Dedicated to the memory of Mihnea Colţoiu

DIMENSIONS OF MEASURES, DEGREES, AND FOLDING ENTROPY IN DYNAMICS

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In this survey, we present some methods in the dynamics and dimension theory for invariant measures of hyperbolic endomorphisms (smooth non-invertible maps), and for conformal iterated function systems with overlaps. For endomorphisms, we recall the notion of asymptotic degree of an equilibrium measure, which is shown to be related to the folding entropy; this degree is then applied to dimension estimates. For finite iterated function systems, we present the notion of overlap number of a measure, which is related to the folding entropy of a lift transformation, and also give some examples when it can be computed or estimated. We apply overlap numbers to prove the exact dimensionality of invariant measures, and to obtain a geometric formula for their dimension. Then, for countable conformal iterated function systems with overlaps, the projections of ergodic measures are shown to be exact dimensional, and we give a dimension formula. Relations with ergodic number theory, continued fractions, and random dynamical systems are also presented.

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1. **INTRODUCTION**

In this survey, we present some relatively recent results in the dynamics and dimension theory of invariant measures for hyperbolic endomorphisms (smooth non-invertible maps), and also for conformal nonlinear iterated function systems (finite or countable), and iterated function systems with place-dependent probabilities.

Dimension theory for invariant measures in dynamical systems and thermodynamic formalism have a rich history; some references are [2], [4], [5], [16], [37], [39], [43]. Young proved in [45] that the Hausdorff dimension of a hyperbolic invariant measure μ for a surface diffeomorphism, is given by a formula involving entropy and the Lyapunov exponents, namely

$$HD(\mu) = h(\mu)(\frac{1}{\chi_u(\mu)} - \frac{1}{\chi_s(\mu)}).$$

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In [23], Manning showed that for an Axiom A diffeomorphism of a surface which preserves an ergodic measure μ, the entropy $h(\mu)$ is in fact equal to the product of the positive Lyapunov exponent of μ and the dimension of the set of μ-generic points from an unstable manifold. Then in [18], Ledrappier and Young established a formula for the entropy of an invariant hyperbolic measure μ for a diffeomorphism of a compact Riemannian manifold, involving the Lyapunov exponents of μ and the dimensions of μ in the stable and unstable directions. In [22], Mañe proved the exact dimensionality for ergodic measures which are invariant to rational maps. Later on, Barreira, Pesin and Schmeling [3] showed that every hyperbolic measure μ invariant under a $C^{1+\varepsilon}$ diffeomorphism of a smooth Riemannian manifold has asymptotically almost local product structure; this, in turn, was used to prove the famous Eckmann–Ruelle Conjecture (see [7]), namely that μ is exact dimensional, and that the Hausdorff dimension of μ is equal to its pointwise dimension and equal to the sum between the dimensions in the stable direction and in the unstable direction. For hyperbolic endomorphisms, it was shown in [28] that conditional measures on stable manifolds are exact dimensional.

In the case of finite conformal iterated function systems with overlaps, Feng and Hu proved in [11] that the projection of any ergodic measure from the shift space, is exact-dimensional on the fractal limit set; they also gave a formula for the Hausdorff dimension of this projection measure, by using the projection entropy and the Lyapunov exponent. Later, in [25] we gave another proof of the exact dimensionality of self-conformal measures (in fact, the exact dimensionality of a larger class of measures) by employing methods from the dynamics of hyperbolic endomorphisms; in particular, a geometric formula for the dimension of the measure was given by using overlap numbers. The case of countable conformal iterated function systems with overlaps was solved by Mihailescu and Urbański in [32], where the projection of any ergodic measure from the shift space to the limit set was shown to be exact dimensional, and a dimension formula was found. In fact, we proved a more general result for random countable conformal iterated function systems with overlaps. In the countable IFS case there are several important differences from the finite case, and the methods are different. For example, the limit set of a countable IFS is not necessarily compact; also the behaviour near the boundary of the countably many system maps plays a significant role. Iterated functions systems are important also for the study of examples from ergodic number theory (for *e.g.* [13], [15], [21], [24], [31], [38]).

The structure of this survey is the following:

First, we recall the notion of folding entropy of an invariant measure for an endomorphism, introduced by Ruelle [40]. Then, we recall the notion defined in [33] of the asymptotic degree for an arbitrary equilibrium measure for a hyperbolic non-degenerate endomorphism; this is in fact an average rate of growth of the number of generic n-preimages of points. It was shown to be related also to the folding entropy. We employ this asymptotic degree for dimension estimates.

Then, we present the case of finite conformal iterated function systems $\mathcal{S}$ and the notion of overlap numbers for projection measures on the fractal limit set. The overlap number is shown to be related to the folding entropy of a lift measure for the lift endomorphism of $\mathcal{S}$. We recall that overlap numbers are used to prove the exact dimensionality for certain projection measures, in particular for self-conformal measures; and to obtain a geometric formula for their Hausdorff (and pointwise) dimension. In some cases, it is possible to compute or estimate the overlap numbers.

Next, we study the significantly different case of countable conformal iterated function systems with overlaps, and recall the results on exact dimensionality of projection measures, and the dimension formula obtained. An application is to invariant measures for random continued fractions. Moreover, we present an application to countable iterated function systems with overlaps and place-dependent probabilities, a case related to random systems with complete connections.

2. ASYMPTOTIC DEGREES OF MEASURES FOR HYPERBOLIC ENDOMORPHISMS

Let us consider a smooth ($\mathcal{C}^2$) map $f : M \to M$ on a $\mathcal{C}^2$ manifold M, and take a compact basic (locally maximal) set Λ (for the definition of basic set see below, or [16]). We assume that f is non-invertible and hyperbolic on Λ, i.e there exists a continuous splitting of the tangent bundle of M over the inverse limit $\hat{\Lambda}$ of Λ and f. The inverse limit of Λ with respect to f is defined by:

$$\hat{\Lambda} := \{\hat{x} = (x, x_{-1}, x_{-2}, \ldots), f(x_j) = x_{j+1}, x_j \in \Lambda, j \leq -1\}.$$

If we want to emphasize the dependence of $\hat{\Lambda}$ on f, we write $\hat{\Lambda}_f$. The notation $\hat{M}$ is similar.

