Dedicated to the memory of Mihnea Colţoiu

ON HYPERCONVEXITY AND TOWARDS BUNDLE-VALUED KERNEL ASYMPTOTICS ON LOCALLY PSEUDOCONVEX DOMAINS

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After recalling basic results on the L^2 $\bar{\partial}$-cohomology groups and known existence criteria for bounded plurisubharmonic exhaustion functions on locally pseudoconvex bounded domains, results on the Bergman kernel on hyperconvex domains will be reviewed. Then, on locally pseudoconvex domains with certain regularity constraints on the boundary, a result on the asymptotics of the Bergman kernel is proved without assuming the existence of plurisubharmonic exhaustion functions, as an application of the finite-dimensionality of L^2 $\bar{\partial}$-cohomology groups.

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INTRODUCTION

Convexity notions arose naturally in the development of several complex variables. Pseudoconvexity, which is somewhat weaker than the geometric convexity, appeared in the pioneering works of Hartogs [38] and Levi [50, 51] to describe geometric properties of the domains of holomorphy. Recall that a domain Ω over $\mathbb{C}^n$ is said to be a domain of holomorphy if Ω is equivalent to a connected component of the structure sheaf $\mathcal{O} \to \mathbb{C}^n$. Hartogs observed that every holomorphic map $\{(z,w) \in \mathbb{D}^2; |z| > \frac{1}{2} \text{ or } |w| < \frac{1}{2}\}$ to Ω is extendable to a holomorphic map $\mathbb{D}^2 \to \Omega$ ($\mathbb{D} := \{z \in \mathbb{C}; |z| < 1\}$). Levi [50, 51] found that any domain of holomorphy $\Omega \subset \mathbb{C}^n$ with C^2-smooth boundary has a defining function ρ (i.e. $\Omega = \{z; \rho(z) < 0\}$ and $\mathrm{d}\rho|_{\partial\Omega} \neq 0$) whose complex Hessian is positive semi-definite on the complex tangent spaces of $\partial\Omega$. $\partial\bar{\partial}\rho$ or its restriction to the complex tangents of $\partial\Omega$ is called the Levi form. Poincaré's remark in [72] had already shown the importance of the Levi form by examples. Another convexity notion, called holomorphic convexity, was introduced by Cartan and Thullen [16]. A complex manifold M is called holomorphically convex if M can be mapped onto a closed complex analytic subset of $\mathbb{C}^N$ by a proper holomorphic map. Relations between these convexity notions have been clarified by Oka based on the study of plurisubharmonic(=psh) functions introduced in [68] and independently by Lelong [48]. In particular, it was first established in [69] that every $\Omega \in \pi_0(\mathcal{O})$ is holomorphically convex and satisfies $H^1(\Omega, \mathcal{O}) = 0$. In Oka's theory, important existence theorems are tied together by an approximation theorem of Runge type, so that the existence of a psh exhaustion function is crucial to apply a limiting argument by approximating a given domain by strongly pseudoconvex subdomains. Y.-T. Siu's paper [75] is a well-written survey on the construction of psh functions in this context.

Local geometry of the boundary of pseudoconvex domains is reflected on the space of holomorphic functions in a somewhat subtler way. In 1933, Bergman [11] observed that the reproducing kernel $B_\Omega(z, w)$ of the space of square integrable (=L^2) holomorphic functions on some class of bounded pseudoconvex domains $\Omega \subset \mathbb{C}^2$ satisfies

$$\delta_\Omega(z)^{-2} \lesssim B_\Omega(z, z) \lesssim \delta_\Omega(z)^{-3} \quad (\delta_\Omega(z) := \inf_{w \notin \Omega} \|z - w\|)$$

i.e. $c\delta_\Omega(z)^{-2} < B_\Omega(z, z) < C\delta_\Omega(z)^{-3}$ for some positive constants c and C.

In 1965, Hörmander proved in [41] that, given a domain $\Omega \subset \mathbb{C}^n$,

$$\lim_{z \to z_0} B_\Omega(z, z)\delta_\Omega(z)^{n+1}$$

exists and is > 0 if the range of the $\bar\partial$-operator $L^{0,0}_{(2)}(\Omega) \to L^{0,1}_{(2)}(\Omega)$ is closed and $\partial\Omega$ is strongly pseudoconvex at z_0. Here $L^{p,q}_{(2)}(\Omega)$ denotes the space of L^2 (p, q)-forms on Ω. Fefferman [31] refined it for bounded pseudoconvex domains Ω with C^∞-smooth boundary by showing that

$$B_\Omega(z, z) = \varphi(z)\delta_\Omega(z)^{-n-1} + \psi(z) \log \delta(z)$$

holds for some C^∞ functions φ and ψ on $\overline{\Omega}$. Recall that the main result of [31] is the C^∞ extendability of any biholomorphic map between two strongly pseudoconvex domains Ω_1 and Ω_2 with C^∞-smooth boundary, as a diffeomorphism from $\overline{\Omega_1}$ to $\overline{\Omega_2}$. On the other hand, by an extension theorem in [67], $B_\Omega(z, z) \gtrsim \delta_\Omega(z)^{-2}$ turned out to be true for any bounded pseudoconvex domain Ω with Lipschitz continuous boundary.

In these decades, many results have been obtained for B_Ω under various positivity assumptions on $\partial\bar\partial\rho$ and under weaker regularity assumptions on $\partial\Omega$. In many of such works the ingredient of the methods has been the combination of constructing psh functions involving geometric data and solving the $\bar\partial$-equations with corresponding L^2 estimates.

The purpose of the present article is to review some of such results at first and proceed to study questions on the space of weighted L^2 holomorphic sections of vector bundles on bounded locally pseudoconvex domains Ω in a complex manifold X. Here the vector bundles with fiber metrics are given over X. Our main interest is to know how far one can replace the existence of psh exhaustion functions on Ω by the combination of the mere local pseudoconvexity of Ω and the positivity of the curvature form of the bundle on $\partial\Omega$. A result to which we would like to give the outline of a proof is the following.

