

*Dedicated to the memory of Mihnea Colţoiu*

# EMBEDDING BORDERED RIEMANN SURFACES IN STRONGLY PSEUDOCONVEX DOMAINS

FRANC FORSTNERIČ

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We show that every bordered Riemann surface,  $M$ , with smooth boundary  $bM$  admits a proper holomorphic map  $M \rightarrow \Omega$  into any bounded strongly pseudoconvex domain  $\Omega$  in  $\mathbb{C}^n$ ,  $n > 1$ , extending to a smooth map  $f : \overline{M} \rightarrow \overline{\Omega}$  which can be chosen an immersion if  $n \geq 3$  and an embedding if  $n \geq 4$ . Furthermore,  $f$  can be chosen to approximate a given holomorphic map  $\overline{M} \rightarrow \Omega$  on compacts in  $M$  and interpolate it at finitely many given points in  $M$ .

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## 1. THE MAIN RESULT

A bordered Riemann surface is an open relatively compact domain,  $M$ , in another Riemann surface whose nonempty boundary  $bM$  is the union of finitely many Jordan curves.

According to Stout [39, Theorem 8.1], such a surface is conformally equivalent to a domain of the form

$$R \setminus \bigcup_{i=1}^l \Delta_i$$

where  $R$  is a compact Riemann surface and  $\Delta_1, \dots, \Delta_l$  are pairwise disjoint closed discs with real analytic boundaries in  $R$ .

A perennial problem in complex geometry is to understand the space of proper complex curves in a given complex manifold. We contribute to this topic with the following result.

**THEOREM 1.1.** *Let  $\Omega$  be a bounded strongly pseudoconvex domain with boundary of class  $\mathcal{C}^{k+1}$  for some integer  $k \geq 3$  in a complex Euclidean space  $\mathbb{C}^n$  for  $n > 1$ , and let  $M$  be a bordered Riemann surface with real analytic boundary. Given a continuous map  $f_0 : \overline{M} \rightarrow \Omega$  which is holomorphic in  $M$ , a compact set  $K$  in  $M$ , and a number  $\epsilon > 0$  there exists a proper holomorphic map  $f : M \rightarrow \Omega$ , extending to a map  $f : \overline{M} \rightarrow \overline{\Omega}$  of Hölder class  $\mathcal{C}^{k,\alpha}(\overline{M})$  for any  $\alpha \in (0,1)$  (and holomorphic on a neighbourhood of  $\overline{M}$  if  $b\Omega$  is real analytic) such that  $\sup_K |f - f_0| < \epsilon$  and  $f$  agrees with  $f_0$  to a given order at finitely many given points in  $M$ . If  $n \geq 3$  then  $f$  can be chosen an immersion on  $\overline{M}$ , and if  $n \geq 4$  then  $f$  can be chosen an embedding on  $\overline{M}$  provided that the interpolation conditions allow it.*

Recall that a bounded domain  $\Omega$  in  $\mathbb{C}^n$  is *strongly pseudoconvex* if it is of the form  $\Omega = \{\rho < 0\}$ , where  $\rho$  is a strongly plurisubharmonic function on a neighbourhood  $U \subset \mathbb{C}^n$  of  $\overline{\Omega}$ , i.e.,  $\rho$  has positive definite Levi form:

$$(1.1) \quad \mathcal{L}_\rho(z, w) = \sum_{i,j=1}^n \frac{\partial^2 \rho(z)}{\partial z_i \partial \bar{z}_j} w_i \bar{w}_j > 0 \quad \text{for all } z \in U \text{ and } w \in \mathbb{C}^n \setminus \{0\},$$

and it satisfies the condition  $d\rho \neq 0$  on the boundary  $b\Omega = \{\rho = 0\}$ .

The main new point is that Theorem 1.1 holds for bordered Riemann surfaces of any given conformal type. If one only wishes to control the topological type of  $M$ , then the conclusion follows easily from the known results, and in this case we can find proper holomorphic immersions if  $n = 2$  and proper holomorphic embeddings if  $n \geq 3$ .

As pointed in Remark 2.2, Theorem 1.1 holds under a weaker geometric assumption on  $\Omega$ . It likely holds for any relatively compact, strongly pseudoconvex domain  $\Omega$  in a Stein manifold  $X$  of appropriate dimension; we leave this generalization to an interested reader.

