Dedicated to the memory of Mihnea Colţoiu

ON THE JORDAN STRUCTURE OF HOLOMORPHIC MATRICES

JÜRGEN LEITERER

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Let $X \subset \mathbb{C}^N$ be open, and let A be an $n \times n$ matrix of holomorphic functions on X. We call a point $\xi \in X$ **Jordan stable** for A if ξ is not a splitting point of the eigenvalues of A and, moreover, there is a neighborhood U of ξ such that, for each $1 \leq k \leq n$, the number of Jordan blocks of size k in the Jordan normal forms of $A(\zeta)$ is the same for all $\zeta \in U$. H. Baumgärtel [4, S 3.4] proved that there is a nowhere dense closed analytic subset of X, which contains the set of all non-Jordan stable points. We give a new proof of this result. This proof shows that the set of non-Jordan stable points ist not only contained in a nowhere dense closed analytic subset, but it is itself such a set, and can be defined by holomorphic functions, the growth of which is bounded by some power (depending only on n) of the growth of A. Also, this proof applies to arbitrary (possibly non-smooth) reduced complex spaces X.

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1. INTRODUCTION

Let X be a connected open subset of $\mathbb{C}^N$, and let A be an $n \times n$ matrix of holomorphic functions on X, $N, n = 1, 2, \ldots$.

We call $\xi \in X$ a **splitting point of the eigenvalues of** A if, for each neighborhood $U \subseteq X$ of ξ, there is a point $\zeta \in U$ such that $A(\zeta)$ has more distinct eigenvalues than $A(\xi)$. It is well-known (cp. Remark 3.6) that the set of splitting points of the eigenvalues of A is a nowhere dense closed analytic subset of X.[1]

We call $\xi \in X$ **Jordan stable for** A if ξ is not a splitting point of the eigenvalues of A and, moreover, there is a neighborhood U of ξ such that, for each $1 \leq k \leq n$, the number of Jordan blocks of size k in the Jordan normal forms of $A(\zeta)$ does not depend on $\zeta \in U$. Let $\operatorname{Jst} A$ be the set of Jordan stable points of A.

[1] $Y \subseteq X$ is called a **closed analytic subset of** X if, for each $\xi \in X$, there exist a neighborhood $U \subseteq X$ of ξ and holomorphic functions $f_1, \ldots, f_\ell$ on U such that $Y \cap U = \{f_1 = \ldots = f_\ell = 0\}$. For $N = 1$ this means that Y is closed and discrete in X.

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H. Baumgärtel proved that, if $X \setminus \operatorname{Jst} A \neq X$, then $X \setminus \operatorname{Jst} A$ is contained in some nowhere dense closed analytic subset of X, see [1], [2, Kap. V, §7], [4, 5.7] for $N = 1$, and [3], [4, S 3.4] for arbitrary N.

In the present paper, we give a new proof for this, which leads to more precise results. For example, Theorem 6.5 says that, if $X \setminus \operatorname{Jst} A \neq X$, then $X \setminus \operatorname{Jst} A$ is not only *contained* in a nowhere dense closed analytic subset of X, but it is itself such a set. Moreover, there exist holomorphic functions $h_1, \ldots, h_\ell$ on X such that

$$X \setminus \operatorname{Jst} A = \big\{h_1 = \ldots = h_\ell = 0\big\}$$

and

$$|h_j(\zeta)| \le (2n)^{7n^2} 2^{n^3} (1 + \|A(\zeta)\|)^{3n^3} \quad \text{for all} \quad \zeta \in X \text{ and } 1 \le j \le \ell.$$

This implies, for example,

– If A is bounded, then $X \setminus \operatorname{Jst} A$ can be defined by bounded functions.

– If X is the unit disk and A is bounded, then $X \setminus \operatorname{Jst} A$ satisfies the Blaschke condition.

– If $X = \mathbb{C}^N$, and the coefficients of A are holomorphic polynomials, then $X \setminus \operatorname{Jst} A$ can be defined by finitely many holomorphic polynomials. For $N = 1$ this means that $X \setminus \operatorname{Jst} A$ is finite.