THEOREM 1. *Let Ω be a bounded locally pseudoconvex domain with C^2-smooth boundary in a complex manifold X and let $L \to X$ be a holomorphic line bundle with a C^∞ fiber metric h whose curvature form is positive at every point of $\partial\Omega$. Then, for any $\varepsilon > 0$ one can find $\nu_0 \in \mathbb{N}$ such that*

$$\liminf_{z \to \partial\Omega} B_{\Omega, L^\nu}(z) \cdot \rho(z)^{2-\varepsilon} > 0. \tag{0.1}$$

holds for any $\nu \geq \nu_0$. Here B_{Ω,L^ν} denotes the Bergman kernel for the L^2 L^ν-valued holomorphic n-forms with respect to h^ν.

Studies in this direction were suggested by Grauert [34, 35, 36] and results preceding Theorem 1 have been obtained by Pinney [71] and Asserda [8]. The methods in [71] and [8] apply to yield (0.1) with $\varepsilon = 0$ if X is compact and F is a positive line bundle on X. It is very natural to expect that this best estimate holds in general, but a better understanding of the $\bar\partial$-equation seems to be necessary to verify it.

1. NOTATION AND PRELIMINARIES

We shall recall a basic result on the L^2 $\bar\partial$-cohomology group, which is a generalization of Kodaira's vanishing theorem to complete Kähler manifolds, initiated by Andreotti and Vesentini [5, 6, 7] and finalized by Demailly [22].

Let M be a connected and paracompact complex manifold of dimension n, let $C^\infty(M) := \{f \in \mathbb{C}^M; f \text{ is of class } C^\infty\}$, let $\mathscr{O}(M) := \{f \in C^\infty(M); \bar\partial f = 0\}$ and let $\mathscr{O}_M \to M$ be the structure sheaf of M. Here $\bar\partial$ denotes the complex exterior derivative of type (0,1).

Given a holomorphic vector bundle $\pi : E \to M$, we put

$$C^{p,q}(M,E) = \{E\text{-valued } C^\infty \ (p,q)\text{−forms on } M\}$$

and

$$H^{p,q}(M,E) = \text{ the } E\text{-valued } \bar\partial\text{-cohomology group of } M \text{ of type } (p,q)$$

$$:= \frac{\mathrm{Ker}(\bar\partial|_{C^{p,q}(M,E)})}{\mathrm{Im}(\bar\partial|_{C^{p,q-1}(M,E)})}.$$

Note that $\mathscr{O}(M) = H^{0,0}(M)$, where $H^{p,q}(M) := H^{p,q}(M,E)$ with $E = \mathbb{C} \times M \to M$.

By the Dolbeault isomorphism, $H^{p,q}(M,E)$ is canonically isomorphic to the sheaf cohomology group $H^q(M,\Omega^p(E))$, where $\Omega^p(E)$ denotes the sheaf of germs of E-valued holomorphic p-forms. By an abuse of notation, vector bundles are identified with locally free sheaves and $H^q(M,\Omega^p(E))$ will be denoted also by $H^q(M,\wedge^p(T_M^{1,0})^*\otimes E)$, where $T_M^{1,0}$ denotes the holomorphic tangent bundle of M.

We put

$$L^{p,q}_{(2)}(M,E,g,h)=\{E\text{-valued } L^2\ (p,q)\text{- forms on } M \text{ w.r.t. } (g,h)\}$$

and

$$\begin{aligned} H^{p,q}_{(2)}(M,E,g,h) &= \text{the } E\text{-valued } \bar\partial\text{-cohomology group of } M \\ &\qquad \text{of type } (p,q) \text{ w.r.t. } (g,h) \\ &:= \frac{\operatorname{Ker}(\bar\partial|_{L^{p,q}_{(2)}(M,E,g,h)})}{\bar\partial(L^{p,q-1}_{(2)}(M,E,g,h))\cap L^{p,q}_{(2)}(M,E,g,h)}. \end{aligned}$$

By an abuse of notation, for any open set $\Omega\subset M$ we put

$$H^{p,q}_{(2)}(\Omega,E,g,h):=H^{p,q}_{(2)}(\Omega,\pi^{-1}(\Omega),g|_\Omega,h|_{\pi^{-1}(\Omega)}).$$

$H^{p,q}_{(2)}(M)(=H^{p,q}_{(2)}(M,g))$ will stand for $H^{p,q}_{(2)}(M,E,g,h)$ if $E=\mathbb{C}\times M$ and $h=1$.

Theorem 2 ([22]. See also [58, 60]). *If g is Kähler,* $\operatorname{rank} E=1$ *and the curvature form Θ of h is positive, then $H^{n,q}_{(2)}(M,E,\Theta,h)=0$ for all $q\geq 1$. Here Θ is identified with a metric on M (by an abuse of notation).*

Let $\mathbb{D}$ be the unit disc centered at 0 in $\mathbb{C}$. We define the set of subharmonic functions on $\mathbb{D}$ by

$$SH(\mathbb{D}):=\left\{u:\mathbb{D}\underset{u.s.c.}{\longrightarrow}[-\infty,\infty);u(z)\leq\frac{1}{2\pi}\int_0^{2\pi}u(z+re^{i\theta})\mathrm{d}\theta\ \forall z\in\mathbb{D}\right\},$$

where $0<r<1-|z|$ and u.s.c.=upper semicontinuous.

Let $\mathscr{O}(\mathbb{D},M)$ be the set of holomorphic maps from $\mathbb{D}$ to M, i.e.

$$\mathscr{O}(\mathbb{D},M):=\{f:\mathbb{D}\underset{cont.}{\longrightarrow}M;s\circ f\in\mathscr{O}_\mathbb{D}\text{ for any }s\in\mathscr{O}_M\}.$$

Then we set

$$PSH(M):=\left\{u:M\underset{u.s.c.}{\longrightarrow}[-\infty,\infty);f\in\mathscr{O}(\mathbb{D},M)\Rightarrow u\circ f\in SH(\mathbb{D})\right\}.$$

Recall that

$$f\in\mathscr{O}(M)\ \Rightarrow\ \log|f|,\ |f|^p\in PSH(M)\ \ (p>0)$$

and that $\lambda \circ \varphi \in PSH(M)$ holds for any $\varphi \in PSH(M) \cap L^\infty_{loc}$ and any convex increasing function λ on $\mathbb{R}$. Elements of $PSH(M)$ are called plurisubharmonic (=psh) functions on M. A subset A of M is called *pluripolar* if $\varphi|_A = -\infty$ for some $\varphi \in PSH(M) \setminus \{-\infty\}$.

