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STABLE RANK 3 VECTOR BUNDLES ON $\mathbb{P}^3$ WITH $c_1 = 0$, $c_2 = 3$

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We clarify the undecided case $c_2 = 3$ of a result of Ein, Hartshorne and Vogelaar [8] about the restriction of a stable rank 3 vector bundle with $c_1 = 0$ on the projective 3-space to a general plane. It turns out that there are more exceptions to the stable restriction property than those conjectured by the three authors. One of them is a Schwarzenberger bundle (twisted by -1); it has $c_3 = 6$. There are also some exceptions with $c_3 = 2$ (plus, of course, their duals).

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INTRODUCTION

A basic technique for studying stable vector bundles on projective spaces (over an algebraically closed field k of characteristic 0) is to investigate their restrictions to general linear subspaces. The prototypes are the Grauert-Mülich-Spindler theorem [15] asserting that if E is a semistable vector bundle on $\mathbb{P}^n$, $n \geq 2$, then, for the general line $L \subset \mathbb{P}^n$, one has $E_L \simeq \bigoplus_{i=1}^{r} \mathscr{O}_L(a_i)$, with $a_1 \leq \cdots \leq a_r$ verifying $a_{i+1} - a_i \leq 1$, $i = 1, \ldots, r-1$ and Barth's restriction theorem [3] asserting that if E is a stable rank 2 vector bundle on $\mathbb{P}^n$, $n \geq 3$, then its restriction to a general hyperplane is stable unless $n = 3$ and E is a (twist of a) nullcorrelation bundle. After Gruson and Peskine reinterpreted Barth's arguments, Ein, Hartshorne and Vogelaar [8] were able to prove a similar result for stable rank 3 vector bundles (actually, even reflexive sheaves) on $\mathbb{P}^3$. Their result is the following one:

THEOREM 0.1 (Ein-Hartshorne-Vogelaar). *If E is a stable rank 3 vector bundle with $c_1 = 0$ on $\mathbb{P}^3$ then the restriction of E to a general plane is stable unless one of the following holds:*

(1) $c_2 \leq 3$;

(2) $E \simeq S^2N$, *for some nullcorrelation bundle N (in which case $c_2 = 4$ and $c_3 = 0$)*;

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(3) *There is an exact sequence*:

$$0 \longrightarrow \Omega_{\mathbb{P}^3}(1) \longrightarrow E' \longrightarrow \mathscr{O}_{H_0}(-c_2+1) \longrightarrow 0\,,$$

for some plane $H_0 \subset \mathbb{P}^3$, *where* E' *is either* E *or its dual* $E^\vee$.

(The easier cases where E has rank 3 and $c_1 = -1$ or -2 had been settled earlier by Schneider [14].) The three authors also show, in [8, Thm. 4.2], that, under the hypothesis of the above theorem, $c_2 \geq 2$ and $c_3 \leq c_2^2 - c_2$. Moreover, for $c_2 = 2$ they prove that the restriction of E to any plane is not stable but in the case $c_2 = 3$ they assert, after the statement of [8, Thm. 0.1], that they "do not know exactly which bundles with $c_2 = 3$ have stable restrictions" and conjecture that the only exceptions are again as in (3).

The aim of this paper is to clarify the case $c_2 = 3$ of the theorem of Ein, Hartshorne and Vogelaar. Our main result is expressed by the next theorem. As one can see from its statement, there are more exceptions than those conjectured by the three authors (which shows that this case needs a special treatment).

THEOREM 0.2. *Let* E *be a stable rank* 3 *vector bundle on* $\mathbb{P}^3$ *with* $c_1 = 0$, $c_2 = 3$ *and* $c_3 \geq 0$.

(a) *If* $\mathrm{H}^0(E_H) \neq 0$ *for every plane* $H \subset \mathbb{P}^3$ *then* $c_3 = 6$ *and there is an exact sequence*:

$$0 \longrightarrow \Omega_{\mathbb{P}^3}(1) \longrightarrow E \longrightarrow \mathscr{O}_{H_0}(-2) \longrightarrow 0\,,$$

for some plane H_0.

(b) *If* $\mathrm{H}^0(E_H^\vee) \neq 0$ *for every plane* $H \subset \mathbb{P}^3$ *then one of the following holds*:

(i) $c_3 = 6$ *and, up to a linear change of coordinates in* $\mathbb{P}^3$, E *is the cokernel of the morphism* $\alpha\colon 3\mathscr{O}_{\mathbb{P}^3}(-2) \to 6\mathscr{O}_{\mathbb{P}^3}(-1)$ *defined by the transpose of the matrix*:

$$\begin{pmatrix} X_0 & X_1 & X_2 & X_3 & 0 & 0 \\ 0 & X_0 & X_1 & X_2 & X_3 & 0 \\ 0 & 0 & X_0 & X_1 & X_2 & X_3 \end{pmatrix};$$

(ii) $c_3 = 2$ *and, up to a linear change of coordinates in* $\mathbb{P}^3$, E *is the cohomology sheaf of a monad of the form*:

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^3}(-2) \xrightarrow{\alpha} 6\mathscr{O}_{\mathbb{P}^3} \xrightarrow{\beta} 2\mathscr{O}_{\mathbb{P}^3}(1) \longrightarrow 0\,,$$

with $\alpha = (X_2^2\,, X_3^2\,, -X_0X_2\,, -X_1X_3\,, X_0^2\,, X_1^2)^{\mathrm{t}}$ *and with* β *defined by the matrix*:

$$\begin{pmatrix} X_0 & a_1X_1 & X_2 & a_1X_3 + a_3X_1 & 0 & a_3X_3 \\ 0 & b_1X_1 & X_0 & b_1X_3 + b_3X_1 & X_2 & b_3X_3 \end{pmatrix},$$

where a_1, a_3, b_1, b_3 *are scalars satisfying* $a_1b_3 - a_3b_1 \neq 0$.

It is clear that the above theorem answers the question of Ein, Hartshorne and Vogelaar because if $c_3 < 0$ then $c_3(E^\vee) = -c_3 > 0$.

We recall now, briefly, the results from [8] that we use in the proof of Theorem 0.2 and then outline the extra arguments that we need to complete its proof. A more detailed description of our method can be found in Section 1.

Let E be a stable rank 3 vector bundle on $\mathbb{P}^3$ with Chern classes $c_1 = 0$, c_2, c_3. The Riemann-Roch theorem asserts, in this case, that $\chi(E(l)) = \chi(3\mathscr{O}_{\mathbb{P}^3}(l)) - (l+2)c_2 + c_3/2$, $\forall\, l \in \mathbb{Z}$. In particular, c_3 must be even. According to a theorem of Spindler [17] (see, also, [8, Cor. 3.5]), for the general plane $H \subset \mathbb{P}^3$ one has $\mathrm{h}^0(E_H) \leq 1$ and $\mathrm{h}^0(E_H^\vee) \leq 1$. One deduces that $c_2 \geq 2$ and $c_3 \leq c_2^2 - c_2$ (see [8, Thm. 4.2]). Assume, now, that $c_2 \geq 3$ and that, for the general plane $H \subset \mathbb{P}^3$, one has $\mathrm{H}^0(E_H) = 0$ and $\mathrm{h}^0(E_H^\vee) = 1$. Applying to $E^\vee$ [8, Prop. 7.4(a)] and the first part of the proof of [8, Prop. 6.4], it follows that, for the general plane $H \subset \mathbb{P}^3$, there is an exact sequence:

$$0 \longrightarrow (\Omega_{\mathbb{P}^3}(1))_H \longrightarrow E_H^\vee \longrightarrow \mathscr{O}_{L_0}(-c_2+1) \longrightarrow 0\,,$$

for some line $L_0 \subset H$. This implies that $\mathrm{h}^1(E_{L_0}^\vee) \geq c_2 - 2$.

Conversely, if a plane $H \subset \mathbb{P}^3$ contains a line L_0 such that $\mathrm{h}^1(E_{L_0}^\vee) \geq c_2 - 2$ then E_H is not stable. Indeed, if E_H were stable then one would have $\mathrm{h}^1(E_H^\vee) = c_2 - 3$, by Riemann-Roch, and $\mathrm{h}^2(E_H^\vee(-1)) = \mathrm{h}^0(E_H(-2)) = 0$ and this would imply that $\mathrm{h}^1(E_{L_0}^\vee) \leq c_2 - 3$.

This observation allows one to show quickly that the bundles from Theorem 0.2(b) are exceptions to the stable restriction property. Indeed, if E is the bundle from Theorem 0.2(b)(i) let (t_0, t_1) and (u_0, u_1) be two linearly independent elements of k^2 and let $L \subset \mathbb{P}^3$ be the line of equations $\sum t_0^{3-i}t_1^iX_i = \sum u_0^{3-i}u_1^iX_i = 0$. Since the kernel of $\mathrm{H}^0(\alpha_L^\vee(-1))$ contains the linearly independent elements $(t_0^5, \dots, t_1^5)^{\mathrm{t}}$ and $(u_0^5, \dots, u_1^5)^{\mathrm{t}}$ it follows that $\mathrm{h}^0(E_L^\vee(-1)) \geq 2$. Taking into account that $E_L^\vee(-1)$ is a subbundle of $6\mathscr{O}_L$, one deduces that $E_L^\vee(-1) \simeq 2\mathscr{O}_L \oplus \mathscr{O}_L(-3)$ hence $E_L^\vee \simeq 2\mathscr{O}_L(1) \oplus \mathscr{O}_L(-2)$. Since any plane $H \subset \mathbb{P}^3$ contains such a line [the planes of equation of the form $\sum t_0^{3-i}t_1^iX_i = 0$, $(t_0, t_1) \in k^2 \setminus \{(0,0)\}$, form a twisted cubic curve Γ in the dual projective space $\mathbb{P}^{3\vee}$ and the secants of Γ fill the whole of $\mathbb{P}^{3\vee}$], E is an exception to the stable restriction property.

If E is one of the bundles from Theorem 0.2(b)(ii), let L be a line joining a point P_0 of the line $X_0 = X_2 = 0$ and a point P_1 of the line $X_1 = X_3 = 0$. Since X_0 and X_2 (resp., X_1 and X_3) vanish at P_0 (resp., P_1) one deduces, using the restriction to L of the dual of the monad defining E, that $\mathrm{h}^1(E_L^\vee) = 1$. Since a general plane contains a line of this kind, E is an exception to the stable restriction property, too.

Now, in order to show that the bundles listed in Theorem 0.2 are the only exceptions to the stable restriction property (in the case $c_1 = 0$, $c_2 = 3$), we proceed as follows. Using the properties of the *spectrum* of a stable rank 3 vector bundle on $\mathbb{P}^3$ (see Remark 1.6) we describe the Horrocks monad of any such bundle E with $c_1 = 0$, $c_2 = 3$, $c_3 \geq 0$ (hence $c_3 \in \{0\,,\,2\,,\,4\,,\,6\}$) that is not isomorphic to one of the bundles from Theorem 0.2(a). Then we show that, for the general plane $H \subset \mathbb{P}^3$, one has $\mathrm{H}^0(E_H) = 0$. We use, for that, the map $\mu\colon \mathrm{H}^1(E(-1)) \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}(-1) \to \mathrm{H}^1(E) \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}$ deduced from the multiplication map $\mathrm{H}^1(E(-1)) \otimes \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^3}(1)) \to \mathrm{H}^1(E)$. If one would have $\mathrm{H}^0(E_H) \neq 0$, for the general plane $H \subset \mathbb{P}^3$, then μ would have, generically, corank 1. Since $\mathrm{H}^1(E)$ and $\mathrm{H}^1(E(-1))$ are k-vector spaces of small dimension (equal, for both, to $3 - c_3/2$) one gets readily a contradiction. Note that this settles, already, the case $c_3 = 0$.

Finally, if E (as above) does not satisfy the stable restriction property then, for the general plane $H \subset \mathbb{P}^3$, one must have $\mathrm{H}^0(E_H) = 0$ and $\mathrm{h}^0(E_H^\vee) = 1$. If $c_3 = 2$ and E has an unstable plane or if $c_3 = 4$ one gets a contradiction by showing that the general plane $H \subset \mathbb{P}^3$ contains no line L_0 such that $\mathrm{h}^1(E_{L_0}^\vee) \geq 1$. The argument for the case $c_3 = 4$ is a little bit lengthy because the best we were able to do was to split it into several cases. The analysis of each case is, however, easy.

If $c_3 = 2$ and E has no unstable plane (resp., $c_3 = 6$) we show that E is as in Theorem 0.2(b)(ii) (resp., Theorem 0.2(b)(i)). We use, for that, the morphism $\mu\colon \mathrm{H}^1(E^\vee(-1)) \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}(-1) \to \mathrm{H}^1(E^\vee) \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}$ deduced from the multiplication map $\mathrm{H}^1(E^\vee(-1)) \otimes \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^3}(1)) \to \mathrm{H}^1(E^\vee)$ and, in the case $c_3 = 6$, the main result of Vallès [18].

Notation. (i) We denote by $\mathbb{P}^n$ the projective n-space over an algebraically closed field k of characteristic 0. We use the classical definition $\mathbb{P}^n = \mathbb{P}(V) := (V \setminus \{0\})/k^*$, where $V := k^{n+1}$. If $e_0, \dots, e_n$ is the canonical basis of V and $X_0, \dots, X_n$ the dual basis of $V^\vee$ then the homogeneous coordinate ring of $\mathbb{P}^n$ is the symmetric algebra $S := S(V^\vee) \simeq k[X_0, \dots, X_n]$.

(ii) If $\mathscr{F}$ is a coherent sheaf on $\mathbb{P}^n$ and $i \geq 0$ an integer, we denote by $\mathrm{H}^i_*(\mathscr{F})$ the graded k-vector space $\bigoplus_{l \in \mathbb{Z}} \mathrm{H}^i(\mathscr{F}(l))$ endowed with its natural structure of graded S-module. We also denote by $\mathrm{h}^i(\mathscr{F})$ the dimension of $\mathrm{H}^i(\mathscr{F})$ as a k-vector space.

