Dedicated to the memory of Mihnea Colţoiu

ON THE EMBEDDING OF LEVI-FLAT HYPERSURFACES IN THE COMPLEX PROJECTIVE PLANE (AND AN APPENDIX WITH LÁSZLÓ LEMPERT)

ANDREI IORDAN

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Let L be a hypothetical smooth Levi flat hypersurface in $\mathbb{CP}_2$ and r the signed distance to L by means of the Fubini-Study metric g. Denote $\mathcal{L}_r u = c_r u$ the second order elliptic equation for the infinitesimal Levi-flat deformations of L, where $c_r = \mathrm{d}_b J\mathfrak{b}_r + \mathfrak{b}_r \wedge J\mathfrak{b}_r$, $\mathfrak{b}_r = \iota_{X_r} \mathrm{d}\gamma_r$, $X_r = \mathrm{grad}_g r / \left\| \mathrm{grad}_g r \right\|_g^2$, γ_r is the restriction of $\mathrm{d}^c r$ to L and d_b is the differentiation along the leafs of the Levi foliation. Then $-c_r \geq H$ as leaf-wise $(1,1)$-forms, where H is the holomorphic bisectional curvature of $\mathbb{CP}_2$. We give also an example of a Levi-flat manifold L of dimension 3 verifying that there exists a $(1,0)$-form α on L such that $\overline{\partial}\alpha$ is a Kähler form on every leaf of the Levi foliation, but L is not embeddable in $\mathbb{CP}_2$.

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1. INTRODUCTION

A classical theorem of Poincaré-Bendixson states that every leaf of a foliation on the real projective plane accumulates on a compact leaf or on a singularity of the foliation. As a codimension 1 holomorphic foliation $\mathcal{F}$ on $\mathbb{CP}_n$, $n \geqslant 2$, does not contain any compact leaf and its singular set $Sing\ \mathcal{F}$ is not empty, a major problem in foliation's theory is the following: can $\mathcal{F}$ contain a leaf F such that $\overline{F} \cap Sing\ \mathcal{F} \neq \emptyset$? If this is the case, then there exists a nonempty compact set K called exceptional minimal, invariant by $\mathcal{F}$ and minimal for the inclusion such that $K \cap Sing\ \mathcal{F} = \emptyset$. The problem of the existence of an exceptional minimal in $\mathbb{CP}_n$, $n \geqslant 2$, is implicit in [5].

In [6], D. Cerveau proved a dichotomy under the hypothesis of the existence of a codimension 1 foliation $\mathcal{F}$ on $\mathbb{CP}_n$ which admits an exceptional minimal $\mathfrak{M}$: $\mathfrak{M}$ is a real analytic Levi-flat hypersurface in $\mathbb{CP}_n$, or there exists $p \in \mathfrak{M}$ such that the leaf through p has a hyperbolic holonomy and the range of the holonomy morphism is a linearisable abelian group. This gave rise to the conjecture of the nonexistence of smooth Levi-flat hypersurface in $\mathbb{CP}_n$, $n \geqslant 2$.

This conjecture was proved for $n \geqslant 3$ by A. Lins Neto [10] for real analytic Levi-flat hypersurfaces and by Y.-T. Siu [11] for smooth Levi-flat hypersurfaces. However, the conjecture is still open for $n = 2$.

In the paper [7], P. de Bartolomeis and A. Iordan studied deformations of Levi-flat structures in complex manifolds and proved as an application the nonexistence of transversally parallelizable Levi-flat hypersurfaces in $\mathbb{CP}_2$. By using a parametrization of the Levi-flat hypersurfaces near a Levi-flat hypersurface in a complex manifold, they obtained a second order elliptic partial differential equation for the tangent to curves representing the infinitesimal Levi-flat deformations of a Levi-flat hypersurface (see also [9]). This equation depends on a defining function r of the Levi flat hypersurface and it is of the form $\mathcal{L}_r u = c_r u$, where $\mathcal{L}_r$ contains the leafwise derivatives of order 1 and 2 of u. In this paper, we prove a positivity property of c_r for smooth Levi-flat hypersurface L in $\mathbb{CP}_2$ with r the signed geodesic distance to L for the Fubini-Study metric.

More precisely, suppose that Y is a holomorphic vector field on $\mathbb{CP}_2$. We consider the Levi-flat deformation of L given by $\left(\Psi_t^Y(L)\right)_{t\in]-\varepsilon,\varepsilon[}$, where Ψ^Y is the flow of Y. Following the parametrization of hypersurfaces near L, this deformation is given by a family of smooth real valued functions $\left(a_t^{r,Y}\right)_{t\in]-\varepsilon,\varepsilon[}$ on L. Then the function $p_r^Y = \frac{\mathrm{d}a_t^{r,Y}}{\mathrm{dt}}_{|t=0}$, which represents the infinitesimal Levi-flat deformation of L defined by Y verifies the equation $\mathcal{L}_r p_r^Y = c_r p_r^Y$. By using the symmetries of $\mathbb{CP}_2$, we show that for every point $x \in L$ there exists a holomorphic vector field Y_x on $\mathbb{CP}_2$ such that x is a strict extremum point for $p_x^{Y_x}$. By using that $-i\partial\overline{\partial}\log\left|p^{r,Y}\right|(x)$ is the curvature of the normal bundle to the leaf of the Levi foliation through x, we show that $-c_r \geqslant H$ as leaf-wise $(1,1)$-forms on L, where H is the bisectional curvature of $\mathbb{CP}_2$. We mention also that M. Adachi and J. Brinkschulte proved that the totally real Ricci curvature of a Levi-flat hypersurface in $\mathbb{CP}_2$ is smaller than H ([2]).

For a hypothetical Levi-flat hypersurface L in $\mathbb{CP}_2$, this result suggests the existence of a $(1,0)$-form α on L such that $\overline{\partial}\alpha$ is a Kähler form on every leaf of the Levi foliation. This is indeed the case (see, for example [3]). We give a simple proof of this fact for Levi-flat hypersurfaces in $\mathbb{CP}_2$ and we give an example of a Levi-flat manifold of dimension 3 verifying this property, but not embeddable in $\mathbb{CP}_2$.

2. PRELIMINARIES

Definition 1. Let M be a complex manifold and L a smooth real hypersurface of M. L is called Levi-flat if $TL \cap JTL$ is integrable, where TL is the tangent bundle to M and J is the complex structure of M.

Remark 1. L is a Levi-flat hypersurface in a complex manifold M if and only if it admits a foliation by complex hypersurfaces of M. We will call this foliation the Levi foliation of L.

All the Levi-flat hypersurfaces we will consider will be of class C^k, $k \geqslant 3$.

In this paragraph, we will recall several definitions and results from [7]:

Let M be a complex manifold and L a C^k Levi-flat hypersurface in M. In a neighborhood $U \subset M$ of L, there exists a defining function r of classes C^k for L verifying $L = \{z \in M : \ r(z) = 0\}$, $dr \neq 0$ on L.

Then the distribution $\xi = T(L) \cap JT(L)$ is integrable and $\xi = \ker \gamma_r$, where $\gamma_r = j^*(\mathrm{d}^c_J r)$, $\mathrm{d}^c_J r = -J\mathrm{d}r$ and $j : L \to M$ is the inclusion.

Let g be a fixed Hermitian metric on M and $Z_r = \mathrm{grad}_g r / \left\|\mathrm{grad}_g r\right\|_g^2$. Then the vector field $X_r = JZ_r$ is tangent to L and verifies

$$\gamma_r(X) = \mathrm{d}^c_J r(JZ_r) = 1.$$

We will call (γ_r, X_r) the canonical DGLA defining couple associated to the defining function r.

Let U be a tubular neighborhood of L in M and $\pi_r : U \to L$ the projection on L along the integral curves of Z_r. Because we work locally around L, we may assume that $U = M$.

We will now parametrize the real hypersurfaces near L and diffeomorphic to L as graphs over L:

Let $a \in C^k(L;\mathbb{R})$. Denote

$$L_a = \{z \in M : \ r(z) = a(\pi_r(z))\}.$$

Since Z_r is transverse to L, L_a is a hypersurface in M. Consider the map $\Phi_a : M \to M$ defined by $\Phi_a(p) = q$, where

$$\pi_r(q) = \pi_r(p), \ r(q) = r(p) + a(\pi_r(p)). \tag{2.1}$$

In particular, for every $x \in L$ we have

$$r(\Phi_a(x)) = a(x). \tag{2.2}$$

U is a tubular neighborhood of L, so Φ_a is a diffeomorphism of M such that $\Phi_a(L) = L_a$ and $\Phi_a^{-1} = \pi_r|_{L_a}$.