Definition 1. M is called *weakly 1-complete* if there exists a C^∞ psh function Φ on M such that $M_c := \{x; \Phi(x) < c\} \Subset M$ for all c.

The function Φ will be referred to simply as an exhaustion function of M. We shall say also that (M, Φ) is a weakly 1-complete manifold.

For any $\varphi \in L^1_{loc}(M)$ a subsheaf $\mathscr{I}_\varphi \subset \mathscr{O}_M$ is defined as the germs of functions f such that $e^{-\varphi}|f|^2$ is locally integrable. We put

$$H^{n,0}_{(2)}(M, E, g, e^{-\varphi}h) := \left\{ f \in H^{n,0}(M, E); \left| \int_M e^{-\varphi} h(f) \wedge \overline{f} \right| < \infty \right\}.$$

Here, in this notation $h \in \mathrm{Hom}(E, \overline{E}^*)$, which is applied to E-valued differential forms coefficientwise.

Note that $H^{n,0}_{(2)}(M, E, g, e^{-\varphi}h)$ does not depend on the choice of g.

$H^{p,q}_{(2)}(M, E, g, h)$ is similarly generalized to $H^{p,q}_{(2)}(M, E, g, e^{-\varphi}h)$ for any (p, q) and $\varphi \in L^1_{loc}(M)$. By analyzing $H^{n,q}_{(2)}(M, E, g, e^{-\varphi}h)$ similarly as in Theorem 2, one has the following.

THEOREM 3 (cf. [52] and [24]). *Let (M, g) be a weakly 1-complete Kähler manifold and let (E, h) be a holomorphic Hermitian line bundle and let $\varphi \in L^1_{loc}(M)$. If $e^{-\varphi}h$ is locally equal to $e^{-\psi}\tilde{h}$ for some plurisubharmonic ψ and smooth $\tilde{h}$ such that the curvature form of $\tilde{h}$ is positive,*

$$H^q(M, K_M \otimes E \otimes \mathscr{I}_\varphi) = 0$$

holds for $q \geq 1$. Here K_M denotes the canonical bundle of M.

Remark 1. Predecessors of Theorem 2 are Kodaira's vanishing theorem in [46] and its generalizations [4], [53], [5, 6, 7] as well as Hörmander's L^2 solution of the Cousin's problem in [41]. Generalization to weakly 1-complete manifolds as in Theorem 3 was initiated by Nakano [54, 55, 56]. (See also [28] and [76].)

Grauert [36] showed that weakly 1-complete manifolds need not be holomorphically convex. For instance, if M is the total space of a holomorphic line bundle F over a compact and connected Kähler manifold of dimension ≥ 1 such that $c_1(F) = 0$ (i.e. F is topologically trivial) and that F^k is not trivial for any $k \in \mathbb{N}$, it is seen from the maximum principle that M does not admit any nonconstant holomorphic function. Nevertheless, an existence theorem analogous to [35] holds on weakly 1-complete manifolds. Namely, given a holomorphic Hermitian line bundle L with curvature form Θ over a weakly 1-complete manifold (M, Φ) and $c \in \mathbb{R}$, the equivalence of the following 1) and 2) is an immediate consequence of [57].

1) $\forall d > c \; \exists \; m \in \mathbb{N}$ and a meromorphic map η from $M_d := \{x; \Phi(x) < d\}$ to $\mathbb{CP}^N$ $(N \gg 1)$ by the ratio of holomorphic sections of $L^m|_{M_d}$ s.t. $\eta|_{M_d \setminus M_c}$ is a holomorphic embedding.

2) $\Theta|_{M \setminus M_c}$ is positive.

Results in [57] implying the equivalence of 1) and 2) can be summarized as follows.

THEOREM 4 (Grauert-type approximation theorem). *Let (M, φ) be an n-dimensional weakly 1-complete manifold and let $L \to X$ be a holomorphic Hermitian line bundle whose curvature form is positive on the complement of M_c for some $c < \infty$. Then*

$$\dim H^{n,q}(M, L) < \infty$$

holds for $q \geq 1$ and the restriction homomorphism

$$H^{n,q}(M, L) \to H^{n,q}(M_c, L)$$

has dense image for $q \geq 0$.

Recently, Theorem 4 has been extended to describe a relation between the positivity of $\Theta|_{\partial\Omega}$ and a convexity property of Ω with respect to sections of $L^m|_\Omega$, where Ω is a relatively compact locally pseudoconvex domain with C^2-smooth boundary in a complex manifold X and (L, h) is a Hermitian holomorphic line bundle over X (cf. [65]). Theorem 1 will be proved in §4 by applying the results in [65] which will be recalled there.

Remark 2. As for a condition for weakly 1-complete manifolds to be holomorphically convex, Takayama [78] showed that the negativity of K_M of M suffices. It was shown in [64] that M is holomorphically convex if K_M is only assumed to be negative outside a compact subset of M. Given a complex manifold X and a locally pseudoconvex bounded domain Ω in X with C^2 boundary, it had been shown in [63] that Ω can be mapped properly and holomorphically onto a locally Stein closed complex analytic subset of an open set of $\mathbb{C}^N$. (Locally closed complex submanifolds of $\mathbb{C}^N$ need not be Stein even if they are locally Stein, as was known by [21, Remark 3]. See also [20] and [74].) Structure theorems have been obtained under curvature conditions on the tangent bundle, when M is compact (cf. [40]). In these works, Theorem 1, Theorem 2 and their refinements play important roles.

2. HYPERCONVEXITY AND THE BERGMAN KERNEL

In the following, we shall restrict ourselves to the results on a special class of Stein manifolds which are modelled on homogeneous bounded domains in $\mathbb{C}^n$. Simple examples are the open balls $\mathbb{B}^n := \{z \in \mathbb{C}^n; \|z\| < 1\}$ and polydiscs $\mathbb{D}^n$.

For any complex manifold M we put

$$SPSH(M) := \left\{u \in PSH(M); u \in C^2 \text{ and } \partial\bar{\partial}u > 0\right\}.$$

Here $\partial\bar{\partial}u$ is identified with the complex Hessian of u by an abuse of notation. Recall that a weakly 1-complete manifold (M, Φ) is a Stein manifold if $\Phi \in SPSH(M)$ (cf. [35]).