(iii) If $Y \subset X$ are closed subschemes of $\mathbb{P}^n$, we denote by $\mathscr{I}_X$ the ideal sheaf of $\mathscr{O}_{\mathbb{P}^3}$ defining X and by $\mathscr{I}_{Y,X}$ the ideal sheaf of $\mathscr{O}_X$ defining Y as a closed subscheme of X, i.e., $\mathscr{I}_{Y,X} = \mathscr{I}_Y/\mathscr{I}_X$. If $\mathscr{F}$ is a coherent sheaf on $\mathbb{P}^n$, we denote the tensor product $\mathscr{F} \otimes_{\mathscr{O}_{\mathbb{P}}} \mathscr{O}_X$ by $\mathscr{F}_X$ and identify it with the restriction $\mathscr{F} \mid X$ of $\mathscr{F}$ to X. In particular, if x is a (closed) point of $\mathbb{P}^n$ then

$\mathscr{F}_{\{x\}}$ is just the *reduced stalk* $\mathscr{F}(x) := \mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x$ of $\mathscr{F}$ at x.

(iv) A *monad* with cohomology sheaf $\mathscr{F}$ is a bounded complex $K^\bullet$ (usually, with only three non-zero terms) of vector bundles on $\mathbb{P}^n$ such that $\mathscr{H}^0(K^\bullet) \simeq \mathscr{F}$ and $\mathscr{H}^i(K^\bullet) = 0$ for $i \neq 0$. For *Horrocks monads*, see Barth and Hulek [4].

1. PRELIMINARIES

This section is devoted to recalling some well-known facts about stable rank 3 vector bundles with $c_1 = 0$ on $\mathbb{P}^2$ and $\mathbb{P}^3$. In particular, we recall the definition and properties of the *spectrum* of a stable rank 3 vector bundle on $\mathbb{P}^3$ and Beilinson's theorem. Then we introduce the new ingredient used in the proof of Theorem 0.2, namely the morphism $\mu\colon \mathrm{H}^1(E'(-1)) \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}(-1) \to \mathrm{H}^1(E') \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}$ deduced from the multiplication map $\mathrm{H}^1(E'(-1)) \otimes S_1 \to \mathrm{H}^1(E')$, where E' is a stable rank 3 vector bundle on $\mathbb{P}^3$ with $c_1 = 0$, $c_2 = 3$.

Lemma 1.1. *Let F be a semistable rank 3 vector bundle on $\mathbb{P}^2$ with Chern classes $c_1 = 0$ and $c_2 \geq 1$. Then $\mathrm{h}^0(F) \leq 2$ and if $\mathrm{h}^0(F) = 2$ then F can be realized as an extension:*

$$0 \longrightarrow 2\mathscr{O}_{\mathbb{P}^2} \longrightarrow F \longrightarrow \mathscr{I}_Z \longrightarrow 0\,,$$

for some 0-dimensional subscheme Z of $\mathbb{P}^2$.

Proof. Since F has rank 3 and $c_1 = 0$, it is semistable if and only if $\mathrm{H}^0(F(-1)) = 0$ and $\mathrm{H}^0(F^\vee(-1)) = 0$. It is well-known that such a bundle has $c_2 \geq 0$ and if $c_2 = 0$ then $F \simeq 3\mathscr{O}_{\mathbb{P}^2}$.

Now, under the hypothesis of the lemma, assume that $\mathrm{h}^0(F) \geq 2$ and consider two linearly independent global sections s_1 and s_2 of F. We assert that the global section $s_1 \wedge s_2$ of $\bigwedge^2 F$ is non-zero. Indeed, since $\mathrm{H}^0(F(-1)) = 0$ the zero scheme Z_1 of s_1 has codimension at least 2 in $\mathbb{P}^2$. If $s_1 \wedge s_2 = 0$ then there exists a regular function f on $\mathbb{P}^2 \setminus Z_1$ such that $s_2 = fs_1$. But the only regular functions on $\mathbb{P}^2 \setminus Z_1$ are the constant ones hence s_1 and s_2 are linearly dependent, which contradicts our assumption.

We have $\bigwedge^2 F \simeq F^\vee$. Since $\mathrm{H}^0(F^\vee(-1)) = 0$, the zero scheme Z of the global section $s_1 \wedge s_2$ of $\bigwedge^2 F$ has codimension at least 2 in $\mathbb{P}^2$. It follows that the *Eagon-Northcott complex*:

$$0 \longrightarrow 2\mathscr{O}_{\mathbb{P}^2} \xrightarrow{(s_1\,,\,s_2)} F \xrightarrow{s_1\wedge s_2 \wedge *} \mathscr{I}_Z \longrightarrow 0$$

is exact. Since length $Z = c_2 \geq 1$ one deduces that $\mathrm{h}^0(F) = 2$. $\square$

Lemma 1.2. *Let F be a rank 3 vector bundle on $\mathbb{P}^2$ with $c_1 = 0$ and such that $\mathrm{H}^0(F(-2)) = 0$ and $\mathrm{H}^0(F^\vee(-2)) = 0$. Then, for the general line $L \subset \mathbb{P}^2$, one has $F_L \simeq 3\mathscr{O}_L$ or $F_L \simeq \mathscr{O}_L(1) \oplus \mathscr{O}_L \oplus \mathscr{O}_L(-1)$.*

Proof. If $\mathrm{H}^0(F(-1)) = 0$ and $\mathrm{H}^0(F^\vee(-1)) = 0$ then F is semistable and one can apply the theorem of Grauert-Mülich-Spindler [15]. If $\mathrm{H}^0(F^\vee(-1)) \neq 0$ or $\mathrm{H}^0(F(-1)) \neq 0$ then one has an exact sequence:

$$0 \longrightarrow G(1) \longrightarrow F' \longrightarrow \mathscr{I}_Z(-1) \longrightarrow 0\,,$$

where F' is either F or $F^\vee$, Z is a 0-dimensional (or empty) closed subscheme of $\mathbb{P}^2$ and G is a rank 2 vector bundle with $c_1(G) = -1$. Since $\mathrm{H}^0(F'(-2)) = 0$ it follows that $\mathrm{H}^0(G(-1)) = 0$. If $\mathrm{H}^0(G) = 0$ then G is stable. If $\mathrm{H}^0(G) \neq 0$ then G can be realized as an extension $0 \to \mathscr{O}_{\mathbb{P}^2} \to G \to \mathscr{I}_W(-1) \to 0$, for some 0-dimensional subscheme W of $\mathbb{P}^2$. Using the theorem of Grauert-Mülich (for the former case) one gets that, for the general line $L \subset \mathbb{P}^2$ one has $G_L \simeq \mathscr{O}_L \oplus \mathscr{O}_L(-1)$. If, moreover, $L \cap Z = \emptyset$ then $F'_L \simeq \mathscr{O}_L(1) \oplus \mathscr{O}_L \oplus \mathscr{O}_L(-1)$. $\square$

Remark 1.3. We recall, here, a formula that we shall need a couple of times. If $\mathscr{F}$ is a coherent torsion sheaf on $\mathbb{P}^n$ then:

$$c_1(\mathscr{F}) = \sum (\mathrm{length}\, \mathscr{F}_\xi) \deg X\,,$$

where the sum is indexed by the 1-codimensional irreducible components X of $\mathrm{Supp}\, \mathscr{F}$ and ξ is the generic point of X (notice that $\mathscr{F}_\xi$ is an Artinian $\mathscr{O}_{\mathbb{P}^n,\xi}$-module).

Remark 1.4. Let E be a stable rank 3 vector bundle on $\mathbb{P}^3$ with $c_1 = 0$. We recall that a plane $H_0 \subset \mathbb{P}^3$ is an *unstable plane* for E if $\mathrm{H}^0(E^\vee_{H_0}(-1)) \neq 0$. The largest integer $r \geq 1$ for which $\mathrm{H}^0(E^\vee_{H_0}(-r)) \neq 0$ is the *order* of H_0. Ein, Hartshorne and Vogelaar show, in [8, Prop. 5.1], that the following conditions are equivalent:

(i) $\mathrm{H}^0(E_H) \neq 0$ for every plane H and there is an unstable plane for E;

(ii) There exists a plane $H_0 \subset \mathbb{P}^3$ and an exact sequence:

$$0 \longrightarrow \Omega_{\mathbb{P}^3}(1) \longrightarrow E \longrightarrow \mathscr{O}_{H_0}(-c_2 + 1) \longrightarrow 0\,;$$

(iii) E has an unstable plane of order $c_2 - 1$.

If the above conditions are satisfied then: (iv) $c_3 = c_2^2 - c_2$. Moreover, if $c_2 \geq 4$ then (iv) $\Rightarrow$ (iii) (hence all four conditions are equivalent). We assert that conditions (i)–(iii) above are also equivalent (for $c_2 \geq 2$) to the condition:

(v) There is a non-zero morphism $\phi\colon \Omega_{\mathbb{P}^3}(1) \to E$.

Indeed, let us show that (v) $\Rightarrow$ (ii). Since $\Omega_{\mathbb{P}^3}(1)$ and E are stable vector bundles with $c_1(\Omega_{\mathbb{P}^3}(1)) = -1$ and $c_1(E) = 0$, ϕ must have, generically, rank 3. It follows that $\bigwedge^3 \phi \colon \mathscr{O}_{\mathbb{P}^3}(-1) \to \mathscr{O}_{\mathbb{P}^3}$ is defined by a non-zero linear form h_0. Let $H_0 \subset \mathbb{P}^3$ be the plane of equation $h_0 = 0$. $\operatorname{Coker} \phi$ is annihilated by h_0 hence it is an $\mathscr{O}_{H_0}$-module. The Auslander-Buchsbaum relation shows that $\operatorname{depth}(\operatorname{Coker} \phi)_x \geq 2$, $\forall\, x \in H_0$, hence $\operatorname{Coker} \phi$ is a locally free $\mathscr{O}_{H_0}$-module. One deduces, from Remark 1.3, that it has rank 1, i.e., that $\operatorname{Coker} \phi \simeq \mathscr{O}_{H_0}(a)$ for some $a \in \mathbb{Z}$. One has $1 = c_2(\Omega_{\mathbb{P}^3}(1)) = c_2 + a$ hence $a = -c_2 + 1$.

Notice that condition (v) above is equivalent to the existence of a non-zero element ξ of $\mathrm{H}^1(E(-1))$ such that $S_1 \xi = 0$ in $\mathrm{H}^1(E)$ (use the exact sequence $0 \to \Omega_{\mathbb{P}^3}(1) \to S_1 \otimes \mathscr{O}_{\mathbb{P}^3} \to \mathscr{O}_{\mathbb{P}^3}(1) \to 0$).

Lemma 1.5. *Let E be a stable rank 3 vector bundle on $\mathbb{P}^3$ with $c_1 = 0$, $c_2 = 3$ and $c_3 \geq 0$. Assume that $\mathrm{H}^0(E_H) = 0$ for the general plane $H \subset \mathbb{P}^3$. Assume, also, that any line $L \subset \mathbb{P}^3$ for which $\mathrm{h}^1(E_L^\vee) \geq 1$ either passes through one of finitely many points, or is contained in one of finitely many planes, or belongs to a family of dimension at most 1. Then, for the general plane $H \subset \mathbb{P}^3$, one has $\mathrm{H}^0(E_H^\vee) = 0$.*

Proof. E does not satisfy the equivalent conditions (i)–(iii) from Remark 1.4 (because $\mathrm{H}^0(E_H) = 0$ for the general plane $H \subset \mathbb{P}^3$) and nor does $E^\vee$ (because $c_3 \neq -6$). One deduces that, for any plane $H \subset \mathbb{P}^3$, one has $\mathrm{H}^0(E_H^\vee(-2)) = 0$ and $\mathrm{H}^0(E_H(-2)) = 0$. Lemma 1.2 implies that, for any plane H, the family of lines $L \subset H$ for which $\mathrm{h}^1(E_L^\vee) \geq 1$ has dimension at most 1. It follows that the general plane $H \subset \mathbb{P}^3$ contains no line L_0 for which $\mathrm{h}^1(E_{L_0}^\vee) \geq 1$. As we noticed in the Introduction (right after the statement of Theorem 0.2), this implies the conclusion of the lemma. $\square$

Remark 1.6. Let E be a stable rank 3 vector bundle on $\mathbb{P}^3$ with $c_1 = 0$. One of the most important applications of Theorem 0.1 (and of the generalized Grauert-Mülich theorem of Spindler [15]) is the existence of a non-decreasing sequence of integers $k_E = (k_1, \ldots, k_m)$, called the *spectrum* of E, such that, putting $K := \bigoplus_{i=1}^m \mathscr{O}_{\mathbb{P}^1}(k_i)$, one has:

(I) $\mathrm{h}^1(E(l)) = \mathrm{h}^0(K(l+1)$, for $l \leq -1$;

(II) $\mathrm{h}^2(E(l)) = \mathrm{h}^1(K(l+1))$, for $l \geq -3$.

We shall need the following properties of the spectrum: $m = c_2$, $-2\Sigma k_i = c_3$, the spectrum is connected, i.e., $k_{i+1} - k_i \leq 1$, for $1 \leq i \leq m-1$, and if 0 does not occur in the spectrum then either $k_{m-2} = k_{m-1} = k_m = -1$ or $k_1 = k_2 = k_3 = 1$. Moreover, by Serre duality, the spectrum of $E^\vee$ is

$(-k_m, \ldots, -k_1)$. Details can be found in the papers of Okonek and Spindler [12], [13], and in the papers [6], [7] of the author. These papers use the approach of Hartshorne [10], [11] who treated the rank 2 case. Details can be also found in Appendix A of the e-print version of this paper `arXiv:2103.11723`.

It is easy to show that conditions (i)–(iii) from Remark 1.4 are also equivalent to the condition :

(vi) The spectrum of E is $(-c_2+1, \ldots, -1, 0)$.