Conversely, if Ψ is a diffeomorphism of M in a suitable neighborhood $\mathcal{U}$ of Id_M, there exists $a \in C^k(L;\mathbb{R})$ such that $\Psi(L) = L_a$. Indeed, for $x \in L$, let

$q(x) \in \Psi(L)$ such that $\pi_r(q(x)) = x$. By defining $a(x) = r(q(x))$, we obtain $\Psi(L) = L_a$.

So for every $\Psi \in \mathcal{U}$ there exists a unique $a \in C^k(L;\mathbb{R})$ such that $\Psi(L) = L_a$ and it follows that a neighborhood $\mathcal{V}$ of 0 in $C^k(L;\mathbb{R})$ is a set of parametrization of hypersurfaces close to L.

For $a \in \mathcal{V}$, let us consider the almost complex structure $J_a = \left(\Phi_a^{-1}\right)_* \circ J \circ (\Phi_a)_*$ on M and denote

$$\alpha_a = \left(\mathrm{d}^c_{J_a} r(X)\right)^{-1} j^*\left(\mathrm{d}^c_{J_a} r\right) - \gamma. \tag{2.3}$$

Then α_a is the unique form in $\Lambda^1(L)$, $\iota_X \alpha_a = 0$, verifying

$$\ker(\gamma + \alpha_a) = (\pi_r)_*(TL_a \cap JTL_a).$$

Definition 2. A 1-dimensional Levi-flat deformation of L is a smooth mapping $\Psi : I \times M \to M$, where I is an interval in $\mathbb{R}$ containing the origin, such that $\Psi_t = \Psi(t,\cdot)$ is a C^k diffeomorphism of M, $L_t = \Psi_t L$ is a Levi-flat hypersurface in M for every $t \in I$ and $L_0 = L$.

Remark 2. If $\Psi : I \times M \to M$ is a Levi-flat deformation of a Levi-flat hypersurface L given by a defining function r, there exists a family $(a_t^r)_{t\in I}$ in a neighborhood of $0 \in C^k(L;\mathbb{R})$ such that $\Psi_t(L) = L_{a_t^r}$, $\pi_*\left(TL_{a_t^r} \cap JTL_{a_t^r}\right) = \ker\left(\gamma + \alpha_{a_t^r}\right)$. We will say that the family $(a_t^r)_{t\in I}$ is a family in $C^k(L;\mathbb{R})$ defining a Levi-flat deformation of L.

THEOREM 1 ([7]). *Let L be a Levi-flat hypersurface in a complex manifold M and r a defining function for L. Let (γ_r, X_r) be the canonical DGLA defining couple associated to r and $(a_t^r)_{t\in I}$ a family in $C^3(L;\mathbb{R})$ defining a Levi-flat deformation of L. Let $p_r = \frac{\mathrm{d}a_t^r}{\mathrm{d}t}_{|t=0}$. Then*

$$\mathrm{d}_b\mathrm{d}_b^c p_r - \mathrm{d}_b p_r \wedge J\mathfrak{b}_r - \mathrm{d}_b^c p_r \wedge \mathfrak{b}_r - p_r \mathrm{d}_b J\mathfrak{b}_r - p_r \mathfrak{b}_r \wedge J\mathfrak{b}_r = 0 \tag{2.4}$$

where J is the complex structure of M, d_b is the differentiation along the leafs of the Levi foliation, $\mathrm{d}_b^c = -J\mathrm{d}_b$ and $\mathfrak{b}_r = \iota_{X_r}\mathrm{d}\gamma_r$.

Notation 1. Under the hypothesis of Theorem 1, we denote

$$\mathfrak{L}_r p_r = \mathrm{d}_b\mathrm{d}_b^c p_r - \mathrm{d}_b p_r \wedge J\mathfrak{b}_r - \mathrm{d}_b^c p_r \wedge \mathfrak{b}_r$$

and

$$c_r = \mathrm{d}_b J\mathfrak{b}_r + \mathfrak{b}_r \wedge J\mathfrak{b}_r. \tag{2.5}$$

So (2.4) is written

$$\mathfrak{L}_r p_r = c_r p_r. \tag{2.6}$$

Notation 2. Let L be a C^3 Levi-flat hypersurface in a complex manifold M and g a Hermitian metric on M. Let r be a C^3 defining function for L and Y a holomorphic vector field on M. We denote by Ψ^Y the flow of Y, $\left(a_t^{r,Y}\right)_{t\in I}$ the family of $C^3\left(L;\mathbb{R}\right)$ which defines the Levi-flat deformation $\left(\Psi_t^Y\left(L\right)\right)_{t\in I}$ of L and $p_r^Y = \frac{\mathrm{d}a_t^{r,Y}}{\mathrm{d}t}_{|t=0}$.

Remark 3 ([7]).

$$p_r^Y = \left\langle \operatorname{Re} Y, \operatorname{grad}_g r\right\rangle_g = \operatorname{Re} Y\left(r\right).$$

3. STRICT EXTREMUM POINTS FOR INFINITESIMAL DEFORMATIONS OF LEVI-FLAT HYPERSURFACES IN COMPLEX SURFACES

A natural question is to find the relationship between c_r and $c_{\widetilde{r}}$ if r and $\widetilde{r}$ are defining functions of L. The following lemma answers this question and was obtained in collaboration with Paolo de Bartolomeis.

LEMMA 1. *Under the hypothesis of Theorem 1, we consider a smooth defining function $\widetilde{r}$ for L given by $\widetilde{r} = e^{\lambda} r$, where λ is a smooth function on M. Denote J the complex structure of M, (γ, X), respectively $\left(\widetilde{\gamma}, \widetilde{X}\right)$, the canonical defining couple corresponding to the defining function r, respectively $\widetilde{r}$. Set $\mathfrak{b} = \iota_X \mathrm{d}\gamma$, $\widetilde{\mathfrak{b}} = \iota_{\widetilde{X}} \mathrm{d}\widetilde{\gamma}$, $p_{\widetilde{r}} = p_{e^{\lambda} r}$. Then:*

1. $\operatorname{grad}_g \widetilde{r} = e^{\lambda} \operatorname{grad}_g r$, $\widetilde{\gamma} = e^{\lambda}\gamma$, $\widetilde{X} = e^{-\lambda} X$;
2. $\mathfrak{b}_{\widetilde{r}} = \mathfrak{b}_r - \mathrm{d}\lambda$;
3. *Let U be an open set of L, $x \in U$, and V a section of $TL_x \cap JTL_x$ on $U \cap L_x$ such that $[V, JV] = 0$. Then*

$$c_{\widetilde{r}}\left(V, JV\right) = c_r\left(V, JV\right) - \left(V\left(\lambda\right)^2 + JV\left(\lambda\right)^2\right) + \left(V^2\left(\lambda\right) + JV^2\left(\lambda\right)\right)$$
$$+ 2\left(\mathfrak{b}_r\left(V\right)V\left(\lambda\right) + \mathfrak{b}_r\left(JV\right)JV\left(\lambda\right)\right)$$

on $U \cap L_x$, where c_r is defined in (2.5).

Proof. 1) We have

$$\mathrm{d}^c\widetilde{r} = e^{\lambda}(r\mathrm{d}^c\lambda + \mathrm{d}^c r)$$

so

$$\gamma_{\widetilde{r}} = e^{\lambda}\gamma_r.$$

Since

$$X_{\widetilde{r}} = J\frac{\mathrm{grad}_g \widetilde{r}}{\left\|\mathrm{grad}_g \widetilde{r}\right\|^2}$$

where g is a Hermitian metric on M, for a vector field W on L we have

$$\mathrm{d}\widetilde{r}\,(W) = e^{\lambda}\mathrm{d}r\,(W) = e^{\lambda} g\left(\mathrm{grad}_g r, W\right) = g\left(e^{\lambda}\mathrm{grad}_g r, W\right) = g\left(\mathrm{grad}_g \widetilde{r}, W\right)$$

so

$$\mathrm{grad}_g \widetilde{r} = e^{\lambda}\mathrm{grad}_g r$$

and

$$\left\|\mathrm{grad}_g \widetilde{r}\right\|^2 = \left\|e^{\lambda}\mathrm{grad}_g r\right\|^2 = e^{2\lambda}\left\|\mathrm{grad}_g r\right\|^2.$$

Therefore,

$$X_{\widetilde{r}} = J\frac{\mathrm{grad}_g \widetilde{r}}{\left\|\mathrm{grad}_g \widetilde{r}\right\|^2} = \frac{e^{\lambda}\mathrm{grad}_g r}{e^{2\lambda}\left\|\mathrm{grad}_g r\right\|^2} = e^{-\lambda}X_r.$$