Also, our proof applies to the more general situation when X is a connected reduced complex space (possibly not smooth).

2. NOTATION

$\mathbb{N}$ denotes the set of natural numbers including 0. $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

If $n, m \in \mathbb{N}^*$, then $\operatorname{Mat}_{n\times m}(\mathbb{C})$ is the space of complex $n \times m$ matrices (n rows, m columns). We write $\operatorname{Mat}_n(\mathbb{C}) := \operatorname{Mat}_{n\times n}(\mathbb{C})$ and $\operatorname{GL}(n, \mathbb{C})$ is the group of invertible matrices in $\operatorname{Mat}_n(\mathbb{C})$.

The matrices $\Phi \in \operatorname{Mat}_{n\times m}(\mathbb{C})$ are often interpreted as linear operators from $\mathbb{C}^n$ to $\mathbb{C}^m$ acting by multiplication from the left. Then by $\|\Phi\|$ we mean the operator norm of Φ (and not the Hilbert-Schmidt norm), where $\mathbb{C}^n$ and $\mathbb{C}^m$ are considered as Hilbert spaces endowed with the Euclidean norm.

If $\Phi \in \operatorname{Mat}_{n\times m}(\mathbb{C})$, then $\operatorname{Ker} \Phi$, $\operatorname{Im} \Phi$ and $\operatorname{rank} \Phi$ are the kernel, the image and the rank of Φ, respectively.

The unit matrix in $\operatorname{Mat}_n(\mathbb{C})$ will be denoted by I_n or simply by I. For $\Phi \in \operatorname{Mat}_n(\mathbb{C})$ and $\lambda \in \mathbb{C}$, we usually write $\lambda - \Phi$ instead of $\lambda I_n - \Phi$.

By a **complex space**, we always mean a *reduced* complex space in the sense of, e.g., [9], which is the same as an *analytic* space in the sense of, e.g.,

[11]. For example, each complex manifold and each analytic subset of a complex manifold is a complex space in this sense.

By an **irreducible** complex space, we mean a *globally* irreducible complex space, i.e., a complex space, for which the manifold of smooth points is connected, see, e.g., [11, Ch. V.4.5] or [9, Ch. 9, §1]. For example, each connected complex manifold is an irreducible complex space.

If X is a topological space and $Y \subseteq X$, then we denote by $\overline{Y}$ the closure of Y in X, and we set $\partial Y = \overline{X} \setminus X$.

3. SPLITTING POINTS OF THE ZEROS OF MONIC POLYNOMIALS

First, we collect some (known) facts on the behavior of the zeros of polynomials depending on a parameter. For convenience of the reader, we supply proofs or precise references.

Definition 3.1. By a polynomial, we mean a function $p : \mathbb{C} \to \mathbb{C}$ of the form $p(\lambda) = p_0 + p_1\lambda + \ldots + p_n\lambda^n$, where $n \in \mathbb{N}$ and $p_0, \ldots, p_n \in \mathbb{C}$. If $p_n \neq 0$, then n is called the **degree** of p, denoted by $\deg p$. If $n \geq 1$ and $p_n = 1$, then P is called **monic**. If $p_0 = \ldots = p_n = 0$, then p is called the **zero polynomial** (which does not have a degree and is not monic).

Let $n \in \mathbb{N}^*$. Then we denote by $\mathcal{P}_n$ the complex vector space which consists of the zero polynomial and all polynomials P which are not identically zero such that $0 \leq \deg P \leq n$. Note that the complex dimension of $\mathcal{P}_n$ is $n+1$. For example, the polynomials λ^ℓ, $\ell = 0, 1, \ldots, n$, form a complex linear basis of $\mathcal{P}_n$.