Indeed, (ii) $\Rightarrow$ (vi) by the above definition of the spectrum. On the other hand, (vi) $\Rightarrow$ (iii) because the spectrum of $E^\vee$ is $(0, 1, \ldots, c_2-1)$ hence $\mathrm{h}^1(E^\vee(-c_2)) = 1$ and $\mathrm{h}^1(E^\vee(-c_2+1)) = 3$ hence there exists a non-zero linear form h_0 such that multiplication by $h_0 \colon \mathrm{H}^1(E^\vee(-c_2)) \to \mathrm{H}^1(E^\vee(-c_2+1))$ is the zero map which implies that $\mathrm{h}^0(E^\vee_{H_0}(-c_2+1)) = 1$, H_0 being the plane of equation $h_0 = 0$.

Remark 1.7. Let E be a stable rank 3 vector bundle on $\mathbb{P}^3$ with $c_1 = 0$, $c_2 = 3$. According to the previous remark, if $c_3 = 6$ then the possible spectra of E are $(-2,-1,0)$ and $(-1,-1,-1)$, if $c_3 = 4$ the spectrum of E is $(-1,-1,0)$, if $c_3 = 2$ the spectrum of E is $(-1,0,0)$, and if $c_3 = 0$ the possible spectra of E are $(0,0,0)$ and $(-1,0,1)$. Assuming that neither E nor $E^\vee$ satisfy the equivalent conditions (i)–(iii) from Remark 1.4 (i.e., that the spectrum of E is neither $(-2,-1,0)$ nor $(0,1,2)$), one has $\mathrm{H}^1(E(l)) = 0$ for $l \leq -3$ and $\mathrm{H}^2(E(l)) = 0$ for $l \geq -1$. In this case, *Beilinson's theorem* [5], with the improvements of Eisenbud, Fløystad and Schreyer [9, (6.1)] (these results are recalled in [1, 1.23–1.25]), implies that E is the cohomology sheaf of a monad that can be described as the total complex of a double complex with the following (possibly) non-zero terms :

$$\begin{tikzcd}
\mathrm{H}^1(E(-2)) \otimes \Omega^2_{\mathbb{P}^3}(2) \arrow[r] & \mathrm{H}^1(E(-1)) \otimes \Omega^1_{\mathbb{P}^3}(1) \arrow[r] & \mathrm{H}^1(E) \otimes \mathscr{O}_{\mathbb{P}^3} \\
0 \arrow[r] \arrow[u] & \mathrm{H}^2(E(-3)) \otimes \Omega^3_{\mathbb{P}^3}(3) \arrow[r] \arrow[u] & \mathrm{H}^2(E(-2)) \otimes \Omega^2_{\mathbb{P}^3}(2) \arrow[u]
\end{tikzcd}$$

such that the term $\mathrm{H}^1(E(-1)) \otimes \Omega^1_{\mathbb{P}^3}(1)$ has bidegree $(0,0)$. The horizontal differentials of this double complex are equal to $\sum_{i=0}^3 X_i \otimes e_i$, X_i acting to the left on $\mathrm{H}^p(E(-l))$ via the S-module structure of $\mathrm{H}^p_*(E)$ and e_i acting to the right on $\Omega^l_{\mathbb{P}^3}(l)$ by contraction (recall that $\Omega^l_{\mathbb{P}^3}(l)$ embeds canonically into $\mathscr{O}_{\mathbb{P}^3} \otimes \bigwedge^l V^\vee$).

Lemma 1.8. *Let E be a stable rank* 3 *vector bundle on* $\mathbb{P}^3$ *with* $c_1 = 0$, $c_2 = 3$. *Assume that neither E nor $E^\vee$ satisfy the equivalent conditions* (i)–

(iii) *from Remark* 1.4. *If* $\mathrm{H}^2(E(-2)) = 0$ *then, for any vector subspace* N_{-1} *of codimension* 1 *of* $\mathrm{H}^1(E(-1))$*, one has* $S_1 N_{-1} = \mathrm{H}^1(E)$.

Proof. Since $\mathrm{H}^2(E(-2)) = 0$, the Beilinson monad of E shows that the morphism $\delta\colon \mathrm{H}^1(E(-1)) \otimes \Omega_{\mathbb{P}^3}(1) \to \mathrm{H}^1(E) \otimes \mathscr{O}_{\mathbb{P}^3}$ deduced from the map $\mathrm{H}^1(E(-1)) \otimes S_1 \to \mathrm{H}^1(E)$ is an epimorphism. Assume, by contradiction, that there exists a proper subspace N_0 of $\mathrm{H}^1(E)$ such that $S_1 N_{-1} \subseteq N_0$. Then δ maps $N_{-1} \otimes \Omega_{\mathbb{P}^3}(1)$ into $N_0 \otimes \mathscr{O}_{\mathbb{P}^3}$. Since there is no epimorphism $\Omega_{\mathbb{P}^3}(1) \to \mathscr{O}_{\mathbb{P}^3}$ one gets a contradiction. □

Remark 1.9. Let E be a stable rank 3 vector bundle on $\mathbb{P}^3$ with $c_1 = 0$, $c_2 = 3$. The new ingredient that we use in the proof of Theorem 0.2 is the analysis of the map:

$$\mu\colon \mathrm{H}^1(E'(-1)) \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}(-1) \longrightarrow \mathrm{H}^1(E') \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}$$

deduced from the multiplication map $\mathrm{H}^1(E'(-1)) \otimes S_1 \to \mathrm{H}^1(E')$, where E' is either E or $E^\vee$. Notice that the kernel of the reduced stalk of μ at a point $[h] \in \mathbb{P}^{3\vee}$ corresponding to a plane $H \subset \mathbb{P}^3$ of equation $h = 0$ is isomorphic to $\mathrm{H}^0(E'_H)$.

If the spectrum of E' is not $(-2, -1, 0)$ then $\mathrm{H}^2(E'(l)) = 0$ for $l \geq -1$ hence, by Riemann-Roch, $\mathrm{H}^1(E'(-1))$ and $\mathrm{H}^1(E')$ have the same dimension, namely $d := 3 - c_3'/2$ (with $c_3' = \pm c_3$). If one fixes k-bases of $\mathrm{H}^1(E'(-1))$ and of $\mathrm{H}^1(E')$ then $\mathrm{H}^0(\mu(1))\colon \mathrm{H}^1(E'(-1)) \to \mathrm{H}^1(E') \otimes \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^{3\vee}}(1))$ is represented by a $d \times d$ matrix $\mathcal{M}$ with entries in $\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^{3\vee}}(1)) = V$. Then the $d \times d$ matrix M_i with scalar entries associated to the multiplication map $X_i\colon \mathrm{H}^1(E'(-1)) \to \mathrm{H}^1(E')$ is obtained by evaluating X_i at the entries of $\mathcal{M}$. It follows that $\mathcal{M} = \sum_{i=0}^{3} M_i e_i$.

Notice that the same matrix $\mathcal{M}$ defines the horizontal differential

$$\mathrm{H}^1(E(-1)) \otimes \Omega^1_{\mathbb{P}^3}(1) \to \mathrm{H}^1(E) \otimes \mathscr{O}_{\mathbb{P}^3}$$

of the Beilinson monad of E (recall that $\mathrm{Hom}(\Omega^1_{\mathbb{P}^3}(1), \mathscr{O}_{\mathbb{P}^3})$ can be identified with V).

Assume, now, that $\mathrm{H}^0(E'_H) \neq 0$, for every plane $H \subset \mathbb{P}^3$. Then μ has, generically, corank 1 (by the theorem of Spindler [17] recalled in the Introduction). Since $\mathrm{Ker}\,\mu$ is reflexive of rank 1 it must be invertible, i.e., $\mathrm{Ker}\,\mu \simeq \mathscr{O}_{\mathbb{P}^{3\vee}}(a)$, for some integer a. Moreover, one must have an exact sequence:

$$0 \longrightarrow (\mathrm{Coker}\,\mu)_{\mathrm{tors}} \longrightarrow \mathrm{Coker}\,\mu \longrightarrow \mathscr{I}_Y(b) \longrightarrow 0\,,$$

for some integer b and some closed subscheme Y of $\mathbb{P}^{3\vee}$, of codimension at least 2. One has the relation:

$$a = -d + b + c_1\left((\mathrm{Coker}\,\mu)_{\mathrm{tors}}\right). \tag{1.1}$$

By Remark 1.3, $c_1((\operatorname{Coker}\mu)_{\mathrm{tors}}) \geq 0$. We conclude with three easy observations.

(i) One has $a \leq -2$. Indeed, if $a = -1$ then there exists a non-zero element ξ of $\mathrm{H}^1(E'(-1))$ such that $S_1\xi = (0)$ in $\mathrm{H}^1(E')$ which contradicts the last assertion in Remark 1.4.

(ii) If $S_1\mathrm{H}^1(E'(-1)) = \mathrm{H}^1(E')$ then $b \geq 1$. Indeed, if $b = 0$ then one must have $Y = \emptyset$. The kernel of the epimorphism $\mathrm{H}^1(E') \otimes \mathscr{O}_{\mathbb{P}^{3\vee}} \to \mathscr{O}_{\mathbb{P}^{3\vee}}$ has the form $N_0 \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}$, for some 1-codimensional subspace N_0 of $\mathrm{H}^1(E')$. Since the image of μ is contained in $N_0 \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}$ it follows that $S_1\mathrm{H}^1(E'(-1)) \subseteq N_0$, which contradicts our assumption.

(iii) Let L be a line and $\mathbb{P}^3$ and let $L^\vee$ be the line in $\mathbb{P}^{3\vee}$ whose points correspond to the planes containing L. The restriction $\mu \mid L^\vee : \mathrm{H}^1(E'(-1)) \otimes \mathscr{O}_{L^\vee}(-1) \to \mathrm{H}^1(E') \otimes \mathscr{O}_{L^\vee}$ is defined by the multiplication map $\mathrm{H}^1(E'(-1)) \otimes \mathrm{H}^0(\mathscr{I}_L(1)) \to \mathrm{H}^1(E')$. Assume that $\mathrm{H}^2(E'(-2)) = 0$. Tensorizing by E' the exact sequence $0 \to \mathscr{O}_{\mathbb{P}^3}(-2) \to 2\mathscr{O}_{\mathbb{P}^3}(-1) \to \mathscr{O}_{\mathbb{P}^3} \to \mathscr{O}_L \to 0$, one gets an exact sequence:

$$\mathrm{H}^1(E'(-1)) \otimes \mathrm{H}^0(\mathscr{I}_L(1)) \longrightarrow \mathrm{H}^1(E') \longrightarrow \mathrm{H}^1(E'_L) \longrightarrow 0\,.$$

If $\mathrm{h}^1(E'_L) \geq 1$ and if L is contained in a plane H with $\mathrm{h}^0(E'_H) = 1$ (hence $\mathrm{h}^1(E'_H) = 1$, by Riemann-Roch) one deduces that $\operatorname{Coker}(\mu \mid L^\vee) \simeq \mathscr{O}_{L^\vee} \oplus \mathscr{T}$, where $\mathscr{T}$ is a torsion $\mathscr{O}_{L^\vee}$-module. This implies that if $b \geq 1$ then L must intersect Y. Consequently, if $b \geq 1$ and the general plane $H \subset \mathbb{P}^3$ contains a line L_0 with $\mathrm{h}^1(E'_{L_0}) \geq 1$ then $\dim Y = 1$.

2. THE CASE $c_3 = 0$

Lemma 2.1. *Let E be a stable rank 3 vector bundle on $\mathbb{P}^3$ with $c_1 = 0$, $c_2 = 3$, $c_3 = 0$ and spectrum $(0,0,0)$. Then E is the cohomology sheaf of a Beilinson monad of the form:*

$$0 \longrightarrow 3\Omega^3_{\mathbb{P}^3}(3) \xrightarrow{\gamma} 3\Omega^1_{\mathbb{P}^3}(1) \xrightarrow{\delta} 3\mathscr{O}_{\mathbb{P}^3} \longrightarrow 0\,,$$

and of a Horrocks monad of the form:

$$0 \longrightarrow 3\mathscr{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} 9\mathscr{O}_{\mathbb{P}^3} \xrightarrow{\beta} 3\mathscr{O}_{\mathbb{P}^3}(1) \longrightarrow 0\,.$$

Proof. One has, by Riemann-Roch, $\mathrm{h}^1(E) = 3$. For the first monad see Remark 1.7 while the second monad can be deduced from the first one and the exact sequence $0 \to \Omega^1_{\mathbb{P}^3}(1) \to 4\mathscr{O}_{\mathbb{P}^3} \to \mathscr{O}_{\mathbb{P}^3}(1) \to 0$. □

Proposition 2.2. *Let E be a stable rank 3 vector bundle on $\mathbb{P}^3$ with $c_1 = 0$, $c_2 = 3$, $c_3 = 0$ and spectrum $(0,0,0)$. Then the restriction of E to a general plane is stable.*

Proof. Since $E^\vee$ has the same Chern classes and spectrum as E, it suffices to show that, for the general plane $H \subset \mathbb{P}^3$, one has $\mathrm{H}^0(E_H) = 0$. Assume, by contradiction, that $\mathrm{H}^0(E_H) \neq 0$, for every plane H. Then the morphism $\mu\colon \mathrm{H}^1(E(-1)) \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}(-1) \to \mathrm{H}^1(E) \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}$ from Remark 1.9 has, generically, corank 1. Using the notation from that remark, one has $a \leq -2$ (by observation (i)) and $b \geq 1$ (by observation (ii) because, using the Horrocks monad of E, one sees that $S_1\mathrm{H}^1(E(-1)) = \mathrm{H}^1(E)$). It follows, from relation (1.1) (in which one has $d = 3$), that $a = -2$ and $b = 1$. Since $\mathscr{I}_Y(1)$ is globally generated by at most 3 linear forms, Y must be a point or a line. Y cannot be a point because, by Lemma 1.1, μ has corank ≤ 2 at every point of $\mathbb{P}^{3\vee}$ (E_H is semistable for every plane H because $\mathrm{H}^1(E(-2)) = 0$ and $\mathrm{H}^1(E^\vee(-2)) = 0$).