2) Since $r = 0$ on L, we have

$$\begin{aligned}
\mathfrak{b}_{\widetilde{r}}\,(V) &= \iota_{X_{\widetilde{r}}}\mathrm{d}\mathrm{d}^c\widetilde{r}\,(V) = \mathrm{d}\left(e^{\lambda}(r\mathrm{d}^c\lambda + \mathrm{d}^c r)\right)(X_{\widetilde{r}}, V)\\
&= \left(e^{\lambda}\mathrm{d}r\wedge\mathrm{d}^c\lambda + e^{\lambda}\mathrm{d}\lambda\wedge\mathrm{d}^c r + e^{\lambda}\mathrm{d}\mathrm{d}^c r\right)\left(e^{-\lambda}X_r, V\right)\\
&= \left(\mathrm{d}r\wedge\mathrm{d}^c\lambda + \mathrm{d}\lambda\wedge\mathrm{d}^c r + \mathrm{d}\mathrm{d}^c r\right)(X_r, V)\\
&= \mathrm{d}r\,(X)\,\mathrm{d}^c\lambda\,(V) - \mathrm{d}r\,(V)\,\mathrm{d}^c\lambda\,(X_r)\\
&\quad + \mathrm{d}\lambda\,(X_r)\,\mathrm{d}^c r\,(V) - \mathrm{d}\lambda\,(V)\,\mathrm{d}^c r\,(X_r) + \iota_X\mathrm{d}\mathrm{d}^c r\,(V).
\end{aligned}$$

But X_r and V are tangent to L, $\mathrm{d}^c r\,(V) = 0$ and $\mathrm{d}^c r\,(X_r) = 1$ so

$$\mathfrak{b}_{\widetilde{r}} = \mathfrak{b}_r - \mathrm{d}\lambda.$$

3) We have

$$\mathrm{d}_b J\mathfrak{b}_{\widetilde{r}}\,(V, JV) = \mathrm{d}J\,(\mathfrak{b}_r - \mathrm{d}\lambda)\,(V, JV) = \mathrm{d}J\mathfrak{b}_t\,(V, JV) - \mathrm{d}_b J\,(\mathrm{d}\lambda)\,(V, JV).$$

Since V is tangent to the Levi foliation and $[V, JV] = 0$ it follows that

$$\begin{aligned}
\mathrm{d}J\,(\mathrm{d}\lambda)\,(V, JV) &= V\,(J\,(\mathrm{d}\lambda)\,(JV)) - JV\,(J\,(\mathrm{d}\lambda)\,(V)) - J\,(\mathrm{d}\lambda)\,[V, JV]\\
&= -V\,((\mathrm{d}\lambda)\,(V)) - JV\,((\mathrm{d}\lambda)\,(JV)) = -\left(V^2\,(\lambda) + JV^2\,(\lambda)\right),
\end{aligned}$$

so

$$\mathrm{d}_b J\mathfrak{b}_{\widetilde{r}}\,(V, JV) = \mathrm{d}_b J\,(\mathfrak{b}_r)\,(V, JV) + \left(V^2\,(\lambda) + JV^2\,(\lambda)\right). \tag{3.1}$$

Since

$$\begin{aligned}
(\mathfrak{b}_{\widetilde{r}}\wedge J\mathfrak{b}_{\widetilde{r}})\,(V, JV) &= (\mathfrak{b}_r - \mathrm{d}\lambda)\wedge J\,(\mathfrak{b}_r - \mathrm{d}\lambda)\,(V, JV)\\
&= (\mathfrak{b}_r\wedge J\mathfrak{b}_r - \mathfrak{b}_r\wedge J\mathrm{d}\lambda - \mathrm{d}\lambda\wedge J\mathfrak{b}_r + \mathrm{d}\lambda\wedge J\mathrm{d}\lambda)\,(V, JV)\\
&= (\mathfrak{b}_r\wedge J\mathfrak{b}_r)\,(V, JV) - \mathfrak{b}_r\,(V)\,(J\mathrm{d}\lambda)\,(JV)
\end{aligned}$$

$$
\begin{aligned}
(3.2) \qquad & + \mathfrak{b}_r (JV) (J\mathrm{d}\lambda) (V) - (\mathrm{d}\lambda) (V) J\mathfrak{b}_r (JV) \\
& + (\mathrm{d}\lambda) (JV) (J\mathfrak{b}_r) (V) + \mathrm{d}\lambda (V) J\mathrm{d}\lambda (JV) \\
& - \mathrm{d}\lambda (JV) J\mathrm{d}\lambda (V) \\
& = (\mathfrak{b}_r \wedge J\mathfrak{b}_r) (V, JV) + \mathfrak{b}_r (V) (\mathrm{d}\lambda) (V) + \mathfrak{b}_r (JV) \mathrm{d}\lambda (JV) \\
& + (\mathrm{d}\lambda) (V) \mathfrak{b}_r (V) + (\mathrm{d}\lambda) (JV) (\mathfrak{b}_r) (JV) \\
& - (\mathrm{d}\lambda (V))^2 - (\mathrm{d}\lambda (JV))^2 \\
& = (\mathfrak{b}_r \wedge J\mathfrak{b}_r) (V, JV) + 2 (\mathfrak{b}_r (V) V (\lambda) + \mathfrak{b}_r (JV) JV (\lambda)) \\
& - \left(V (\lambda)^2 + JV (\lambda)^2\right),
\end{aligned}
$$

by adding (3.1) and (3.2) the third point is proved. □

An immediate consequence of Lemma 1 is:

COROLLARY 1. *Under the hypothesis of Lemma 1 we have* $p_{\widetilde{r}}^Y = e^\lambda p_r^Y$.

In the sequel, we will suppose that $\dim M = 2$.

Remark 4. Let L be a Levi-flat hypersurfaces of M and $s \in L$. There exists local holomorphic coordinates $\zeta_s = (z, w)$ in a neighborhood U of s given by $z = x + iy$, $w = u + iv$, $x, y, u, v \in \mathbb{R}$, such that the leaf L_s of the Levi foliation through $s = 0$ is given by $\{(z, w) \in U : w = 0\}$, $T_s L = \{v = 0\}$, $L = \{z \in U : v = \varphi(z, u)\}$. $V = \frac{\partial}{\partial x}$ and $JV = \frac{\partial}{\partial y}$ are tangent to L on $U \cap L_s$. We will call ζ_s adapted coordinates in a neighborhood of s. If ω is a 2-form on L_s and ζ_s a fixed adapted coordinates in a neighborhood of s, we have $\omega = \omega^{\zeta_s}(z)\, \mathrm{d}x \wedge \mathrm{d}y$ on $U \cap L_s$ and we will denote $\omega(s) = \omega^{\zeta_s}(s)$. We will say $\omega \geqslant 0$ if the $(1,1)$-form $\omega^{1,1}$corresponding to ω is positive. In local coordinates the $(1,1)$-form corresponding to ω is $\frac{i}{2}\omega^{\zeta_s}(z)\, \mathrm{d}z \wedge \mathrm{d}\overline{z}$. In order to simplify notations, we will denote ∂ and $\overline{\partial}$ instead the tangential operators ∂_b and $\overline{\partial}_b$ on L.

COROLLARY 2. *Under the hypothesis of Lemma 1 suppose that x is a critical point of λ. By using the notations of Remark 4, we have*

$$
\begin{aligned}
c_{\widetilde{r}} (V, JV) (x) &= c_r (V, JV) (x) + \left(V^2 (\lambda) + JV^2 (\lambda)\right) (x) \\
&= c_r (V, JV) (x) + \Delta_z \lambda (x)
\end{aligned}
$$

where Δ is the Laplace operator.

PROPOSITION 1. *Under the hypothesis of Theorem 1, let $m_r = \inf_L p_r$, $K_{m,r} = \{x \in L : \ p_r(x) = m_r\}$, $M_r = \sup_L p_r$, $K_{M,r} = \{x \in L : \ p_r(x) = M_r\}$. Suppose that $m_r < 0$ and $M_r > 0$. Then $c_r^{\zeta_x} \leqslant 0$ for every $x \in K_{m,r} \cup K_{M,r}$ and ζ_x adapted coordinates in a neighborhood of x.*

Proof. By Theorem 1 and using Notation 1 , p_r verifies the equation (2.6).

$$\mathfrak{L}_r p_r = c_r p_r.$$

Since r is of class C^3, p_r is of class C^2 and c_r is continuous.