The Hopf–Oleĭnik lemma [34, 36], applied to the negative subharmonic function  $\rho \circ f$  on  $M$  which vanishes on  $bM$ , shows that a map  $f$  as in the theorem is an immersion along the boundary  $bM$  which intersects  $b\Omega$  transversely. Conversely, if  $C$  is a smooth curve in  $b\Omega$  lying in the boundary of a complex analytic curve  $A \subset \Omega$  such that  $\overline{A} \cap b\Omega \subset C$  then the pair  $(A, C)$  is a local smooth manifold with boundary at every point  $p \in C$ ; see [7] and [24, Theorem 1.1].

*Remark 1.2.* A generic holomorphic map  $M \rightarrow \mathbb{C}^n$  from an open Riemann surface is an immersion if  $n \geq 2$ , and is an injective immersion if  $n \geq 3$ . Hence, the conclusion of the theorem is not optimal in this respect. This loss of the optimal immersion and embedding dimension by one may be due to the method of proof. On the other hand, it is not clear whether the general position

argument gives the optimal conclusion within the space of holomorphic maps  $M \rightarrow \Omega \subset \mathbb{C}^n$  which extend smoothly to  $\overline{M}$  and map  $bM$  to  $b\Omega$ . We do not investigate this question here.

There are many proper holomorphic discs in the ball  $\mathbb{B}^n$  of  $\mathbb{C}^n$  given by rational and also by polynomial maps; see [11, 12]. We ask the following question.

*Problem 1.3.* Let  $M$  be a bordered Riemann surface with real analytic or real algebraic boundary in a compact Riemann surface  $R$ . Is there a proper holomorphic embedding  $M \hookrightarrow \mathbb{B}^n$  for some  $n > 1$  given by a meromorphic map of  $R$  to  $\mathbb{C}^n$  without poles in  $\overline{M}$ ?

Let us place our theorem in the context of known results.

A partial result for the disc  $\Delta = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ , with interpolation at one point, was obtained by Forstnerič and Globevnik [28] in 1992. The proof in this case is much simpler since the disc carries a unique conformal structure. The main contribution of [28] was the technique of pushing the boundary of an analytic disc in a strongly pseudoconvex domain  $\Omega$  into the boundary  $b\Omega$  by using the Riemann-Hilbert boundary value problem.

Without asking for approximation, interpolation or general position properties of the map  $f$  in Theorem 1.1, the result follows from the disc case. Indeed, by Ahlfors [1] every bordered Riemann surface,  $M$ , admits many inner functions, i.e., nonconstant continuous functions on  $\overline{M}$  which are holomorphic on  $M$  and have modulus one on the boundary  $bM$ . Any such function gives a proper holomorphic map  $M \rightarrow \Delta$ . The smallest degree of such a map is  $2g + m$  where  $g$  is the genus of  $M$  and  $m$  is the number of its boundary components. If the boundary  $bM$  is real analytic then, by the reflection principle, every inner function on  $M$  extends to a holomorphic function on a neighbourhood of  $\overline{M}$  in the ambient Riemann surface. Composing an inner function  $\overline{M} \rightarrow \overline{\Delta}$  with a proper holomorphic map  $\Delta \rightarrow \Omega$  extending smoothly to  $\overline{\Delta}$  gives the result.

For maps to the polydisc there is a result of Stout from 1966 [40, IV.1 Theorem] saying the following. Let  $T = b\Delta = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ . On every bordered Riemann surface,  $M$ , there exist three inner functions  $f_1, f_2, f_3$  such that the map  $f = (f_1, f_2, f_3) : \overline{M} \rightarrow \overline{\Delta^3}$  into the closed 3-polydisc separates the points in  $\overline{M}$ , it maps the boundary  $bM$  to the torus  $T^3$  (the distinguished boundary of the polydisc  $\Delta^3$ ), and  $f : M \hookrightarrow \Delta^3$  is a holomorphic embedding. Including  $\Delta^3$  in the ball centred at 0 of radius  $\sqrt{3}$ , the torus  $T^3$  sits in the sphere, so this also gives proper holomorphic embeddings of  $M$  into the 3-ball. However, approximation and interpolation of maps as in Theorem 1.1 are impossible by this method.