Y cannot be a line, either. Indeed, if Y is a line then the kernel of the epimorphism $\mathrm{H}^1(E) \otimes \mathscr{O}_{\mathbb{P}^{3\vee}} \to \mathscr{I}_Y(1)$ is isomorphic to $\mathscr{O}_{\mathbb{P}^{3\vee}} \oplus \mathscr{O}_{\mathbb{P}^{3\vee}}(-1)$, hence μ factorizes as:

$$\mathrm{H}^1(E(-1)) \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}(-1) \xrightarrow{\overline{\mu}} \mathscr{O}_{\mathbb{P}^{3\vee}} \oplus \mathscr{O}_{\mathbb{P}^{3\vee}}(-1) \longrightarrow \mathrm{H}^1(E) \otimes \mathscr{O}_{\mathbb{P}^{3\vee}} .$$

The kernel of the component $\mathrm{H}^1(E(-1)) \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}(-1) \to \mathscr{O}_{\mathbb{P}^{3\vee}}(-1)$ of $\overline{\mu}$ has the form $N_{-1} \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}(-1)$, for some 1-codimensional subspace N_{-1} of $\mathrm{H}^1(E(-1))$. The direct summand $\mathscr{O}_{\mathbb{P}^{3\vee}}$ of $\mathscr{O}_{\mathbb{P}^{3\vee}} \oplus \mathscr{O}_{\mathbb{P}^{3\vee}}(-1)$ corresponds to a 1-dimensional subspace N_0 of $\mathrm{H}^1(E)$. Since μ maps $N_{-1} \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}(-1)$ into $N_0 \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}$, it follows that $S_1N_{-1} \subseteq N_0$, which contradicts Lemma 1.8. $\square$

Proposition 2.3. *Let E be a stable rank 3 vector bundle on $\mathbb{P}^3$ with $c_1 = 0$, $c_2 = 3$, $c_3 = 0$ and spectrum $(-1, 0, 1)$. Then:*

(a) *E has an unstable plane of order 1.*

(b) *The restriction of E to a general plane is stable.*

(c) *E is the cohomology sheaf of a Horrocks monad of the form:*

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^3}(-2) \xrightarrow{\alpha} \mathscr{O}_{\mathbb{P}^3}(1) \oplus 3\mathscr{O}_{\mathbb{P}^3} \oplus \mathscr{O}_{\mathbb{P}^3}(-1) \xrightarrow{\beta} \mathscr{O}_{\mathbb{P}^3}(2) \longrightarrow 0 .$$

Proof. (a) The spectrum of $E^\vee$ is $(-1, 0, 1)$, as well. It follows that $\mathrm{h}^1(E^\vee(-2)) = 1$ and $\mathrm{h}^1(E^\vee(-1)) = 3$. One deduces that there exists a non-zero linear form h_0 such that the multiplication by $h_0\colon \mathrm{H}^1(E^\vee(-2)) \to \mathrm{H}^1(E^\vee(-1))$ is the zero map. If H_0 is the plane of equation $h_0 = 0$ then $\mathrm{h}^0(E^\vee_{H_0}(-1)) = 1$.

(b) It follows from (a) and from [8, Prop. 5.1] (recalled in Remark 1.4) that $\mathrm{H}^0(E_H) = 0$, for the general plane $H \subset \mathbb{P}^3$. Since $E^\vee$ has the same Chern classes and spectrum as E, one deduces that, for the general plane $H \subset \mathbb{P}^3$, one has $\mathrm{H}^0(E^\vee_H) = 0$ as well.

(c) We assert that the graded S-module $\mathrm{H}^1_*(E)$ is generated by $\mathrm{H}^1(E(-2))$. Indeed, using the spectrum one sees that $\mathrm{H}^1(E(l)) = 0$ for $l \leq -3$, $\mathrm{h}^1(E(-2)) = 1$, $\mathrm{h}^1(E(-1)) = 3$ and that $\mathrm{H}^2(E(l)) = 0$ for $l \geq -1$. Moreover, $\mathrm{h}^1(E) = 3$ by

Riemann-Roch. Since $\mathrm{H}^2(E(-1)) = 0$ and $\mathrm{H}^3(E(-2)) = 0$, the slightly more general variant of the Castelnuovo-Mumford Lemma recalled in [1, Lemma 1.21] implies that $\mathrm{H}^1_*(E)$ is generated in degrees ≤ 0.

Since $\mathrm{H}^0(E) = 0$, $\mathrm{H}^1(E(-2))$ cannot be annihilated by two linearly independent linear forms (because if it would be, denoting by L the line defined by these forms and using the exact sequence $0 \to \mathscr{O}_{\mathbb{P}^3}(-2) \to 2\mathscr{O}_{\mathbb{P}^3}(-1) \to \mathscr{I}_L \to 0$, one would have $\mathrm{H}^0(\mathscr{I}_L \otimes E) \neq 0$). It follows that $S_1\mathrm{H}^1(E(-2)) = \mathrm{H}^1(E(-1))$. On the other hand, by (b), if h is a general linear form then, denoting by H the plane of equation $h = 0$, one has $\mathrm{H}^0(E_H) = 0$ hence multiplication by $h\colon \mathrm{H}^1(E(-1)) \to \mathrm{H}^1(E)$ is injective hence bijective. Our assertion is proven.

Since $E^\vee$ has the same Chern classes and spectrum as E it follows that $\mathrm{H}^1_*(E^\vee)$ is generated by $\mathrm{H}^1(E^\vee(-2))$. One deduces (see Barth and Hulek [4]) that E is the cohomology sheaf of a Horrocks monad of the form $0 \to \mathscr{O}_{\mathbb{P}^3}(-2) \to B \to \mathscr{O}_{\mathbb{P}^3}(2) \to 0$, where B is a direct sum of line bundles. B has rank 5, $\mathrm{H}^0(B(-2)) = 0$ and $\mathrm{h}^0(B(-1)) = \mathrm{h}^0(\mathscr{O}_{\mathbb{P}^3}(1)) - \mathrm{h}^1(E(-1)) = 1$. Analogously, $\mathrm{H}^0(B^\vee(-2)) = 0$ and $\mathrm{h}^0(B^\vee(-1)) = 1$. It follows that $B \simeq \mathscr{O}_{\mathbb{P}^3}(1) \oplus 3\mathscr{O}_{\mathbb{P}^3} \oplus \mathscr{O}_{\mathbb{P}^3}(-1)$. $\square$

3. THE CASE $c_3 = 2$

LEMMA 3.1. *Let E be a stable rank 3 vector bundle on $\mathbb{P}^3$ with $c_1 = 0$, $c_2 = 3$, $c_3 = 2$. Then:*

(a) *E is the cohomology sheaf of a monad of the form:*

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^3}(-1) \oplus \mathscr{O}_{\mathbb{P}^3}(-2) \xrightarrow{\alpha} 6\mathscr{O}_{\mathbb{P}^3} \oplus \mathscr{O}_{\mathbb{P}^3}(-1) \xrightarrow{\beta} 2\mathscr{O}_{\mathbb{P}^3}(1) \longrightarrow 0\,.$$

(b) *If E has no unstable plane then it is the cohomology sheaf of a monad of the form:*

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^3}(-2) \xrightarrow{\alpha} 6\mathscr{O}_{\mathbb{P}^3} \xrightarrow{\beta} 2\mathscr{O}_{\mathbb{P}^3}(1) \longrightarrow 0\,.$$

Proof. (a) The spectrum of E must be $(-1, 0, 0)$. One deduces that $\mathrm{H}^1(E(l)) = 0$ for $l \leq -2$, $\mathrm{h}^1(E(-1)) = 2$, $\mathrm{h}^2(E(-3)) = 4$, $\mathrm{h}^2(E(-2)) = 1$, and $\mathrm{H}^2(E(l)) = 0$ for $l \geq -1$. By Riemann-Roch, $\mathrm{h}^1(E) = 2$.

CLAIM 1. *$\mathrm{H}^1_*(E)$ is generated by $\mathrm{H}^1(E(-1))$.*

Indeed, since $\mathrm{H}^2(E(-1)) = 0$ and $\mathrm{H}^3(E(-2)) = 0$, the Castelnuovo-Mumford Lemma (in its slightly more general form recalled in [1, Lemma 1.21]) implies that $\mathrm{H}^1_*(E)$ is generated in degrees ≤ 0. It remains to show that the multiplication map $S_1 \otimes \mathrm{H}^1(E(-1)) \to \mathrm{H}^1(E)$ is surjective. Assume, by contradiction, that it is not. Then its image is contained in a 1-dimensional subspace A of $\mathrm{H}^1(E)$. Consider the Beilinson monad of E (see Remark 1.7):

$$0 \longrightarrow \mathrm{H}^2(E(-3)) \otimes \Omega^3_{\mathbb{P}^3}(3) \xrightarrow{\gamma} \begin{matrix} \mathrm{H}^2(E(-2)) \otimes \Omega^2_{\mathbb{P}^3}(2) \\ \oplus \\ \mathrm{H}^1(E(-1)) \otimes \Omega^1_{\mathbb{P}^3}(1) \end{matrix} \xrightarrow{\delta} \mathrm{H}^1(E) \otimes \mathscr{O}_{\mathbb{P}^3} \longrightarrow 0\,.$$

By our assumption, the image of the restriction δ_2 of δ to $\mathrm{H}^1(E(-1)) \otimes \Omega^1_{\mathbb{P}^3}(1)$ is contained in $A \otimes \mathscr{O}_{\mathbb{P}^3}$. Let A' denote the quotient $\mathrm{H}^1(E)/A$. Denoting by γ_1 the component $\mathrm{H}^2(E(-3)) \otimes \Omega^3_{\mathbb{P}^3}(3) \to \mathrm{H}^2(E(-2)) \otimes \Omega^2_{\mathbb{P}^3}(2)$ of γ, one deduces that one has an epimorphism $\mathrm{Coker}\,\gamma_1 \to A' \otimes \mathscr{O}_{\mathbb{P}^3}$. But the multiplication map $S_1 \otimes \mathrm{H}^2(E(-3)) \to \mathrm{H}^2(E(-2))$ is surjective (because $\mathrm{H}^3(E(-4)) = 0$) hence the morphism γ_1 is non-zero. Since there is no complex $\mathscr{O}_{\mathbb{P}^3}(-1) \xrightarrow{\phi} \Omega^2_{\mathbb{P}^3}(2) \xrightarrow{\pi} \mathscr{O}_{\mathbb{P}^3}$ with $\phi \neq 0$ and π an epimorphism, we have got the desired contradiction.

Claim 2. *$\mathrm{H}^1_*(E^\vee)$ has one minimal generator of degree -2 and at most one of degree -1.*

Indeed, since the spectrum of $E^\vee$ is $(0,0,1)$, it follows that $\mathrm{H}^1(E^\vee(l)) = 0$ for $l \leq -3$, $\mathrm{h}^1(E^\vee(-2)) = 1$, $\mathrm{h}^1(E^\vee(-1)) = 4$, and $\mathrm{H}^2(E^\vee(l)) = 0$ for $l \geq -2$. One deduces that $\mathrm{H}^1_*(E^\vee)$ is generated in degrees ≤ -1 (because $\mathrm{H}^2(E^\vee(-2)) = 0$ and $\mathrm{H}^3(E^\vee(-3)) = 0$). Moreover, the multiplication map $S_1 \otimes \mathrm{H}^1(E^\vee(-2)) \to \mathrm{H}^1(E^\vee(-1))$ cannot have rank ≤ 2 because in that case it would exist a line $L \subset \mathbb{P}^3$ such that $\mathrm{H}^0(\mathscr{I}_L \otimes E^\vee) = 0$, which is not true.

The two claims above imply that E is the cohomology sheaf of a (not necessarily minimal) Horrocks monad of the form

$$0 \to \mathscr{O}_{\mathbb{P}^3}(-1) \oplus \mathscr{O}_{\mathbb{P}^3}(-2) \to B \to 2\mathscr{O}_{\mathbb{P}^3}(1) \to 0,$$

with B a direct sum of line bundles. B has rank 7, $\mathrm{H}^0(B(-1)) = 0$, $c_1(B) = -1$ hence $B \simeq 6\mathscr{O}_{\mathbb{P}^3} \oplus \mathscr{O}_{\mathbb{P}^3}(-1)$.

(b) If E has no unstable plane then the multipliction map

$$S_1 \otimes \mathrm{H}^1(E^\vee(-2)) \to \mathrm{H}^1(E^\vee(-1))$$

is injective and therefore bijective hence $\mathrm{H}^1_*(E^\vee)$ has no minimal generator of degree -1. $\square$

Lemma 3.2. *Let E be a stable rank 3 vector bundle on $\mathbb{P}^3$ with $c_1 = 0$, $c_2 = 3$, $c_3 = 2$. Then, for the general plane $H \subset \mathbb{P}^3$, one has $\mathrm{H}^0(E_H) = 0$.*

Proof. Assume, by contradiction, that this is not the case. Consider the morphism $\mu \colon \mathrm{H}^1(E(-1)) \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}(-1) \to \mathrm{H}^1(E) \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}$ from Remark 1.9. Using the notation from that remark, one has $a \leq -2$ (by observation (i)) and $b \geq 1$ (by observation (ii) because, by Claim 1 in the proof of Prop. 2.2, one has $S_1\mathrm{H}^1(E(-1)) = \mathrm{H}^1(E)$). But this contradicts relation (1.1) (in which one has $d = 2$). $\square$

Proposition 3.3. *Let E be a stable rank 3 vector bundle on $\mathbb{P}^3$ with $c_1 = 0$, $c_2 = 3$, $c_3 = 2$. Assume that E has an unstable plane. Then, for the general plane $H \subset \mathbb{P}^3$, one has $\mathrm{H}^0(E_H^\vee) = 0$.*

Proof. We intend to apply Lemma 1.5. According to Lemma 3.1, E is the cohomology sheaf of a minimal Horrocks monad of the form:

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^3}(-1) \oplus \mathscr{O}_{\mathbb{P}^3}(-2) \xrightarrow{\alpha} 6\mathscr{O}_{\mathbb{P}^3} \oplus \mathscr{O}_{\mathbb{P}^3}(-1) \xrightarrow{\beta} 2\mathscr{O}_{\mathbb{P}^3}(1) \longrightarrow 0\,.$$

The component $\mathscr{O}_{\mathbb{P}^3}(-1) \to \mathscr{O}_{\mathbb{P}^3}(-1)$ of α is zero and the component $\mathscr{O}_{\mathbb{P}^3}(-2) \to \mathscr{O}_{\mathbb{P}^3}(-1)$ is defined by a linear form h_0. Since $\mathrm{H}^0(\alpha^\vee)$ is injective, one has $h_0 \neq 0$. Let $H_0 \subset \mathbb{P}^3$ be the plane of equation $h_0 = 0$.

