Dedicated to the memory of Mihnea Colţoiu

ON SINGULAR MAPS WITH LOCAL FIBRATION

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Communicated by Cezar Joiţa

We discuss the most general condition under which a singular local tube fibration exists. We give an application to composition of map germs.

AMS 2020 Subject Classification: 14D06, 58K05, 57R45, 14P10, 32S20, 32S60, 58K15, 32C18.

Key words: real map germs, fibrations.

1. STATING A MEANINGFUL LOCAL FIBRATION THEOREM

Let $G : (\mathbb{K}^m, 0) \to (\mathbb{K}^p, 0)$ be an analytic map germ, where $m \geq p \geq 1$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. In case $p = 1$ and $\mathbb{K} = \mathbb{C}$, it is well-known that holomorphic function germs have a local Milnor fibration. Whenever $\mathbb{K} = \mathbb{R}$, removing the origin disconnects the real line and, by similar arguments, one obtains a Milnor fibration $G : (\mathbb{R}^m, 0) \to (\mathbb{R}, 0)$ which is a trivial fibration over each of the two connected components of the set germ $(\mathbb{R} \setminus \{0\}, 0)$.

The Milnor's local fibration theorem for holomorphic function germs [12] makes sense only if it is independent on the choices of ε and δ (see Definition 2.12). In case of map germs, complex or real, the main problem is to produce the necessary conditions for the existence of the local fibration independently on the chosen neighbourhoods, provided they are small enough. Some part of the recent literature misses the hot core of the local fibration problem by not caring about this independency condition.

It is of course the setting $p \geq 2$ to which we focus here. Let us look at two examples in order to get a first impression on the problems that may occur when varying neighbourhoods, and on their complexity.

Example 1.1 ([13]). $F : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)$, $(x, y, z) \mapsto (x^2 - y^2 z, y)$. The singular locus of F is included in the central fibre $F^{-1}(0, 0)$.

Ying Chen acknowledges the support from the National Natural Science Foundation of China (NSFC) (Grant no. 11601168). Mihai Tibăr acknowledges the support of the Labex CEMPI (ANR-11-LABX-0007-01).

doi: 10.59277/RRMPA.2023.9.17

The image of F is open in 0, namely we have the equality of set germs $(\operatorname{Im} F, 0) = (\mathbb{C}^2, 0)$. Nevertheless this map germ F does not have a locally trivial fibration over the set germ $(\mathbb{C}^2 \setminus \{0\}, 0)$, as shown by Sabbah [13]. In particular, one cannot apply Ehresmann Theorem due to the local non-properness of the map.

Example 1.2. $F : (\mathbb{K}^2, 0) \to (\mathbb{K}^2, 0)$, $F(x, y) = (x, xy)$. The image $F(B_\varepsilon)$ of the ball B_ε centred at 0 depends heavily on its radius $\varepsilon > 0$. Here, not only that the image $F(B_\varepsilon)$ does not contain a neighbourhood of the origin, but it turns out that the image of F is not even well-defined as a set germ.

We discuss here the sharpest conditions under which the analytic map germs define local fibrations. We find a general condition under which the composition of map germs has a local fibration. Our main result is Theorem 3.2, and we discuss one of its consequences.

2. TAME MAP GERMS

2.1. Map germs having germ image sets

In Example 1.2, the image of the map germs G is not well-defined as a set germ. Let us define this condition more carefully. We state this in case $\mathbb{K} = \mathbb{R}$ but everything holds over $\mathbb{C}$.

Remark 2.1. Let $A, A' \subset \mathbb{R}^p$ be two subsets containing the origin. By definition, one has the equality of set germs $(A, 0) = (A', 0)$ if and only if there exists some open ball $B_\varepsilon \subset \mathbb{R}^p$ centred at 0 and of radius $\varepsilon > 0$ such that $A \cap B_\varepsilon = A' \cap B_\varepsilon$.

Definition 2.2 (*Nice map germs* [5], [8]). Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m \geq p > 0$, be a continuous map germ. One says that the image $G(K)$ of a subset $K \subset \mathbb{R}^m$ containing 0 is a *well-defined set germ* at $0 \in \mathbb{R}^p$ if for all small enough open balls $B_{\varepsilon_1}, B_{\varepsilon_2}$ centred at 0, with $\varepsilon_1, \varepsilon_2 > 0$, the equality of germs $(G(B_{\varepsilon_1} \cap K), 0) = (G(B_{\varepsilon_2} \cap K), 0)$ holds.

