ should be given by an exact sequence

$$0 \to \mathcal{J}_3/\mathcal{I}\mathcal{J}_2 \to \mathcal{J}_2/\mathcal{I}\mathcal{J}_2 \xrightarrow{p_2} F(\cong \mathcal{O}_X(-2) \oplus \mathcal{O}_X(-1) \oplus \mathcal{O}_X) \to 0\,.$$

Taking p_2 to be the projection on a "well chosen" direct summand of $\mathcal{J}/\mathcal{I}\mathcal{I}_2$, one gets $\mathcal{J}_3 =$ the ideal $(ux + vy) + \mathcal{I}^3$. One gets $\mathcal{J}_3/\mathcal{I}\mathcal{J}_2 \cong \mathcal{O}_X(-2)$. Direct computation shows that $\mathcal{I}\mathcal{J}_2/\mathcal{I}\mathcal{J}_3 \cong \mathcal{O}_X(-3) \oplus \mathcal{O}_X(-2) \oplus \mathcal{O}_X(-1) \oplus \mathcal{O}_X$ and from the exact sequence

$$0 \to \mathcal{I}\mathcal{J}_2/\mathcal{I}\mathcal{J}_3 \to \mathcal{J}_3/\mathcal{I}\mathcal{J}_3 \to \mathcal{J}_3/\mathcal{I}\mathcal{J}_2) \to 0$$

one gets $\mathcal{J}_3/\mathcal{I}\mathcal{J}_3 \cong \mathcal{O}_X(-3) \oplus 2\mathcal{O}_X(-2) \oplus \mathcal{O}_X(-1) \oplus \mathcal{O}_X$.

3. $\mathcal{J}_4$ should be given by an exact sequence

$$0 \to \mathcal{J}_4/\mathcal{I}\mathcal{J}_3 \to \mathcal{J}_3/\mathcal{I}\mathcal{J}_3 \xrightarrow{p_3} G(\cong \mathcal{O}_X(-2) \oplus \mathcal{O}_X(-1)) \to 0\,.$$

As above, one can choose p_3 to be a projection on a direct summand and one gets $\mathcal{J}_4 =$ the ideal $(ux + vy, x^3, x^2y, xy^2, y^3, z^3, x^2z^2, xyz^2, y^2z^2)$. Then one computes $\mathcal{J}_4/\mathcal{I}\mathcal{J}_3 \cong \mathcal{O}_X(-3) \oplus \mathcal{O}_X(-2) \oplus \mathcal{O}_X$. Direct computation shows $\mathcal{I}\mathcal{J}_3/\mathcal{I}\mathcal{J}_4 \cong \mathcal{O}_X(-2)$ and so $\mathcal{J}_4/\mathcal{I}\mathcal{J}_4 \cong \mathcal{O}_X(-3) \oplus 2\mathcal{O}_X(-2) \oplus \mathcal{O}_X$.

4. $\mathcal{J}_5 = \mathcal{J}$ is obtained from an exact sequence

$$0 \to \mathcal{J}_5/\mathcal{I}\mathcal{J}_4 \to \mathcal{J}_4/\mathcal{I}\mathcal{J}_4 \xrightarrow{p_4} L(\cong \mathcal{O}_X(-2)) \to 0\,.$$

Again, p_4 is chosen to be a suitable projection and one gets $\mathcal{J} =$ the ideal $(ux + vy, x^3, x^2y, xy^2, y^3, z^3)$.

Remark 4. The above example can be constructed directly from a triple structure X' in the hyperplane $\mathbb{P}^3 \subset \mathbb{P}^4$ given by $z = 0$, namely take a triple line of ideal $(ux + vy, x^3, x^2y, xy^2, y^3)$ in $\mathbb{P}^3$. Its ideal in $\mathbb{P}^4$ is $(ux + vy, x^3, x^2y, xy^2, y^3, z)$ and our multiplicity 9 structure is evidently a simple tripling of Y'.

Remark 5. An easy generalization of the above example is given by $\mathcal{J} =$ the ideal $(ax + by, x^3, x^2y, xy^2, y^3, z^3)$, where a, b are form in u, v of the same degree, say $d+1 \geq 1$, without common zeros on X. This example corresponds to $E \cong \mathcal{O}_X(-1) \oplus \mathcal{O}_X(d)$; one has $F \cong \mathcal{O}_X(-2) \oplus \mathcal{O}_X(d-1) \oplus \mathcal{O}_X(2d)$, $L \cong \mathcal{O}_X(2d-2)$, $G \cong \mathcal{O}_X(d-2) \oplus \mathcal{O}_X(2d-1)$. When a, b have degree 2, Y is in the Hilbert scheme of a disjoint union of 9 lines and the curves in the canonical stratification of Y are in the Hilbert scheme of 3, 6, respectively 8 disjoint lines.

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REFERENCES

[1] C. Bănică and O. Forster, *Sur les structures multiples.* Manuscript, 1981.

[2] C. Bănică and O. Forster, *Multiplicity structures on space curves.* Contemp. Math. **58** (1986), 47–64.

[3] M. Boratynsky, *Locally complete intersection multiple structures on smooth algebraic curves*. Proc. Amer. Math. Soc. **115** (1992), 877–879.

[4] D. Ferrand, *Courbes gauches et fibrés de rang 2*. C. R. Math. Acad. Sci. Paris **281** (1975), 345–347.

[5] R. Fossum, *Commutative extensions by canonical modules are Gorenstein rings*. Proc. Amer. Math. Soc. **40** (1973), 395–400.

[6] N. Manolache, *Cohen-Macaulay nilpotent structures*. Rev. Roumaine Math. Pures Appl. **31** (1986), 563–575.

[7] N. Manolache, *Multiple structures on smooth support*. Math. Nachr. **167** (1994), 157–202.

[8] N. Manolache, *Gorenstein multiple structures on smooth algebraic varieties*. Preprint, arXiv:00706.2204v1 (math.AG).

[9] N. Manolache, *Linkage extensions*. Rev. Roumaine Math. Pures Appl. **52** (2007), 397–403.

[10] C. Peskine and L. Szpiro, *Liaison des variétes algébriques*. Invent. Math. **26** (1977), 271–302.

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