Dualizing the above monad, one deduces that $E^\vee$ is the middle cohomology sheaf of a complex:

$$0 \longrightarrow 2\mathscr{O}_{\mathbb{P}^3}(-1) \longrightarrow 2\mathscr{O}_{\mathbb{P}^3} \oplus \Omega_{\mathbb{P}^3}(1) \xrightarrow{\varepsilon} \mathscr{O}_{H_0}(2) \longrightarrow 0\,.$$

ε is defined by two elements f_0, f_1 of $\mathrm{H}^0(\mathscr{O}_{H_0}(2))$ and by a morphism $\Omega_{\mathbb{P}^3}(1) \to \mathscr{O}_{H_0}(2)$ which can be written as a composite map $\Omega_{\mathbb{P}^3}(1) \to \mathscr{O}_{\mathbb{P}^3} \otimes V^\vee \xrightarrow{\phi} \mathscr{O}_{H_0}(2)$. Since $\mathrm{H}^0(E^\vee) = 0$ it follows that f_0 and f_1 are linearly independent. Put $Z := \{x \in H_0 \mid f_0(x) = f_1(x) = 0\}$.

If L is a line not contained in H_0 then $\mathrm{H}^1(E_L^\vee)$ is isomorphic to the cokernel of:

$$\mathrm{H}^0(\varepsilon_L)\colon \mathrm{H}^0\left(2\mathscr{O}_L \oplus (\Omega_{\mathbb{P}^3}(1) \mid L)\right) \longrightarrow \mathrm{H}^0(\mathscr{O}_{L\cap H_0}(2))\,.$$

If $L \cap Z = \emptyset$ then $\mathrm{H}^0(\varepsilon_L)$ is surjective hence $\mathrm{H}^1(E_L^\vee) = 0$. Assume, now, that $L \cap H_0$ consists of a point x belonging to Z. Since ε is an epimorphism, the map $(\Omega_{\mathbb{P}^3}(1))(x) \to (\mathscr{O}_{H_0}(2))(x)$ is surjective. But $(\Omega_{\mathbb{P}^3}(1))(x) = \mathrm{H}^0(\mathscr{I}_{\{x\}}(1)) \subset V^\vee$ and $\mathrm{H}^0(\Omega_{\mathbb{P}^3}(1) \mid L) = \mathrm{H}^0(\mathscr{I}_L(1)) \subset \mathrm{H}^0(\mathscr{I}_{\{x\}}(1))$. One deduces that for all the lines L passing through x and not contained in H_0, except at most one, the map $\mathrm{H}^0(\Omega_{\mathbb{P}^3}(1) \mid L) = \mathrm{H}^0(\mathscr{I}_L(1)) \to \mathscr{O}_{H_0}(2)(x) = \mathrm{H}^0(\mathscr{O}_{L\cap H_0}(2))$ induced by ϕ is surjective. Consequently, any line $L \subset \mathbb{P}^3$ for which $\mathrm{h}^1(E_L^\vee) \geq 1$ is either contained in H_0 or belongs to a family of dimension at most 1 (because $\dim Z \leq 1$). The conclusion of the proposition follows, now, from Lemma 1.5. $\square$

Lemma 3.4. *Let V be a 4-dimensional k-vector space and let W be a 4-dimensional subspace of $\bigwedge^2 V$. Then there exists a k-basis $v_0, \dots, v_3$ of V such that W admits one of the following bases:*

(i) $v_0 \wedge v_1,\ v_1 \wedge v_2,\ v_2 \wedge v_3,\ v_0 \wedge v_3$;

(ii) $v_0 \wedge v_1,\ v_1 \wedge v_2,\ v_2 \wedge v_3,\ v_0 \wedge v_2 + v_1 \wedge v_3$;

(iii) $v_0 \wedge v_1,\ v_1 \wedge v_2,\ v_2 \wedge v_3,\ v_0 \wedge v_2$.

Proof. Consider the canonical pairing $\langle *, * \rangle \colon \bigwedge^2 V^\vee \times \bigwedge^2 V \to k$ and let $W^\perp \subset \bigwedge^2 V^\vee$ consist of the elements α with $\langle \alpha, \omega \rangle = 0$, $\forall\, \omega \in W$. Since $W^\perp$ has dimension 2, a well-known result (see, for example, [2, Lemma G.4]) says that there exists a basis $h_0, \dots, h_3$ of $V^\vee$ such that $W^\perp$ admits one of the following bases:

(1) $h_0 \wedge h_2,\ h_1 \wedge h_3$;

(2) $h_0 \wedge h_3,\ h_0 \wedge h_2 - h_1 \wedge h_3$;

(3) $h_0 \wedge h_3,\ h_1 \wedge h_3$.

Let $v_0, \dots, v_3$ be the dual basis of V. If $W^\perp$ admits the basis (1) (resp., (2), resp., (3)) then V admits the basis (i) (resp., (ii), resp., (iii)). $\square$

PROPOSITION 3.5. *Let E be a stable rank 3 vector bundle on $\mathbb{P}^3$ with $c_1 = 0$, $c_2 = 3$, $c_3 = 2$. Assume that E has no unstable plane. If $\mathrm{H}^0(E_H^\vee) \neq 0$, for every plane $H \subset \mathbb{P}^3$, then E is as in Theorem 0.2(b)(ii).*

Proof. According to Lemma 3.1(b), E is the cohomology sheaf of a monad of the form:
$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^3}(-2) \xrightarrow{\alpha} 6\mathscr{O}_{\mathbb{P}^3} \xrightarrow{\beta} 2\mathscr{O}_{\mathbb{P}^3}(1) \longrightarrow 0\,.$$
The spectrum of the dual vector bundle $E^\vee$ is $(0,0,1)$. It follows that $\mathrm{H}^1(E^\vee(l)) = 0$ for $l \leq -3$, $\mathrm{h}^1(E^\vee(-2)) = 1$, $\mathrm{h}^1(E^\vee(-1)) = 4$, $\mathrm{h}^2(E^\vee(-3)) = 2$, $\mathrm{H}^2(E^\vee(l)) = 0$ for $l \geq -2$. Moreover, by Riemann-Roch, $\mathrm{h}^1(E^\vee) = 4$. By Remark 1.7, $E^\vee$ is the cohomology sheaf of a Beilinson monad of the form:
$$0 \longrightarrow \begin{array}{c} \mathrm{H}^1(E^\vee(-2)) \otimes \Omega^2_{\mathbb{P}^3}(2) \\ \oplus \\ \mathrm{H}^2(E^\vee(-3)) \otimes \Omega^3_{\mathbb{P}^3}(3) \end{array} \xrightarrow{\gamma} \mathrm{H}^1(E^\vee(-1)) \otimes \Omega^1_{\mathbb{P}^3}(1) \xrightarrow{\delta} \mathrm{H}^1(E^\vee) \otimes \mathscr{O}_{\mathbb{P}^3} \to 0\,.$$
Consider, now, the morphism
$$\mu \colon \mathrm{H}^1(E^\vee(-1)) \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}(-1) \to \mathrm{H}^1(E^\vee) \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}$$
from Remark 1.9. Using the notation from that remark, one has $a \leq -2$ (by observation (i)) and $b \geq 1$ (by observation (ii) because one sees easily, dualizing the above Horrocks monad, that $S_1\mathrm{H}^1(E^\vee(-1)) = \mathrm{H}^1(E^\vee)$).

Claim 1. $b \geq 2$.

Indeed, assume, by contradiction, that $b = 1$. As we noticed in the Introduction (after the statement of Theorem 0.2) the general plane $H \subset \mathbb{P}^3$ contains a line L_0 with $\mathrm{h}^1(E^\vee_{L_0}) \geq 1$. Observation (iii) in Remark 1.9 implies that Y has dimension 1. Since $\mathscr{I}_Y(1)$ is globally generated, Y must be a line. In this case, μ factorizes as:

$$\mathrm{H}^1(E^\vee(-1)) \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}(-1) \xrightarrow{\overline{\mu}} (N_0 \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}) \oplus \mathscr{O}_{\mathbb{P}^{3\vee}}(-1) \longrightarrow \mathrm{H}^1(E^\vee) \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}\,,$$

for some subspace N_0 of $\mathrm{H}^1(E^\vee)$ of dimension 2. The kernel of the component $\mathrm{H}^1(E^\vee(-1)) \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}(-1) \to \mathscr{O}_{\mathbb{P}^{3\vee}}(-1)$ of $\overline{\mu}$ has the form $N_{-1} \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}(-1)$ for some 1-codimensional subspace N_{-1} of $\mathrm{H}^1(E^\vee(-1))$.

Since μ maps $N_{-1} \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}(-1)$ into $N_0 \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}$ it follows that $S_1 N_{-1} \subseteq N_0$ which contradicts Lemma 1.8.

It follows, now, from Claim 1 and from relation (1.1) (with $d = 4$), that $b = 2$ and $a = -2$. In this case, we have an exact sequence:

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^{3\vee}}(-2) \xrightarrow{\kappa} \mathrm{H}^1(E^\vee(-1)) \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}(-1) \xrightarrow{\mu} \mathrm{H}^1(E^\vee) \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}\,.$$

Choosing a basis of $\mathrm{H}^1(E^\vee(-1))$, κ is defined by four vectors $u_0, \ldots, u_3 \in V = \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^{3\vee}}(1))$. We assert that $u_0, \ldots, u_3$ are *linearly independent*.

Indeed, if $ku_0 + \ldots + ku_3$ has dimension $c < 4$ then there is a decomposition $\mathrm{H}^1(E^\vee(-1)) = N_{-1} \oplus N'_{-1}$, with N_{-1} of dimension c, such that $\mathrm{Im}\,\kappa \subset N_{-1} \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}(-1)$. It follows that $\mathrm{Coker}\,\kappa \simeq \mathscr{F} \oplus (N'_{-1} \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}(-1))$, where $\mathscr{F}$ is a sheaf defined by an exact sequence:

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^{3\vee}}(-2) \longrightarrow N_{-1} \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}(-1) \longrightarrow \mathscr{F} \longrightarrow 0\,.$$

One cannot have $c = 1$ because, in this case, $\mathscr{F}$ is a torsion sheaf and this contradicts the fact that $\mathrm{Coker}\,\kappa \simeq \mathrm{Im}\,\mu$. If $c \in \{2, 3\}$ then, dualizing the above sequence, one sees that $\mathrm{Hom}_{\mathscr{O}_{\mathbb{P}^{3\vee}}}(\mathscr{F}, \mathscr{O}_{\mathbb{P}^{3\vee}})$ has dimension 1 for $c = 2$ and dimension 3 for $c = 3$. One deduces that there exists a subspace N_0 of $\mathrm{H}^1(E^\vee)$, of dimension 1 if $c = 2$ and of dimension 3 if $c = 3$, such that μ maps $N_{-1} \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}(-1)$ into $N_0 \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}$. This means that $S_1 N_{-1} \subseteq N_0$ and this contradicts the following assertion:

Claim 2. *If N_{-1} is a subspace of $\mathrm{H}^1(E^\vee(-1))$, of dimension 2 or 3, then the dimension of the subspace $S_1 N_{-1}$ of $\mathrm{H}^1(E^\vee)$ is $> \dim_k N_{-1}$.*

Indeed, the image of $\mathrm{H}^0(\alpha^\vee)\colon \mathrm{H}^0(6\mathscr{O}_{\mathbb{P}^3}) \to \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^3}(2))$ is a 6-dimensional, base point free subspace U of $\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^3}(2))$. One has $\mathrm{H}^1(E^\vee(-1)) \simeq \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^3}(1))$ and $\mathrm{H}^1(E^\vee) \simeq \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^3}(2))/U$. By the first isomorphism, N_{-1} is identified with $\mathrm{H}^0(\mathscr{I}_\Lambda(1))$, for some linear subspace Λ of $\mathbb{P}^3$ with $\mathrm{codim}(\Lambda, \mathbb{P}^3) = \dim_k N_{-1}$. Then $S_1 N_{-1} \simeq (\mathrm{H}^0(\mathscr{I}_\Lambda(2)) + U)/U$ hence $\mathrm{H}^1(E^\vee)/S_1 N_{-1}$ is isomorphic to the cokernel of the restriction map $U \to \mathrm{H}^0(\mathscr{O}_\Lambda(2))$. Since U is base point free, the dimension of this cokernel is 0 if $\dim \Lambda = 0$ and ≤ 1 if $\dim \Lambda = 1$. The claim is proven.

It remains that $u_0, \dots, u_3$ are linearly independent. This implies that $\mathrm{Coker}\,\kappa \simeq \mathrm{T}_{\mathbb{P}^{3\vee}}(-2)$. μ induces a morphism $\overline{\mu}\colon \mathrm{T}_{\mathbb{P}^{3\vee}}(-2) \to \mathrm{H}^1(E^\vee) \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}$. Since one has $S_1\mathrm{H}^1(E^\vee(-1)) = \mathrm{H}^1(E^\vee)$, the map

$$\mathrm{H}^0(\mu^\vee)\colon \mathrm{H}^1(E^\vee)^\vee \to \mathrm{H}^1(E^\vee(-1))^\vee \otimes \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^{3\vee}}(1))$$

is injective. It follows that the map $\mathrm{H}^0(\overline{\mu}^\vee)\colon \mathrm{H}^1(E^\vee)^\vee \to \mathrm{H}^0(\Omega_{\mathbb{P}^{3\vee}}(2))$ is injective, too. Its image is a 4-dimensional subspace W of $\mathrm{H}^0(\Omega_{\mathbb{P}^{3\vee}}(2)) \simeq \bigwedge^2 V$. Using Lemma 3.4 one sees, now, that, choosing convenient k-bases of $\mathrm{H}^1(E^\vee(-1))$ and $\mathrm{H}^1(E^\vee)$, μ is represented by some concrete 4×4 matrix $\mathcal{M}$ with entries in V.