Whenever both images $\operatorname{Im} G$ and $G(\operatorname{Sing} G)$ are well-defined as set germs, one says that G is a *nice map germ*, and we abbreviate this by writing NMG.

It turns out that the NMG condition implies that the *discriminant*

$$\operatorname{Disc} G := G(\operatorname{Sing} G)$$

is a well-defined closed subanalytic set germ, see [8].

Let us point out the following significant cases of a nice map germ, complex and real, respectively:

Proposition 2.3 ([8]). *Let* $F : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$*,* $n \ge p > 0$*, be a holomorphic map germ. If the fibre* $F^{-1}(0)$ *has dimension* $n-p$ *then* $(\operatorname{Im} F, 0) = (\mathbb{C}^p, 0)$*. If moreover* $\operatorname{Sing} F \cap F^{-1}(0) = \{0\}$ *then* F *is a NMG.*

Proof. We fix some small enough open ball at the origin $B_\varepsilon \subset \mathbb{C}^n$ where the holomorphic map F is well-defined. If $Z \subset B_\varepsilon$ be a general complex p-plane passing through 0, then 0 is an isolated point of the slice $Z \cap F^{-1}(0)$. For any small enough open neighbourhood U_ε of 0 in Z, the induced map $F_{|U_\varepsilon} : U_\varepsilon \to \mathbb{C}^p$ is then finite-to-one, and by the Open Mapping Theorem, this implies that $F(U_\varepsilon)$ is open, which shows the equality of germs $(\operatorname{Im} F, 0) = (\mathbb{C}^p, 0)$.

The condition $\operatorname{Sing} F \cap F^{-1}(0) = \{0\}$ means that the set germ $(F^{-1}(0), 0)$ is an ICIS = *isolated complete intersection singularity*, and in this case Looijenga proved in [10] that the image $F(\operatorname{Sing} F)$ is a hypersurface germ at $0 \in \mathbb{C}^p$. □

One has the following real counterpart of the last assertion of Proposition 2.3:

Proposition 2.4 ([8]). *Let* $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$*,* $m \ge p > 0$ *be an analytic map germ. If* $\operatorname{Sing} G \cap G^{-1}(0) = \{0\}$*, then* G *is a NMG.*

The question whether a given map germ has a well-defined image as a set germ seems to be wide open. A first classification has been given in [9] for the case of holomorphic map germs with target $(\mathbb{C}^2, 0)$.

2.2. Milnor set and tame map germs

We will give here a convenient condition which implies NMG in full generality.

Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be a non-constant analytic map germ, $m \ge p \ge 1$. Let $U \subset \mathbb{R}^m$ be a manifold, let $\rho := \| \cdot \|$ be the Euclidean distance function, and let $\rho_{|U}$ be its restriction to U. The set of *ρ-nonregular points* of $G_{|U}$, also called *the Milnor set of* $G_{|U}$, is defined as:

$$M(G_{|U}) := \left\{ x \in U \mid \rho_{|U} \not\pitchfork_x G_{|U} \right\}.$$

It turns out from the definition that $M(G_{|U})$ is real analytic whenever U is supposed analytic. By definition $M(G_{|U})$ coincides with the singular set $\operatorname{Sing}(\rho, G)_{|U}$, which is itself defined as the set of points $x \in U$ such that either $x \in \operatorname{Sing}(G_{|U})$, or $x \notin \operatorname{Sing}(G_{|U})$ and $\operatorname{rank}_x(\rho_{|U}, G_{|U}) = \operatorname{rank}_x(G_{|U})$. We will actually consider the germ at 0 of $M(G_{|U})$, and we will denote it by $M(G)$.

Definition 2.5 (*The Milnor set in the stratified setting*). Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, with $m \ge p > 1$, be a non-constant analytic map germ. We say that a finite semi-analytic Whitney (a)-regular stratification $\mathcal{W}$ of $\mathbb{R}^m$ is a stratification of G if $\operatorname{Sing} G$ is a union of strata, and such that the restriction $G_{|W}$ has constant rank for any $W \in \mathcal{W}$.

Let $W \in \mathcal{W}$ be the germ at 0 of the stratum W, and let $M(G_{|W})$ be the Milnor set of $G_{|W}$, as defined above. One then calls

$$M(G) := \bigsqcup_{W \in \mathcal{W}} M(G_{|W_\alpha})$$

the set of *stratwise ρ-nonregular points* of G with respect to the stratification $\mathcal{W}$.

