Dedicated to the memory of my PhD advisor, Professor Mihnea Colţoiu, whose wise guidance and unwavering support helped to shape the early years of my research career.

LCK SPACES. A SHORT SURVEY

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In this short survey, we present the results obtained so far in the geometry of lcK singular spaces.

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1. **INTRODUCTION**

Problems arising in differential geometry are often related to finding a good metric, in some sense, according to the given context of the problem. For complex analytic geometry, the metrics with the most convenient properties are Kähler metrics. However, there are topological obstructions to existence of Kähler metrics, which come from the Hodge decomposition theorem: for a compact complex space to admit a Kähler metric, it has to have the odd Betti numbers $b_{2k+1} \equiv 0 \pmod 2$, for all $k \geq 0$. Hence, in non-Kähler geometry, one has to find a good substitute for Kähler metrics in the class of Hermitian metrics.

The notion of *locally conformally Kähler (lcK)* manifolds was introduced by Vaisman [9]. An lcK manifold is a complex manifold endowed with a Hermitian metric whose associated 2-form ω verifies $d\omega = \theta \wedge \omega$, where θ is a closed 1-form, called the Lee form of ω. Equivalently, we can say that locally, there exists a smooth function f such that $e^{-f}\omega$ is Kähler, which explains the name. Then, the 1-form θ is given by the local 1-forms df. If f can be defined globally, which means that θ is exact, then ω is called *globally conformally Kähler (gcK)*. If ω is not gcK, then we call it *pure lcK*. Another alternative definition of lcK manifolds can be given *via* the universal cover: a complex manifold M

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admits an lcK metric if and only if the universal cover $\widetilde{M}$ admits a Kähler metric $\widetilde{\omega}$ such that for every $\gamma \in \mathrm{Deck}(\widetilde{M}/M)$, we have $\gamma^*\widetilde{\omega} = c_\gamma \widetilde{\omega}$, where c_γ is a positive constant, and in this case we say that $\mathrm{Deck}(\widetilde{M}/M)$ acts on $\widetilde{\omega}$ by positive homotheties. This last characterization is often used to prove that a complex manifold admits lcK metrics.

Some years after introducing the class of lcK manifolds, Vaisman [10] proved that on compact complex manifolds, Kähler and pure lcK metrics with respect to the same complex structure cannot both exist. Hence, for compact non-Kähler manifolds, it makes sense to ask if there exist at least lcK metrics.

Since then, many more results about lcK metrics have been proved. For an up-to-date reference on the theme of lcK manifolds, one can check the book by Ornea and Verbitsky [3].

Kähler forms on singular complex spaces were first introduced by Grauert [2], using families of locally defined strictly plurisubharmonic functions and compatibility conditions. In the spirit of Grauert's idea, we can define lcK forms on singular spaces, as in [8]:

Definition 1.1. Let X be a complex space.

(K) A *Kähler metric* on X is the equivalence class $\widehat{(U_i, \varphi_i)_{i\in I}}$ of a family such that $(U_i)_{i\in I}$ is an open cover of X, $\varphi_i : U_i \longrightarrow \mathbb{R}$ is $\mathcal{C}^\infty$ and strictly psh, and $\mathrm{i}\partial\overline{\partial}\varphi_i = \mathrm{i}\partial\overline{\partial}\varphi_j$ on $U_i \cap U_j \cap X_{\mathrm{reg}}$, for every $i, j \in I$. Two such families are equivalent if their union verifies the compatibility condition on the intersections, described above.

(lcK) An *lcK metric* on X is the equivalence class $\widehat{(U_i, \varphi_i, f_i)_{i\in I}}$ of a family such that $(U_i)_{i\in I}$ is an open cover of X, $\varphi_i : U_i \longrightarrow \mathbb{R}$ is $\mathcal{C}^\infty$ and strictly psh, $f_i : U_i \longrightarrow \mathbb{R}$ is smooth, and $\mathrm{i}e^{f_i}\partial\overline{\partial}\varphi_i = \mathrm{i}e^{f_j}\partial\overline{\partial}\varphi_j$ on $U_i \cap U_j \cap X_{\mathrm{reg}}$, for every $i, j \in I$. Again, two such families are equivalent if their union verifies the compatibility condition written above.

Since for lcK forms on singular spaces we also want to define its associated Lee form, we have the following:

Definition 1.2. Let X be a topological space and consider $(U_i, f_i)_{i\in I}$, consisting of an open cover $(U_i)_{i\in I}$ of X and a family of continuous functions $f_i : U_i \longrightarrow \mathbb{R}$ such that $f_i - f_j$ is locally constant on $U_i \cap U_j$, for all $i, j \in I$. The class

$$\theta = \widehat{(U_i, f_i)_{i\in I}} \in \check{\mathrm{H}}^0\left(X, \mathscr{C}/\mathbb{R}\right)$$

is called a *topologically closed* 1*-form* (*TC* 1*-form*).