Definition 2. M is said to be hyperconvex if there exists a function $\Phi \in SPSH(M)$ which maps M properly onto $[-1, 0)$.

If M is hyperconvex, it is easy to show by the L^2 method, by applying Theorem 2 for instance, that the correspondence

$$M \ni x \mapsto \{f; f(x) = 0\} \subset H^{n,0}_{(2)}(M),$$

which shall be denoted by ι_M, is an embedding into the projective space

$$(H^{n,0}_{(2)}(M))^*/(\mathbb{C} \setminus \{0\})$$

if M is hyperconvex.

S. Kobayashi [45] observed that the Fubini-Study metric on

$$(H^{n,0}_{(2)}(M))^*/(\mathbb{C} \setminus \{0\})$$

induces the Bergman metric on M by such an embedding. This characterization of the Bergman metric implies the following criterion for its completeness.

THEOREM 5 ([45, Theorem 9.1]). *The embedding ι_M induces a complete metric on M if the following is satisfied: for every infinite sequence S of points of M which has no adherent point in M and for each $f \in H^{n,0}_{(2)}(M)$, there exists a subsequence S' of S such that*

$$\lim_{S'} \frac{f \wedge \overline{f}}{B_M} = 0,$$

where B_M denotes the Bergman kernel of M restricted to the diagonal.

If M is a bounded domain in $\mathbb{C}^n$, Kobayashi's criterion is satisfied if the function $|B_M/(\mathrm{d}z \wedge \mathrm{d}\overline{z})|$ is exhaustive and $\mathscr{O}(M) \cap L^\infty$ is dense in $\mathscr{O}(M) \cap L^2$. Based on this observation, it was shown in [59] that bounded Stein domains Ω with C^1 boundary in Stein manifolds are complete with respect to the

Bergman metrics $\partial\bar{\partial}\log B_\Omega(z,z)$. Since it was shown by Kerzman and Rosay [44] that such domains are hyperconvex, it was natural to ask whether or not the Bergman metrics on hyperconvex manifolds are complete. This was settled by B.-Y. Chen [17]. Namely,

THEOREM 6. *The Bergman metrics on hyperconvex manifolds are complete.*

Let us recall the works on hyperconvexity before and after [17].

First, we shall recall some basic results on hyperconvexity.

Clearly, $(M, -\log(-\Psi))$ is Stein if (M,Ψ) is hyperconvex. Conversely, if (M,Φ) is Stein and the length of $\partial\Phi$ w.r.t. $\partial\bar{\partial}\Phi$ is bounded, then it is easy to see that one can find two constants $a,b>0$ such that $\left(M, \dfrac{-b}{a+\Phi}\right)$ is hyperconvex. In other words, a hyperconvex manifold can be identified with a complete Kähler manifold (M,g) of the form $g=\partial\bar{\partial}\Phi$ for an exhaustion function Φ with bounded gradient. Bounded homogeneous domains in $\mathbb{C}^n$ are of this class (cf. [43]). It is known that

$$H^{p,q}_{(2)}(M)=0 \quad \text{if } p+q\neq n$$

holds for such (M,g) (cf. [29]). An open question here is whether or not $H^{p,q}_{(2)}(M)\neq 0 \quad \text{if } p+q=n$.

If $n=1$, a theorem of Bouligand [13] in 1926 implies that a bounded domain $\Omega\Subset\mathbb{C}$ is hyperconvex if and only if every element of $C^0(\partial\Omega)$ is extendable to an element of $C^0(\overline{\Omega})$ harmonically on Ω.

In several complex variables, the notion of hyperconvexity was introduced by Stehlé [77] in 1975, where the Steinness of analytic fiber bundles over Stein manifolds was proved for hyperconvex fibers.

After Stehlé's work, hyperconvexity has been verified under various regularity assumptions on the boundary. In the study of smoothly bounded pseudoconvex domains, Diederich and Fornaess [26] obtained the following in 1977.

THEOREM 7. *Let X be a Stein manifold and $\Omega\subset X$ a relatively compact pseudoconvex domain with C^r-boundary, $2\le r\le\infty$. Then there is a C^r defining function ρ on a neighborhood U of $\overline{\Omega}$, such that for any number η with $0<\eta<1$ and η small enough, the function $\hat{\rho}=-(-\rho)^\eta$ belongs to $SPSH(\Omega)$.*

By Stehlé's method of patching psh functions, the following was established in [44].

PROPOSITION 1. *A bounded domain $\Omega\Subset\mathbb{C}^n$ is hyperconvex if and only if each boundary point of Ω has a neighborhood U in $\mathbb{C}^n$ such that $U\cap\Omega$ is hyperconvex.*

Theorem 7 was extended in [44] as follows.

THEOREM 8. *Bounded Stein domains with C^1-boundary in Stein manifolds are hyperconvex.*

Theorem 8 was further extended by Demailly [23] to the following.

THEOREM 9. *Bounded Stein domains with Lipschitz continuous boundary in Stein manifolds are hyperconvex.*

In [62] the following was proved by combining the L^2 extension theorem in [67] and the symmetry property of the Green function.

THEOREM 10. *Let $\Omega \Subset M$ be a bounded hyperconvex domain in a Stein manifold M. Then*

$$\lim_{z \to \partial\Omega} B_\Omega(z) = \infty.$$

In 1994, Diederich told the author that Herbort had proved the Bergman completeness of 1-dimensional hyperconvex manifolds.

In [66], based on a work of Takeuchi [79] and Elencwajg [30], Sibony and the author extended Theorem 7 as follows.

THEOREM 11. *Let $\Omega \Subset M$ be a pseudoconvex domain with C^2 boundary in a complete Kähler manifold (M, g). Assume that the holomorphic bisectional curvature of M is strictly positive. Let $r(z) = -\mathrm{dist}(z, \partial\Omega) =: \delta(z)$ where δ is computed with respect to the Kähler metric. Then there exists $\varepsilon > 0$ such that $\varphi = -(-r)^\varepsilon$ is plurisubharmonic on Ω and there is a constant c_ε such that $\partial\bar\partial\varphi \geq c_\varepsilon|\varphi|g$.*

Harrington [37] generalized Theorems 9 and 11 by showing the following.