Suppose that there exists $x_{m,r} \in K_{m,r}$ (respectively, $x_{M,r} \in K_{M,r}$) such that $c_r^{\xi_{x_{m,r}}}(x_{m,r}) > 0$ (respectively, $c_r^{\xi_{x_{M,r}}}(x_{M,r}) > 0$).

Since $m_r = p_r(x_{m,r}) < 0$, (respectively, $p_r(x_{M,r}) > 0$) we have

$$\Delta_{z_1} p_r(x_{m,r}) = c_r^{\xi_{x_{m,r}}}(x_{m,r})\, p_r(x_{m,r}) < 0,$$

(respectively, $\Delta_{z_1} p_r(x_{M,r}) = c_r^{\xi_{x_{M,r}}}(x_{M,r})\, p_r(x_{M,r}) > 0$). Contradiction. □

Lemma 2. *Let U be a neighborhood of 0 in $\mathbb{C}$ and $p : U \to \mathbb{R}$, $p \in C^2(U)$ such that 0 is a nondegenerate critical point of p. The following are equivalent:*

i) 0 *is a strict local minimum (respectively maximum) for p, i.e.* $\min p = p(0)$ *and $p(z) > p(0)$ (respectively,* $\max_U p = p(0)$ *and $p(z) < p(0)$) for every $z \in U \setminus \{0\}$.*

ii)

$$\frac{\partial^2 p}{\partial z \partial \overline{z}}(0) > 2\left|\frac{\partial^2 p}{\partial z^2}(0)\right|,$$

(respectively,

$$\frac{\partial^2 p}{\partial z \partial \overline{z}}(0) < -2\left|\frac{\partial^2 p}{\partial z^2}(0)\right|).$$

Proof. i) $\Longrightarrow$ ii) We have $p(z) = p(0) + az^2 + \overline{az}^2 + b|z|^2 + O\left(|z|^3\right)$. Since 0 is a nondegenerate critical point of p,

$$az^2 + \overline{az}^2 + b|z|^2 > 0 \text{ (respectively, } az^2 + \overline{az}^2 + b|z|^2 < 0), \text{if } z \neq 0,$$

so

$$ae^{2i\theta} + \overline{a}e^{-2i\theta} + b > 0 \text{ (respectively, } ae^{2i\theta} + \overline{a}e^{-2i\theta} + b < 0),\ \forall \theta \in \mathbb{R}.$$

Therefore

$$b > -2\operatorname{Re}\left(ae^{2i\theta}\right) \text{ (respectively, } b < -2\operatorname{Re}\left(ae^{2i\theta}\right)),\ \forall \theta \in \mathbb{R}$$

and ii) follows.

ii) $\Longrightarrow$ i) Conversely, if

$$b = \frac{\partial^2 p}{\partial z \partial \overline{z}}(0) > 2\left|\frac{\partial^2 p}{\partial z^2}(0)\right| = 2|a|,$$

respectively,

$$b = \frac{\partial^2 p}{\partial z \partial \overline{z}}(0) < -2\left|\frac{\partial^2 p}{\partial z^2}(0)\right| = -2|a|,$$

then for every $z = re^{i\theta}$, $r > 0$, $\theta \in \mathbb{R}$,

$$az^2 + \overline{az}^2 + b|z|^2 = r^2 \left(b + 2\operatorname{Re}\left(ae^{2i\theta}\right)\right) \geqslant r^2 (b - 2|a|) > 0,$$

respectively,

$$az^2 + \overline{az}^2 + b|z|^2 = r^2 \left(b + 2\operatorname{Re}\left(ae^{2i\theta}\right)\right) \leqslant r^2 (b + 2|a|) < 0$$

and i) follows. □

THEOREM 2. *Let L be a Levi-flat hypersurface in a complex surface M. Let Y be a holomorphic vector field Y on M and $x_m \in L$ such that $\min_L p_r^Y = p_r^Y(x_m) < 0$. Suppose that x_m is a strict minimum point of $p_r = p_r^Y$. Then for any ζ_{x_m} adapted coordinates in a neighborhood of x_m*

$$\frac{\frac{\partial^2 p_r}{\partial z \partial \overline{z}}(x_m)}{-p_r(x_m)} \leqslant -\frac{c_r^{\zeta_{x_m}}}{4}(x_m).$$

Proof. By Proposition 1, we have $c_r^{\zeta_{x_m}}(x_m) \leqslant 0$. Suppose that

$$\frac{\frac{\partial^2 p_r}{\partial z \partial \overline{z}}(x_m)}{-p_r(x_m)} > -\frac{c_r^{\zeta_{x_m}}}{4}(x_m). \tag{3.3}$$

Then x_m is a nondegenerate critical point of p_r.

Let $\widetilde{r} = e^{\lambda} r$ with λ a smooth function on M such that the restriction of λ to $U \cap L_{x_m}$ is $\lambda(z) = \alpha z^2 + \overline{\alpha z} + \beta|z|^2$ with $\alpha \in \mathbb{C}$, $\beta \in \mathbb{R}$.

By Corollary 1 $p_{\widetilde{r}}(z) = p_r(z) e^{\alpha z^2 + \overline{\alpha z} + \beta|z|^2}$ for $z \in U \cap L_{x_m}$, so

$$\frac{\partial^2 p_{\widetilde{r}}}{\partial z^2}(0) = \frac{\partial^2 p_r}{\partial z^2}(0) + 2\alpha p_r(0)$$

and

$$\frac{\partial^2 p_{\widetilde{r}}}{\partial z \partial \overline{z}}(0) = \frac{\partial^2 p_r}{\partial z \partial \overline{z}}(0) + \beta p_r(0).$$

By Lemma 2, we have

$$\frac{\partial^2 p_r}{\partial z \partial \overline{z}}(0) > 2\left|\frac{\partial^2 p_r}{\partial z^2}(0)\right|, \tag{3.4}$$

so $\frac{\partial^2 p_r}{\partial z \partial \overline{z}}(0) > 0$. Choose α such that

$$\frac{\partial^2 p_r}{\partial z^2}(0) + 2\alpha p_r(0) = 0 \quad \text{and} \quad 0 < \beta < \frac{\frac{\partial^2 p_r}{\partial z \partial \overline{z}}(0)}{-p_r(0)}.$$

Then

$$\frac{\partial^2 p_r}{\partial z \partial \overline{z}}(0) + \beta p_r(0) = \frac{\partial^2 p_{\widetilde{r}}}{\partial z \partial \overline{z}}(0) > \left|\frac{\partial^2 p_{\widetilde{r}}}{\partial z^2}(0)\right| = \frac{\partial^2 p_r}{\partial z^2}(0) + 2\alpha p_r(0) = 0$$

and it follows by Lemma 2 that 0 is a local strict minimum for $p_{\widetilde{r}}$.

By using (3.3), we may choose

$$-\frac{c_r^{\zeta_{x_m}}}{4}(0) < \beta < \frac{\frac{\partial^2 p_r}{\partial z \partial \overline{z}}(0)}{-p_r(0)}$$

and by Corollary 2 it follows that

$$c_{\widetilde{r}}^{\zeta_{x_m}}(0) = c_r^{\zeta_{x_m}}(0) + \left(\frac{\partial^2 \lambda}{\partial x^2} + \frac{\partial^2 \lambda}{\partial y^2}\right)(0) = c_r^{\zeta_{x_m}}(0) + 4\beta > 0.$$

But this contradicts Proposition 1. □

Definition 3. Let L be a compact Levi-flat hypersurface in a complex surface M, r a defining function for L, Y a holomorphic vector field on M and $\mathcal{T}_Y$ the set of points where Y is tangent to L. Define $I_{Y,r} : L \backslash \mathcal{T}_Y \to \mathbb{R}$,

$$I_{Y,r} = i\partial\overline{\partial} \log \left| p_r^Y \right| - \frac{1}{2} c_r$$

($p_r^Y \neq 0$ on $L \backslash \mathcal{T}_Y$ by Remark 3).