PROPOSITION 3.2. *Let X be a connected topological space and $P : X \to \mathcal{P}_n$, $n \in \mathbb{N}^*$, a continuous map all values of which are of degree n and monic. Suppose, there exists $m \in \{1, \ldots, n\}$ such that, for each $\zeta \in X$, $P(\zeta)$ has m distinct zeros. Moreover, assume that there are continuous functions $\lambda_1, \ldots, \lambda_m : X \to \mathbb{C}$ such that, for each $\zeta \in X$, $\lambda_1(\zeta), \ldots, \lambda_m(\zeta)$ are the distinct zeros of $P(\zeta)$.*[2] *Then:*

(i) *For each $1 \leq j \leq m$, the order of $\lambda_j(\zeta)$ as a zero of $P(\zeta)$ is the same for all $\zeta \in X$.*

(ii) *Suppose that there is a second collection of continuous functions $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_m : X \to \mathbb{C}$ such that also, for each $\zeta \in X$, $\widetilde{\lambda}_1(\zeta), \ldots, \widetilde{\lambda}_m(\zeta)$ are the distinct zeros of $P(\zeta)$, and moreover, for at least one $\xi \in X$, $\widetilde{\lambda}_j(\xi) = \lambda_j(\xi)$ for all $1 \leq j \leq m$. Then $\widetilde{\lambda}_j(\zeta) = \lambda_j(\zeta)$ for all $\zeta \in X$ and $1 \leq j \leq m$.*

[2] This means, $\{\lambda_1(\zeta), \ldots, \lambda_m(\zeta)\}$ is the set of zeros of $P(\zeta)$ and $\lambda_i(\zeta) \neq \lambda_j(\zeta)$ if $i \neq j$.

Proof of part (i). Since X is connected, we only have to prove that, for each $\xi \in X$, there is a neighborhood U of ξ such that, for each $j = 1, \ldots, m$, the order of $\lambda_j(\zeta)$ as a zero of $P(\zeta)$ is the same for all $\zeta \in U$.

Let $\xi \in X$ be given. Choose $\varepsilon > 0$ so small that the disks

$$\mathbb{D}_j := \big\{\lambda \in \mathbb{C} \,\big|\, |\lambda - \lambda_j(\xi)| < \varepsilon\big\}, \quad 1 \le j \le m, \tag{3.1}$$

are pairwise disjoint. Then $P(\xi)(\lambda) \neq 0$ for

$$\lambda \in \partial\mathbb{D}_1 \cup \ldots \cup \partial\mathbb{D}_m.$$

Since P is continuous and the set

$$\partial\mathbb{D}_1 \cup \ldots \cup \partial\mathbb{D}_m$$

is compact, it follows: there is a neighborhood U of ξ such that $P(\zeta)(\lambda) \neq 0$ for

$$\zeta \in U \quad \text{and} \quad \lambda \in \partial\mathbb{D}_1 \cup \ldots \cup \partial\mathbb{D}_m.$$

Therefore, if n_j is the order of $\lambda_j(\xi)$ as a zero of $P(\xi)$, by Rouché's theorem:

(*) for all $\zeta \in U$ and $j = 1, \ldots, m$, <u>counting multiplicities</u>, $P(\zeta)$ has exactly n_j zeros in $\mathbb{D}_j$.

On the other hand, also the functions λ_j are continuous. Therefore, shrinking U, we can achieve that

$$|\lambda_j(\zeta) - \lambda_j(\xi)| < \varepsilon$$

for all $\zeta \in U$ and $1 \le j \le m$. Since the disks (3.1) are pairwise disjoint and, for each ζ, all zeros of $P(\zeta)$ lie in $\big\{\lambda_1(\zeta), \ldots, \lambda_m(\zeta)\big\}$, this implies that, for all $\zeta \in U$ and $1 \le j \le m$, $\lambda_j(\zeta)$ is the only zero of $P(\zeta)$ which lies in the disk

$$\big\{\lambda \in \mathbb{C} \,\big|\, |\lambda - \lambda_j(\xi)| < \varepsilon\big\}.$$

Together with (*), this implies that, for all $\zeta \in U$, n_j is the order of $\lambda_j(\zeta)$ as a zero of $P(\zeta)$. In particular, for $j = 1, \ldots, m$, the order of $\lambda_j(\zeta)$ as a zero of $P(\zeta)$ is the same for all $\zeta \in U$.