THEOREM 12. *Let $\Omega \subset \mathbb{CP}^n$ be a pseudoconvex domain with Lipschitz boundary. There exist a Lipschitz defining function ρ and an exponent $0 < \eta < 1$ such that $i\partial\bar\partial(-(-\rho)^\eta) \geq C(-\rho)^\eta\omega$ in the sense of currents where $C > 0$ and ω is the Kähler form for the Fubini-Study metric on $\mathbb{CP}^n$.*

Combining Theorem 12 with Theorem 6, one knows that Stein domains with Lipschitz continuous boundary in $\mathbb{CP}^n$ have complete Bergman metrics.

Remark 3. Theorem 11 was originally meant to explore the geometry of pseudoconvex domains in $\mathbb{CP}^2$ to verify the nonexistence of Levi flat hypersurfaces, which still remains as an open question.

Remark 4. Quite recently, Chen [19] proved that $B_\Omega(z,z) \gtrsim \delta_\Omega(z)^{-2}$ holds if $\Omega = \{z; \rho(z) < 0\} \Subset \mathbb{C}^n$ for some continuous psh function ρ defined on a neighborhood of $\overline{\Omega}$, by exploiting the idea of [15].

We note that even the quantitative hyperconvexity achieved in Theorem 11 does not seem to imply the Bergman completeness of Ω in a straightforward way. This difficulty has been overcome by the development of the study of pluricomplex Green function, the higher dimensional analogue of the Green function which is most natural in the context of several complex variables. We shall sketch its elements next.

3. PLURICOMPLEX GREEN FUNCTION AND THE BERGMAN METRIC

In several complex variables, the following is the most natural generalization of the Green function in one complex variable.

Definition 3.

$$G_M(z,w) := \sup\{u \in PSH(M); u < 0 \text{ and } u(z) - \log \mathrm{dist}_g(z,w) \in L^\infty_{loc}\}$$

is called *the pluricomplex Green function* of M.

If $n = 1$, $G_M(z,w) = G_M(w,z)$ follows from Gauss-Green's formula and it is known by Poisson, Schwarz and Perron that

$$\lim_{z\to w}\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\frac{\partial}{\partial r}G_{\mathbb{D}}(z, re^{i\theta})\mathrm{d}\theta = u(w)$$

holds for all $u \in C^0(\partial\mathbb{D})$ and $w \in \partial\mathbb{D}$.

According to Lempert [49],

$$G_\Omega(z,w) \equiv G_\Omega(w,z)$$

holds if Ω is a bounded convex domain in $\mathbb{C}^n$. However, Bedford and Demailly [10] showed that

$$G_\Omega(z,w) \not\equiv G_\Omega(w,z)$$

if Ω is a hyperconvex domain $\{z = (z_1, z_2) \in \mathbb{C}^2; \rho(z_1,z_2) < 0\}$ with

$$\rho(z_1,z_2) = \max\left(\frac{1}{2}\log\left(\frac{|z_2^2 - z_1^2(z_1-a)|}{\epsilon^2}\right), \log|z_1|\right) \quad (0 < \epsilon \ll |a| < 1).$$

In spite of such an disadvantage, useful properties of G_M have been found. Of particular importance for the Bergman metric is the behavior of the sublevel sets $A(w,M) := \{z; G_M(z,w) \leq -1\}$, which was explored by virtue of the following.

Proposition 2 (Carlehed-Cegrell-Wikström [14]). *Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain and $(w_m)_m \subset \Omega$ a sequence, that converges to a boundary point. Then there exists a pluripolar set $A \subset \Omega$ such that*

$$\limsup_{m\to\infty} G_\Omega(z, w_m) = 0$$

for all $z \in \Omega \setminus A$.

The following information on the volume (=:Vol) of $A(w, \Omega)$ for the domains $\Omega \subset \mathbb{C}^n$ can be deduced from Proposition 2.

Lemma. *Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain and $(y_j)_j$ a sequence that converges to a point $y_0 \in \partial\Omega$. Then there exists a subsequence $(y_{j_\ell})_\ell$ of $(y_j)_j$ such that* $\mathrm{Vol}(A(y_{j_\ell}, \Omega)) \to 0$, *as $\ell \to \infty$.*

Then, by combining Theorem 5 with a standard method in [41] one can prove

Theorem 13 (Błocki-Pflug [12], Herbort [39]). *The Bergman metric is complete on bounded hyperconvex domains in $\mathbb{C}^n$.*

By a similar method based also on a result in [23], Theorem 13 was generalized to Theorem 6 in [17].

There exist Bergman complete but non-hyperconvex domains.

Theorem 14 (cf. [39]). *The domain*

$$\Omega = \left\{ (z_1, z_2) \in \mathbb{C}^2; 0 < |z_1| < 1, |z_2|^2 \exp\left(\frac{1}{|z_1|^2}\right) < 1 \right\}$$

is Bergman complete, but not hyperconvex.

More counterexamples are in [42] and [80].

Recently, Chen [18] showed the following.

Theorem 15. *Let Ω be a bounded Stein domain in a Stein manifold. If $\partial\Omega$ is Hölder continuous, then Ω is hyperconvex.*

At this point, the difference between $\mathbb{C}^n$ and $\mathbb{CP}^n$ became apparent, because there exist non-hyperconvex Stein domains in $\mathbb{CP}^n$ which have Hölder continuous boundary if $n \geq 3$ (cf. [27]).

Remark 5. Adachi [1] has discovered a hyperconvex manifold M without nonconstant bounded holomorphic functions. It is remarkable that M is an analytic $\mathbb{D}$-bundle over a compact Riemann surface and function spaces on some of such manifolds can be analyzed in great detail (cf. [2, 3]). Other kind of examples have been known in the context of characterizing Stein manifolds without nonconstant bounded holomorphic functions (*parabolic Stein manifolds*). Sibony and Wong [73] showed that if the growth of the volume of an analytic set $A \subset \mathbb{C}^n$ is slow enough, then every bounded holomorphic function on A is constant. Such examples contained infinitely many copies of $\mathbb{C}$. (See also [9].)