Corollary 3. *Let L be a Levi-flat hypersurface in a complex surface M, r a defining function for L, Y a holomorphic vector field on M, and x_m a local strict minimum for $p_r = p_r^Y$ such that $p_r(x_m) < 0$. Then*

i) *If $\widetilde{r} = e^{\lambda} r$ and x_m is a critical point of λ, $I_{Y,\widetilde{r}}(x_m) = I_{Y,r}(x_m)$.*

ii)

$$I_{Y,r}(x_m) \geqslant 0.$$

Proof. i) If $\widetilde{r} = e^{\lambda} r$ with λ a smooth real function on M, by Corollary 1 we have $p_{\widetilde{r}}^Y = e^{\lambda} p_r^Y$, so

$$\begin{aligned} i\partial\overline{\partial} \log \left| p_{\widetilde{r}}^Y \right| (x_m) &= i\partial\overline{\partial} \log \left| e^{\lambda} p_r^Y \right| (x_m) = i\partial\overline{\partial}\lambda + i\partial\overline{\partial} \log \left| p_r^Y \right| (x_m) \\ &= \frac{1}{2} \left(\Delta\lambda(x_m) \right) dx dy + i\partial\overline{\partial} \log \left| p_r^Y \right| (x_m) \end{aligned}$$

Since by Remark 4

$$c_{\widetilde{r}}(x_m) = c_r(x_m) + \Delta\lambda(x_m)$$

it follows that

$$\begin{aligned} I_{Y,\widetilde{r}}(x_m) &= i\partial\overline{\partial} \log \left| p_{\widetilde{r}}^Y \right| (x_m) - \frac{1}{2} c_{\widetilde{r}}(x_m) \\ &= i\partial\overline{\partial} \log \left| p_r^Y \right| (x_m) + i\partial\overline{\partial}\lambda(x_m) - \frac{1}{2} c_r(x_m) - \frac{1}{2} \Delta\lambda(x_m) \\ &= i\partial\overline{\partial} \log \left| p_r^Y \right| (x_m) - \frac{1}{2} c_r(x_m) = I_{Y,r}(x_m). \end{aligned}$$

ii) By Theorem 2

$$\frac{\frac{\partial^2 p_r}{\partial z \partial \overline{z}}(x_m)}{-p_r(x_m)} \leqslant -\frac{c_r^{\zeta_{x_m}}}{4}(x_m)$$

so

$$\begin{aligned} I_{Y,r}(x_m) &= i\partial\overline{\partial}\log\left|p_r^Y\right|(x_m) - \frac{1}{2}c_r(x_m) = \frac{i\partial\overline{\partial}p_r(x_m)}{p_r(x_m)} - \frac{c_r}{2}(x_m) \\ &= 2\left(\frac{\frac{\partial^2 p_r}{\partial z \partial \overline{z}}(0)(x_m)}{p_r(x_m)} - \frac{c_r^{\zeta_{x_m}}}{4}(x_m)\right) \geqslant 0. \end{aligned}$$

□

4. STRICT MAXIMUM AND ORTHOGONALITY OF HOLOMORPHIC VECTOR FIELDS IN $\mathbb{CP}_2$

Notation 3. We denote by g_{FS} the Fubini-Study metric on $\mathbb{CP}_2$.

LEMMA 3. *Let a, b distinct points of $\mathbb{CP}_2$ and $v \in T_a^{1,0}(\mathbb{CP}_2)$ and $w \in T_b^{1,0}(\mathbb{CP}_2)$ such that v and w have the same g_{FS} length. There exists Φ a biholomorphic isometry by means of g_{FS} such that $\Phi(a) = b$ and $\Phi_{*,a}(v) = w$.*

Proof. We may suppose that $a = [1:0:0]$ and $b = [0:1:0]$, where $[z_0, z_1, z_2]$ are homogeneous coordinates in $\mathbb{CP}_2$. We denote by $\widehat{a} = (1,0,0)$, $\widehat{b} = (0,1,0) \in \mathbb{C}^3$ and let $U : \mathbb{C}^3 \to \mathbb{C}^3$ be the isometry of $\mathbb{C}^3$ whose matrix in the canonical basis of $\mathbb{C}^3$ is

$$U = \begin{pmatrix} 0 & \cos\alpha & -\sin\alpha \\ 1 & 0 & 0 \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \ \alpha \in \mathbb{R}.$$

Then U induces a biholomorphic isometry Φ of $\mathbb{CP}_2$ by means of g_{FS} such that $\Phi(a) = b$, $\Phi([z_0, z_1, z_2]) = ([z_1\cos\alpha - z_2\sin\alpha : z_0 : z_1\sin\alpha + z_2\cos\alpha])$.

By choosing non-homogeneous coordinates coordinates $\zeta_1 = \frac{z_1}{z_0}, \zeta_2 = \frac{z_2}{z_0}$ in a neighborhood of a and $\zeta_1' = \frac{z_0}{z_1}, \zeta_2' = \frac{z_2}{z_1}$ in a neighborhood of b, Φ is given by

$$\zeta_1' = \zeta_1\cos\alpha - \zeta_2\sin\alpha, \ \zeta_2' = \zeta_1\sin\alpha + \zeta_2\cos\alpha$$

and so

$$\Phi_{*,a} \approx \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

We can therefore choose α such that $\Phi_{*,a}(v) = w$. □

Lemma 4. *There exists a holomorphic vector field Y on $\mathbb{CP}_2$ and $a \in \mathbb{CP}_2$ such that $\|Y(a)\|_{g_{FS}} = \max_{x \in \mathbb{CP}_2} \|Y(x)\|_{g_{FS}}$ and a is a strict maximum for $\|Y(\cdot)\|_{g_{FS}}$.*

Proof. Consider the vector field $\widetilde{Y} = -z_0 \frac{\partial}{\partial z_0}$ on $\mathbb{C}^3 \setminus \{0\}$ and let $Y = \pi_* \left(\widetilde{Y}\right)$, where $\pi : \mathbb{C}^3 \setminus \{0\} \to \mathbb{CP}_2$ is the canonical map. If $z_0 \neq 0$ and $\zeta_1 = z_1/z_0, \zeta_2 = z_2/z_0$ are non-homogeneous coordinates

$$\pi_* \left(\widetilde{Y}\right) = \zeta_1 \frac{\partial}{\partial \zeta_1} + \zeta_2 \frac{\partial}{\partial \zeta_2}.$$

Since for $z_0 \neq 0$

$$\begin{aligned} g_{FS} = & \frac{1 + |\zeta_2|^2}{\left(1 + |\zeta_1|^2 + |\zeta_2|^2\right)^2} \mathrm{d}\zeta_1 \otimes \mathrm{d}\overline{\zeta}_1 + \frac{1 + |\zeta_1|^2}{\left(1 + |\zeta_1|^2 + |\zeta_2|^2\right)^2} \mathrm{d}\zeta_2 \otimes \mathrm{d}\overline{\zeta}_2 \\ & - \frac{\zeta_1 \overline{\zeta}_2}{\left(1 + |\zeta_1|^2 + |\zeta_2|^2\right)^2} \mathrm{d}\zeta_2 \otimes \mathrm{d}\overline{\zeta}_1 - \frac{\overline{\zeta}_1 \zeta_2}{\left(1 + |\zeta_1|^2 + |\zeta_2|^2\right)^2} \mathrm{d}\zeta_1 \otimes \mathrm{d}\overline{\zeta}_2, \end{aligned}$$

we have

$$\begin{aligned} \|Y\|_{FS}^2 &= g_{FS} \left(\zeta_1 \frac{\partial}{\partial \zeta_1} + \zeta_2 \frac{\partial}{\partial \zeta_2}, \zeta_1 \frac{\partial}{\partial \zeta_1} + \zeta_2 \frac{\partial}{\partial \zeta_2} \right) \\ &= 2 \operatorname{Re} \left(\frac{1 + |\zeta_2|^2}{\left(1 + |\zeta_1|^2 + |\zeta_2|^2\right)^2} |\zeta_1|^2 + \frac{1 + |\zeta_1|^2}{\left(1 + |\zeta_1|^2 + |\zeta_2|^2\right)^2} |\zeta_2|^2 \right) \\ &- 2 \operatorname{Re} \left(\frac{\zeta_1 \overline{\zeta}_2}{\left(1 + |\zeta_1|^2 + |\zeta_2|^2\right)^2} \zeta_2 \overline{\zeta}_1 + \frac{\overline{\zeta}_1 \zeta_2}{\left(1 + |\zeta_1|^2 + |\zeta_2|^2\right)^2} \zeta_1 \overline{\zeta}_2 \right) \\ &= 2 \left(\frac{1 + |\zeta_2|^2}{\left(1 + |\zeta_1|^2 + |\zeta_2|^2\right)^2} |\zeta_1|^2 + \frac{1 + |\zeta_1|^2}{\left(1 + |\zeta_1|^2 + |\zeta_2|^2\right)^2} |\zeta_2|^2 \right) \\ &- 2 \left(\frac{|\zeta_1|^2 |\zeta_2|^2}{\left(1 + |\zeta_1|^2 + |\zeta_2|^2\right)^2} + \frac{|\zeta_1|^2 |\zeta_2|^2}{\left(1 + |\zeta_1|^2 + |\zeta_2|^2\right)^2} \right) \\ &= \frac{4}{\left(1 + |\zeta_1|^2 + |\zeta_2|^2\right)^2}. \end{aligned}$$