Proof of part (ii). Let M be the set of points $\zeta \in X$ with $\widetilde{\lambda}_j(\zeta) = \lambda_j(\zeta)$ for $1 \le j \le m$. Since λ_j and $\widetilde{\lambda}_j$ are continuous, M is closed. By hypothesis, $M \neq \emptyset$. Therefore, (X is connected) it remains to prove that M is open. Let $\xi \in M$ be given. Choose $\varepsilon > 0$ so small that the disks

$$\mathbb{D}_j := \big\{\lambda \in \mathbb{C} \,\big|\, |\lambda - \lambda_j(\xi)| < \varepsilon\big\}, \quad 1 \le j \le m,$$

are pairwise disjoint. Since the functions λ_j and $\widetilde{\lambda}_j$ are continuous, then we can find a neighborhood U_ξ of ξ so small that $\lambda_j(\zeta), \widetilde{\lambda}_j(\zeta) \in \mathbb{D}_j$ for all $\zeta \in U_\xi$ and $1 \le j \le m$. Since all zeros of $P(\zeta)$ lie in the set

$$\big\{\lambda_1(\zeta), \ldots, \lambda_m(\zeta)\big\} = \big\{\widetilde{\lambda}_1(\zeta), \ldots, \widetilde{\lambda}_m(\zeta)\big\},$$

then, for all $\zeta \in U_\xi$ and $1 \le j \le m$, we have the following two statements.

- $\lambda_j(\zeta)$ is the only zero of $P(\zeta)$ which lies in $\mathbb{D}_j$.
- $\widetilde{\lambda}_j(\zeta)$ is the only zero of $P(\zeta)$ which lies in $\mathbb{D}_j$.

Hence $\lambda_j(\zeta) = \widetilde{\lambda}_j(\zeta)$ for all $\zeta \in U_\xi$ and $1 \le j \le m$, i.e., $U_\xi \subseteq M$. □

Definition 3.3. Let X be a topological space and $P : X \to \mathcal{P}_n$, $n \in \mathbb{N}^*$, a continuous map, all values of which are of degree n and monic. Then $\xi \in X$ is called a **splitting point of the zeros of** P if, for each neighborhood U of ξ, there exists $\zeta \in U$ such that $P(\zeta)$ has more zeros than $P(\xi)$ (not counting multiplicities).

PROPOSITION 3.4. *Let X be a topological space and $P : X \to \mathcal{P}_n$, $n \in \mathbb{N}^*$, a continuous map, all values of which are monic of degree n. Then $\xi \in X$ is not a splitting point of the zeros of P if and only if there exist a neighborhood U of ξ and continuous functions $\lambda_1, \ldots, \lambda_m : U \to \mathbb{C}$ such that, for each $\zeta \in U$, $\lambda_1(\zeta), \ldots, \lambda_m(\zeta)$ are the distinct zeros of $P(\zeta)$.*[3] *Moreover, if X is a complex space and P is holomorphic, then these functions are holomorphic.*

Proof. It is clear that the condition is sufficient. To prove the necessity, assume that ξ is not a splitting point of the zeros of P, and let m be the number of zeros of $P(\xi)$ (not counting multiplicities). Then, by definition, there is a neighborhood U_ξ of ξ such that

(3.2)
$$\forall \zeta \in U_\xi : \; m \ge \text{the number of zeros of } P(\zeta), \text{ not counting multiplicities.}$$

Let $w_1, \ldots, w_m$ be some enumeration of the distinct zeros of $P(\xi)$, and let n_j be the order of w_j as a zero of $P(\xi)$. Choose $\varepsilon > 0$ so small that the closed disks $\{\lambda \in \mathbb{C} \,\big|\, |\lambda - w_j| \le \varepsilon\}$, $1 \le j \le m$, are pairwise disjoint. Then each of these disks contains precisely one zero of $P(\xi)$, namely its center w_j. Therefore, by the Rouché theorem, shrinking U_ξ, we can achieve that, *counting multiplicities*, for each $\zeta \in U_\xi$, the number of zeros of $P(\zeta)$ which lie in $\{\lambda \in \mathbb{C} \,\big|\, |\lambda - w_j| \le \varepsilon\}$ is equal to n_j. In particular, each of these discs contains *at least one* zero of $P(\zeta)$, which means, by (3.2), that each of these disks contains *precisely one* zero of $P(\zeta)$, and the order of this zero is n_j. We denote it by $\lambda_j(\zeta)$.