4. FINITE-DIMENSIONALITY AND HARMONIC REPRESENTATION FOR THE L^2 COHOMOLOGY

Let (X, g) be a complete Hermitian manifold of dimension n and let (E, h) be a holomorphic Hermitian vector bundle over X. For any $u \in L^{p,q}_{(2)}(X, E, g, h)$, the pointwise norm of u with respect to (g, h) is denoted by $|u|$. Let ω_g denote the fundamental form of g and set $\mathrm{d}V_g = \frac{1}{n!}\omega_g^n$. We put

$$\|u\|^2 = \int |u|^2 \mathrm{d}V_g.$$

Let

$$\bar\partial : \bigoplus_{p,q} L^{p,q}_{(2)}(X, E, g, h) \to \bigoplus_{p,q} L^{p,q+1}_{(2)}(X, E, g, h)$$

be the maximal closed extension of the complex exterior differentiation

$$\bar\partial : \bigoplus_{p,q} C^{p,q}_0(X, E, g, h) \to \bigoplus_{p,q} C^{p,q+1}_0(X, E, g, h)$$

of type (0,1), where $C^{p,q}_0(X, E)$ denotes the set of compactly supported E-valued C^∞ forms on X, equipped with the structure of pre-Hilbert space with respect to the metrics g and h. $\mathrm{Dom}\bar\partial$, $\mathrm{Im}\bar\partial$ and $\mathrm{Ker}\bar\partial$ will stand for the domain, the image and the kernel of $\bar\partial$ on $\oplus L^{p,q}_{(2)}(X, E, g, h)$, respectively.

Let

$$\bar\partial^* : \bigoplus_{p,q} L^{p,q}_{(2)}(X, E, g, h) \to \bigoplus_{p,q} L^{p,q-1}_{(2)}(X, E, g, h)$$

be the adjoint of $\bar\partial$. We put $\mathscr{H}(X, E, g, h) = \mathrm{Ker}\bar\partial \cap \mathrm{Ker}\bar\partial^*$.

$$\mathscr{H}(X, E, g, h) \cap L^{p,q}_{(2)}(X, E, g, h) \cong \frac{\mathrm{Ker}\bar\partial \cap L^{p,q}_{(2)}(X, E, g, h)}{[\mathrm{Im}\bar\partial] \cap L^{p,q}_{(2)}(X, E, g, h)},$$

where $[\mathrm{Im}\bar\partial]$ denotes the closure of $\mathrm{Im}\bar\partial$.

The following is most basic for our purpose.

PROPOSITION 3. $\dim H^{p,q}_{(2)}(X, E, g, h) < \infty$ *if there exist a compact set* $K \subset X$ *and a constant* $C > 0$ *such that*

$$\|u\|^2 \le C\left(\int_K |u|^2 dV_g + \|\bar\partial u\|^2 + \|\bar\partial^* u\|^2\right) \tag{4.1}$$

holds for all $u \in \mathrm{Dom}\bar\partial \cap \mathrm{Dom}\bar\partial^* \cap L^{p,q}_{(2)}(X,E,g,h)$.

Recall that the proof of Proposition 3 is done by combining Rellich's lemma and the finite-dimensionality of Banach spaces whose bounded sets are relatively compact. Note that no estimate for $\dim H^{p,q}_{(2)}(X,E,g,h)$ is obtained from Proposition 3.

Recall also that (4.1) follows immediately if there exists a compact set K_1 in the interior of K such that $d\omega_g = 0$ on $X \setminus K_1$ and the curvature form of h denoted by Θ_h satisfies $\Theta_h \geq Id_E \otimes g$ on $X \setminus K_1$. In fact, that (4.1) holds for $u \in C^{n,q}_0(X,E)$ ($q \geq 1$) follows from the equality

$$((\bar\partial\bar\partial^* + \bar\partial^*\bar\partial - \partial_h\partial_h^* - \partial_h^*\partial_h)u,u) = \|\bar\partial u\|^2 + \|\bar\partial^* u\|^2 - \|\partial_h u\|^2 - \|\partial_h^* u\|^2$$

applied to $u \in C^{n,q}_0(X \setminus K_1, E)$ through integration by parts, and by combining this equality with Nakano's formula

$$\bar\partial\bar\partial^* + \bar\partial^*\bar\partial - \partial_h\partial_h^* - \partial_h^*\partial_h = [\sqrt{-1}\Theta_h, \omega_g^*] \tag{4.2}$$

and the inequality $([\sqrt{-1}\Theta_h,\omega_g^*]u,u) \geq \|u\|^2$ coming from the pointwise one. Here $(,)$ denotes the inner product, ∂_h denotes the (1,0)-part of the Chern connection for h, ∂_h^* the adjoint of ∂_h, and ω_g^* denotes the pointwise adjoint of $u \mapsto \omega_g \wedge u$. The (1,1)-form Θ_h is identified with the exterior multiplication by it from the left hand side.

In general, once the estimate (4.1) is known to hold for $C^{p,q}_0(X,E)$, it extends to $\mathrm{Dom}\bar\partial \cap Dom\bar\partial^* \cap L^{p,q}_{(2)}(X,E)$ by the completeness of g. (cf. [32, 33]. See also [5, 6, 7].) One can deduce Theorem 4 by combining this reasoning with a limiting argument in the proof of Proposition 3.4.5 in [41], which is similar as in the following.

PROPOSITION 4 (cf. [62]). *Let* φ_μ ($\mu = 1,2,\dots$) *be an increasing sequence of* C^∞ *functions on* X *converging to a* C^∞ *function* φ *on* X *such that* (4.1) *for fixed* p,q, K *and* C *holds for all* $u \in rmDom\bar\partial \cap \mathrm{Dom}\bar\partial^* \cap L^{p,q}_{(2)}(X,E,g,he^{-\varphi_\mu})$. *Then there exist* $\mu_0 > 0$ *and* $C_1 > 0$ *such that*

$$\|u\|^2 \leq C_1(\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2) \tag{4.3}$$

holds for those $u \in \mathrm{Dom}\bar\partial \cap Dom\bar\partial^* \cap L^{p,q}_{(2)}(X,E,g,he^{-\varphi_\mu})$ *which are orthogonal to* $\mathscr{H}(X,E,g,he^{-\varphi})$ *in* $L^{p,q}_{(2)}(X,E,g,he^{-\varphi})$, *if* $\mu \geq \mu_0$.