Hence,

$$\max_{x\in\mathbb{CP}_2} \|Y(x)\|_{g_{FS}} = \|Y([1:0:0])\|_{g_{FS}} = 4$$

and $[1:0:0]$ is a strict maximum for $\|Y(\cdot)\|_{g_{FS}}$. $\square$

COROLLARY 4. *Let $a \in \mathbb{CP}_2$ and H a complex hyperplane such that $a \in H$. Then there exists a holomorphic vector field Y_a on $\mathbb{CP}_2$ such that $\|Y_a(a)\|_{g_{FS}} = \max_{x\in\mathbb{CP}_2} \|Y_a(x)\|_{g_{FS}}$, a is a strict maximum for $\|Y_a(\cdot)\|_{g_{FS}}$ and $Y_a(a) \perp_{g_{FS}} H$.*

Proof. By Lemma 4, there exists a vector field T on $\mathbb{CP}_2$, $T \neq 0$ and let $b \in \mathbb{CP}_2$ such that $\|T(b)\|_{g_{FS}} = \max_{x\in\mathbb{CP}_2} \|T(x)\|_{g_{FS}}$ and $\|T(b)\|_{g_{FS}} > \|T(x)\|_{g_{FS}}$ if $x \neq b$. By Lemma 3, there exists Φ a biholomorphic isometry by means of g_{FS} such that $\Phi(b) = a$ and $\Phi_{*,a}(Ta) \perp_{g_{FS}} H$. Then we can choose $Y_a = \Phi_* T$. $\square$

In the sequel, L is a C^3 Levi-flat hypersurface in $\mathbb{CP}_2$ and r is the signed geodesic distance to L by means of g_{FS}.

Remark 5. Let Y a holomorphic vector field on $\mathbb{CP}_2$. Then $\operatorname{Re} Y(r)$ cannot take only strictly positive (negative) values on L.

Indeed, let $\Omega_\pm = \{z \in \mathbb{CP}_2 : \ r(z) < 0\}$ and suppose that $\operatorname{Re} Y(r) > 0$ on L. Let Ψ^Y be the flow of Y. Then $L_t = \Psi^Y_t(L)$ is a compact Levi-flat hypersurface of Ω_+ for $t > 0$. But Ω_+ is Stein and $\varphi = -\log r$ is strongly plurisubharmonic on Ω_+. Let $x \in L_t$ such that $x = \sup_L \varphi$. Then the restriction of φ to the leaf of the Levi foliation through x is strongly plurisubharmonic and has a maximum at x. Contradiction.

PROPOSITION 2. *For every point $x \in L$, there exists a holomorphic vector field Y_x on $\mathbb{CP}_2$ such that $\max_L \left|p_r^{Y_x}\right| = \left|p_{rx}^{Y_x}(x)\right|$ and x is a strict maximum for $\left|p_r^{Y_x}\right|$.*

Proof. Let $x \in L$. By Corollary 4, there exists a holomorphic vector field Y_x on $\mathbb{CP}_2$ such that $\|Y_x(x)\|_{g_{FS}} = \max_{y\in\mathbb{CP}_2} \|Y_x(y)\|_{g_{FS}}$, x is a strict maximum for $\|Y_x(\cdot)\|_{g_{FS}}$ and $Y_x(x) \perp_{g_{FS}} T_x^{\mathbb{C}} L$.

Since Y_x is a holomorphic vector field and r is the geodesic distance by means of g_{FS}, by Remark 3 we have

$$\left|p_r^{Y_x}(y)\right| = \left|\operatorname{Re} Y_x r(y)\right| \leqslant \|\operatorname{Re} Y_x(y)\|_{g_{FS}} = \frac{\|Y_x(y)\|_{g_{FS}}}{\sqrt{2}} \leqslant \frac{\|Y_x(x)\|_{g_{FS}}}{\sqrt{2}}.$$

for every $y \in L$.

But $Y_x(x) \perp_{g_{FS}} T_x^{\mathbb{C}} L$, so

$$\left|\operatorname{Re} Y_x r(x)\right| = \|\operatorname{Re} Y_x(x)\|_{g_{FS}}$$

and

$$\left|p_r^{Y_x}(x)\right| = \left|\operatorname{Re} Y_x r(x)\right| = \frac{\|Y_x(x)\|_{g_{FS}}}{\sqrt{2}},$$

which implies

$$\left|p_r^{Y_x}(x)\right| = \max_{y\in L}\left|p_r^{Y_x}(y)\right| \text{ and } \left|p_r^{Y_x}(x)\right| > \left|p_r^{Y_x}(y)\right| \text{ for } y \neq x.$$

□

THEOREM 3. *Let $x \in L$ and Y a holomorphic vector field on $\mathbb{CP}_2$ such that $\|Y(x)\|_{g_{FS}} = \max_{y\in\mathbb{CP}_2}\|Y(y)\|_{g_{FS}}$, x is a strict maximum for $\|Y(\cdot)\|_{g_{FS}}$ and $Y(x)\perp_{g_{FS}} T_x^{\mathbb{C}}L$. Then*

$$-2i\partial\overline{\partial}\log p_{\delta_{FS}}^{Y}(x) \geqslant Hx,$$

where H is bisectional curvature of $\mathbb{CP}_2$ endowed with the Fubini-Study metric.

Proof. We have $p_r^{-Y}(x) = -p_r^{Y}(x)$, so

$$\min_L p_r^{-Y} = -p_r^{Y}(x) < 0.$$

By Corollary 3, we have

$$I_{Y,r}(x) = i\partial\overline{\partial}\log\left|p_r^{-Y}\right|(x) - \frac{1}{2}c_r(x) \geqslant 0$$

so

$$\frac{i\partial\overline{\partial}p_r^{-Y}(x)}{-p_r^{-Y}(x)} \leqslant -\frac{c_r}{2}(x)\,. \tag{4.1}$$

But

$$i\partial\overline{\partial}\log\left|p_{\delta_{FS}}^{-Y}\right|(x) = \frac{i\partial\overline{\partial}p_r^{-Y}(x)}{p_r^{-Y}(x)}$$

We denote $\widetilde{p} = |Yr|$ and L_x the leaf of the Levi foliation through x. We consider the holomorphic line bundle

$$N_{L_x} = T^{1,0}(\mathbb{CP}_2)\,|L_x\,/T^{1,0}(L_x) \leftrightarrows T(\mathbb{CP}_2)\,|L_x\,/T^{\mathbb{C}}(L_x) \leftrightarrows \left(T^{\mathbb{C}}(L_x)\right)^{\perp_{g_{FS}}}$$

endowed with the metric induced by g_{FS}. Then $Y\,|L_x$ is a holomorphic section of this bundle and $i\Theta(N_{L_x}) = -i\partial\overline{\partial}\log|Y\,|L_x|^2_{g_{FS}}$. Since $Y(x)\perp_{g_{FS}} T_x^{\mathbb{C}}L$, it follows that $Y(x) \in \left(T^{\mathbb{C}}(L_x)\right)^{\perp_{g_{FS}}}$ and since r is the signed distance to L,

$$|Y(x)|_{g_{FS}} = |Yr(x)| = \widetilde{p}(x)\,.$$

Since

$$|Yr| \geqslant |(\operatorname{Re} Y)\,r|$$

and

$$|Yr(x)| = |(\operatorname{Re} Y)\,r(x)|$$

it follows that the function

$$\log|Yr| - \log|(\operatorname{Re} Y)\, r|$$

has a minimum at x and, consequently,

$$i\partial\overline{\partial}\log|Yr|^2(x) \geqslant i\partial\overline{\partial}\log|(\operatorname{Re} Y)\, r|^2(x).$$

So

(4.2)

$$i\Theta(N_{L_x})(x) = -i\partial\overline{\partial}\log|Y|L_x|^2_{g_{FS}}(x) = -2i\partial\overline{\partial}\log\widetilde{p}(x) = -i\partial\overline{\partial}\log|Yr|^2(x)$$
$$\leqslant -i\partial\overline{\partial}\log|(\operatorname{Re} Y)\, r|^2(x) = -i\partial\overline{\partial}\log\left|p_r^{-Y}(x)\right|^2.$$

Since N_{L_x} is a quotient of $T(\mathbb{CP}_2)$, we have

$$i\Theta(N_{L_x}) \geqslant i\Theta_{g_{FS}}(T(\mathbb{CP}_2)) \tag{4.3}$$

and it is known that

$$i\Theta_{g_{FS}}(T(\mathbb{CP}_2)) \geqslant H. \tag{4.4}$$

From (4.2), (4.3) and (4.4) it follows that

$$-i\partial\overline{\partial}\log\left|p_{\delta_{FS}}^{-Y}(x)\right|^2 \geqslant Hx.$$ □

Corollary 5. *For every $x \in L$ we have*

$$-c_r(x) \geqslant H(x).$$

Proof. Let $x \in L$. By Corollary 4 there exists a holomorphic vector field Y_x on $\mathbb{CP}_2$ such that $\|Y_x(x)\|_{g_{FS}} = \max_{x\in\mathbb{CP}_2}\|Y_x(y)\|_{g_{FS}}$, x is a strict maximum for $\|Y_x(\cdot)\|_{g_{FS}}$ and $Y_x(x) \perp_{g_{FS}} T_x^{\mathbb{C}}(L)$.