In 2004, it was shown by Černe [5, Corollary 1.4] that every finitely bordered planar domain with smooth boundary embeds properly holomorphically and smoothly up to the boundary in any smoothly bounded convex domain in  $\mathbb{C}^n$  for  $n \geq 2$ . The technique developed in his paper (see also the sequel [6] by Černe and Flores) is the basis for our construction in the present paper. It uses solutions to certain Riemann-Hilbert boundary value problems on  $M$ , similarly to what was done by the author in [23] and used in [28] in the special case when  $M$  is the disc  $\Delta$ .

More recently, Coupet, Sukhov, and Tumanov [10] constructed proper  $J$ -holomorphic discs in almost Stein manifolds  $(\Omega, J)$  of real dimension 4. Their approach follows the same geometric scheme of reducing to a Riemann-Hilbert problem, but the details are considerably more difficult in the case of a nonintegrable almost complex structure  $J$ .

For general bordered Riemann surfaces  $M$  and strongly pseudoconvex domains  $\Omega$  as in Theorem 1.1, the extant methods on the one hand provide proper holomorphic maps  $M \rightarrow \Omega$  extending to continuous maps  $\overline{M} \rightarrow \overline{\Omega}$  (see [2, Theorem 8.3.13], which also gives an asymptotic estimate of the distance between the initial holomorphic map  $f_0 : \overline{M} \rightarrow \Omega$  and the final map  $f : \overline{M} \rightarrow \overline{\Omega}$  with  $f(bM) \subset b\Omega$  in terms of the distance from  $f_0(bM) \subset \Omega$  to  $b\Omega$ ), and on the other hand give maps as in Theorem 1.1 but with a slightly changed conformal structure on the underlying surface  $M$ . More precisely, if  $M \subset M'$  are bordered Riemann surfaces such that  $M$  is relatively compact in  $M'$  and smoothly diffeotopic to  $M'$ , there are a bordered Riemann surface  $M_0$  diffeomorphic to  $M$ , with  $M \subset M_0 \subset M'$ , and a proper holomorphic map  $M_0 \rightarrow \Omega$  as in Theorem 1.1 extending to a smooth map  $\overline{M}_0 \rightarrow \overline{\Omega}$ . This can be done by using the main result of [15] as will become clear in the proof of Theorem 1.1, given in Section 2.

For domains  $\Omega \subset \mathbb{C}^n$  of dimension  $n > 2$ , Theorem 1.1 holds under weaker geometric (pseudoconvexity) assumptions on  $b\Omega$ ; see Remark 2.2. An indication that this is possible is seen from the paper [15] by Drinovec Drnovšek and Forstnerič. In this paper, the authors constructed proper holomorphic immersions of any bordered Riemann surface into an arbitrary complex manifold  $\Omega$  of dimension  $n \geq 2$  (embeddings if  $n \geq 3$ ) with a smooth exhaustion function  $\rho : \Omega \rightarrow \mathbb{R}_+$  whose Levi form has at least two positive eigenvalues at every point. By a different technique, using holomorphic peak functions, their construction was extended in [16] to proper holomorphic maps and embeddings of higher dimensional strongly pseudoconvex domains to  $q$ -convex manifolds for suitable values of  $q$ . (These are complex manifolds satisfying a condition on the minimum number of positive eigenvalues of the Levi form of an exhaustion function; see Andreotti and Grauert [3] and the survey by Colţoiu [9].) Earlier

results for maps to  $q$ -convex domains in  $\mathbb{C}^n$  are due to Dor [14]. The main difference from Theorem 1.1 is that, in the results mentioned above, there is no boundary condition to be taken care of. For recent surveys, see [26, Chapter 9] and [27].

Finally, we mention the most recent work by Drinovec Drnovšek and Forstnerič [18] who combined the peak function technique with Oka theory to construct proper holomorphic embeddings  $X \hookrightarrow \mathbb{C}^n$  from an arbitrary Stein manifold  $X$  with  $2 \dim X < n$  such that the image lies in a given concave domain  $\Omega \subset \mathbb{C}^n$  which is only slightly bigger than a halfspace, and we can approximate a given holomorphic embedding  $K \hookrightarrow \Omega$  on a compact holomorphically convex subset of  $X$ .