We say that a TC 1-form θ is *exact* if $\theta = \widehat{(X, f)}$ for a continuous function $f : X \longrightarrow \mathbb{R}$. In this case, we make the notation $\theta = df$.

Let $\omega = \widehat{(U_i, \varphi_i, f_i)_{i\in I}}$ be an lcK metric on a complex space X. Then, the TC 1-form $\theta = \widehat{(U_i, f_i)_{i\in I}}$ is called the *Lee form* of ω. If θ is exact, then ω is called *globally conformally Kähler (gcK)*.

In this survey, we present the progress that has been made so far in the study of lcK spaces. These are mainly results which show that the fundamental theorems of lcK manifolds are still true for lcK spaces, under some reasonable additional conditions.

2. KNOWN RESULTS

The first result is the characterization theorem for lcK spaces [6, Theorem 3.10], whose statement is the exact analogue of the one for the smooth case.

THEOREM 2.1. *Let X be a complex space. Then X admits an lcK metric if and only if its universal covering $\widetilde{X}$ admits a Kähler metric $\widetilde{\omega}$ such that the deck group acts on $\widetilde{\omega}$ by positive homothethies.*

The *if* part is a straightforward adaptation of the proof for manifolds, so we only have to give an explanation for the *only if* part. In [6], this is done only with elementary tools, by defining a function which generalizes the integral of a closed 1-form along a submanifold of dimension 1, to TC 1-forms along a given path. However, the most simple proof is given in [7, Theorem 2.6], and is based on the fact that every TC 1-form on the simply connected complex space $\widetilde{X}$ is exact. For this, we consider the short exact sequence

$$0 \longrightarrow \underline{\mathbb{R}} \longrightarrow \mathscr{C} \longrightarrow \mathscr{C}/\underline{\mathbb{R}} \longrightarrow 0,$$

where $\underline{\mathbb{R}}$ is the sheaf of locally constant functions and $\mathscr{C}$ is the sheaf of continuous functions. We then pass to the long exact sequence in cohomology:

$$0 \longrightarrow H^0(\widetilde{X}, \underline{\mathbb{R}}) \longrightarrow H^0(\widetilde{X}, \mathscr{C}) \longrightarrow H^0(\widetilde{X}, \mathscr{C}/\underline{\mathbb{R}}) \longrightarrow H^1(\widetilde{X}, \underline{\mathbb{R}}) \longrightarrow \ldots$$

Now, denote by $\widetilde{\omega}$ and $\widetilde{\theta}$ the pull-back to $\widetilde{X}$ of ω and θ. Then, $\widetilde{\theta}$ is the Lee form of $\widetilde{\omega}$. We have $\widetilde{\theta} \in \check{\mathrm{H}}^0\left(\widetilde{X}, \mathscr{C}/\underline{\mathbb{R}}\right)$. Moreover, as $\widetilde{X}$ is simply connected, $H^1(\widetilde{X}, \underline{\mathbb{R}}) = 0$. We then obtain immediately that $\widetilde{\theta}$ is exact, which means that $\widetilde{\omega}$ is gcK, thus there exists a smooth function $\widetilde{f} : \widetilde{X} \to \mathbb{R}$ such that $\widetilde{\theta} = d\widetilde{f}$. Then, $e^{-\widetilde{f}}\widetilde{\omega}$ is a Kähler metric and since $d(\widetilde{f} - \gamma^*\widetilde{f}) = 0$, we get

$$\gamma^*(e^{-\widetilde{f}}\widetilde{\omega}) = e^{-\gamma^*\widetilde{f}}\widetilde{\omega} = e^{a_\gamma - \widetilde{f}}\widetilde{\omega} = c_\gamma e^{-\widetilde{f}}\widetilde{\omega},$$

which shows that we also have the action by positive homotheties on $e^{-\widetilde{f}}\widetilde{\omega}$.

Remark 2.2. Theorem 2.1 remains true, with the same proof, if instead of the universal cover, there exists a cover $\widetilde{X}$ with a Kähler metric $\widetilde{\omega}$ such that the deck group acts by positive homotheties on $\widetilde{\omega}$.

The next important result is a generalization of Vaisman's theorem to complex spaces. For compact complex manifolds, Vaisman [10] proved that there cannot exist both Kähler and pure lcK metrics with respect to the same complex structure. Any known proof to this theorem uses the Hodge decomposition or the $\partial\overline{\partial}$-lemma, which both are unavailable in the context of singular complex spaces.