The case of a hyperbolic non-invertible transformation is very different from the expanding case and from the case of hyperbolic diffeomorphisms, for *e.g.* [7], [39], [40], [6], [12], [20], [27], [28]. In the hyperbolic non-invertible case, for every prehistory $\hat{x} \in \hat{\Lambda}$ there exist a local stable manifold $W^s_r(x)$ and a local unstable manifold $W^u_r(\hat{x})$. Notice that the local stable manifold depends only on the base point x, since it is defined as the set of points $y \in M$ such that

$d(f^j(y), f^j(x)) < r, j \geq 0$; so it depends only on the forward trajectory of x. By contrast, the local unstable manifold is

$$W^u_r(\hat{x}) := \{y, \exists \hat{y} \in \hat{M} \text{ with } \mathrm{d}(y_{-j}, x_{-j}) < r, j \geq 0\},$$

thus the local unstable manifold depends on the whole past trajectory, *i.e.* on the prehistory $\hat{x} \in \hat{\Lambda}$. There are examples where there exist uncountably many local unstable manifolds going through a point x, and this happens actually for most points $x \in \Lambda$ (see [27]).

Moreover, another complication is that, in general, the map f is not constant-to-1 on the fractal set Λ. Hence a good notion of "degree" of $f|_\Lambda$ with respect to an invariant measure on Λ is necessary. The number of preimages belonging to Λ is important in dimension estimates for the invariant set and invariant measures (see for *e.g.* [29]). We defined in [33] a notion of asymptotic degree with respect to an arbitrary equilibrium measure μ_ϕ (where ϕ is a Hölder continuous potential on Λ). In particular, for the measure of maximal entropy μ_0 on Λ, we obtain the average logarithmic growth of the number of n-preimages that remain in Λ (when $n \to \infty$), which can be considered as the "topological degree" of f on Λ. This asymptotic degree was then used in [33] to obtain dimension estimates of stable sections through basic sets.

By *basic set* (or *locally maximal set* [16]), we understand a compact f-invariant set $\Lambda \subset M$ such that f is topologically transitive on Λ and there exists a neighbourhood U of Λ such that,

$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U).$$

The *Jacobian* of an f-invariant measure μ with respect to the endomorphism f (see Parry, [36]) is defined μ-a.e by

$$J_f(\mu)(x) = \lim_{r \to 0} \frac{\mu(f(B(x,r))}{\mu(B(x,r)}.$$

The limit above can be shown to exist locally as the Radon-Nikodym derivative between two absolutely continuous measures.

If μ is an f-invariant probability measure on M, then Ruelle introduced in [40] the *folding entropy* $F_f(\mu)$, as being the conditional entropy $H_\mu(\epsilon|f^{-1}\epsilon)$, where ϵ is the single point partition and $f^{-1}\epsilon$ is the fiber partition associated to f on M. It can be shown (see [20]) that,

$$F_f(\mu) := H_\mu(\epsilon|f^{-1}\epsilon) = \int J_f(\mu) d\mu. \tag{1}$$

The folding entropy $F_f(\mu)$ is bounded above by the metric entropy $h_f(\mu)$, and bounded below by $h_f(\mu)$ plus the sum of negative Lyapunov exponents of μ (see [19]). It was studied also in [33], [44]. A related notion is that of entropy production for an f-invariant measure μ, namely $e_f(\mu) := F_f(\mu) - \int \log|\det Df(x)| d\mu(x)$ (see [40], and the references in [34]).

Let now $\phi : \Lambda \to \mathbb{R}$ be a Hölder continuous potential on the compact basic set Λ; so there exists a unique equilibrium (Gibbs) measure μ_ϕ on Λ, which is the only ergodic probability f-invariant measure for which the supremum is attained in Variational Principle for pressure ([4], [43]). Hence

$$P(\phi) = h(\mu_\phi) + \int_\Lambda \phi \, d\mu_\phi.$$

Since the set Λ is not necessarily totally invariant, the measurable function

$$d_n(x) := \mathrm{Card}(f^{-n}(f^n(x)) \cap \Lambda), \ x \in \Lambda,$$

may be non-constant on Λ (see the class of examples in [27]). Define now the finite set of n-preimages of $f^n(x)$ which are τ-"generic" with respect to ϕ, namely let

$$G_n(x, \mu, \tau) := \{y \in f^{-n}(f^n x) \cap \Lambda, \text{ s.t } |\frac{S_n\phi(y)}{n} - \int \phi \, d\mu| < \tau\}, \tag{2}$$

Definition 1. Denote by $d_n(x, \mu, \tau) :=$ Card $G_n(x, \mu, \tau), x \in \Lambda, n > 0$, $\tau > 0$. The function $d_n(\cdot, \mu, \tau)$ is measurable, nonnegative and bounded on Λ (if f is locally injective on Λ).

In the following Theorem 1 proved in [33], we gave the basis for the definition of a *measure-theoretic asymptotic degree* with respect to an equilibrium measure μ_ϕ as above. The measure-theoretic degree represents the asymptotic rate of growth of the number of generic n-preimages from Λ, when $n \to \infty$, and we proved that it is connected to the folding entropy of μ_ϕ.

Theorem 1 (Measure-theoretic asymptotic degree, [33]). *Let $f : M \to M$ be a $\mathcal{C}^2$ non-invertible map and let Λ be a basic set for f, such that f is hyperbolic on Λ and it does not have critical points in Λ. Let also ϕ be a Hölder continuous potential on Λ and μ_ϕ be the equilibrium measure associated to ϕ. Then we have the following formula:*

$$\lim_{\tau \to 0} \lim_{n \to \infty} \frac{1}{n} \int_\Lambda \log d_n(x, \mu_\phi, \tau) \, d\mu_\phi(x) = F_f(\mu_\phi).$$

An interesting particular case is when we consider the measure of maximal entropy, which is in fact the equilibrium measure μ_0 of $\phi \equiv 0$ (or equivalently to a constant potential). In this case, every n-preimage from Λ is generic with respect to μ_0. Thus we obtain a "topological degree" of the restriction $f|_\Lambda$:

Corollary 1 (Topological degree, [33]). *In the setting of Theorem 1, denote by μ_0 the unique measure of maximal entropy of $f|_\Lambda$.*

If $d_n(x) := \mathrm{Card}(f^{-n}(f^n x) \cap \Lambda)$ *for* $n \geq 1$*, then we have:*

$$\lim_{n\to\infty} \frac{1}{n} \log d_n(x) = F_f(\mu_0),\ \mu_0 - a.e\ x \in \Lambda,$$

and

$$\lim_{n\to\infty} \frac{1}{n} \int_\Lambda \log d_n(x)\ \mathrm{d}\mu_0(x) = F_f(\mu_0).$$