Remark 2.6. The Milnor set $M(G)$ is closed because $\mathcal{W}$ is a Whitney (a)-stratification. Also notice that if $\operatorname{rank} G_{|W} = \dim W$, then $W \subset M(G)$.

Because of the next definition, we are especially interested in the intersection $M(G) \cap G^{-1}(0)$. The following inclusion holds:

$$M(G) \cap G^{-1}(0) \subset \operatorname{Sing} G \cap G^{-1}(0). \tag{1}$$

Indeed, by Milnor's classical result on the local conical structure of semi-analytic sets [12], [7], there exists $\varepsilon_0 > 0$ such that the manifold $G^{-1}(0) \setminus \operatorname{Sing} G$ is transversal to the sphere S_ε^{m-1} centred at 0, for any $0 < \varepsilon < \varepsilon_0$. For any fixed point $a \in G^{-1}(0) \setminus \operatorname{Sing} G$, a whole open ball B centred at a does not intersect $\operatorname{Sing} G$, and it then follows that the nearby fibres of G inside B are also transversal to the levels of the distance function ρ, provided that B is small enough.

This implies that $M(G) \cap (G^{-1}(0) \setminus \operatorname{Sing} G) = \emptyset$, which proves our claim.

Definition 2.7 (*Tame map germs* [8]). Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, with $m > p \geq 2$, be a non-constant analytic map germ. We say that G is *tame* with respect to the stratification $\mathcal{W}$ if the following inclusion of set germs holds:

$$\overline{M(G) \setminus G^{-1}(0)} \cap G^{-1}(0) \subset \{0\}. \tag{2}$$

It follows from the definition that if G is tame then the closure of the strata of $\mathcal{W}$ of dimensions $\leq p$ intersect $G^{-1}(0)$ only at $\{0\}$.

Remark 2.8. The identity map is not tame. As one may easily check, Examples 1.1 and 1.2 are not tame.

As we work in a highly singular situation, we have to insure the existence of the images as set germs because this is a preliminary condition in the problem of the existence of a singular local fibration. The "tame" condition just does this job:

Theorem 2.9 ([8]). *Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, with $m \geq p \geq 2$, be a non-constant analytic map germ. If G is tame then:*

(a) *G is a NMG at the origin.*

(b) *the image $G(W_\alpha)$ is a well-defined set germ at the origin, for any stratum $W_\alpha \in \mathcal{W}$.*

Remark 2.10. Let us point out two very particular cases where $\operatorname{Im} G$ is well-defined as a set germ:

(i) $\operatorname{Sing} G \subset G^{-1}(0)$ and G satisfies condition (2). It was shown by Massey [11] that under these hypotheses one has $(\operatorname{Im} G, 0) = (\mathbb{R}^p, 0)$.

(ii) $\operatorname{Sing} G \cap G^{-1}(0) \neq G^{-1}(0)$. This is the hypothesis of Proposition 2.4, and condition (2) is not required. In this case, we also have $(\operatorname{Im} G, 0) = (\mathbb{R}^p, 0)$.

2.3. **Singular stratified fibration theorem**

After [8], we show that *tame* is the most suitable condition under which one can prove the existence of a local singular fibration. We focus on the general case $\dim \operatorname{Disc} G > 0$. We first need two definitions.

Definition 2.11 (Regular stratification of the map germ G). Let

$$G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$$

be a non-constant analytic map germ, $m > p \geq 2$, and let $\mathcal{W}$ be a Whitney (b)-regular stratification of G at 0, as defined above. We assume that G is tame. Then Theorem 2.9 tells that the images of all strata of $\mathcal{W}$ are well-defined as set germs at 0. By the classical stratification theory, there exists a germ of a finite subanalytic stratification $\mathcal{S}$ of the target such that $\operatorname{Disc} G$ (which is a closed subanalytic set, as remarked after Definition 2.2) is a union of strata, and such that G is a stratified submersion relative to the couple of stratifications $(\mathcal{W}, \mathcal{S})$, meaning that the image by G of a stratum $W_\alpha \in \mathcal{W}$ is a single stratum $S_\beta \in \mathcal{S}$, and that the restriction $G_| : W_\alpha \to S_\beta$ is a submersion.

We call the couple $(\mathcal{W}, \mathcal{S})$ a *regular stratification of the map germ G*.

Definition 2.12 (Singular Milnor tube fibration, [8]). Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m \geq p > 1$, be a non-constant analytic map germ. Assume that there exists some regular stratification $(\mathcal{W}, \mathcal{S})$ of G.