In [7], the authors proved that for compact complex spaces the theorem holds, but with the strong additional assumption of local irreducibility. The statement is the following:

THEOREM 2.3. *Let (X, ω, θ) be a compact, locally irreducible, lcK space. If X admits a Kähler metric, then (X, ω, θ) is gcK.*

The proof relies on the original Vaisman theorem for compact manifolds, the existence of a resolution of singularities by Hironaka, and a theorem of Fujiki which says that the the blow-up of a Kähler space along a compact closed subspace admits Kähler metrics. An essential step in the proof is based on the fact that the blow-up of a locally irreducible complex space is also locally irreducible.

Furthermore, [7] contains a counterexample constructed by Vuletescu which shows that the requirement of local irreducibility cannot be dropped. This counterexample is a compact complex space with only one singular point, obtained by identifying two distinct points on $\mathbb{P}^n$.

With regard to blow-ups, Ornea, Verbitsky and Vuletescu [4] proved that the blow-up of an lcK manifold (M, ω, θ) along a closed compact submanifold Z admits lcK metrics if and only if $\omega_{\restriction Z}$ is gcK. This was generalized to lcK spaces in [8], with some additional conditions, in the following form:

THEOREM 2.4. *Let (X, ω, θ) be an lcK space, and $Z \subset X$ a compact complex subspace, which is normal and is a locally complete intersection.*

Then, the blow-up of X along Z, denoted $\widehat{X}$, admits an lcK metric if and only if $\omega_{\restriction Z}$ is gcK.

The proof for the direct implication makes use of Varouchas' results [11] which give sufficient conditions under which the image of a Kähler space under a holomorphic map is also of Kähler type. For this, we need Z to be normal and the fibers of the canonical projection of the blow-up to be compact complex manifolds of equal dimension. The latter condition is satisfied if Z is a locally complete intersection. We also use Vaisman's theorem for lcK spaces (2.3), for which we need local irreducibility of Z, guaranteed by the normality assumption.

Another key element for the proof of the direct implication is [8, Lemma 3.1] about fibrations, which is an adaptation to the singular setting of the analogue result for the smooth case [4, Lemma 3.1], whose proof involves integrating along the fibers and is based on the fact that the fibers along the center of the blow-up are biholomorphic. To still have regular fibers which are biholomorphic for the singular case, we need again the additional condition that Z is a locally complete intersection.

The next two results show the importance of the characterization 2.1, as it is used to prove the existence of lcK metrics without effectively constructing them. The first one, [6, Theorem 4.1], which the lcK version of [12, Theorem 1], says that the image of an lcK space under a holomorphic map with discrete fibers admits lcK metrics. Obviously, it is nontrivial only when the map is a ramified covering, otherwise we can just take the pull-back of the lcK metric. Its statement is the following:

THEOREM 2.5. *Let $g : X \to Y$ be a holomorphic map between complex spaces with discrete fibers and assume (Y, ω, θ) is lcK. Then, X also admits an lcK metric.*

Its proof is done by passing to the universal cover, then following the same main steps as in [12] for constructing a Kähler metric on the universal cover of X. However, all the elements which are part of this construction must be chosen such that the deck group of the universal cover of X acts by positive homotheties on the constructed Kähler metric. This is done by carefully adding to the proof of [12, Theorem 1] some topological arguments.

An important mention is that the construction of the Kählerian metric in [12] is not canonical, as it depends on many choices along the way. Since Theorem 2.5 uses the proof from [12], the construction of the lcK metric on X is also non-canonical.

The second results which uses an adapted version of 2.1 is about modifications of compact lcK spaces. Theorem 2.4 shows that blow-up of an lcK space does not necessarily admit lcK metrics. Hence, we are interested to find a larger class of metrics which are stable under blow-ups and, more generally, under modifications. It turns out that this is achieved simply by working with a more general definition than that of lcK metrics and allowing the strictly plurisubharmonic functions which locally define the metric to take the value $-\infty$ on a small set of points. This type of metric will be called a *quasi-lcK metric*, inspired by the notion of *quasi-Kähler metric* introduced by Colţoiu [1], and later used by Popa-Fischer [5] under the name of *generalized Kähler metric*. For our definitions, we also require $\mathcal{C}^\infty$-regularity, as in [5].

Definition 2.6. Let X be a complex space. For $U \subset X$ and a psh function $\varphi : U \longrightarrow \mathbb{R}$, we denote $\{\varphi = -\infty\} = A_\varphi$.