The above Theorem and discussion permit us to state now the following:

Definition 2. In the setting of Theorem 1, define the **asymptotic logarithmic degree of** $f|_\Lambda$ (with respect to the measure of maximal entropy μ_0) by: $a_l(f,\Lambda) := \lim\limits_n \frac{1}{n} \int_\Lambda \log d_n(x) \mathrm{d}\mu_0(x)$. The **asymptotic degree** of $f|_\Lambda$ is then defined as the number

$$d_\infty(f,\Lambda) := e^{a_l(f,\Lambda)}.$$

Similarly, we define the **asymptotic degree with respect to the measure** μ_ϕ on Λ, as

$$d_\infty(f,\mu_\phi) := \exp\big(\lim_{\tau\to 0}\lim_{n\to\infty} \frac{1}{n} \int_\Lambda \log d_n(x,\mu_\phi,\tau)\ \mathrm{d}\mu_\phi(x)\big).$$

In particular, if $f|_\Lambda$ is d-to-1, then $d_\infty(f,\Lambda) = d$, and $F_f(\mu_0) = \log d$.

Moreover, the measure degree was used to obtain **estimates** for the Hausdorff dimension of slices through the set Λ, when f is hyperbolic on Λ. If we consider the potential $\Phi^s(x) := \log |Df_s(x)|,\ x \in \Lambda$, then for any fixed number $\gamma \leq h_{top}(f|_\Lambda)$, we have that the function

$$t \mapsto P(t\Phi^s - \gamma),$$

is strictly decreasing and convex, it is non-negative for $t = 0$, and converges to $-\infty$ if $t \to \infty$. So this pressure function has a unique zero, which will appear in the next Theorem. Denote by E^s_x the stable tangent space at $x \in \Lambda$, and by $W^s_r(x)$ the local stable manifold at x.

Theorem 2 (Dimension estimates, [33]). *In the setting of Theorem 1, assume that* f *is conformal on local stable manifolds on the saddle basic set* Λ*, and that* μ_ϕ *is the equilibrium measure of a Hölder continuous potential* ϕ *on* Λ*; denote* $\Phi^s(y) := \log |Df|_{E^s(y)}|,\ y \in \Lambda$*. Then there exists a Borel set* $\mathcal{K}(\mu_\phi) \subset \Lambda$ *such that* $\mu_\phi(\mathcal{K}(\mu_\phi)) = 1$*, and for every* $x \in \Lambda$ *we have:*

$$HD\big(W^s_r(x) \cap \mathcal{K}(\mu_\phi)\big) \leq t^s_{d_\infty(f,\mu_\phi)},$$

where $t^s_{d_\infty(f,\mu_\phi)}$ *is the unique zero of the pressure function*

$$t \to P(t\Phi^s - \log d_\infty(f,\mu_\phi)).$$

3. FINITE CONFORMAL ITERATED SYSTEMS AND OVERLAP NUMBERS

Let us take a finite set I and a function system $\mathcal{S} = \{\phi_i, i \in I\}$, with $\phi_i : \bar{U} \to \mathbb{R}^d, i \in I$, being conformal and injective maps on the closure $\bar{U}$ of a bounded open set $U \subset \mathbb{R}^d$, which are uniformly contracting on $\bar{U}$, i.e $\exists \gamma \in (0,1)$ with $|\phi_i'| < \gamma, \forall i \in I$ (for *e.g.* [8]). Let Σ_I^+ denote the 1-sided symbolic space $\{\omega = (\omega_1, \omega_2, \ldots), \omega_i \in I, i \geq 1\}$, with canonical metric and topology, equipped with the canonical shift transformation $\sigma : \Sigma_I^+ \to \Sigma_I^+$. Also, denote by $[\omega_1 \ldots \omega_n]$ the cylinder $\{\eta \in \Sigma_I^+, \eta_1 = \omega_1, \ldots, \eta_n = \omega_n\}$. In the sequel, let us denote by

$$\phi_{i_1 \ldots i_p} := \phi_{i_1} \circ \phi_{i_2} \circ \ldots \circ \phi_{i_p},$$

for $p \geq 1, i_1, \ldots, i_p \in I$, and by $\phi_{i_1 i_2 \ldots}$ the point given as intersection of the descending sets $\phi_{i_1 \ldots i_p}(U)$, for $p \to \infty$. Denote by Λ the set of points of type $\phi_{i_1 i_2 \ldots}$, called the *limit set* of $\mathcal{S}$,

$$\Lambda = \pi(\Sigma_I^+),$$

where $\pi : \Sigma_I^+ \to \Lambda,\ \pi(\omega) = \phi_{\omega_1 \omega_2 \ldots},\ \omega \in \Sigma_I^+$, is the canonical projection. In general, the image sets $\phi_i(\Lambda), i \in I$ may intersect each other. If there exists an open nonempty set V (a neighbourhood of Λ) so that $\underset{i \in I}{\cup} \phi_i(V) \subset V$ and the sets $\phi_j(V), j \in I$ are mutually disjoint, then we say that the system $\mathcal{S}$ satisfies the Open Set Condition. In this case, the dimension of Λ and of invariant measures on Λ can be computed relatively easily (see [8]). Some less restrictive separation conditions were studied for example in [17]. However, if the sets $\phi_i(\Lambda), i \in I$ intersect arbitrarily, then the problem of dimension for invariant measures on Λ is much more difficult. We are going to present results in this direction for systems with arbitrary overlaps. They use the notion of *overlap number of a measure* μ, which is an average rate of growth of the number of μ-generic n-preimages; the overlap number of μ was shown to be related to the folding entropy of μ. Thus, define first the following skew product transformation, called *the lift endomorphism* of $\mathcal{S}$,

$$\Phi : \Sigma_I^+ \times \Lambda \to \Sigma_I^+ \times \Lambda,\ \Phi(\omega, x) = (\sigma\omega, \phi_{\omega_1}(x)),\ \text{for } (\omega, x) \in \Sigma_I^+ \times \Lambda.$$

For any integer $n \geq 1$, we have

$$\Phi^n(\omega, x) = (\sigma^n(\omega), \phi_{\omega_n \ldots \omega_1}(x)),\ (\omega, x) \in \Sigma_I^+ \times \Lambda.$$