We say that G has a *(singular) Milnor tube fibration* relative to $(\mathcal{W}, \mathcal{S})$ if for any small enough $\varepsilon > 0$ there exists $0 < \eta \ll \varepsilon$ such that the restriction:

$$G_| : B^m_\varepsilon \cap G^{-1}(B^p_\eta \setminus \{0\}) \to B^p_\eta \setminus \{0\} \tag{3}$$

is a stratified locally trivial fibration which is independent, up to stratified homeomorphisms, of the choices of ε and η.

By *stratified locally trivial fibration*, we mean that for any stratum S_β of $\mathcal{S}$, the restriction $G_{|G^{-1}(S_\beta)}$ is a locally trivial *stratwise fibration.*

What means more precisely "independent, up to stratified homeomorphisms, of the choices of ε and η"? It means that when replacing the Milnor data (ε, η) by another Milnor data (ε', η'), for some $\varepsilon' < \varepsilon$, and suitably smaller $\eta' < \eta$, then the two fibrations (3) have the same stratified image in the smaller ball $B^p_{\eta'}$, and these fibrations are stratified diffeomorphic over $B^p_{\eta'} \setminus \{0\}$.

The non-empty fibres of (3) are those over some connected stratum $S_\beta \subset \operatorname{Im} G$ of $\mathcal{S}$, and such fibre $G^{-1}(s)$, for some $s \in S_\beta$, is the union $\sqcup_{W_\alpha} G^{-1}(s)$ over all strata $W_\alpha \subset G^{-1}(S_\beta)$.

One has the following fundamental result:

THEOREM 2.13 ([8]). *Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m > p \geq 2$, be a non-constant analytic map germ. If G is tame, then G has a singular Milnor tube fibration* (3).

The proof consists in showing that the restriction map

$$G_| : W_\alpha \cap \overline{B^m_\varepsilon} \cap G^{-1}(S_\beta \cap B^p_\eta \setminus \{0\}) \to S_\beta \cap B^p_\eta \setminus \{0\} \tag{4}$$

is well-defined, it is a submersion from a manifold with boundary, hence, it is therefore a locally trivial stratified fibration by Thom-Mather Isotopy Theorem. The tame condition (2) also implies that this fibration is independent of ε and η up to stratified homeomorphisms.

Remark 2.14. The relation between *tame* and the Thom regularity illustrates once more why *tame* is a universal condition, more precisely, we have the implication: if G has a Whitney stratification $\mathcal{W}$ (cf Definition 2.5) such that it is Thom regular at all the strata included in $G^{-1}(0)$ then G is tame. We refer to [8] for details, and for several examples.

3. APPLICATIONS OF THE TUBE FIBRATION

We consider here the particular setting of analytic map germs

$$G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$$

with *isolated singular value*, which means $\operatorname{Sing} G \subset G^{-1}(0)$. This is more general than what was usually considered in the literature after Milnor's [12] and before 2008, namely map germs with an isolated singularity. With [2], [3], [11] and [4] this more general setting started to be studied in more detail.

Remark 3.1. A map germ $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m \geq p \geq 2$, with *isolated singular value* has the following properties:

(a) The stratification of the target has only two strata: $\mathbb{R}^p \setminus \{0\}$, and the origin. A stratification of the source means some Whitney (a)-regular stratification of the singular locus $\operatorname{Sing} G \subset G^{-1}(0)$ without any other conditions, and $\mathbb{R}^m \setminus \operatorname{Sing} G$ is one stratum, if connected, or it is the union of the connected components of $\mathbb{R}^m \setminus \operatorname{Sing} G$ as strata.

(b) The *tame* condition for G amounts to the *ρ-regularity* in our setting of isolated singular value, which reads (compare to Definitions 2.5 and 2.7):

$$\overline{M(G_{|\mathbb{R}^m \setminus G^{-1}(0)})} \cap G^{-1}(0) \subset \{0\}. \tag{5}$$

Whenever (5) holds, we will say for short: *G is ρ-regular.*

(c) If the map germ G is tame and with isolated singular value, then the image of G is locally open, in other words, we have the equality of set germs $(\operatorname{Im} G, 0) = (\mathbb{R}^p, 0)$. This was remarked by Massey [11, Cor. 4.7], see also [3], [8]. Moreover, since G has tube fibration by Theorem 2.13, and since $\mathbb{R}^p \setminus \{0\}$ is connected for $p \geq 2$, the Milnor fibre $\operatorname{Fib}(G)$ of G is independent of the point, in the sense that one has a diffeomorphism $G^{-1}(a) \simeq G^{-1}(b)$ for any $a, b \neq 0$ in a small disk at 0 in $\mathbb{R}^p$.