Theorem 1.1 has no analogue if  $M$  is a strongly pseudoconvex domain of higher dimension due to rigidity of CR maps. Indeed, by [22, Theorem 2.2] the set of germs of smooth strongly pseudoconvex real hypersurfaces in  $\mathbb{C}^n$  which admit a smooth (or even just a formal) Cauchy-Riemann (CR) embedding into a sphere of any dimension (the boundary of the ball in some  $\mathbb{C}^N$ ) is of the first category in a suitable Fréchet space topology. The same holds for all strongly pseudoconvex real algebraic hypersurfaces as targets; see [25]. (Recall that a CR map is one that satisfies the tangential Cauchy-Riemann equations. In the case of a strongly pseudoconvex source hypersurface, such a map extends locally holomorphically to the pseudoconvex side.) The question which (strongly) pseudoconvex hypersurfaces admit local CR embeddings into various model hypersurfaces has attracted considerable attention; see the surveys by Baouendi, Ebenfelt and Rothschild [4], Pinchuk, Shafikov, and Sukhov [37], and (for maps to balls) by D'Angelo [11]. On the other hand, there are proper holomorphic maps and embeddings of strongly pseudoconvex domains to balls and other model domains which extend continuously to the closure of the domain; see the papers by Løw [35], Globevnik [31], Hakim [32], and Dor [13], among others.

There are also results in the literature in which one deforms either the domain or the codomain. A famous example of the latter is the theorem, due to Fornæss [20] and Chirka and Henkin [33], which says that every relatively compact strongly pseudoconvex domain in a Stein manifold embeds properly holomorphically and smoothly up to the boundary in some strongly convex domain in  $\mathbb{C}^N$  for a sufficiently big  $N$ . Concerning the former, for every pair of Stein manifolds  $(X, J_X)$  and  $(Y, J_Y)$  with  $\dim Y > 2 \dim X$  there is a Stein structure  $J$  on  $X$ , homotopic to the original structure  $J_X$ , such that  $(X, J)$  admits a proper holomorphic embedding into  $(Y, J_Y)$  (see [27, Theorem 5.1]). This is based on the work of Eliashberg [19] and Forstnerič and Slapar [30, 29]; see also the monograph [8] by Cieliebak and Eliashberg.

## 2. PROOF OF THEOREM 1.1

Recall that  $\Delta = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$  denotes the open unit disc.

Let  $\Omega$  be a bounded domain in  $\mathbb{C}^n$  for  $n > 1$  with a smooth strongly plurisubharmonic defining function  $\rho$  on a neighbourhood of  $\bar{\Omega}$ . Let  $z = (z_1, z_2, \dots, z_n)$  denote the coordinates on  $\mathbb{C}^n$ . We shall need the following lemma.

**LEMMA 2.1.** *There is an open neighbourhood  $U$  of  $b\Omega$  and a smooth family of embedded holomorphic discs  $g_{z,v} : \Delta \rightarrow \mathbb{C}^n$  for  $z \in U \cap \Omega$  and  $v = (v_1, \dots, v_n)$  a unit vector in  $\mathbb{C}^n$  satisfying*

$$\partial\rho(z)v := \sum_{i=1}^n \frac{\partial\rho}{\partial z_i}(z) v_i = 0$$

such that  $g_{z,v}(0) = z$ ,  $g'_{z,v}(0) = cv$  for some  $c > 0$  independent of  $(z, v)$ , and  $g_{z,v}(\Delta)$  intersects  $b\Omega$  transversely in a closed smooth Jordan curve.

*Proof.* This construction can be found in several papers cited in the introductory section (see e.g. [28] or [17, Sect. 3]). Nevertheless, we give an outline of proof for completeness. Fix a point  $z \in \mathbb{C}^n$  in the domain of  $\rho$ . The Taylor expansion of  $\rho$  around this point equals

$$(2.1) \quad \rho(z+w) = \rho(z) + \Re\left(2 \sum_{j=1}^n \frac{\partial\rho(z)}{\partial z_j} w_j + \sum_{j,k=1}^n \frac{\partial^2\rho(z)}{\partial z_j \partial z_k} w_j w_k\right) + \mathcal{L}_\rho(z, w) + o(|w|^2),$$

where  $\mathcal{L}_\rho$  is the Levi form (1.1). If  $z$  is close enough to  $b\Omega = \{\rho = 0\}$  then the differential  $\partial\rho(z)$  is nonvanishing. For such  $z$ , the set

$$(2.2) \quad \Sigma_z = \left\{ w = (w_1, \dots, w_n) \in \mathbb{C}^n : 2 \sum_{j=1}^n \frac{\partial\rho(z)}{\partial z_j} w_j + \sum_{j,k=1}^n \frac{\partial^2\rho(z)}{\partial z_j \partial z_k} w_j w_k = 0 \right\}$$

is a quadratic complex hypersurface which is smooth (nonsingular) near the origin  $0 \in \mathbb{C}^n$ , and it follows from (2.1) that near  $w = 0$  we have that

$$\rho(z+w) = \rho(z) + \mathcal{L}_\rho(z, w) + o(|w|^2) \quad \text{for } w \in \Sigma_z.$$