Since σ expands distances locally and $\phi_i, i \in I$ are contractions, it follows that Φ has a hyperbolic-type behaviour and a stable foliation with leaves $\{\omega\} \times \Lambda$, $\omega \in \Sigma_I^+$. Let now a Hölder continuous potential $\psi : \Sigma_I^+ \times \Lambda \to \mathbb{R}$. Then as in the case of hyperbolic diffeomorphisms, there exists a unique *equilibrium* *measure* μ_ψ on $\Sigma_I^+ \times \Lambda$, (for *e.g.* [4], [16], [43]). Denote also the projection of μ_ψ on Λ by

$$\nu_\psi := \pi_{2*}\mu_\psi.$$

In general, for a Φ-invariant probability μ on $\Sigma_I^+ \times \Lambda$, its *Lyapunov exponent* is defined by,

$$\chi(\mu) = \int_{\Sigma_I^+\times\Lambda} -\log|\phi'_{\omega_1}(x)| \, \mathrm{d}\mu(\omega, x) \ > 0.$$

In [34], we defined a notion of *overlap number* $o(\mathcal{S}, \mu_\psi)$ for an equilibrium measure μ_ψ of a Hölder continuous potential on $\Sigma_I^+ \times \Lambda$. It represents the average rate of growth of the number of μ_ψ-generic n-preimages in Λ, given that points in Λ can be covered many times by the images $\phi_{i_1\dots i_m}(\Lambda)$ if the system $\mathcal{S}$ has overlaps. For an arbitrary small $\tau > 0$, and integer $n \geq 1$ denote:

$$\Delta_n((\omega, x), \tau, \mu_\psi) = \Big\{(\eta_1, \dots, \eta_n) \in I^n, \exists y \in \Lambda, \phi_{\omega_n\dots\omega_1}(x) = \phi_{\eta_n\dots\eta_1}(y), \\ |\frac{S_n\psi(\eta, y)}{n} - \int_{\Sigma_I^+\times\Lambda} \psi \, \mathrm{d}\mu_\psi| < \tau\Big\},$$

where $(\omega, x) \in \Sigma_I^+ \times \Lambda$ and $S_n\psi(\eta, y)$ is the consecutive sum of ψ with respect to Φ. Denote by

$$b_n((\omega, x), \tau, \mu_\psi) := \mathrm{Card}\Delta_n((\omega, x), \tau, \mu_\psi.$$

We thus defined the overlap number of μ_ψ by the following limit, which was shown to exist:

Definition 3. The **overlap number** of $\mathcal{S}$ with respect to the measure μ_ψ is defined by:

$$o(\mathcal{S}, \mu_\psi) = \lim_{\tau\to 0}\lim_{n\to\infty} \frac{1}{n}\int_{\Sigma_I^+\times\Lambda} \log b_n((\omega, x), \tau, \mu_\psi) \, \mathrm{d}\mu_\psi(\omega, x).$$

For the measure of maximal entropy μ_0, the **topological overlap number** $o(\mathcal{S})$ is equal to $o(\mathcal{S}, \mu_0)$.

The topological overlap number gives the rate of growth of the number of overlappings at step n in Λ, when $n \to \infty$. We showed in [34] that, if $\pi : \Sigma_I^+ \to \Lambda$ is the canonical projection to Λ and

$$\beta_n(x) := \mathrm{Card}\{(\eta_1, \dots, \eta_n) \in I^n, x \in \phi_{\eta_1\dots\eta_n}(\Lambda)\}, \ n \geq 1,$$

then the topological overlap number of $\mathcal{S}$ is given by the formula:

$$o(\mathcal{S}) = \exp\big(\lim_{n\to\infty}\frac{1}{n}\int_{\Sigma_I^+} \log\beta_n(\pi\omega) \, \mathrm{d}\mu^+_{max}(\omega)\ \big). \tag{3}$$

In [34] we proved that $o(\mathcal{S}, \mu_\psi)$ is related to the folding entropy of μ_ψ with respect to the skew-product endomorphism Φ (recall the definition in (1)).

Theorem 3 ([34]). *In the above setting, we have*

$$o(\mathcal{S}, \mu_\psi) = \exp(F_\Phi(\mu_\psi)).$$

The overlap number of a measure represents an average rate of growth of the number of generic overlaps in $\mathcal{S}$ (with respect to the measure), and it can be used to estimate the Hausdoff dimension of sections through the fractal limit set Λ. Such a result is the following:

Theorem 4 ([34]). *Consider a finite conformal iterated function system $\mathcal{S} = \{\phi_i\}_{i\in I}$ with limit set Λ, $\pi : \Sigma_I^+ \to \Lambda$ be the canonical projection, and let a Hölder continuous potential $\psi : \Sigma_I^+ \times \Lambda \to \mathbb{R}$, with its unique equilibrium measure μ_ψ; and denote by $\nu_\psi := \pi_{2*}\mu_\psi$ the projection of the measure μ_ψ on the second coordinate. Then,*

$$HD(\nu_\psi) \leq t(\mathcal{S}, \psi),$$

where $t(\mathcal{S}, \psi)$ is the unique zero of the pressure function with respect to the shift $\sigma : \Sigma_I^+ \to \Sigma_I^+$, $t \to P_\sigma(t \log|\phi'_{\omega_1}(\pi(\sigma\omega))| - \log o(\mathcal{S}, \mu_\psi))$.

It is possible to compute or estimate the overlap numbers of measures, and thus to obtain also dimension estimates. Especially, we estimate topological overlap numbers in several concrete algebraic cases for Bernoulli convolutions. Such results were obtained in [26].