Proof. The proof is done by contradiction. Assume on the contrary that there exists no such C_1. Then one can find a sequence $u_\mu \in \mathrm{Dom}\bar\partial \cap \mathrm{Dom}\bar\partial^* \cap L^{p,q}_{(2)}(X,E,g,he^{-\varphi_\mu})$ ($\mu = 1,2,\dots$) with $\|u_\mu\| = 1$ which has a subsequence u_{μ_k} ($k = 1,2,\dots$) such that $\lim_{k\to\infty}\|\bar\partial u_{\mu_k}\| = 0$, $\lim_{k\to\infty}\|\bar\partial^* u_{\mu_k}\| = 0$ and $u_{\mu_k} \perp \mathscr{H}(X,E,g,he^{-\varphi})$. By Rellich's lemma (or by Sobolev's embedding theorem) one has a strongly locally convergent subsequence of u_{μ_k} whose limit, say u_∞ is an element of $L^{p,q}_{(2)}(X,E,g,he^{-\varphi})$ satisfying $\bar\partial u_\infty = 0$, $\bar\partial^* u_\infty = 0$ and $u_\infty \perp \mathscr{H}(X,E,g,he^{-\varphi})$. But (4.1) for fixed K and C for all u_{μ_k} implies that $u_\infty \neq 0$, which is clearly a contradiction. $\square$

Hence we obtain the following by virtue of a fundamental theorem of Hörmander (cf. [41, Theorem 1.1.4]).

Proposition 5. *In the situation of Proposition 4,*

$$\dim H^{p,q}_{(2)}(X,E,g,he^{-\varphi_\mu}) < \infty$$

for all μ and the homomorphisms $H^{p,q}_{(2)}(X,E,g,he^{-\varphi_\mu}) \to H^{p,q}_{(2)}(X,E,g,he^{-\varphi_\nu})$ $(\nu \geq \mu)$ induced by the inclusions are bijective for sufficiently large μ.

5. APPLICATION OF THE FINITE-DIMENSIONALITY TO BUNDLE-CONVEXITY

It was observed by Grauert [34] that, for any Stein manifold X and any point $x \in X$, $X \setminus \{x\}$ admits a complete Kähler metric of the form $\partial\bar\partial\varphi + \partial\bar\partial\psi$, where φ is a C^∞ plurisubharmonic function on X and ψ is a C^∞ function on $X \setminus \{x\}$ such that $\operatorname{supp}\psi$ is relatively compact in X. In order to apply Proposition 5 to obtain nontrivial existence results, we need some additional information on the metric for suitable choices of φ and ψ, which can be formulated as follows.

Proposition 6. *Let Ω be a bounded domain in $\mathbb{C}^n$ which admits a C^∞ psh exhaustion function $\varphi : \Omega \to [1,\infty)$ satisfying $\partial\bar\partial\varphi \geq \partial\varphi\bar\partial\varphi$ and*

$$\lim_{c\to\infty} \sup\left\{R^{\varphi}_{z,w};\, z,w \in \Omega \text{ and } |\varphi(z)-\varphi(w)| > c\right\} = 0, \tag{5.1}$$

where $R^{\varphi}_{z,w} := \dfrac{\log(|\log\|z-w\|| + 1)}{\varphi(w)}$.

Then there exists a C^∞ strictly psh function Φ on a neighborhood of $\overline{\Omega}$ and constants $A > 0$ and $B > 0$ such that, for any sufficiently large c and any $z_0 \in \Omega$ satisfying $2c < \varphi(z_0) < 3c$, one can find a C^∞ function $\psi : \Omega \setminus \{z_0\} \to (-\infty, 0]$ satisfying the following conditions.

1) *$\partial\bar\partial(\Phi + \psi + A\varphi)$ is a complete Kähler metric on $\Omega \setminus \{z_0\}$ satisfying*

$$\partial\bar\partial(\Phi + \psi + A\varphi) \geq \partial\varphi\bar\partial\varphi + \frac{1}{A}\partial\psi\bar\partial\psi.$$

2) $e^{-\psi(z)} = 2n \log \dfrac{1}{\|z-z_0\|} + B$ *on* $\{z \in \Omega \setminus \{z_0\}; 2c < \varphi(z) < 3c\}$.

3) $\operatorname{supp}\psi \Subset \{z \in \Omega; c < \varphi(z) < 4c\}$.

For the proof, the reader is referred to [65].

Definition 4 (Grauert [36]). Given $E \to M$, M is called **strictly *E*-convex** if

$\forall K \Subset M \; \exists \hat{K} \Subset E$ s.t. $\forall x \in M \setminus 0^{-1}(\hat{K})$ and $\forall v \in E_x$

$\exists s \in \Gamma(M, E)$ s.t. $s(K) \subset \hat{K}$ and $s(x) = v$.

Here, $0 : M \to E$ denotes the zero section.

Definition 5 (Pinney [71], Asserda [8]). Given $E \to M$ and $\Omega \Subset M$, Ω is called ***E*-convex** if

$$\forall \gamma \in \Omega^{\mathbb{N}} \text{ s.t. } \gamma(\mathbb{N}) \not\Subset \Omega \; \exists s \in \Gamma(\Omega, E) \text{ s.t. } s(\gamma(\mathbb{N})) \not\Subset E.$$

THEOREM 16 (Asserda [8]). *Suppose that* $M \Subset M$, $\mathrm{rank} E = 1$ *and* $E > 0$. *Then, for any locally pseudoconvex* $\Omega \Subset M$, *one can find* $\mu_0 \in \mathbb{N}$ *such that* Ω *is* E^μ*-convex for* $\mu \geq \mu_0$.

THEOREM 17 (Bundle-convexity theorem). *Let* $\Omega \Subset X$ *be a bounded locally pseudoconvex domain and let* (L, h) *be a holomorphic Hermitian line bundle over* X *whose curvature form is positive on* $\partial\Omega$. *Assume that* $\partial\Omega$ *is a* C^2 *real hypersurface. Then, for any Hermitian holomorphic vector bundle* (F, h_F) *over* X, Ω *is* $(F \otimes L^\mu, h_F h^\mu)$*-convex for sufficiently large* μ.

Proof. Let X, Ω and (L, h) be as above. Let us fix a fiber metric κ of the canonical bundle K_X of X.

Let us choose a Hermitian metric g on X such that $g = \Theta_h$ on a neighborhood of $\partial\Omega$. Let φ be as in Proposition 6.