Since  $\mathcal{L}_\rho(z, w) \geq C|w|^2$  for some constant  $C > 0$  which can be chosen independent of the point  $z$  in a neighbourhood of  $b\Omega$ , we see that  $\rho$  restricted to the affine complex hypersurface  $W_z = z + \Sigma_z$  (passing through  $z$ ) increases quadratically in a neighbourhood of  $z$ . Hence, if a point  $z \in \Omega$  is close enough to  $b\Omega$  then the connected component  $W_z^0$  of  $W_z \cap \Omega$  containing  $z$  is a topological ball whose smooth boundary  $bW_z^0$  is a connected component of  $W_z \cap b\Omega$  and the intersection is transverse along  $bW_z^0$ . Clearly,  $W_z$  is locally near  $z$  a graph

over the affine tangent plane  $T_z W_z = z + \ker \partial\rho(z)$ . This gives holomorphic discs  $g_{z,v}$  with images in  $W_z$  (graphs over affine linear discs in the hyperplane  $z + \ker \partial\rho(z)$  in directions  $v \in \ker \partial\rho(z)$ ) satisfying the conclusion of the lemma.

Note that the hypersurfaces  $\Sigma_z$  in (2.2) depend on the first and the second order partial derivatives of the defining function  $\rho$ . Hence, if  $\rho$  is of class  $\mathcal{C}^{k+1}$  then the holomorphic discs  $g_{z,v}$  obtained in this way are of class  $\mathcal{C}^{k-1}$  in the parameters  $(z, v)$ .  $\square$

*Proof of Theorem 1.1.* Let  $f_0 : \overline{M} \rightarrow \Omega$  be a continuous map which is holomorphic on  $M$ , and let  $K$  be a compact subset of  $M$ . By [15, Theorem 1.1] we can approximate  $f_0$  uniformly on  $K$  and interpolate it to any given order at finitely many given points in  $M$  by a holomorphic map  $f_1 : \overline{M} \rightarrow \Omega$  satisfying  $f_1(bM) \subset U \cap \Omega$ , where  $U$  is a neighbourhood of  $b\Omega$  for which Lemma 2.1 holds and  $\partial\rho(z) \neq 0$  for each  $z \in U$ . By a general position argument we may further assume that  $f_1$  is an immersion if  $n = 2$  and an embedding (injective immersion) if  $n \geq 3$  provided that this is consistent with the interpolation conditions.

Consider the complex vector bundle  $E = \ker \partial\rho \rightarrow U$ , a complex hyperplane subbundle of class  $\mathcal{C}^k$  of the holomorphic tangent bundle  $TU$ . Note that the pull-back bundle  $(f_1)^*E \rightarrow bM$  by the map  $f_1 : bM \rightarrow U \cap \Omega$  is trivial (since every orientable vector bundle over a circle is trivial and  $bM$  is a union of finitely many circles). This gives a  $\mathcal{C}^k$  map  $v : bM \rightarrow \mathbb{C}^n$  such that for every  $x \in bM$  we have that

$$|v(x)| = 1 \quad \text{and} \quad v(x) \in E_{f_1(x)} = \ker \partial\rho(f_1(x)).$$

Let  $g_{z,v} : \Delta \rightarrow \mathbb{C}^n$  be the family of holomorphic discs given by Lemma 2.1. With the map  $v$  as above, we define the map  $h : bM \times \Delta \rightarrow \mathbb{C}^n$  by

$$(2.3) \quad h(x, \zeta) = g_{f_1(x), v(x)}(\zeta) \quad \text{for } x \in bM \text{ and } \zeta \in \Delta.$$

Note that  $h$  is smooth of class  $\mathcal{C}^{k-1}$  in both variables, and for each  $x \in bM$  we have that  $h(x, 0) = f_1(x)$  and the map  $h(x, \cdot) : \Delta \rightarrow \mathbb{C}^n$  is holomorphic. By the construction, each holomorphic disc  $g_{z,v} : \Delta \rightarrow \mathbb{C}^n$  intersects  $b\Omega$  transversely in the image  $g_{z,v}(\gamma_{z,v})$  of a closed Jordan curve  $\gamma_{z,v} \subset \Delta$  enveloping the origin  $0 \in \Delta$ . It follows that  $h(x, \cdot) : \Delta \rightarrow \mathbb{C}^n$  has the same property for every  $x \in bM$ .