We make, at this point, the following observation: the same matrix $\mathcal{M}$ defines the differential δ of the above Beilinson monad of $E^\vee$ (see Remark 1.9). The component $\gamma_1 : \mathrm{H}^1(E^\vee(-2)) \otimes \Omega^2_{\mathbb{P}^3}(2) \to \mathrm{H}^1(E^\vee(-1)) \otimes \Omega^1_{\mathbb{P}^3}(1)$ of the differential γ of the monad is defined by a 4×1 matrix $(w_0, \dots, w_3)^{\mathrm{t}}$ with entries in V. Since E has no unstable plane, the multiplication map $\mathrm{H}^1(E^\vee(-2)) \otimes S_1 \to \mathrm{H}^1(E^\vee(-1))$ is an isomorphism. It follows that $w_0, \dots, w_3$ *are linearly independent*. Moreover, the fact that $\delta \circ \gamma_1 = 0$ is equivalent to the following relation (for matrices with entries in the exterior algebra $\bigwedge V$):

$$\mathcal{M} \wedge (w_0\,,\, w_1\,,\, w_2\,,\, w_3)^{\mathrm{t}} = 0\,. \tag{3.1}$$

The argument splits, now, according to Lemma 3.4 into three cases.

Case 1. *W is as in Lemma* 3.4(i).

In this case, choosing convenient bases of $\mathrm{H}^1(E^\vee(-1))$ and $\mathrm{H}^1(E^\vee)$, μ is defined by the transpose of the matrix

$$\begin{pmatrix} -v_1 & 0 & 0 & -v_3 \\ v_0 & -v_2 & 0 & 0 \\ 0 & v_1 & -v_3 & 0 \\ 0 & 0 & v_2 & v_0 \end{pmatrix}, \text{ i.e., by the matrix } \mathcal{M} := \begin{pmatrix} -v_1 & v_0 & 0 & 0 \\ 0 & -v_2 & v_1 & 0 \\ 0 & 0 & -v_3 & v_2 \\ -v_3 & 0 & 0 & v_0 \end{pmatrix}.$$

Recall that the same matrix defines the differential

$$\delta\colon \mathrm{H}^1(E^\vee(-1)) \otimes \Omega^1_{\mathbb{P}^3}(1) \to \mathrm{H}^1(E^\vee) \otimes \mathscr{O}_{\mathbb{P}^3}$$

of the Beilinson monad of $E^\vee$.

It is an elementary fact that if $u_1, \dots, u_p$ are linearly independent vectors and if $u'_1, \dots, u'_p$ are some other vectors satisfying $\sum_{i=1}^p u_i \wedge u'_i = 0$ then there exists a $p \times p$ symmetric matrix A such that:

$$(u'_1, \dots, u'_p) = (u_1, \dots, u_p)A\,.$$

In particular, $u'_i \in ku_1 + \ldots + ku_p$, $i = 1, \ldots, p$.

One sees, now, easily that relation (3.1) implies that $w_i \in kv_i$, $i = 0, \ldots, 3$, i.e., that $w_i = a_i v_i$, $i = 0, \ldots, 3$. One deduces, from the same relation, that $a_i = (-1)^i a_0$, $i = 1, 2, 3$. Consequently, we can assume that $w_i = (-1)^i v_i$, $i = 0, \ldots, 3$. Moreover, after a linear change of coordinates in $\mathbb{P}^3$, we can assume that $v_i = (-1)^i e_i$, $i = 0, \ldots, 3$, where $e_0, \ldots, e_3$ is the canonical basis of $V = k^4$.

Now, the Beilinson monad of $E^\vee$ shows that one has an exact sequence:

$$0 \longrightarrow 2\mathscr{O}_{\mathbb{P}^3}(-1) \longrightarrow K \longrightarrow E^\vee \longrightarrow 0\,,$$

where K is the cohomology sheaf of the monad:

$$0 \longrightarrow \Omega^2_{\mathbb{P}^3}(2) \xrightarrow{\gamma_1} 4\Omega^1_{\mathbb{P}^3}(1) \xrightarrow{\delta} 4\mathscr{O}_{\mathbb{P}^3} \longrightarrow 0\,,$$

with δ and γ_1 defined by the matrices:

$$\delta = \begin{pmatrix} e_1 & e_0 & 0 & 0 \\ 0 & -e_2 & -e_1 & 0 \\ 0 & 0 & e_3 & e_2 \\ e_3 & 0 & 0 & e_0 \end{pmatrix}, \ \gamma_1 = \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

We assert that K is isomorphic to the kernel of the epimorphism $\pi\colon 6\mathscr{O}_{\mathbb{P}^3} \to \mathscr{O}_{\mathbb{P}^3}(2)$ defined by $(X_2^2\,, X_3^2\,, -X_0X_2\,, -X_1X_3\,, X_0^2\,, X_1^2)$. Indeed, let K' be the kernel of π. The only non-zero cohomology groups $\mathrm{H}^p(K'(l))$ in the range $-3 \leq l \leq 0$ are $\mathrm{H}^1(K'(-2)) \simeq S_0$, $\mathrm{H}^1(K'(-1)) \simeq S_1$ and $\mathrm{H}^1(K') \simeq S_2/I_2$, where I_2 is the subspace of S_2 generated by the monomials defining π. Choosing the canonical bases of S_0 and S_1 and the basis of S_2/I_2 consisting of the classes of the monomials X_0X_1, $-X_1X_2$, X_2X_3, X_0X_3 one sees that the Beilinson monad of K' is precisely the above monad (the linear part $\mathrm{H}^1(K'(-l)) \otimes \Omega^l_{\mathbb{P}^3}(l) \to \mathrm{H}^1(K'(-l+1)) \otimes \Omega^{l-1}_{\mathbb{P}^3}(l-1)$ of a differential of the Beilinson monad is defined by $\sum_{i=0}^3 X_i \otimes e_i$). It follows that $K' \simeq K$.

Consequently, $E^\vee$ is the cohomology sheaf of a monad of the form:

$$0 \longrightarrow 2\mathscr{O}_{\mathbb{P}^3}(-1) \xrightarrow{\rho} 6\mathscr{O}_{\mathbb{P}^3} \xrightarrow{\pi} \mathscr{O}_{\mathbb{P}^3}(2) \longrightarrow 0\,,$$

with π the morphism considered above. $\mathrm{H}^0(\pi(1))\colon \mathrm{H}^0(6\mathscr{O}_{\mathbb{P}^3}(1)) \to \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^3}(3))$ is obviously surjective hence its kernel has dimension 4. It is, therefore, generated by the elements:

$$(X_0\,, 0\,, X_2\,, 0\,, 0\,, 0)^{\mathrm{t}}\,,\ (0\,, X_1\,, 0\,, X_3\,, 0\,, 0)^{\mathrm{t}}\,,$$
$$(0\,, 0\,, X_0\,, 0\,, X_2\,, 0)^{\mathrm{t}}\,,\ (0\,, 0\,, 0\,, X_1\,, 0\,, X_3)^{\mathrm{t}}\,.$$

One deduces that ρ must be defined by the transpose of a matrix of the form:

$$\begin{pmatrix} a_0X_0 & a_1X_1 & a_0X_2 + a_2X_0 & a_1X_3 + a_3X_1 & a_2X_2 & a_3X_3 \\ b_0X_0 & b_1X_1 & b_0X_2 + b_2X_0 & b_1X_3 + b_3X_1 & b_2X_2 & b_3X_3 \end{pmatrix}.$$

Since $\rho^\vee$ is surjective at the point $[1:0:0:0]$, one has $\begin{vmatrix} a_0 & a_2 \\ b_0 & b_2 \end{vmatrix} \neq 0$. Permuting, if necessary, the rows of the above matrix, one can assume that $a_0 \neq 0$. Substracting from the second row the first row multiplied by $b_0 a_0^{-1}$, one gets the matrix:

$$\begin{pmatrix} a_0X_0 & a_1X_1 & a_0X_2 + a_2X_0 & a_1X_3 + a_3X_1 & a_2X_2 & a_3X_3 \\ 0 & b_1'X_1 & b_2'X_0 & b_1'X_3 + b_3'X_1 & b_2'X_2 & b_3'X_3 \end{pmatrix},$$

where $b_i' = a_0^{-1}\begin{vmatrix} a_0 & a_i \\ b_0 & b_i \end{vmatrix}$, $i = 1, 2, 3$. Notice that $b_2' \neq 0$. Substracting from the first row of the new matrix the second row multiplied by $a_2 b_2'^{-1}$, one gets the matrix:

$$\begin{pmatrix} a_0X_0 & a_1'X_1 & a_0X_2 & a_1'X_3 + a_3'X_1 & 0 & a_3'X_3 \\ 0 & b_1'X_1 & b_2'X_0 & b_1'X_3 + b_3'X_1 & b_2'X_2 & b_3'X_3 \end{pmatrix},$$

where $a_i' = b_2'^{-1}\begin{vmatrix} a_i & a_2 \\ b_i' & b_2' \end{vmatrix}$. Finally, multiplying the first (resp., second) row of the last matrix by a_0^{-1} (resp., $b_2'^{-1}$), one gets the matrix:

$$\begin{pmatrix} X_0 & a_1''X_1 & X_2 & a_1''X_3 + a_3''X_1 & 0 & a_3''X_3 \\ 0 & b_1''X_1 & X_0 & b_1''X_3 + b_3''X_1 & X_2 & b_3''X_3 \end{pmatrix}.$$

Since $\rho^\vee$ is surjective at the point $[0:1:0:0]$, it follows that $\begin{vmatrix} a_1'' & a_3'' \\ b_1'' & b_3'' \end{vmatrix} \neq 0$. Conversely, if this determinant is non-zero then:

$$X_0^2\,,\ X_2^2\,,\ \begin{vmatrix} a_1'' & a_3'' \\ b_1'' & b_3'' \end{vmatrix} X_1^2\,,\ \begin{vmatrix} a_1'' & a_3'' \\ b_1'' & b_3'' \end{vmatrix} X_3^2$$

are among the 2×2 minors of the above matrix hence this matrix defines an epimorphism $6\mathscr{O}_{\mathbb{P}^3} \to 2\mathscr{O}_{\mathbb{P}^3}(1)$.

Case 2. *W is as in Lemma* 3.4(ii).

In this case, μ is defined by the matrix:

$$\mathcal{M} := \begin{pmatrix} -v_1 & v_0 & 0 & 0 \\ 0 & -v_2 & v_1 & 0 \\ 0 & 0 & -v_3 & v_2 \\ -v_2 & -v_3 & v_0 & v_1 \end{pmatrix}.$$

Recall relation (3.1). One deduces, from this relation, using the elementary fact recalled at the beginning of Case 1, that $w_0 \in kv_0 + kv_1$, $w_1 \in (kv_1 + kv_0) \cap (kv_2 + kv_1) = kv_1$, $w_2 \in kv_2$, and $w_3 \in kv_2 + kv_3$. Moreover, the relation $-v_2 \wedge w_1 + v_1 \wedge w_2 = 0$ implies that $w_1 = -av_1$ and $w_2 = av_2$, for some $a \in k$

and the relation $-v_3 \wedge w_2 + v_2 \wedge w_3 = 0$ implies that $w_3 = -av_3 + bv_2$, for some $b \in k$. The coefficient of $v_1 \wedge v_3$ in the left hand side of the relation:

$$-v_2 \wedge w_0 - v_3 \wedge w_1 + v_0 \wedge w_2 + v_1 \wedge w_3 = 0$$

is $-2a$ hence $a = 0$ and this contradicts the fact that $w_0, \ldots, w_3$ are linearly independent. Consequently, this case cannot occur.

Case 3. *W is as in Lemma* 3.4(iii).

In this case, μ is defined by the matrix:

$$\mathcal{M} := \begin{pmatrix} -v_1 & v_0 & 0 & 0 \\ 0 & -v_2 & v_1 & 0 \\ 0 & 0 & -v_3 & v_2 \\ -v_2 & 0 & v_0 & 0 \end{pmatrix}.$$

Let h be a non-zero element of $V^\vee$ vanishing in v_0, v_1, v_2 and let $H \subset \mathbb{P}^3$ be the plane of equation $h = 0$. The matrix of the multiplication by $h\colon \mathrm{H}^1(E^\vee(-1)) \to \mathrm{H}^1(E^\vee)$ is obtained by applying h to the entries of $\mathcal{M}$. This matrix has rank 1 hence $\mathrm{h}^0(E_H^\vee) = 3$. But this contradicts Lemma 1.1. Consequently, this case cannot occur. $\square$

4. THE CASE $c_3 = 4$

Lemma 4.1. *Let E be a stable rank 3 vector bundle on $\mathbb{P}^3$ with $c_1 = 0$, $c_2 = 3$, $c_3 = 4$. Then E is the cohomology sheaf of a Horrocks monad of the form:*

$$0 \longrightarrow 2\mathscr{O}_{\mathbb{P}^3}(-2) \xrightarrow{\alpha} 3\mathscr{O}_{\mathbb{P}^3} \oplus 3\mathscr{O}_{\mathbb{P}^3}(-1) \xrightarrow{\beta} \mathscr{O}_{\mathbb{P}^3}(1) \longrightarrow 0.$$

Proof. The spectrum of E must be $(-1,-1,0)$ hence $\mathrm{H}^1(E(l)) = 0$ for $l \leq -2$, $\mathrm{h}^1(E(-1)) = 1$, $\mathrm{H}^2(E(l)) = 0$ for $l \geq -1$. Moreover, by Riemann-Roch, $\mathrm{h}^1(E) = 1$. Since $\mathrm{H}^2(E(-1)) = 0$ and $\mathrm{H}^3(E(-2)) = 0$, it follows that $\mathrm{H}^1_*(E)$ is generated in degrees ≤ 0 (by the Castelnuovo–Mumford Lemma, as formulated in [1, Lemma 1.21]). But, by the last part of Remark 1.4, one has $S_1\mathrm{H}^1(E(-1)) \neq (0)$ hence $S_1\mathrm{H}^1(E(-1)) = \mathrm{H}^1(E)$ hence $\mathrm{H}^1_*(E)$ is generated by $\mathrm{H}^1(E(-1))$.