We have $p_r^{-Y_x}(x) = -p_r^{Y_x}(x)$, so

$$\min_L p_r^{-Y_x} = -p_r^{Y_x}(x) < 0.$$

By Corollary 3, we have

$$I_{Y_x,r}(x) = i\partial\overline{\partial}\log\left|p_r^{-Y_x}\right|(x) - \frac{1}{2}c_r(x) \geqslant 0$$

and by Theorem 3

$$-2i\partial\overline{\partial}\log p_r^{Y_x}(x) \geqslant Hx.$$

So

$$-\frac{1}{2}c_r(x) \geqslant -i\partial\overline{\partial}\log p_r^{Y_x}(x) \geqslant \frac{H(x)}{2}. \tag{4.5}$$ □

Remark 6. The form c_r is defined for vectors in $T(L) \cap JTL$. But there exists a unique $\widetilde{c}_r \in \Lambda^2(L)$ such that $\widetilde{c}_r = c_r$ on every leaf of the Levi foliation. Indeed if (γ, X) is a DGLA defining couple and $V \in TL$, then $V = W + \lambda X$, with $W \in T(L) \cap JTL$. Then we define $\widetilde{c}_r(V) = c_r(W)$.

5. APPENDIX WITH LÁSZLÓ LEMPERT

In this appendix, we will consider a family of examples of three dimensional Levi-flat CR manifolds and address the question whether they can be embedded into $\mathbb{CP}_2$. We start by introducing some notation, in greater generality than needed here.

A CR manifold is a couple (M, P), where M is a smooth manifold and P is a subbundle of $\mathbb{C}TM$ such that $P \cap \overline{P} = \{0\}$ and P is involutive, i.e., the Lie bracket of any two smooth sections of P is again a section of P.

We denote by $\mathcal{E}(M)$ the set of smooth functions on M. If $r = 1, 2, \ldots$, an r-form on (M, P) is a smooth map

$$\lambda : (P \oplus \overline{P})^{\oplus r} \to \mathbb{C}$$

which is alternating r-linear on each fiber $P_x \oplus \overline{P}_x$, $x \in M$. Given $p, q = 0, 1, \ldots$ with $p + q = r$, this λ is a (p, q)-form if $\lambda(v_1, \ldots, v_r) = 0$ whenever $v_1, \ldots, v_s \in P$, $v_{s+1}, \ldots, v_r \in \overline{P}$ and $s \neq p$.

We denote by $\mathcal{E}_P^r(M)$ the set of r-forms on (M, P) and by $\mathcal{E}_P^{p,q}(M)$ the set of (p, q)-forms. If $f \in \mathcal{E}(M)$, then $\overline{\partial}_P f \in \mathcal{E}_P^{0,1}(M)$ stands for the form whose restriction to $\overline{P}$ agrees with df. If E is a complex vector bundle over M, we define similarly E-valued forms and we denote by $\mathcal{E}_P^r(M, E)$ the set of E-valued r-forms on (M, P) and by $\mathcal{E}_P^{p,q}(M, E)$ the set of E-valued (p, q)-forms. For example, $\lambda \in \mathcal{E}_P^r(M, E)$ is a smooth fiber map $(P \oplus \overline{P})^{\oplus r} \to E$ that restricts to r-linear maps $(P_x \oplus \overline{P}_x)^{\oplus r} \to E_x$. When $r = 0$, we just write $\mathcal{E}(M, E)$ for the space of smooth sections.

Let us return to a hypothetical smooth Levi-flat hypersurface L in the complex projective plane; it inherits from the plane $\mathbb{CP}_2$ a CR structure (L, P). Theorem 3 might suggest that there is a form $\beta \in \mathcal{E}_P^{1,0}(L)$ such that $\overline{\partial}_P \beta > 0$.

This is indeed so. As M. Adachi pointed out to us, for example the D'Angelo $(1, 0)$-form in [3] (defined in greater generality than the Levi-flat submanifolds of $\mathbb{CP}_2$) has this property. In the Levi-flat case in $\mathbb{CP}_2$, it can be obtained without much calculation:

Let $N^{1,0} \to L$ be the $(1, 0)$-normal bundle to the leaves of the Levi foliation endowed with the Hermitian metric induced by the Fubini-Study metric on $\mathbb{CP}_2$. This is a smooth line bundle, whose restrictions to the leaves are holomorphic, and the curvature of the metric is positive since $N^{1,0}$ is a quotient of $T^{1,0}\mathbb{CP}_2$. $N^{1,0}$ admits a leafwise Chern connection and a Bott connection. Both are partial connections, because they allow differentiation only in directions tangent to leaves. The difference of the two is represented by a form $\lambda \in \mathcal{E}_P^{1,0}(L)$. Since the curvature of the Bott connection vanishes, the curvature of the Chern connection is given by $\overline{\partial}_P \lambda$, and it follows that $\overline{\partial}_P (i\lambda) > 0$.

The first question that arises is whether there are such Levi-flat CR manifolds at all, not necessarily embedded in $\mathbb{CP}_2$. It turns out that there are. The examples that we give below have already appeared in works of Diedrich–Ohsawa, Brunella, and Adachi [8], [4], [1] in the study of several properties of compact Levi-flat manifolds.

Let $\mathbb{D}$ be the unit disc in the complex plane and $\mathcal{G}$ the group of biholomorphic self maps of $\mathbb{D}$, a subgroup of biholomorphic self maps of $\mathbb{CP}_1$. Fix a discrete subgroup $\Gamma \subset \mathcal{G}$ such that $\mathbb{D}/\Gamma$ is a compact Riemann surface, and a homomorphism $\rho : \Gamma \to \mathcal{G}$. The group Γ acts on the manifolds $N = \mathbb{D} \times \partial\mathbb{D}$ and $M = \mathbb{D} \times \mathbb{CP}_1$ by

$$(z, \zeta) \mapsto (gz, \rho(g)\zeta), \quad g \in \Gamma, \tag{5.1}$$

and the quotients are

$$N/\Gamma = L \subset M/\Gamma = X. \tag{5.2}$$

L is a Levi-flat hypersurface in the complex surface X, the leaves of the Levi foliation of L being the projections of the surfaces $\mathbb{D}\times\{\zeta\}$, $\zeta \in \partial\mathbb{D}$. We denote the CR structure of L by $P = \mathbb{C}TL \cap T^{1,0}X$. Finally, L and X are compact since L (respectively, X) is the image of $\overline{W} \times \partial\mathbb{D}$ (respectively, $\overline{W} \times \mathbb{CP}_1$) under the quotient map, where $W \subset \mathbb{D}$ is a relatively compact fundamental domain for the action of Γ.

Proposition 3. *If ρ is the inclusion $\Gamma \to \mathcal{G}$, then there exists $\beta \in \mathcal{E}_P^{1,0}(L)$ such that $\overline{\partial}_P\beta > 0$.*

Proof. The action (5.1) above is now a subaction of the action of $\mathcal{G}$ on N,

$$(z, \zeta) \mapsto (gz, g\zeta), \ g \in \mathcal{G},$$

and this action is simply transitive. Indeed, any element $g \in \mathcal{G}$ acts on $\mathbb{CP}_1$ as $\varepsilon(z-a)/(1-\overline{a}z)$ with uniquely determined $\varepsilon \in \partial\mathbb{D}$, $a \in \mathbb{D}$. Thus g^{-1} maps $(0,1) \in \mathbb{D}\times\partial\mathbb{D}$ to $(b, \delta) \in \mathbb{D}\times\partial\mathbb{D}$ if and only if $a = b$ and $\varepsilon = \delta\left(\overline{\delta} - \overline{a}\right)(\delta - a)$.