Consider the Taylor expansion of  $h$  in the second variable:

$$h(x, \zeta) = f_1(x) + \sum_{j=1}^{\infty} h_j(x) \zeta^j \quad \text{for } x \in bM \text{ and } \zeta \in \Delta.$$

The  $\mathbb{C}^n$ -valued functions  $h_j$  are of class  $\mathcal{C}^{k-1}(bM)$ . By the Mergelyan theorem

we can approximate them in  $\mathcal{C}^{k-1}(bM)$  by holomorphic functions  $\tilde{h}_j$  on an open collar  $V \subset R$  around  $bM$  in the ambient compact Riemann surface  $R$ .

By a theorem of Royden [38, Theorem 10], every holomorphic function on an open neighbourhood of a compact set  $L$  in a compact Riemann surface  $R$  can be approximated uniformly on  $L$  by meromorphic functions on  $R$  without poles on  $L$ . (For more precise theorems of this kind, see the surveys [21] and [2, Sect. 1.12].) Applying Royden's result to the functions  $\tilde{h}_j$  on a compact neighbourhood  $L \subset V$  of  $bM$  gives a neighbourhood  $M' \subset R$  of  $\overline{M}$  and a meromorphic map on  $M' \times \mathbb{C}$  with values in  $\mathbb{C}^n$  of the form

$$(2.4) \quad H(x, \zeta) = f_1(x) + \sum_{j=1}^N H_j(x) \zeta^j, \quad x \in M', \zeta \in \mathbb{C}$$

for some large  $N \in \mathbb{N}$  such that the coefficients  $H_j$  have no poles on  $L \cap M'$  and  $H$  approximates  $h$  (2.3) as closely as desired in  $\mathcal{C}^{k-1}(bM \times r\Delta)$  for any given  $r \in (0, 1)$ . If the approximation of  $h$  by  $H$  is close enough, there is for each  $x \in bM$  a closed smooth Jordan curve  $\gamma_x \subset r\Delta$  bounding a simply connected domain  $0 \in D_x \subset r\Delta$  such that

$$(2.5) \quad H(x, D_x) \subset \Omega \quad \text{and} \quad H(x, \gamma_x) \subset b\Omega \quad \text{for all } x \in bM.$$

By the construction, the family of Jordan curves  $\{\gamma_x\}_{x \in bM}$  in  $\Delta$  depends smoothly of class  $\mathcal{C}^{k+1}$  (the smoothness class of  $\rho$ ) on the point  $x \in bM$ . More precisely, the torus  $\{(x, \gamma_x) : x \in bM\}$  in  $bM \times \Delta$  fibred by these curves is of class  $\mathcal{C}^{k+1}$ .

We now apply the main result of Černe's paper [5], according to which there is a function  $\zeta : \overline{M} \rightarrow \Delta$  of class  $\mathcal{C}^{k,\alpha}$  for any  $\alpha \in (0, 1)$  which is holomorphic in  $M$ , it solves the Riemann-Hilbert boundary value problem

$$(2.6) \quad \zeta(x) \in \gamma_x \quad \text{for all } x \in bM,$$

and it vanishes to a given order at any given finite set of points in  $M$ . More precisely, given an effective divisor  $D$  on  $M$  with finite support, we can choose  $\zeta$  as above such that  $(\zeta) \geq D$ . In particular, we may choose  $\zeta$  to vanish at all poles in  $M$  of the coefficients  $H_j$  of the map  $H$  in (2.4) to the same or higher order so that the product  $\zeta(x)H_j(x)$  is holomorphic on  $M$  for every  $j \in \{1, \dots, N\}$ . Furthermore, by choosing  $\zeta$  to have sufficiently many zeros, the normal families argument shows that  $\zeta$  will be arbitrarily uniformly close to zero on any given compact set in  $M$ . For some such  $\zeta$ , the map  $f : \overline{M} \rightarrow \mathbb{C}^n$  defined by

$$(2.7) \quad f(x) = H(x, \zeta(x)) \quad \text{for } x \in \overline{M}$$

satisfies the conclusion of the theorem. Indeed,  $f \in \mathcal{C}^{k,\alpha}(\overline{M})$ , it is holomorphic in  $M$ , and for every  $x \in bM$  we have that  $f(x) \in b\Omega$  by (2.5) and (2.6).