On the other hand, the spectrum of $E^\vee$ is $(0,1,1)$ hence $\mathrm{H}^1(E^\vee(l)) = 0$ for $l \leq -3$, $\mathrm{h}^1(E^\vee(-2)) = 2$, $\mathrm{h}^1(E^\vee(-1)) = 5$, $\mathrm{H}^2(E^\vee(l)) = 0$ for $l \geq -2$. Moreover, by Riemann-Roch, $\mathrm{h}^1(E^\vee) = 5$. Since $\mathrm{H}^2(E^\vee(-2)) = 0$ and $\mathrm{H}^3(E^\vee(-3)) = 0$, $\mathrm{H}^1_*(E^\vee)$ is generated in degrees ≤ -1.

Now, assume that $\mathrm{H}^1_*(E^\vee)$ has m minimal generators of degree -1, for some $m \geq 0$. Then E is the cohomology sheaf of a monad of the form:

$$0 \longrightarrow m\mathscr{O}_{\mathbb{P}^3}(-1) \oplus 2\mathscr{O}_{\mathbb{P}^3}(-2) \longrightarrow B \longrightarrow \mathscr{O}_{\mathbb{P}^3}(1) \longrightarrow 0,$$

where B is a direct sum of line bundles. B has rank $m+6$, $c_1(B) = -m-3$, $\mathrm{H}^0(B(-1)) = 0$ and $\mathrm{H}^0(B^\vee(-2)) = 0$. It follows that $B \simeq 3\mathscr{O}_{\mathbb{P}^3} \oplus (m+3)\mathscr{O}_{\mathbb{P}^3}(-1)$. The component $m\mathscr{O}_{\mathbb{P}^3}(-1) \to (m+3)\mathscr{O}_{\mathbb{P}^3}(-1)$ of the left differential of the monad is zero, by the minimality of m. Since there is no locally split monomorphism $\mathscr{O}_{\mathbb{P}^3}(-1) \to 3\mathscr{O}_{\mathbb{P}^3}$ it follows that $m = 0$. □

LEMMA 4.2. *Let E be a stable rank 3 vector bundle on $\mathbb{P}^3$ with $c_1 = 0$, $c_2 = 3$, $c_3 = 4$. Then there is a point $x \in \mathbb{P}^3$ such that, for every plane $H \subset \mathbb{P}^3$, $\mathrm{H}^0(E_H) = 0$ if $x \notin H$ and $\mathrm{h}^0(E_H) = 1$ if $x \in H$.*

Proof. The component $\beta_1 \colon 3\mathscr{O}_{\mathbb{P}^3} \to \mathscr{O}_{\mathbb{P}^3}(1)$ of the differential β of the monad of E from Lemma 4.1 is defined by three linearly independent linear forms (because $\mathrm{H}^0(E) = 0$). x is the point where these three forms vanish simultaneously. □

LEMMA 4.3. *Let E be a stable rank 3 vector bundle on $\mathbb{P}^3$ with $c_1 = 0$, $c_2 = 3$, $c_3 = 4$ and let α_2 be the component $2\mathscr{O}_{\mathbb{P}^3}(-2) \to 3\mathscr{O}_{\mathbb{P}^3}(-1)$ of the differential α of the monad of E from Lemma 4.1. Then, up to automorphisms of $\mathbb{P}^3$, $3\mathscr{O}_{\mathbb{P}^3}$ and $2\mathscr{O}_{\mathbb{P}^3}(1)$, $\alpha_2^\vee(-1)$ is defined by one of the following matrices:*

$$(1)\begin{pmatrix} X_0 & X_1 & X_2 \\ 0 & X_0 & X_1 \end{pmatrix};\ (2)\begin{pmatrix} X_0 & X_1 & 0 \\ 0 & X_0 & X_2 \end{pmatrix};\ (3)\begin{pmatrix} X_0 & 0 & X_2 \\ 0 & X_1 & X_2 \end{pmatrix};$$
$$(4)\begin{pmatrix} X_0 & X_1 & X_2 \\ 0 & X_0 & X_3 \end{pmatrix};\ (5)\begin{pmatrix} X_0 & 0 & X_2 \\ 0 & X_1 & X_3 \end{pmatrix};\ (6)\begin{pmatrix} X_0 & X_1 & X_2 \\ 0 & X_2 & X_3 \end{pmatrix};$$
$$(7)\begin{pmatrix} X_0 & X_1 & X_2 \\ X_1 & X_2 & X_3 \end{pmatrix}.$$

Proof. The argument is standard. Let us denote $\alpha_2^\vee(-1) \colon 3\mathscr{O}_{\mathbb{P}^3} \to 2\mathscr{O}_{\mathbb{P}^3}(1)$ by ϕ. The morphism ϕ is uniquely determined by

$$\mathrm{H}^0(\phi) \colon \mathrm{H}^0(3\mathscr{O}_{\mathbb{P}^3}) \to \mathrm{H}^0(2\mathscr{O}_{\mathbb{P}^3}(1))$$

which can be viewed as a linear map $\rho \colon k^3 \to (k^2)^\vee \otimes V^\vee$.

Let $\psi \colon 3\mathscr{O}_{\mathbb{P}^1} \to \mathscr{O}_{\mathbb{P}^1}(1) \otimes V^\vee$ be the unique morphism for which $\mathrm{H}^0(\psi) = \rho$. Let u_0, u_1 be the canonical basis of k^2 and T_0, T_1 the dual basis of $(k^2)^\vee$. ϕ is defined by a 2×3 matrix Φ with entries in $V^\vee$ and ψ is defined by a 4×3 matrix Ψ with entries in $(k^2)^\vee$. Since both of these matrices are derived from ρ, they are related as follows: for $i = 0, 1$, the i th row of Φ is $(X_0, \ldots, X_3)\Psi(u_i)$, where $\Psi(u_i)$ is the 4×3 matrix with entries in k obtained by evaluating the entries of Ψ at u_i. Notice that $\Psi(u_i)$ defines the reduced stalk of the morphism ψ at the point $[u_i]$ of $\mathbb{P}^1$.

Claim. *$\psi\colon 3\mathscr{O}_{\mathbb{P}^1} \to \mathscr{O}_{\mathbb{P}^1}(1) \otimes V^\vee$ has rank ≥ 2 at every point of $\mathbb{P}^1$.*

Indeed, the fact that $\mathrm{H}^0(E^\vee) = 0$ implies that $\mathrm{H}^0(\alpha^\vee)$ is injective. In particular, $\mathrm{H}^0(\phi(1))$ is injective. Assume, by contradiction, that there is a point of $\mathbb{P}^1$ where ψ has rank ≤ 1. Up to an automorphism of $\mathbb{P}^1$, one can assume that this point is $[0 : 1]$. This means that, up to an automorphism of $2\mathscr{O}_{\mathbb{P}^3}(1)$, ϕ is represented by a matrix of the form:

$$\begin{pmatrix} h_{00} & h_{01} & h_{02} \\ h_{10} & h_{11} & h_{12} \end{pmatrix},$$

with $\dim_k(kh_{10} + kh_{11} + kh_{12}) \leq 1$. Up to an automorphism of $3\mathscr{O}_{\mathbb{P}^3}$ one can assume that $h_{11} = h_{12} = 0$ and this contradicts the fact that $\mathrm{H}^0(\phi(1))$ is injective.

Consider, now, the morphism $\psi^\vee\colon \mathscr{O}_{\mathbb{P}^1}(-1) \otimes V \to 3\mathscr{O}_{\mathbb{P}^1}$. Since $\mathbb{P}^1$ has dimension 1 and $\mathrm{H}^1(\mathscr{O}_{\mathbb{P}^1}(-1)) = 0$ it follows that the map $\mathrm{H}^0(3\mathscr{O}_{\mathbb{P}^1}) \to \mathrm{H}^0(\operatorname{Coker}\psi^\vee)$ is surjective hence $\mathrm{h}^0(\operatorname{Coker}\psi^\vee) \leq 3$. One deduces that if $\psi^\vee$ has, generically, rank 3 then $\operatorname{Coker}\psi^\vee$ is a torsion sheaf of length ≤ 3 generated, locally, by one element, and if it has rank 2 everywhere then $\operatorname{Coker}\psi^\vee$ is a line bundle, which must be $\mathscr{O}_{\mathbb{P}^1}(2)$ or $\mathscr{O}_{\mathbb{P}^1}(1)$ (it cannot be $\mathscr{O}_{\mathbb{P}^1}$ because $\mathrm{H}^0(\phi)$ is injective hence so is $\mathrm{H}^0(\psi)$).

Consequently, up to an automorphism of $\mathbb{P}^1$, one can assume that $\operatorname{Coker}\psi^\vee$ is one of the following sheaves:

(i) $\mathscr{O}_{\mathbb{P}^1,P_0}/\mathfrak{m}_{P_0}^3$; (ii) $\mathscr{O}_{\mathbb{P}^1,P_0}/\mathfrak{m}_{P_0}^2 \oplus \mathscr{O}_{\mathbb{P}^1,P_1}/\mathfrak{m}_{P_1}$; (iii) $\bigoplus_{i=0}^2 \mathscr{O}_{\mathbb{P}^1,P_i}/\mathfrak{m}_{P_i}$;
(iv) $\mathscr{O}_{\mathbb{P}^1,P_0}/\mathfrak{m}_{P_0}^2$; (v) $\mathscr{O}_{\mathbb{P}^1,P_0}/\mathfrak{m}_{P_0} \oplus \mathscr{O}_{\mathbb{P}^1,P_1}/\mathfrak{m}_{P_1}$; (vi) $\mathscr{O}_{\mathbb{P}^1,P_0}/\mathfrak{m}_{P_0}$;
(vii) 0; (viii) $\mathscr{O}_{\mathbb{P}^1}(2)$; (ix) $\mathscr{O}_{\mathbb{P}^1}(1)$,

where $P_0 = [0 : 1]$, $P_1 = [1 : 0]$ and $P_2 = [1 : -1]$.

In case (i), choosing the k-basis of $\mathscr{O}_{\mathbb{P}^1,P_0}/\mathfrak{m}_{P_0}^3$ consisting of the classes of the regular functions 1, $-T_0/T_1$, T_0^2/T_1^2, $\psi^\vee$ is defined, up to automorphisms of $3\mathscr{O}_{\mathbb{P}^1}$ and V, by the following matrix:

$$\begin{pmatrix} T_0 & 0 & 0 & 0 \\ T_1 & T_0 & 0 & 0 \\ 0 & T_1 & T_0 & 0 \end{pmatrix}$$

hence the matrix Ψ defining ψ is the dual of this matrix. One deduces that the matrix Φ defining ϕ is as in item (1) from the statement.

Analogously, in the cases (ii)–(vii), Φ is as in the items (2)–(7) from the statement, respectively. We show, now, that the cases (viii) and (ix) cannot occur in our context.

In case (viii), choosing the k-basis T_1^2, $-T_0T_1$, T_0^2 of $\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^1}(2))$, $\psi^\vee$ is defined by the matrix:

$$\begin{pmatrix} T_0 & 0 & 0 & 0 \\ T_1 & T_0 & 0 & 0 \\ 0 & T_1 & 0 & 0 \end{pmatrix} \text{ hence } \Phi = \begin{pmatrix} X_0 & X_1 & 0 \\ 0 & X_0 & X_1 \end{pmatrix}.$$

Consider the line $L \subset \mathbb{P}^3$ of equations $X_0 = X_1 = 0$ and restrict to L the dual of the monad from Lemma 4.1:

$$0 \longrightarrow \mathscr{O}_L(-1) \xrightarrow{\beta_L^\vee} 3\mathscr{O}_L \oplus 3\mathscr{O}_L(1) \xrightarrow{\alpha_L^\vee} 2\mathscr{O}_L(2) \longrightarrow 0\,.$$

Let $\alpha_1 \colon 2\mathscr{O}_{\mathbb{P}^3}(-2) \to 3\mathscr{O}_{\mathbb{P}^3}$ be the other component of α and let $\beta_1 \colon 3\mathscr{O}_{\mathbb{P}^3} \to \mathscr{O}_{\mathbb{P}^3}(1)$ and $\beta_2 \colon 3\mathscr{O}_{\mathbb{P}^3}(-1) \to \mathscr{O}_{\mathbb{P}^3}(1)$ be the components of β. Since $\alpha_2^\vee \,|\, L = 0$, $(\alpha_1^\vee \,|\, L) \colon 3\mathscr{O}_L \to 2\mathscr{O}_L(2)$ is an epimorphism hence its kernel is isomorphic to $\mathscr{O}_L(-4)$. Moreover, $(\alpha_2^\vee \,|\, L) \circ (\beta_2^\vee \,|\, L) = 0$ hence $(\alpha_1^\vee \,|\, L) \circ (\beta_1^\vee \,|\, L) = 0$. It follows that $(\beta_1^\vee \,|\, L) = 0$ and this contradicts the fact that β_1 is defined by three linearly independent linear forms (because $\mathrm{H}^0(E) = 0$).