Theorem 2.13 enables us to treat some significant classes of singular map germs, as follows.

THEOREM 3.2. *Let $F : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ and $G : (\mathbb{R}^p, 0) \to (\mathbb{R}^k, 0)$, $m \geq p \geq k \geq 2$, be analytic map germs, such that F is tame and has isolated singular value, and that G has an isolated singular point at the origin.*

Then $H = G \circ F$ is tame, locally open, and has a local tube fibration.

Proof. Let us first remark that G has a tube fibration since it has an isolated singular point, as Milnor has shown in [12, page 94], see also [4]. By comparing the corresponding Jacobian matrices, we deduce that the hypotheses $\operatorname{Sing} F \subset F^{-1}(0)$ and $\operatorname{Sing} G \subset \{0\}$ imply $\operatorname{Sing} H \subset F^{-1}(0) \subset H^{-1}(0)$. This shows in particular that H has isolated singular value. It, moreover, shows:

$$\begin{aligned} \overline{M(H) \setminus H^{-1}(0)} \cap H^{-1}(0) &\subset M(H) \cap H^{-1}(0) \\ &\subset \operatorname{Sing} H \cap H^{-1}(0) \subset \operatorname{Sing} H \cap F^{-1}(0), \end{aligned} \tag{6}$$

where the first inclusion is obvious, the second is shown in (1) of Remark 2.6, and the last inclusion was pointed out just above.

We have the inclusions $M(H) \subset M(F)$ and $F^{-1}(0) \subset H^{-1}(0)$, and therefore we obtain the inclusion $M(H) \setminus H^{-1}(0) \subset M(F) \setminus F^{-1}(0)$.

By taking closures and intersecting with $H^{-1}(0)$, we then get the inclusions:

$$\overline{M(H) \setminus H^{-1}(0)} \cap H^{-1}(0) \subset \overline{M(H) \setminus H^{-1}(0)} \cap F^{-1}(0)$$
$$\subset \overline{M(F) \setminus F^{-1}(0)} \cap F^{-1}(0)$$

where the first one uses (6). This shows that the ρ-regularity of F implies the ρ-regularity of H. By the Tube Fibration Theorem 2.13, the map H has a tube fibration. It is also locally open due to Remark 2.10(i). □

One has the following particular case of the above results:

Corollary 3.3. *Let $F = (g_1, \ldots, g_p) : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m \geq p \geq 3$, be an analytic map with an isolated singular value, and ρ-regular.*

Let $H := (g_1, \ldots, g_{p-1}) : (\mathbb{R}^m, 0) \to (\mathbb{R}^{p-1}, 0)$. Then H has a tube fibration and its Milnor fibre of H is homeomorphic to the Milnor fibre of F times an open interval.

In the particular setting "F has an isolated singular point", Corollary 3.3 has been Milnor's question at [12, p.100]. In the above more general setting "F has an isolated critical value", a proof of Corollary 3.3 has been given in [1, Theorem 6.3].

Proof of Corollary 3.3. The map germ G from Theorem 3.2 is here the projection $(\mathbb{R}^p, 0) \to (\mathbb{R}^{p-1}, 0)$ to the first $p-1$ coordinates. As a direct consequence of Theorem 3.2, it follows that $H = F \circ G$ has a tube fibration.

The image by F of the Milnor fibre of H is a line segment I, which is contractible. The restriction of F on the Milnor fibre of H is a locally trivial fibration of base I and fibre the Milnor fibre of F, thus it is a trivial fibration. □

Example 3.4. Let $F : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$, $F(x, y, z) = (y^4 - z^2x^2 - x^4, xy)$. It was shown in [4, Example 5.1] that F is tame with isolated singular value. A direct computation shows that the Milnor fibre of F is a disjoint union of two closed intervals, for instance by considering $F^{-1}(\delta, 0)$ for $\delta > 0$ sufficiently small.

Let $G : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, $G(u, v) = (u^2 - v^2, 2uv)$. We see that G has an isolated singular point at the origin, thus it has a Milnor tube fibration, and one easily computes that the Milnor fibre is two points. By Theorem 3.2, the map germ $H = G \circ F$ has a local tube fibration. Its Milnor fibre is the disjoint union of 4 line segments.

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