It follows that an arbitrary $(1,0)$-form on $T_0^{1,0}\mathbb{D} \subset T_{(0,1)}^{1,0}(\mathbb{D}\times\mathbb{CP}_1)$ has a unique $\mathcal{G}$-invariant extension to a form in $\mathcal{E}_P^{1,0}(N)$. Let α be the extension of the $(1,0)$-form $i\,\mathrm{d}z$. The value of α at any $(a,1) \in \mathbb{D}\times\{1\}$ is $g^*(i\,dz)$, where

$$g = \varepsilon\frac{z-a}{1-\overline{a}z}, \qquad \varepsilon = \frac{1-\overline{a}}{1-a},$$

as calculated above. One finds that α and $\overline{\partial}_P\alpha$ on $\mathbb{D}\times\{1\}$, respectively at $(0,1)$, are given by

$$\frac{i(1-\overline{z})\,\mathrm{d}z}{(1-z)(1-|z|^2)}, \qquad i\frac{\mathrm{d}z\wedge\mathrm{d}\overline{z}}{(1-|z|^2)^2} > 0, \qquad z \in \mathbb{D}.$$

Since $\overline{\partial}_P\alpha$ is also $\mathcal{G}$-invariant, it follows that it is everywhere positive. By invariance, α descends to a form $\beta \in \mathcal{E}_P^{1,0}(L)$ with $\overline{\partial}\beta > 0$. $\square$

The next question is whether the compact Levi flat CR manifolds $L = L_\rho = N/\Gamma$ constructed above can be embedded in $\mathbb{CP}_2$. They cannot be:

THEOREM 4. *For no homomorphism $\rho : \Gamma \to \mathcal{G}$ does L admit a smooth CR embedding in $\mathbb{CP}_2$.*

The proof will involve CR vector bundles and their curvature; since the notion of CR vector bundles does not seem to be universally agreed on in the literature, we start by explaining the notions we will use. Again, we review the relevant notions in greater generality than what is strictly needed here.

Definition 4. A partial connection on a complex vector bundle $E \to (M, P)$ is a $\mathbb{C}$-linear map $\nabla : \mathcal{E}(M, E) \to \mathcal{E}_P^{0,1}(M, E)$ verifying $\nabla(f\sigma) = \left(\overline{\partial}_P f\right)\sigma + f\nabla\sigma$ for every $f \in \mathcal{E}(M)$ and $\sigma \in \mathcal{E}(M, E)$.

If $v \in \overline{P}_x$, we let $\nabla_v\sigma = (\nabla\sigma)(v) \in E_x$.

Definition 5. A CR structure on E is given by a partial connection ∇ on E such that for any $v, w \in \mathcal{E}(M, \overline{P})$,

$$\nabla_v\nabla_w - \nabla_w\nabla_v = \nabla_{[v,w]}, \tag{5.3}$$

where $[\cdot,\cdot]$ is the Lie bracket.

A CR vector bundle is a complex vector bundle endowed with a CR structure; we say that a section $\sigma \in \mathcal{E}(M, E)$ is CR if $\nabla\sigma = 0$.

In this generality, there is no reason why a CR vector bundle should have a nonzero CR section. Note that when P has rank 1, any partial connection will satisfy (5.3), because any two sections v, w are proportional.

Suppose $E, F \to (M, P)$ are CR vector bundles whose CR structures are defined by partial connections ∇^E, ∇^F. A homomorphism $\Phi : E \to F$ is said to be CR if $\nabla^F(\Phi \circ \sigma) = \Phi \circ (\nabla^E\sigma)$ for any $\sigma \in \mathcal{E}(M, E)$. Isomorphisms of CR vector bundles are defined accordingly.

A submanifold $M' \subset M$ inherits a CR manifold structure from (M, P) if $P' = (P|M') \cap \mathbb{C}TM'$ is a subbundle of $\mathbb{C}TM'$. If so, the restriction to M' of a CR vector bundle $E \to (M, P)$ is a CR vector bundle $E|M' \to (M', P')$. In the particular case when $P \oplus \overline{P} = \mathbb{C}TM$, (M, P) is in fact a complex manifold, P being the $(1, 0)$ tangent bundle; CR vector bundles are holomorphic vector bundles, ∇ being the Cauchy–Riemann operator acting on sections; any smooth hypersurface $M' \subset M$ is a CR submanifold, and any holomorphic vector bundle over M restricts to a CR vector bundle over M'.

Lemma 5. *Let (M, P) be a CR manifold. Let E be the subbundle of 1-forms on M which vanish on $\overline{P}$. Then:*

i) $\nabla : \mathcal{E}(M, E) \to \mathcal{E}^{0,1}_P(M, E)$, *given for* $\sigma \in \mathcal{E}(M, E) \subset \mathcal{E}(M, \mathbb{C}T^*M)$ *and* $v \in \overline{P}$ *by* $\nabla_v \sigma = \iota_v \mathrm{d}\sigma$, *defines a CR structure on* E.

ii) *Suppose that M is a real hypersurface embedded in a complex manifold Z. Then the homomorphism $\Lambda^{1,0}T^*Z\,|M \to E$ given by restriction is an isomorphism of CR bundles.*

We emphasize that in the lemma 1-forms refer to honest forms on M and not to forms on (M, P).

Proof. i) (M,P) being a CR manifold, if $v, w \in \mathcal{E}\big(M,\overline{P}\big)$, $[v, w] \in \mathcal{E}\big(M,\overline{P}\big)$, so $\sigma\,(v) = \sigma\,(w) = \sigma\,([v, w]) = 0$ and

$$\iota_w\,(\iota_v \mathrm{d}\sigma) = \mathrm{d}\sigma\,(v, w) = v\,(\sigma\,(w)) - w\,(\sigma\,(v)) - \sigma\,([v, w]) = 0.$$

Hence $\nabla_v \sigma | \overline{P} = 0$, and $\nabla_v \sigma$ is a section of E.

Let $f \in \mathcal{E}\,(M)$ and $\sigma \in \mathcal{E}\,(M, E)$. Then for $v \in \overline{P}$, we have

$$\nabla_v\,(f\sigma) = \iota_v \mathrm{d}\,(f\sigma) = \iota_v\,\big(\overline{\partial}_P f\big)\,\sigma + f \iota_v \mathrm{d}\sigma.$$

or $\nabla_v\,(f\sigma) = \big(\overline{\partial}_P f\big)\,(v)\,\sigma + f\nabla_v\sigma$; therefore, ∇ is a partial connection.

Finally, denoting Lie derivative by $\mathcal{L}$, for $\sigma \in \mathcal{E}\,(M, E)$ and $v \in \mathcal{E}\,\big(M, \overline{P}\big)$ we have

$$\mathcal{L}_v \sigma = \iota_v \mathrm{d}\sigma + \mathrm{d}\iota_v \sigma = \iota_v \mathrm{d}\sigma = \nabla_v \sigma,$$

and (5.3) follows.

ii) Consider a complex vector space V of dimension n, a complex vector subspace of $W \subset V$ of dimension $n - 1$ and a real vector subspace $H \supset W$ of dimension $2n - 1$. It is straightforward that restricting $\mathbb{C}$-linear forms on V to H gives a bijection between V^* and the space of $\mathbb{R}$-linear forms on H which are $\mathbb{C}$-linear on W. Therefore, the $\Lambda^{1,0}T^*Z\,|M \to E$ of the lemma is an isomorphism of complex vector bundles. In fact, it is an isomorphism of CR vector bundles, as one checks from the definition. □

Proof of Theorem 4. With the CR manifold L of the theorem we associate the CR vector bundle E, the subbundle of 1-forms on L which vanish on $(0, 1)$-tangent vectors to the leaves, as in Lemma 5. Pull back a not identically zero holomorphic 1-form on $\mathbb{D}/\Gamma$ by the projection $X = M/\Gamma \to \mathbb{D}/\Gamma$ to obtain a holomorphic section of $\Lambda^{1,0}T^*X$ and denote its restriction to $L = N/\Gamma$ by σ. By Lemma 5, this is a not identically zero CR section of E.

Suppose L is embedded in $\mathbb{CP}_2$. The restriction homomorphism

$$\Lambda^{1,0}T^*\mathbb{CP}_2\,|L \to E,$$

again by Lemma 5, is an isomorphism of CR bundles. The Fubini-Study metric on $\mathbb{CP}_2$ induces on E a metric h whose curvature along the leaves of L is strictly negative. Hence $h \circ \sigma$ is a subharmonic function along the leaves; in fact $h \circ \sigma$ is strictly subharmonic where nonzero. But, since L is compact, this gives a contradiction at the point where $h \circ \sigma$ attains its maximum. □

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Sorbonne Université
Institut de Mathématiques de Jussieu-Paris Rive Gauche
Faculté des Sciences et Ingénierie
(former Université Pierre et Marie Curie)
4 Place Jussieu, 75252 Paris Cedex 05, France
andrei.iordan@imj-prg.fr