It remains to prove that $\lambda_j(\zeta)$ depends continuously resp. holomorphically on $\zeta \in U_\xi$. For that, for a moment, we fix $\zeta \in U_\xi$. Since $\lambda_1(\zeta), \ldots \lambda_m(\zeta)$ are the distinct zeros of $P(\zeta)$, where the order of $\lambda_j(\zeta)$ is n_j, and since $P(\zeta)$ is monic, then

$$P(\zeta)(\lambda) = (\lambda - \lambda_1(\zeta))^{n_1} \cdot \ldots (\lambda - \lambda_m(\zeta))^{n_m}, \quad \lambda \in \mathbb{C},$$

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[3] By Lemma 3.2 (ii), up to the numbering, these functions are uniquely determined on each connected component of U.

and, for the complex derivative $P(\zeta)'$ of $P(\zeta)$, we have

$$P(\zeta)'(\lambda) = \sum_{j=1}^{m} n_j(\lambda - \lambda_j(\zeta))^{n_j-1} \cdot \prod_{k\neq j}(\lambda - \lambda_k(\zeta))^{n_k}, \quad \lambda \in \mathbb{C}.$$

Choose $\delta > 0$ so small that, for $j = 1, \ldots, m$, the closed disk $\{\lambda \in \mathbb{C} \,|\, |\lambda - \lambda_j(\zeta)| \le \delta\}$ is contained in $\{\lambda \in \mathbb{C} \,|\, |\lambda - w_j| < \varepsilon\}$. Then in a neighborhood of $\{\lambda \in \mathbb{C} \,|\, |\lambda - \lambda_j(\zeta)| \le \delta\}$,

$$\lambda \frac{P(\zeta)'(\lambda)}{P(\zeta)(\lambda)} = \frac{\lambda n_j}{\lambda - \lambda_j(\zeta)} + \text{holomorphic terms},$$

which implies that, for $j = 1, \ldots, m$,

$$\frac{1}{n_j} \int\limits_{|\lambda - \lambda_j(\zeta)| = \delta} \lambda \frac{P(\zeta)'(\lambda)}{P(\zeta)(\lambda)} d\lambda = \int\limits_{|\lambda - \lambda_j(\zeta)| = \delta} \frac{\lambda}{\lambda - \lambda_j(\zeta)} d\lambda$$

$$= \int\limits_{|\lambda - \lambda_j(\zeta)| = \delta} \left(\frac{\lambda_j(\zeta)}{\lambda - \lambda_j(\zeta)} + \frac{\lambda - \lambda_j(\zeta)}{\lambda - \lambda_j(\zeta)} \right) d\lambda = \lambda_j(\zeta) 2\pi i.$$

So, for $j = 1, \ldots, m$ and all $\zeta \in U_\xi$, we have proved the formula

$$\lambda_j(\zeta) = \frac{1}{n_j 2\pi i} \int\limits_{|\lambda - w_j| = \varepsilon} \lambda \frac{P(\zeta)'(\lambda)}{P(\zeta)(\lambda)} d\lambda \quad \text{for} \quad 1 \le j \le m.$$

This formula shows the required continuity, resp., holomorphicity. $\square$

The following theorem can be found in [11, Ch. V, §7.1].