Let a probabilistic vector $\mathbf{p} = (\mathbf{p_1}, \dots, \mathbf{p_{|I|}})$ and associate to $\mathbf{p}$ the Bernoulli measure $\mu_{\mathbf{p}}^+$ on Σ_I^+. The projection of $\mu_{\mathbf{p}}^+$ on the limit set Λ of the iterated function system $\mathcal{S}$ is $\pi_*\mu_{\mathbf{p}}^+$. In fact, the Bernoulli measure $\mu_{\mathbf{p}}^+$ is the equilibrium measure of the Hölder continuous potential $g : \Sigma_I^+ \to \mathbb{R}$, $g(\omega) = \log p_{\omega_1}$, $\omega \in \Sigma_I^+$. Thus define the potential $\psi := g \circ \pi_1 : \Sigma_I^+ \times \Lambda \to \mathbb{R}$, which clearly is Hölder continuous; so there exists μ_ψ its unique equilibrium measure with respect to the endomorphism Φ. In [34] it was shown that $\pi_{2*}\mu_\psi = \pi_*\pi_{1*}\mu_\psi$. Hence, for some constant r_0, $\mu_\psi([\omega_1 \dots \omega_n] \times B(x, r_0)) \approx e^{S_n\psi(\omega,x) - nP_\Phi(\psi)}$, with comparability constant independent of n, x, ω. This implies that $\mu_\psi([\omega_1 \dots \omega_n] \times \Lambda) \approx e^{S_n g(\omega) - nP_\sigma(g)}$. Denote $\mu_{g\circ\pi_1}$ by $\mu_{\mathbf{p}}$, which is a probability measure on $\Sigma_I^+ \times \Lambda$. From above, $\pi_{1*}\mu_{\mathbf{p}}$ satisfies the same estimates on cylinders as the Bernoulli measure $\mu_{\mathbf{p}}^+$, and then from the uniqueness of equilibrium measure of ψ, $\pi_{1*}\mu_{\mathbf{p}} = \mu_{\mathbf{p}}^+$. Hence,

$$\pi_{2*}\mu_{\mathbf{p}} = \pi_*\mu_{\mathbf{p}}^+. \tag{4}$$

In particular, if μ_{max}^+ denotes the measure of maximal entropy on Σ_I^+, namely the Bernoulli measure corresponding to the probability vector $(\frac{1}{|I|}, \dots, \frac{1}{|I|})$), then

$$\pi_{2*}\mu_{max} = \pi_*\mu_{max}^+. \tag{5}$$

To give examples of overlap numbers, consider the system $\mathcal{S}_\lambda = \{\phi_{-1}, \phi_1\}$, with $\phi_{-1}(x) = \lambda x - 1$, $\phi_1(x) = \lambda x + 1$. When $\lambda \in (\frac{1}{2}, 1)$, $\mathcal{S}_\lambda$ has overlaps, and its limit set is the whole interval $I_\lambda = [-\frac{1}{1-\lambda}, \frac{1}{1-\lambda}]$. When there is no confusion about λ, this limit set is denoted by Λ.

Example 1. The first example is for reciprocals of *Garsia numbers.* A number γ is called a *Garsia number* if it is an algebraic integer in $(1, 2)$ whose minimal polynomial has a constant coefficient ± 2 and such that γ and all its conjugates have absolute value larger than 1 (see [13]). Examples of such minimal polynomials are $x^{n+p} - x^n - 2$ for $n, p \geq 1$, with $\max\{p, n\} \geq 2$. Thus in particular, $2^{\frac{1}{n}}, n \geq 2$, are Garsia numbers.

THEOREM 5 ([26]). *The topological overlap number of the system $\mathcal{S}_\lambda$ when $\frac{1}{\lambda}$ is a Garsia number, is equal to 2λ. Thus in this case $o(\mathcal{S}_\lambda) = 2\lambda > 1$.*

In particular, for any $n \geq 1$, we obtain from Theorem 5 a system $\mathcal{S}_\lambda$ with $\lambda = 2^{-1/n}$, which is asymptotically $\sqrt[n]{2^{n-1}}$ -to-1; so this system is asymptotically *irrational-to-1.*

Example 2. Consider now systems $\mathcal{S}_\lambda$ associated to reciprocals of *Pisot numbers.* A Pisot number is an algebraic integer greater than 1, all of whose conjugates are strictly less than 1 in absolute value (for *e.g.* [13]). Pisot numbers have remarkable properties, for *e.g.* their powers approach integers at an exponential rate (a converse is also true). An example of Pisot number is the golden ratio $\frac{1+\sqrt{5}}{2}$. In [26] we found a lower estimate for the topological overlap number in this case.

THEOREM 6 ([26]). *The topological overlap number $o(\mathcal{S}_\lambda)$ of the above system $\mathcal{S}_\lambda$ for $\lambda \in (\frac{1}{2}, 1)$ with $\frac{1}{\lambda}$ a Pisot number, satisfies the inequality*

$$o(\mathcal{S}_\lambda) \geq 2\lambda > 1.$$

Example 3. Consider now the case when there are *exact overlaps* in the iterated system $\mathcal{S}$, so

$$\phi_{i_1 \ldots i_p}(\Lambda) = \phi_{j_1 \ldots j_p}(\Lambda),$$

for certain maximal tuples $(i_1, \ldots, i_p), (j_1, \ldots, j_p)$. Exact overlaps may appear after a certain number of iterates, but let us look first at the case $p = 1$; the generalization is straightforward. Thus in the system $\mathcal{S} = \{\phi_i, 1 \leq i \leq m\}$ of conformal injective contractions, assume that

$$\phi_1 = \ldots = \phi_{k_1}, \ \phi_{k_1+1} = \ldots = \phi_{k_2}, \ldots, \ \phi_{k_p} = \phi_m, \tag{6}$$

where there are no overlaps between the different blocks, i.e the system $\{\phi_{k_i}, 1 \leq i \leq p\}$ satisfies the Open Set Condition. If μ_0^+ is the measure of maximal entropy on Σ_m^+, denote the measure of maximal entropy for Φ on $\Sigma_m^+ \times \Lambda$ by

μ_{max}. Then we proved in [34] that

$$o(\mathcal{S}) = \exp\big(\lim_{n\to\infty} \frac{1}{n} \int_{\Sigma_m^+} \log \beta_n(\pi\omega) \, \mathrm{d}\mu_{max}^+(\omega)\big), \tag{7}$$

where $\beta_n(x) := \mathrm{Card}\{(\eta_1, \ldots, \eta_n) \in I^n,\ x \in \phi_{\eta_1 \ldots \eta_n}(\Lambda)\}$. Then we proved the following formula:

Proposition 1 ([26]). *In the above setting from (6), the topological overlap number is,*

$$\begin{aligned} o(\mathcal{S}) &= o(\mathcal{S}, \mu_{max}) \\ &= \exp\Big(\frac{k_1 \log k_1 + (k_2 - k_1)\log(k_2 - k_1) + \ldots + (k_p - k_{p-1})\log(k_p - k_{p-1})}{m}\Big). \end{aligned}$$