Furthermore, if  $\zeta$  is chosen to vanish to order  $d$  at the points  $a_1, \dots, a_l \in M$  which are not the poles of any  $H_j$  in (2.4), then at these points  $f$  agrees with  $f_1$  to order  $d$ . At the poles of  $H_j$  we can ensure the same condition by choosing  $\zeta$  to vanish to a suitably high order. Next, if  $\zeta$  is close enough to 0 on a given compact set  $K \subset M$  then  $f$  is  $\epsilon$ -close to  $f_1$  on  $K$  by the construction. Finally, we can ensure that  $f(\overline{M})$  lies in the domain of the strongly plurisubharmonic defining function  $\rho$  for  $\Omega$ , and hence the maximum principle applied to the subharmonic function  $\rho \circ f$ , along with the fact this function vanishes on  $bM$ , implies  $f(M) \subset \Omega$ .

It was already mentioned in the introduction that a proper holomorphic map  $f : M \rightarrow \Omega$  which is of class  $\mathcal{C}^1(\overline{M})$  is an immersion along  $bM$ . It remains to explain how to ensure that  $f$  is an immersion on  $M$  if  $n = 3$ , and an embedding if  $n \geq 4$ .

Assume that  $n \geq 3$ . Since  $\dim_{\mathbb{R}}(bM \times \Delta) = 3$  and  $f_1 : \overline{M} \rightarrow \Omega$  is an immersion, we can slightly perturb the map  $h$  in (2.3), keeping it fixed on  $bM \times \{0\}$ , so as to make it an immersion. This property is inherited by the map  $H$  in (2.4) on the set  $V \times r\Delta$  provided that the approximation of  $h$  by  $H$  is close enough and the neighbourhood  $V$  of  $bM$  is chosen small enough. From the definition (2.7) it follows that  $f$  is then an immersion on  $\overline{M} \cap V$  for every choice of the function  $\zeta$  in (2.6). By choosing  $\zeta$  to be close enough to zero on the compact set  $K = M \setminus V \subset M$ , we ensure that  $f$  is so close to the immersion  $f_1$  on  $K$  that it is itself an immersion there. Hence, for such  $\zeta$  the map  $f$  is an immersion.

If  $n \geq 4$  then we can perturb the map  $h$  in (2.3) to ensure that it is an embedding. As before, this property is inherited by the map  $H$  in (2.4) on the set  $V \times r\Delta$  provided that the approximation is close enough and the neighbourhood  $V$  of  $bM$  is chosen small enough. We conclude the proof as in the previous case, showing that  $f$  is an embedding provided that the function  $\zeta$  is close enough to 0 on the compact set  $K = M \setminus V \subset M$ .  $\square$

*Remark 2.2.* It is evident from our proof of Theorem 1.1 that the same result holds under the following weaker condition on  $\Omega$ :

- (a) there is a smooth defining function  $\rho$  for  $\Omega$  whose Levi form  $\mathcal{L}_\rho(z, \cdot)$  has at least two positive eigenvalues at every point  $z \in \overline{\Omega}$ , and
- (b) for every closed path  $\gamma : [0, 1] \rightarrow b\Omega$  there is a closed path of nonzero complex tangent vectors  $0 \neq v(t) \in \ker \partial\rho(\gamma(t)) = T_{\gamma(t)}^{\mathbb{C}} b\Omega$  such that

$$\mathcal{L}_\rho(\gamma(t), v(t)) > 0 \quad \text{for all } t \in [0, 1].$$

If  $\Omega$  is strongly pseudoconvex then condition (b) holds since the complex vector bundle  $\ker \partial\rho \rightarrow b\Omega$  is orientable, and hence trivial over any loop in  $b\Omega$ .

This is an open condition, so it also holds for loops in  $\Omega$  close enough to  $b\Omega$ ; note that this was used in the proof.

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*Franc Forstnerič*

*University of Ljubljana*

*Faculty of Mathematics and Physics*

*Jadranska 19, SI-1000 Ljubljana, Slovenia*

*Institute of Mathematics, Physics and Mechanics*

*Jadranska 19, SI-1000 Ljubljana, Slovenia*

*franc.forstneric@fmf.uni-lj.si*