THEOREM 3.5. *Let X be a complex space, and let $P : X \to \mathcal{P}_n$, $n \in \mathbb{N}^*$, be a holomorphic map all values of which are of degree n and monic. Then the set of splitting points of the zeros of P is a nowhere dense closed analytic subset of X.*

Proof. (Cp. [11, Ch. V, §7.1]). Since each complex space is the union of a locally finite family of irreducible complex spaces, see, e.g., [11, Ch. IV, §2.9] or [9, Ch. 9, §2.2]), we may assume that X is irreducible (i.e., the manifold of smooth points of X is connected).

Denote by $k(\zeta)$ the number of distinct zeros of $P(\zeta)$. Let

$$m := \max_{\zeta \in X} k(\zeta),$$

and let A be the set of all $(\lambda_1, \ldots, \lambda_m, \zeta) \in \mathbb{C}^m \times X$ such that $\lambda_1, \ldots, \lambda_m$ are zeros of $P(\zeta)$. Since P is holomorphic, A is a closed analytic subset of $\mathbb{C}^m \times X$. Let $\pi : A \to X$ be the restriction to A of the canonical projection $\mathbb{C}^m \times X \to X$.

We claim that π is proper (i.e., for each $\xi \in X$, there is a neighborhood U_ξ of ξ such that $\pi^{-1}(U_\xi)$ is relatively compact in A).

Indeed, let $\xi \in X$ be given. Take an arbitrary compact neighborhood U_ξ of ξ. Let $p_0, \dots, p_{n-1} : X \to \mathbb{C}$ be the functions with $P(\zeta)(\lambda) = \lambda^n + \sum_{\mu=0}^{n-1} p_\mu(\zeta)\lambda^\mu$, $\zeta \in X$, $\lambda \in \mathbb{C}$. Since P is continuous, then

$$C := 1 + \max_{\zeta \in U_\xi} \sum_{\mu=0}^{n-1} \big|p_\mu(\zeta)\big| < \infty.$$

Hence, if $\zeta \in U_\xi$ and λ is a zero of $P(\zeta)$, then $|\lambda| \le C$. Therefore, $\pi^{-1}(U_\xi)$ is contained in the compact set

$$A \cap \Big\{ (\lambda_1, \dots, \lambda_m, \zeta) \,\Big|\, \zeta \in U_\xi \text{ and } |\lambda_j| \le C \text{ for } 1 \le j \le m \Big\}$$

Now let $M := \big\{ \zeta \in X \,|\, k(\zeta) < m \big\}$, and let M' be the set of all $(\lambda_1, \dots, \lambda_m, \zeta) \in A$ such that at least two of the numbers $\lambda_1, \dots, \lambda_m$ are equal. Then M' is a closed analytic subset of A and $\pi(M') = M$. Since π is proper, this implies by Remmert's proper mapping theorem (see, e.g., [9, Ch. 10, §6.1] or [11, Ch. V, §5.1]) that M is a closed analytic subset of X. Since $M \neq X$ (by definition of m) and X is irreducible, M is nowhere dense in X.

It remains to observe that M is the set of splitting points of the zeros of P. Indeed, if ξ is a splitting point, then, by definition of m, $k(\xi) < m$, i.e., $\xi \in M$. Conversely, let $\xi \in M$, i.e., $k(\xi) < m$. Assume ξ is not a splitting point. Then there is a neighborhood U_ξ of ξ such that $k(\zeta) \le k(\xi)$ for all $\zeta \in U_\xi$. Since $k(\xi) < m$, this implies that $U_\xi \subseteq M$, which is not possible, because M is nowhere dense in X. □

Remark 3.6. If X is a complex manifold, there are many other sources for Theorem 3.5 in the literature, see, e.g., [7, Ch. III, Satz 6.5 and Satz 6.12], [6, Ch. III, Theorems 4.3 and 4.6], [3], [4, S3.1]. There, for the proof, the fact is used that P can be written as a finite product

$$P = \omega_1^{r_1} \cdot \ldots \cdot \omega_\ell^{r_\ell}, \tag{3.3}$$

where $r_i \in \mathbb{N}^*$, each ω_i is a monic polynomial with coefficients from $\mathcal{O}(X)$ of positive degree, each ω_i is prime as an element of the monoid of all monic polynomials with coefficients from $\mathcal{O}(X)$, and $\omega_i \neq \omega_j$ if $i \neq j$. Then it is proved that the discriminant of the polynomial $\omega_1(\zeta) \cdot \ldots \cdot \omega_\ell(\zeta)$, Δ, does not identically vanish, and that $\{\Delta = 0\}$ is the set of splitting points of the zeros of P.