The above estimates can be extended for the system of p-iterations

$$\{\phi_{i_1 \ldots i_p},\ i_j \in I, 1 \le j \le p\}.$$

Corollary 2 ([26]). *Let the system of conformal injective contractions $\mathcal{S} = \{\phi_i, i \in I\}$ with $|I| = m$, with Λ its limit set. Assume that there exists a family $\mathcal{F} \subset I^p$ of p-tuples such that $\phi_{i_p \ldots i_1}(\Lambda) = \phi_{j_p \ldots j_1}(\Lambda)$ for $(i_1, \ldots, i_p)$, $(j_1, \ldots, j_p) \in \mathcal{F}$, and denote* $\mathrm{Card}(\mathcal{F}) = N(\mathcal{F})$. *Then*

$$o(\mathcal{S}) \ge \exp\big(\frac{N(\mathcal{F})\log N(\mathcal{F})}{m^p}\big).$$

However, in general there may exist only *partial overlaps* at the level of p-iterates, a case covered by the next Corollaries. They apply also for Bernoulli convolutions systems $\mathcal{S}_\lambda$, since then the limit set is an interval $\Lambda = I_\lambda$ and we can estimate the proportion of overlaps at some iterate p.

Corollary 3 ([26]). *In the above setting, assume that there is a family $\mathcal{F} \subset I^p$ of p-tuples and $k \ge 1$ so that for any $(i_1, \ldots, i_p) \in \mathcal{F}$, there exists $(j_1 \ldots j_k) \in I^k$ such that $\phi_{i_1 \ldots i_p j_1 \ldots j_k}(\Lambda) \subset \bigcap_{(\ell_1, \ldots \ell_p) \in \mathcal{F}} \phi_{\ell_1 \ldots \ell_p}(\Lambda)$. Then, if $N(\mathcal{F}) := \mathrm{Card}(\mathcal{F})$, we obtain: $o(\mathcal{S}) \ge \exp\big(\frac{N(\mathcal{F})\log N(\mathcal{F})}{m^{p+k}}\big)$.*

Similarly, for a more general case, we obtained the following:

Corollary 4 ([26]). *In the above setting, assume that there are families $\mathcal{F}_1, \ldots, \mathcal{F}_s \subset I^p$ of p-tuples and positive integers $k_1, \ldots, k_s$ such that, for any $1 \le j \le s$ and for any $(i_{j1}, \ldots, i_{jp}) \in \mathcal{F}_j$ there exists some k_j-tuple $(j_1, \ldots, j_{k_j}) \in I^{k_j}$ with $\phi_{i_{j1} \ldots i_{jp} j_1 \ldots j_{k_j}}(\Lambda) \subset \bigcap_{(\ell_1, \ldots, \ell_p) \in \mathcal{F}_j} \phi_{\ell_1 \ldots \ell_p}(\Lambda)$.*

Then, if $N(\mathcal{F}_j) := \mathrm{Card}\mathcal{F}_j$, $1 \le j \le s$, we obtain:

$$o(\mathcal{S}) \ge \exp\big(\frac{N(\mathcal{F}_1)\log N(\mathcal{F}_1)}{m^{p+k_1}} + \ldots + \frac{N(\mathcal{F}_s)\log N(\mathcal{F}_s)}{m^{p+k_s}}\big).$$

Definition 4. Let μ be a probability measure on a metric space X. The upper, respectively lower, pointwise dimension of μ at a point $x \in X$ is the limit,

$$\overline{\delta}(\mu)(x) := \limsup_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}, \ \underline{\delta}(\mu(x)) := \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

If the upper and lower pointwise dimension coincide at x, the common value is called the pointwise dimension of μ at x and it is denoted by $\delta(\mu)(x)$. We say that a measure μ is *exact dimensional* if its pointwise dimension exists at μ-a.e $x \in X$, and it has a constant value.

Exact dimensional measures are very important in dynamics, since for them all their dimensions (Hausdorff, pointwise, box) coincide (see [8], [45]). Many invariant measures were studied from the point of view of dimension in various settings, and several classes of measures were proved to be exact dimensional; see for *e.g.* [3], [7], [8], [9], [10], [11], [28], [31], [37], [45], to name a few.

Feng and Hu proved that the projections of ergodic measures on the limit set of any finite conformal iterated system with overlaps, are exact dimensional, and found a dimension formula.

Theorem 7 (Feng and Hu [11]). *Let a finite conformal IFS $\mathcal{S}$, and μ be an ergodic measure on Σ_I^+, and denote its push-forward through the coding map on the limit set of $\mathcal{S}$ by $\nu := \pi_* \mu$. Then,*

$$HD(\nu) = \frac{h_\mu(\mathcal{S})}{|\chi(\mu)|},$$

where the projection entropy $h_\mu(\mathcal{S}) := H_\mu(\xi|\sigma^{-1}\pi^{-1}(\epsilon_{\mathbb{R}^q})) - H_\mu(\xi|\pi^{-1}(\epsilon_{\mathbb{R}^q}))$, with ξ being the partition of Σ_I^+ into 1-cylinders, $\epsilon_{\mathbb{R}^q}$ is the point partition of $\mathbb{R}^q$, $\pi^{-1}(\epsilon_{\mathbb{R}^q})$ is the fiber partition of Σ_I^+ with respect to π, and $\chi(\mu)$ is the Lyapunov exponent of μ.

Overlap numbers were later used in [25], namely in Theorem 8 below, to prove the exact dimensionality for a class of projection measures, which includes the self-conformal measures (i.e projections of Bernoulli measures). Theorem 8 has a different proof than Theorem 7, and it provides a more geometric formula for the dimension of measures.

Theorem 8 ([25]). *Let $\mathcal{S}$ be a finite conformal iterated function system with limit set Λ, and ψ be a Hölder continuous potential on Σ_I^+ with equilibrium measure μ_ψ, and let $\hat{\mu}_\psi$ be the equilibrium measure of $\psi \circ \pi_1$ on $\Sigma_I^+ \times \Lambda$ with respect to Φ. Denote by $\nu_{2,\psi} := \pi_{2*}\hat{\mu}_\psi$. Then the measure $\nu_{2,\psi}$ is exact* *dimensional, and for* $\nu_{2,\psi}$*-a.e* $x \in \Lambda$,

$$HD(\nu_{2,\psi}) = \delta(\nu_{2,\psi})(x) = \frac{h_\sigma(\mu_\psi) - \log(o(\mathcal{S}, \hat{\mu}_\psi))}{|\chi_s(\hat{\mu}_\psi)|}.$$

Thus, from the formula in Theorem 8, we see that the overlap numbers explain the difference in dimension for projection measures between the case of IFS with overlaps, versus the case of IFS with Open Set Condition (without overlaps).