(q-K) A *quasi-Kähler metric* on X is the equivalence class $\widehat{(U_i, \varphi_i)_{i\in I}}$ of a family such that

(a) $(U_i)_{i\in I}$ is an open cover of X;

(b) $\varphi_i : U_i \longrightarrow [-\infty, \infty)$ is strictly psh, $\varphi_i \not\equiv -\infty$ on any irreducible component of U_i, and φ_i is of class $\mathcal{C}^\infty$ on $U_i \setminus A_{\varphi_i}$;

(c) $\mathrm{i}\partial\overline{\partial}\varphi_i = \mathrm{i}\partial\overline{\partial}\varphi_j$ on $(U_i \cap U_j) \setminus (X_{\text{sing}} \cup A_{\varphi_i} \cup A_{\varphi_j})$, for any $i, j \in I$;

(d) $\varphi_i - \varphi_j$ restricted to $U_i \cap U_j \setminus (A_{\varphi_i} \cup A_{\varphi_j})$ is locally bounded around points of $A_{\varphi_i} \cup A_{\varphi_j}$, for any $i, j \in I$;

Two such families are equivalent if their union still verifies the compatibility conditions (c) and (d).

(q-lcK) A *quasi-lcK metric* on X is the equivalence class $\widehat{(U_i, \varphi_i, f_i)_{i\in I}}$ of a family $(U_i, \varphi_i, f_i)_{i\in I}$ such that $(U_i, \varphi_i)_{i\in I}$ verifies conditions (a) and (b) in 2.6 – (q-K), and moreover:

(e) $f_i : U_i \longrightarrow \mathbb{R}$ is of class $\mathcal{C}^\infty$ for any $i \in I$;

(f) $\mathrm{i}e^{f_i}\partial\overline{\partial}\varphi_i = \mathrm{i}e^{f_j}\partial\overline{\partial}\varphi_j$ on $(U_i \cap U_j) \setminus (X_{\text{sing}} \cup A_{\varphi_i} \cup A_{\varphi_j})$, for any $i, j \in I$;

(g) $(f_i - f_j)\varphi_i - \varphi_j$ restricted to $U_i \cap U_j \setminus (A_{\varphi_i} \cup A_{\varphi_j})$ is locally bounded around points of $A_{\varphi_i} \cup A_{\varphi_j}$, for any $i, j \in I$.

Two such families are equivalent if their union still verifies conditions (f) and (g).

With this definition, we have [8, Theorem 4.1], with the following statement:

THEOREM 2.7. *Let $p : X \to Y$ be a modification of the compact complex space Y. Suppose that Y is quasi-lcK. Then, X also admits a quasi-lcK metric.*

The proof given in [8] is based on the following strategy: we use [5, Theorem 2.5] to construct a quasi-Kähler metric $\widetilde{\omega}$ on the universal cover of $\widetilde{X}$ of X. We show that if all the choices we make in that construction are well related, then $\mathrm{Deck}(\widetilde{X}/X)$ acts by homotheties on $\widetilde{\omega}$. Finally, Theorem 2.1 can be easily adapted to quasi-lcK spaces, so it can be applied to conclude that X admits a quasi-lcK metric.

3. SOME QUESTIONS

In this section we formulate some questions which arise naturally when analyzing the proofs of the theorems stated in the previous section.

Vaisman's theorem is an essential result for lcK geometry. However, the proof given in [6] works only for compact spaces which are locally irreducible, which is a strong restriction. For example, when blowing up a complex space, the exceptional divisor is not always locally irreducible. Also, fibers of holomorphic mappings between complex spaces are not always locally irreducible. [6, Example 4.5] shows that by identifying two distinct points in $\mathbb{P}^n$, we obtain a compact complex space with only one singular point, locally reducible in that point, for which Vaisman's theorem does not hold. It would be important to know if the assumption of local irreducibility can be replaced with some weaker condition. This leads to the following:

Question 3.1. Does every compact, globally irreducible and locally reducible Kähler space admit a pure lcK metric?

A more general construction than identifying two distinct points in a given Kähler manifold, as in [6, Example 4.5], is to identify two biholomorphic closed subspaces. However, in this case it is no longer clear how to find, or even if there exists, a Kähler metric on the new space that is obtained by this method.

Question 3.2. If (X, ω) is a Kähler manifold and $Z_1, Z_2 \subset X$ are closed complex submanifolds such that

- $Z_1 \cap Z_2 = \emptyset$;
- there exists a biholomorphism $f : Z_1 \to Z_2$

and we consider X' to be the complex space obtained by the identification $Z_1 \sim Z_2$ given by f, does X' always admit a Kähler metric?

Theorem 2.4 has two additional conditions on the center of the blow-up, the subspace Z. The condition of normality of Z is necessary for the proof given in [8] when using Varouchas' results [11] on the image of a Kähler space under a holomorphic mapping, and also for the local irreducibility required when applying Theorem 2.3, so it would be difficult to replace with something else. However, it would be interesting to study if the proof can be modified so that the second condition, that of Z to be a locally complete intersection, can be weakened or even dropped.

Question 3.3. Does Theorem 2.4 remain true without the assumption that Z is a locally complete intersection?

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