Note that this proof also shows that the set of splitting points of the zeros of P, at each point of this set, is of codimension 1 in X.

4. **A NEW PROOF OF THEOREM 3.5**

Here, we give a new proof of Theorem 3.5, which results in a more precise statement with estimates. In this proof, we do not use the factorization (3.3) (also not in the case when X is a complex manifold).

Definition 4.1. Let p be a monic polynomial of degree $n \geq 2$. Then we denote by Φ_p the complex linear map from $\mathcal{P}_{n-2} \oplus \mathcal{P}_{n-1}$ to $\mathcal{P}_{2n-2}$ defined by

$$\Phi_p(s, q) = ps - p'q \quad \text{for} \quad (s, q) \in \mathcal{P}_{n-2} \oplus \mathcal{P}_{n-1},$$

where p' denotes the complex derivative of p.

The main tool of our proof is the following lemma, which is known (see [10, §2, 1, VII] or [8, Theorem 0.1]). For convenience of the reader, we give a proof.

LEMMA 4.2. *Let p be a monic polynomial of degree $n \geq 2$, and let m be the number of zeros of p (not counting multiplicities). Then*[4]

$$\operatorname{rank} \Phi_p = n + m - 1. \tag{4.1}$$

Proof. Let $\lambda_1, \ldots, \lambda_m$ be the distinct zeros of p, and let k_j be the order of λ_j as a zero of p. Since p is monic and of degree n, then $k_1 + \ldots + k_m = n$ and

$$p(\lambda) = (\lambda - \lambda_1)^{k_1} \ldots (\lambda - \lambda_m)^{k_m}, \quad \lambda \in \mathbb{C}.$$

Set $q_0(\lambda) = (\lambda - \lambda_1) \ldots (\lambda - \lambda_m)$ and $s_0(\lambda) = \sum_{j=1}^{m} k_j(\lambda - \lambda_1) \ldots \underset{\widehat{j}}{} \ldots (\lambda - \lambda_m)$. Then

$$ps_0 = p'q_0. \tag{4.2}$$

Next, we prove the following

Claim. $\operatorname{Ker} \Phi_p = \big\{(s_0 a, q_0 a) \,\big|\, a \in \mathcal{P}_{n-1-m}\big\}$.

Proof of "$\supseteq$" in the Claim: For $m = n$ this is trivial. Let $1 \leq m \leq n-1$ and $a \in \mathcal{P}_{n-1-m}$. Since s_0 is of degree $m-1$ and q_0 of degree m, then $(s_0 a, q_0 a) \in \mathcal{P}_{n-2} \oplus \mathcal{P}_{n-1}$, and by (4.2), $\Phi_p(s_0 a, q_0 a) = (ps_0 - p'q_0)a = 0$.

Proof of "$\subseteq$" in the Claim: Let $(s, q) \in \operatorname{Ker} \Phi_p$, i.e., $s \in \mathcal{P}_{n-2}$, $q \in \mathcal{P}_{n-1}$ and

$$ps = p'q. \tag{4.3}$$

[4] One can show that $\pm \det \Phi_p$ is the discriminant of p (see, e.g., [14, §35]). Therefore, this lemma in particular contains the well-known fact that p has no multiple zeros if and only if its discriminant is different from zero.