Finally, in case (ix), choosing the k-basis of $\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^1}(1))$ consisting of T_1, $-T_0$, $\psi^\vee$ is defined by the matrix:

$$\begin{pmatrix} T_0 & 0 & 0 & 0 \\ T_1 & 0 & 0 & 0 \\ 0 & T_0 & T_1 & 0 \end{pmatrix} \text{ hence } \Phi = \begin{pmatrix} X_0 & 0 & X_1 \\ 0 & X_0 & X_2 \end{pmatrix}.$$

But $(X_1\,,\, X_2\,,\, -X_0)^{\mathrm{t}}$ belongs to the kernel of the map $3S_1 \to 2S_2$ defined by Φ and this contradicts the fact that $\mathrm{H}^0(\phi(1))$ is injective. $\square$

Proposition 4.4. *Let E be a stable rank 3 vector bundle on $\mathbb{P}^3$ with $c_1 = 0$, $c_2 = 3$, $c_3 = 4$. Then, for the general plane $H \subset \mathbb{P}^3$, one has $\mathrm{H}^0(E_H^\vee) = 0$.*

Proof. We intend to apply Lemma 1.5. Consider the monad of E from Lemma 4.1 and let $\alpha_1 \colon 2\mathscr{O}_{\mathbb{P}^3}(-2) \to 3\mathscr{O}_{\mathbb{P}^3}$ and $\alpha_2 \colon 2\mathscr{O}_{\mathbb{P}^3}(-2) \to 3\mathscr{O}_{\mathbb{P}^3}(-1)$ be the components of α. It follows, from Lemma 4.3, that the degeneracy scheme of α_2 is a locally Cohen-Macaulay subscheme $Y \subset \mathbb{P}^3$ of pure dimension 1, which is locally complete intersection except at finitely many points and has degree 3. More precisely, denoting by L_{ij} the line of equations $X_i = X_j = 0$, $0 \leq i < j \leq 3$, one can assume that one of the following holds:

(1) Y is the Weil divisor $3L_{01}$ on the cone Σ of equation $X_1^2 - X_0X_2 = 0$;

(2) $Y = X \cup L_{01}$, where X is the divisor $2L_{02}$ on the plane $H_0 \colon X_0 = 0$;

(3) $Y = L_{01} \cup L_{02} \cup L_{12}$;

(4) $Y = X \cup L_{01}$, where X is the divisor $2L_{03}$ on the surface

$$\Sigma\colon X_1X_3 - X_0X_2 = 0\,;$$

(5) $Y = L_{01} \cup L_{02} \cup L_{13}\,;$

(6) $Y = C \cup L_{23}$, where C is the conic of equations

$$X_0 = X_1X_3 - X_2^2 = 0\,;$$

(7) Y is a twisted cubic curve.

One deduces, using the Eagon-Northcott complex, an exact sequence:

$$0 \longrightarrow 2\mathscr{O}_{\mathbb{P}^3}(-2) \xrightarrow{\alpha_2} 3\mathscr{O}_{\mathbb{P}^3}(-1) \longrightarrow \mathscr{I}_Y(1) \longrightarrow 0\,,$$

which, by dualization, produces an exact sequence:

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^3}(-1) \longrightarrow 3\mathscr{O}_{\mathbb{P}^3}(1) \xrightarrow{\alpha_2^\vee} 2\mathscr{O}_{\mathbb{P}^3}(2) \xrightarrow{\pi} \omega_Y(3) \longrightarrow 0\,.$$

The image of $\mathrm{H}^0(\pi\circ\alpha_1^\vee)\colon \mathrm{H}^0(3\mathscr{O}_{\mathbb{P}^3}) \to \mathrm{H}^0(\omega_Y(3))$ is a subspace W of $\mathrm{H}^0(\omega_Y(3))$, which has dimension 3 (because $\mathrm{H}^0(E^\vee) = 0$) and generates $\omega_Y(3)$ globally. Denoting by Q the cokernel of α, one has exact sequences:

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^3}(-1) \xrightarrow{\overline{\beta}^\vee} Q^\vee \longrightarrow E^\vee \longrightarrow 0\,,$$
$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^3}(-1) \longrightarrow Q^\vee \longrightarrow W \otimes_k \mathscr{O}_{\mathbb{P}^3} \xrightarrow{\varepsilon} \omega_Y(3) \longrightarrow 0\,,$$

where ε is the evaluation morphism. Let us denote $\omega_Y(3)$ by $\mathscr{L}$. Consider a line L that is not a component of Y and passes through none of the points where Y is not locally complete intersection. One has an exact sequence:

$$0 \longrightarrow \mathscr{O}_L(a-1) \longrightarrow Q_L^\vee \longrightarrow W \otimes_k \mathscr{O}_L \xrightarrow{\varepsilon_L} \mathscr{L}_L \longrightarrow 0\,,$$

where $a = \text{length}\,(L \cap Y)$ (because $\mathscr{L}_L \simeq \mathscr{O}_{L\cap Y}$). Since $\mathrm{H}^1(Q_L^\vee) \xrightarrow{\sim} \mathrm{H}^1(E_L^\vee)$ it follows that $\mathrm{h}^1(E_L^\vee) \geq 1$ if and only if $\mathrm{H}^0(\varepsilon_L)\colon W \to \mathrm{H}^0(\mathscr{L}_L)$ is not surjective. In that case one must have $a \geq 2$, i.e., L must be a *secant* of Y. We split, now, the proof into several cases according to the various possibilities for Y listed above.

Case 1. *Y as in item* (1) *above.*

If L is a secant of Y not passing through the vertex $P_3 := [0:0:0:1]$ of the cone Σ then L is tangent to Σ at a point of $L_{01} \setminus \{P_3\}$ hence L is contained in the plane H_0 of equation $X_0 = 0$. One applies, now, Lemma 1.5.

The cases 2, 3 and 5 where Y is as in one of the items (2), (3), or (5) above can be treated similarly.

Case 4. *Y as in item* (4) *above.*

A secant L of Y not passing through the point $P_2 := [0:0:1:0]$ where L_{03} and L_{01} intersect is either contained in the plane H_0 spanned by L_{03} and L_{01} or is tangent to the nonsingular quadric surface Σ at a point of $L_{03} \setminus \{P_2\}$. Taking into account Lemma 1.5, it suffices to show that if L is a general tangent to Σ at a point of $L_{03} \setminus \{P_2\}$ then the map $\mathrm{H}^0(\varepsilon_L)$ is surjective.

Our argument uses two observations. Firstly, let s_0, s_1, s_2 be a k-basis of W. The morphism $(0,1)\colon 2\mathscr{O}_{\mathbb{P}^3} \to \mathscr{O}_{L_{03}}$ induces an epimorphism $\omega_Y(1) \to \mathscr{O}_{L_{03}}$. Let f_i be the image of s_i into $\mathrm{H}^0(\mathscr{O}_{L_{03}}(2))$. Since f_0, f_1, f_2 generate $\mathscr{O}_{L_{03}}(2)$ globally, one can assume that f_0 and f_1 are linearly independent. Interpreting the restrictions of f_0 and f_1 to $L_{03} \setminus \{P_2\}$ as functions of one variable, one has, for a general point x of $L_{03} \setminus \{P_2\}$, $f_0'(x)f_1(x) - f_0(x)f_1'(x) \neq 0$. This means that the images of f_0 and f_1 in $\mathscr{O}_{L_{03}\,,x}/\mathfrak{m}^2_{L_{03}\,,x}$ are linearly independent.

Secondly, let Γ be the "fat point" on Σ at x defined by the ideal sheaf $\mathscr{I}_\Gamma := \mathscr{I}^2_{\{x\}} + \mathscr{I}_\Sigma$. $\mathscr{I}_{\{x\}}/\mathscr{I}_\Gamma$ is a 2-dimensional vector space. If L is a line tangent to Σ at x then $(\mathscr{I}_L + \mathscr{I}_\Gamma)/\mathscr{I}_\Gamma$ is a 1-dimensional subspace of $\mathscr{I}_{\{x\}}/\mathscr{I}_\Gamma$ and in this way one gets all the 1-dimensional subspaces of $\mathscr{I}_{\{x\}}/\mathscr{I}_\Gamma$. If $L \neq L_{03}$ then $\mathscr{I}_L + \mathscr{I}_\Gamma$ is the ideal sheaf of the scheme $L \cap X$, while $\mathscr{I}_{L_{03}} + \mathscr{I}_\Gamma$ defines the divisor $2x$ on L_{03}. Let us denote the scheme associated to this divisor by D.

Now, according to the first observation, if x is a general point of $L_{03} \setminus \{P_2\}$ then one can assume that the images of s_0 and s_1 in $\mathscr{L}_D$ are linearly independent. It follows, from the second observation, that if L is a general tangent line to Σ at x then the images of s_0 and s_1 in $\mathscr{L}_{L\cap X}$ are linearly independent. This implies that the map $\mathrm{H}^0(\varepsilon_L)$ is surjective.

Case 6. *Y as in item* (6) *above.*

If L is a secant of Y then either L passes through the point $P_1 := [0:1:0:0]$ where C and L_{23} intersect, or it is contained in the plane H_0 of C, or joins a point x of $C \setminus \{P_1\}$ and a point y of $L_{23} \setminus \{P_1\}$. ω_Y is an invertible $\mathscr{O}_Y$-module. Since $\omega_Y(1)$ is globally generated, $\mathscr{L} = \omega_Y(3)$ is very ample. Since W generates $\mathscr{L}$ globally, the restriction maps $W \to \mathrm{H}^0(\mathscr{L} \mid C)$ and $W \to \mathrm{H}^0(\mathscr{L} \mid L_{23})$ must have rank at least 2.

Let x be a point of $C \setminus \{P_1\}$. The space $W' := \{\sigma \in W \mid \sigma(x) = 0\}$ has dimension 2. By what has been said above, the restriction map $W' \to \mathrm{H}^0(\mathscr{L} \mid L_{23})$ is non-zero hence, for a general point $y \in L_{23} \setminus \{P_1\}$, the space $W'' := \{\sigma \in W' \mid \sigma(y) = 0\}$ has dimension 1. If L is the line joining x and y then the map $\mathrm{H}^0(\varepsilon_L)$ is surjective.

Finally, the case 7 where Y is as in item (7) above is quite easy. $\square$

5. THE CASE $c_3 = 6$

Lemma 5.1. *Let E be a stable rank 3 vector bundle on $\mathbb{P}^3$ with $c_1 = 0$, $c_2 = 3$, $c_3 = 6$ and spectrum $(-1,-1,-1)$. Then one has an exact sequence:*

$$0 \longrightarrow 3\mathscr{O}_{\mathbb{P}^3}(-2) \xrightarrow{\alpha} 6\mathscr{O}_{\mathbb{P}^3}(-1) \longrightarrow E \longrightarrow 0\,.$$

Proof. The result is due to Spindler [16]. We include, for completeness, a short argument.

One has $\mathrm{H}^2(E(l)) = 0$ for $l \geq -1$ (by the spectrum) and $\mathrm{H}^3(E(l)) = 0$ for $l \geq -4$ (by Serre duality). Moreover, from Riemann-Roch, $\mathrm{h}^1(E) = 0$. It follows that E is 1-regular. Using the spectrum one deduces that $\mathrm{H}^1_*(E) = 0$.

Since, also by Riemann-Roch, $\mathrm{h}^0(E(1)) = 6$, one has an epimorphism $6\mathscr{O}_{\mathbb{P}^3} \to E(1)$. The kernel K of this epimorphism has $\mathrm{H}^i_*(K) = 0$, $i = 1, 2$, hence it is a direct sum of line bundles. Since K has rank 3, $c_1(K) = -3$ and $\mathrm{H}^0(K) = 0$ it follows that $K \simeq 3\mathscr{O}_{\mathbb{P}^3}(-1)$. $\square$

Corollary 5.2. *Under the hypothesis of Lemma 5.1, $\mathrm{H}^0(E_H) = 0$, for every plane $H \subset \mathbb{P}^3$.*

Proposition 5.3. *Let E be a stable rank 3 vector bundle on $\mathbb{P}^3$ with $c_1 = 0$, $c_2 = 3$, $c_3 = 6$ and spectrum $(-1,-1,-1)$. If $\mathrm{H}^0(E_H^\vee) \neq 0$, for every plane $H \subset \mathbb{P}^3$, then E is as in Theorem 0.2(b)(i).*

Proof. We will show that E has infinitely many unstable planes. Then the main result of Vallès [18, Thm. 3.1] (see, also, the proof of [18, Prop. 2.2]) will imply the conclusion of the proposition.

Assume, by contradiction, that E has only finitely many unstable planes. Let $\Pi \subset \mathbb{P}^{3\vee}$ be a plane containing none of the points of $\mathbb{P}^{3\vee}$ corresponding to the unstable planes of E. Let $H \subset \mathbb{P}^3$ be a plane of equation $h = 0$ such that $[h] \in \Pi$. One has $\mathrm{H}^0(E_H^\vee(-1)) = 0$ and $\mathrm{H}^0(E_H) = 0$ by Cor. 5.2. Applying Lemma 1.1 to $F := E_H^\vee$ (on $H \simeq \mathbb{P}^2$) one gets that $\mathrm{h}^0(E_H^\vee) \leq 1$ hence $\mathrm{h}^0(E_H^\vee) = 1$, due to our hypothesis. One deduces that the kernel $\mathscr{M}$ and the cokernel $\mathscr{L}$ of the restriction to Π of the morphism

$$\mu\colon \mathrm{H}^1(E^\vee(-1)) \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}(-1) \to \mathrm{H}^1(E^\vee) \otimes \mathscr{O}_{\mathbb{P}^{3\vee}}$$

from Remark 1.9 are line bundles on Π ($\mathrm{H}^1(E^\vee(-1))$ and $\mathrm{H}^1(E^\vee)$ have both dimension 6) hence $\mathscr{L} \simeq \mathscr{O}_\Pi(a)$ and $\mathscr{M} \simeq \mathscr{O}_\Pi(b)$, for some integers a, b. One thus has an exact sequence:

$$0 \to \mathscr{O}_\Pi(b) \longrightarrow \mathrm{H}^1(E^\vee(-1)) \otimes \mathscr{O}_\Pi(-1) \xrightarrow{\mu|\Pi} \mathrm{H}^1(E^\vee) \otimes \mathscr{O}_\Pi \longrightarrow \mathscr{O}_\Pi(a) \to 0\,.$$

It follows that $a = b+6$ and $\chi(\mathscr{O}_\Pi(a-1)) = \chi(\mathscr{O}_\Pi(b-1))$, i.e., $a(a+1) = b(b+1)$. Since the equation $(b+6)(b+7) = b(b+1)$ has no integer solution, we have got a contradiction. $\square$

We recall, finally, that, by the proof of the Proposition on page 72 of [7], if E is a stable rank 3 vector bundle on $\mathbb{P}^3$ with $c_1 = 0$, $c_2 = 3$, $c_3 = 6$ and spectrum $(-2, -1, 0)$ then $\mathrm{H}^0(E_H^\vee) = 0$ for a general plane $H \subset \mathbb{P}^3$.

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