Let x_ν ($\nu = 1, 2, \dots$) be a sequence of points in Ω which does not have any accumulation point in Ω. Then, by Proposition 6, for any holomorphic Hermitian vector bundle (F, h_F) on X, one can find a subsequence x_{ν_k} of x_ν, an increasing sequence $c_k \in (1, \infty)$ ($k = 1, 2, \dots$) satisfying $8c_k + 1 < c_{k+1}$, a complete Hermitian metric $\tilde{g}$ on $\Omega \setminus \{x_{\nu_k}; k = 1, 2, \dots\}$ and a C^∞ function $\tilde{\psi} : \Omega \setminus \{x_{\nu_k}; k = 1, 2, \dots\} \to (-\infty, 0)$ such that

$$\mathrm{supp}\tilde{\psi} \cap \bigcup_{k=1}^{\infty} \{x; 8c_k < \varphi(x) < c_{k+1}\} = \emptyset,$$

$e^{e^{-\tilde{\psi}}}$ is not integrable around each x_{ν_k}, and

$$\sqrt{-1}(\Theta_F + Id_F \otimes (-\Theta_\kappa - \partial\bar{\partial} e^{-\tilde{\psi}} + \mu\partial\bar{\partial}(\tilde{\psi} + \varphi) + \mu^2\Theta_h)) \geq Id_F \otimes \omega_{\tilde{g}}$$

holds outside a compact subset of $\Omega \setminus \{x_{\nu_k}; k = 1, 2, \dots\}$ for sufficiently large μ.

Therefore, by Nakano's formular and Proposition 3,

$$\dim H^{n,1}_{(2)}(\Omega', K_X^{-1} \otimes F \otimes L^{\mu^2}, \tilde{g}, \kappa^{-1} \otimes h_F \otimes e^{e^{-\tilde{\psi}}} e^{-\mu(\tilde{\psi}+\varphi)} h^{\mu^2}) < \infty$$

holds for sufficiently large μ. Here $\Omega' = \Omega \setminus \{x_{\nu_k}; k = 1, 2, \dots\}$ and $n = \dim \Omega$.

On the other hand, it is clear in this situation that one can find C^∞ sections s_ℓ ($\ell = 1, 2, \dots$) of $F \otimes L^{\mu^2}$ on Ω satisfying

$$\lim_{k\to\infty} |s_\ell(x_{\nu_k})|_{h_F \otimes h^{\mu^2}} = \infty,$$

$$\lim_{k\to\infty} \left| \frac{s_\ell(x_{\nu_k})}{s_{\ell+1}(x_{\nu_k})} \right| = 0,$$

and

$$\bar{\partial} s_\ell \in L^{n,1}_{(2)}(\Omega', K_X^{-1} \otimes F \otimes L^{\mu^2}, \tilde{g}, \kappa^{-1} \otimes h_F \otimes e^{e^{-\tilde{\psi}}} e^{-\mu(\tilde{\psi}+\varphi)} h^{\mu^2})$$

for all ℓ.

Hence one can find a nontrivial linear combination of s_ℓ, say $\sigma = \sum_{\ell=1}^N a_\ell s_\ell$ such that there exists a solution to $\bar{\partial} s = \bar{\partial}\sigma$ with

$$s \in L^{n,0}_{(2)}(\Omega', K_X^{-1} \otimes F \otimes L^{\mu^2}, \tilde{g}, h_F \otimes e^{e^{-\tilde{\psi}}} e^{-\mu(\tilde{\psi}+\varphi)} h^{\mu^2}).$$

Clearly $\sigma - s$ extends to a holomorphic section $\tilde{\sigma}$ of $F \otimes L^{\mu^2}$ which satisfies

$$\lim_{k\to\infty} |\tilde{\sigma}(x_{\nu_k})| = \infty.$$

□

6. PROOF OF THE KERNEL ASYMPTOTICS

In the situation of Theorem 17, it is natural to ask whether one can see the asymptotic behavior of the Bergman kernels B_{Ω, L^ν} for the Bergman spaces $H^{n,0}_{(2)}(\Omega, L^\nu, h^\nu)$ as $z \to \partial\Omega$ for $\nu \gg 1$. Recall that $B_{\Omega, L^\nu, \mu}(x)$ on Ω defined by

$$B_{\Omega, L^\nu}(x) := \sup \left\{ \frac{|s(x)|^2}{\|s\|^2}; s \in H^{n,0}_{(2)}(\Omega, L^\nu, h^\nu) \setminus \{0\} \right\}.$$

From the proof of Theorem 17 one cannot directly see the boundary behavior of B_{Ω, L^ν}. Nevertheless, after reducing the question to the case where $\sqrt{-1}\Theta_h > 0$ on Ω, by allowing singularities of h along $\partial\Omega$, such asymptotics can be analyzed by solving the $\bar{\partial}$ equations with L^2 norm estimates by a standard technique (cf. [Hm], [Dm-1,3] or [Oh-2,4]). The reduction is done by the following.

Lemma. *Let X be a complex manifold and let $L \to X$ be a holomorphic line bundle. If the meromorphic map*

$$\sigma : X \cdots \to Proj(H^{0,0}(X, L)) := H^{0,0}(X, L)^* / (\mathbb{C} \setminus \{0\})$$

induced from the correspondence $x \mapsto \{s; s(x) = 0\}$ *is proper onto its image, there exists a complex manifold* $\tilde{X}$ *with a proper and bimeromorphic holomorphic map* $\pi : \tilde{X} \to X$ *and a divisor* A *on* $\tilde{X}$ *with compact support such that* $\sigma \circ \pi$ *is holomorphic and*

$$\pi^* L \otimes [A] = (\sigma \circ \pi)^* \mathscr{O}(1),$$

where $\mathscr{O}(1)$ *denotes the hyperplane section bundle over* $Proj(H^{0,0}(X, L))$.

Proof. Given such σ as above, by virtue of a theorem of Hironaka, there exists a succession of blow-ups $X_N \to X_{N-1} \to \cdots X_1 \to X_0 = X$ which ends up with a map $\pi : \tilde{X} \to X$ with the required property. □

Hence, by applying Theorem 2 after modifying (X, L) to $(\tilde{X}, \tilde{L})$ with a positive line bundle $\tilde{L} \to \tilde{X}$ extending $L|_{\partial\Omega}$, it is not difficult to prove Theorem 1. It is very likely that Theorem 1 holds also for $\varepsilon = 0$.

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