In particular, from Theorem 8, we obtain the exact dimensionality of *self-conformal measures*, and a geometric formula for dimension by using their overlap numbers. Indeed let $\mu^+_{\mathbf{p}}$ be the Bernoulli measure on Σ^+_I associated to the probability vector $\mathbf{p}$, and $\hat{\mu}_{\mathbf{p}}$ be the lift of $\mu^+_{\mathbf{p}}$ to $\Sigma^+_I \times \Lambda$. We also showed in [34] that $\nu_{1,\mathbf{p}} = \nu_{2,\mathbf{p}}$, so $\delta(\nu_{1,\mathbf{p}}) = \delta(\nu_{2,\mathbf{p}})$, where $\nu_{1,\mathbf{p}}$ is a self-conformal measure.

4. COUNTABLE CONFORMAL ITERATED FUNCTION SYSTEMS WITH OVERLAPS

In contrast to the case of finite iterated function systems (IFS), the case of countable IFS with overlaps is very different. For instance, the limit set is no longer necessarily compact, and there may exist no zero for the pressure function; also, there are infinitely many images of the contractions of the system. Many of the methods of proof are also different.

On the other hand, there are important examples from Ergodic Number Theory, generated by countable IFS. For instance, regular continued fractions are generated by the countable system of maps on $[0,1]$, given by $x \to \frac{1}{x+n}$, $x \in [0,1], n \geq 1$ (for *e.g.* [15]). Also, one can associate countable conformal iterated function systems to β-maps and to other types of continued fractions ([24], [31]).

Consider thus a compact connected set $X \subset \mathbb{R}^q, q \geq 1$ with $X = \overline{IntX}$. Let a countable alphabet (set) E, and the system $\mathcal{S}$ of conformal injective contractions $\phi_e : X \to X, e \in E$, such that there exists a bounded open connected set $W \subset \mathbb{R}^q$ with $X \subset W$ and all $\phi_e : X \to X$ extend to conformal injective maps from W to W, so that the Lipschitz constants of the maps $\phi^\lambda_e, e \in E$ do not exceed a common value $0 < \gamma < 1$. Recall that Σ^+_E is the 1-sided shift on the countable alphabet E (with its canonical metric), and $\sigma : \Sigma^+_E \to \Sigma^+_E$ is the canonical shift map. Similarly, for the 2-sided shift Σ_E. Denote by C^{n-1}_0 the set of cylinders in Σ_E from 0 to $n-1$ positions, $n \geq 1$. If $\psi : \Sigma_E \to \mathbb{R}$ is a continuous function, then the topological pressure $P(\psi)$ is defined by

$$P(\psi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in C_0^{n-1}} \exp(\sup(S_n\psi|_{[\omega]})), \tag{8}$$

where $S_n\psi$ denotes the n-consecutive sum of ψ; this limit exists due to a subadditivity argument. Thermodynamic formalism for countable shifts was studied for *e.g.* in [24], [31], [41].

Denote by Λ the fractal limit set in $\mathbb{R}^q$ of the system $\mathcal{S}$, namely the set of all points of type $\phi_{\omega_1} \circ \phi_{\omega_2} \circ \ldots$, where $\omega = (\omega_1, \omega_2, \ldots) \in \Sigma_E^+$ (we use that all maps ϕ_i are contractions). Denote by

$$\pi : \Sigma_E^+ \to \Lambda,$$

the canonical projection given by the above formula, namely $\pi(\omega) = \phi_{\omega_1} \circ \phi_{\omega_2} \circ \ldots$, for $\omega \in \Sigma_E^+$.

Definition 5. Given the countable system $\mathcal{S} = \{\phi_e, e \in E\}$, and a σ-invariant probability measure μ on Σ_E^+, define the *projectional entropy* of μ and $\mathcal{S}$, as a difference of conditional entropies, by:

$$h_\mu(\mathcal{S}) := H_\mu(\xi|\sigma^{-1}\pi^{-1}(\epsilon_{\mathbb{R}^q})) - H_\mu(\xi|\pi^{-1}(\epsilon_{\mathbb{R}^q})),$$

where ξ is the partition of Σ_E^+ into 1-cylinders, $\epsilon_{\mathbb{R}^q}$ is the point partition of $\mathbb{R}^q$, and $\pi^{-1}(\epsilon_{\mathbb{R}^q})$ is the fiber partition of Σ_E^+ with respect to the projection π.

Definition 6. In the above setting, define the *Lyapunov exponent* of an ergodic measure μ on Σ_E^+ with respect to the countable iterated function system $\mathcal{S}$ by:

$$\chi_\mu := -\int_{\Sigma_E^+} \log \left|(\phi_{\omega_1})'(\pi(\sigma(\omega))\right| \mathrm{d}\mu(\omega).$$

Then, from Birkhoff's Ergodic Theorem we obtain that for μ-a.e. $\omega \in \Sigma_E^+$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \left|(\phi_{\omega|_n})'(\pi(\sigma^n(\omega))\right| = \chi_\mu. \tag{9}$$

We now state the main Theorem proved in [32] establishing the exact dimensionality of the projection measure on Λ for any ergodic measure μ on Σ_E^+ which satisfies a finiteness condition for the entropy (needed since we work with a countable system). Denote by ϵ_Λ the point partition on the limit set $\Lambda \subset \mathbb{R}^q$ and again by ξ the partition in 1-cylinders of Σ_E^+.

THEOREM 9 ([32]). *In the above setting, if μ is ergodic on Σ_E^+ and $H_\mu(\xi|\pi^{-1}\epsilon_\Lambda)) < \infty$, then for μ-a.e. $\omega \in \Sigma_E^+$, we have the exact dimensionality of the projection measure $\pi_*\mu$ on Λ, and moreover,*

$$\lim_{r\to 0} \frac{\log(\pi_*\mu(B(\pi(\omega), r)))}{\log r} = \frac{h_\mu(\mathcal{S})}{\chi_\mu}.$$

In fact, we proved a more general version of this Theorem for *random countable conformal IFS with arbitrary overlaps.* Theorem 9 has applications to examples from Ergodic Number Theory, such as Kahane-Salem sets, Bernoulli convolutions, random continued fractions, as shown in [32]. To give an example, let us denote the continued fraction with positive integer digits $a_1, a_2, \dots$ by

$$[a_1, a_2, \dots] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots}}}.$$

Then, the *random continued fractions* are defined as $[1, X_1, 1, X_2, \dots]$, where the random variables $X_i, i \geq 1$ are i.i.d and take the values $0, \lambda$ each with probability $1/2$, and where λ is a fixed number in $(0, \infty)$. Denote by ν_λ the distribution of this random continued fraction, which turns out to be an invariant measure for a countable IFS, namely the π_λ-projection of the Bernoulli measure $\mu_{(1/2,1/2)}$. Lyons [21] showed that ν_λ is singular for all $\lambda \in (\alpha_c, 0.5]$, where $\alpha_c \in (0.2688, 0.2689)$; and in [42] it was shown that ν_λ is absolutely continuous for Lebesgue-a.e $\lambda \in (0.215, \alpha_c)$. We showed in [32] that for **all** parameters $\lambda \in (0, \infty)$, the measure ν_λ is exact dimensional, and that for all $\lambda \in (\frac{\sqrt{3}-1}{2}, 0.5)$, its Hausdorff (and pointwise) dimension satisfies the estimate

$$HD(\nu_\lambda) > 0.174. \tag{10}$$

Another interesting case is that of *countable IFS with overlaps and place-dependent probabilities.* Finite systems with place-dependent probabilities were studied for *e.g.* by Barnsley, Demko, Elton and Geronimo ([1]), by using also the method of chains with complete connections of Onicescu and Mihoc ([35]); see also their generalization to random systems with complete connections in [14]. Let thus a system of smooth contractions $\mathcal{S} = \{\phi_i : V \to V\}_{i \in I}$ defined on a compact set $V \subset \mathbb{R}^D$ indexed by countable I, with limit set Λ, and let the weights $p_i : V \to \mathbb{R}$, $i \in I$, such that $\forall x \in V$,

$$\sum_{i \in I} p_i(x) = 1. \tag{11}$$

If $i_1, \dots, i_n \in I, n \geq 1$, denote by $\phi_{i_1 \dots i_n} := \phi_{i_1} \circ \dots \circ \phi_{i_n}$. Recall that $\pi : \Sigma_I^+ \to \Lambda, \pi(\omega) = \lim_{n \to \infty} \phi_{\omega_0 \omega_1 \dots \omega_n}$ if $\omega = (\omega_0, \omega_1, \dots) \in \Sigma_I^+$, is the canonical projection. Assume that $p_i(\cdot)$ are uniformly Hölder continuous, i.e there exist constants $\alpha, C > 0$ so that for all $x, y \in V$ and all $i \in I$,

$$|p_i(x) - p_i(y)| \leq C|x - y|^\alpha. \tag{12}$$

The *transfer probability* is defined by $P(x, B) := \sum_{i \in I} p_i(x) \delta_{\phi_i(x)}(B)$, and the *transfer operator* $\mathcal{L} : \mathcal{C}(\mathcal{V}) \to \mathcal{C}(\mathcal{V})$ is defined by (see for *e.g.* [1]),

$$\mathcal{L}(f)(x) = \int_X f(y) P(x, \mathrm{d}y).$$

A measure μ on V is called *stationary* if it is a fixed point of the dual operator of $\mathcal{L}$,

$$\mathcal{L}^*(\nu)(B) = \int P(x, B)\mathrm{d}\nu(x) = \sum_{i \in I} \int_{\phi_i^{-1}(B)} p_i(x)\mathrm{d}\nu(x).$$

It can be proved that there exist stationary measures, and they are unique (this is basically due to condition (12) and to the uniqueness of equilibrium measures). This stationary measure was shown in [30] to be exact dimensional, and we found an estimate for its Hausdorff (and pointwise) dimension. For an IFS with place-dependent probabilities $\mathcal{S}$ as above, let us define $\psi : \Sigma_I^+ \to \mathbb{R}$,

$$\psi(\omega) := \log p_{\omega_0}(\pi(\sigma\omega)), \omega \in \Sigma_I^+. \tag{13}$$

From (11) and (12) it follows that ψ has a unique equilibrium measure, denoted by μ_ψ on Σ_I^+.

Theorem 10 ([30]). *In the above setting, if the system $\mathcal{S} = \{\phi_i, i \in I\}$ is countable and conformal and if the probabilities $\{p_i(\cdot), i \in I\}$ satisfy (11) and (12), then the stationary measure $\tilde{\mu}_P$ for the system $\mathcal{S}$ with place-dependent probabilities $P = \{p_i(\cdot), i \in I\}$ is exact dimensional, and*

$$HD(\tilde{\mu}_P) = \frac{h_{\mu_\psi}(\mathcal{S})}{\chi_{\mu_\psi}} \leq \frac{h_{\mu_\psi}(\sigma)}{\chi_{\mu_\psi}},$$

where μ_ψ is the equilibrium measure of ψ, and $h_{\mu_\psi}(\mathcal{S})$ is the projection entropy of μ_ψ.

Another type of randomized countable conformal IFS was introduced and studied in [31], namely the *Smale skew product endomorphisms.* For these systems it was proved the exact dimensionality of the projections of fiber measures, and the global exact dimensionality of the projections of equilibrium measures from the shift space onto the global basic set (after establishing certain Volume Lemmas). The methods of proof are different for Smale endomorphisms. These results were applied in [31] to a large class of systems generated by inverse limits of non-invertible systems, and to examples from Ergodic Number Theory, including Lüroth maps, expanding Markov-Rényi maps, continued fractions, Manneville-Pomeau maps, β-maps (for arbitrary $\beta > 1$).

In particular, our results on Smale endomorphisms found some surprising applications in [31] to solving the *Doeblin-Lenstra Conjecture*, about Diophantine approximation for numbers $x \in [0, 1]$ outside of the original set where it was solved (see for *e.g.* [15]), and for a larger class of invariant measures. Moreover, in [30] Smale endomorphisms were associated to random systems with complete connections ([14]), and they were applied to unfold countable conformal IFS with overlaps into families of subfractals.

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