Then each λ_j is a zero of order $\geq k_j$ of $p'q$. Since the order of λ_j as a zero of p' is $< k_j$ (for $k_j = 1$, by this we mean that $p'(\lambda_j) \neq 0$), it follows that each λ_j is a zero of q. Hence, q is of the form

$$q = q_0 a, \tag{4.4}$$

where a is some complex polynomial (possibly, $a \equiv 0$). Then

$$a \in \mathcal{P}_{n-1-m}. \tag{4.5}$$

Indeed, for $a \equiv 0$, this is trivial. If $a \not\equiv 0$, from (4.4) it follows that $\deg a = \deg q - \deg q_0$. Since $\deg q_0 = m$ and $\deg q \leq n-1$, this implies that $\deg a \leq n - 1 - m$. Hence, we have (4.5). Moreover, by (4.2), (4.4) and (4.3),

$$p s_0 a = p' q_0 a = p' q = ps.$$

As $p \not\equiv 0$, this implies that $s = s_0 a$. Together with (4.4) and (4.5) this proves that (s, q) belongs to $\{(s_0 a, q_0 a) \,|\, a \in \mathcal{P}_{n-1-m}\}$. The Claim is proved.

Now, we consider the complex linear map

$$\begin{aligned} \Psi : \mathcal{P}_{n-1-m} &\longrightarrow \mathcal{P}_{n-2} \oplus \mathcal{P}_{n-1} \\ a \quad &\longmapsto \quad (s_0 a, q_0 a). \end{aligned}$$

Since $s_0 \not\equiv 0$ and $q_0 \not\equiv 0$, this map is injective. Hence

$$\dim \operatorname{Im} \Psi = \dim \mathcal{P}_{n-1-m} = n - m.$$

As, by the Claim, $\operatorname{Im} \Psi = \operatorname{Ker} \Phi_p$, it follows that

$$\dim \operatorname{Ker} \Phi_p = \dim \operatorname{Im} \Psi = n - m.$$

As $\operatorname{rank} \Phi_p = 2n - 1 - \dim \operatorname{Ker} \Phi_p$, this proves (4.1). $\square$

Definition 4.3. Let X be a topological space, and let

$$M : X \to \operatorname{Mat}_{n \times m}(\mathbb{C})$$

be a continuous map. A point $\xi \in X$ will be called a **jump point** of the rank of M if, for each neighborhood U of ξ, there exists $\zeta \in U$ such that $\operatorname{rank} M(\zeta) > \operatorname{rank} M(\xi)$.

LEMMA 4.4. *Let X be an irreducible complex space*[5]*, and let $M : X \to \operatorname{Mat}_{n \times m}(\mathbb{C})$ a holomorphic map which is not identically zero. Set*

$$r_{\max} := \max_{\zeta \in X} \operatorname{rank} M(\zeta),$$

and denote by $h_1, \ldots, h_\ell$ be the minors of order $r_{\max}$ of M. Then

$$\{\zeta \in X \,|\, h_1(\zeta) = \ldots = h_\ell(\zeta) = 0\} \tag{4.6}$$

is the set of jump points of the rank of M.

[5]Recall that a complex space is called **irreducible** if the manifold of smooth points of X is connected

Proof. First let ξ be a jump point of the rank of M. Then, by definition, there exists $\zeta \in X$ such that $\operatorname{rank} M(\xi) < \operatorname{rank} M(\zeta)$. In particular, $\operatorname{rank} M(\xi) < r_{\max}$. Hence, all minors of order $r_{\max}$ of $M(\xi)$ vanish, i.e., ξ lies in (4.6).

Now let $\xi \in X$ be a point which lies in (4.6). Since X is irreducible and M is holomorphic and $\not\equiv 0$, and, hence, the set (4.6) is nowhere dense in X, then, for each neighborhood U of ξ, there exists $\zeta \in U$ which does not belong to (4.6), i.e., such that at least one of the minors of order $r_{\max}$ of $M(\zeta)$ is not zero, i.e., such that $\operatorname{rank} M(\zeta) > r_{\max}$. Hence, ξ is a jump point of the rank of M. □

Now, we are ready to give the announced new proof Theorem 3.5. Actually, we prove the following more precise theorem.

THEOREM 4.5. *Let X be a complex space and let $P : X \to \mathcal{P}_n$, $n \in \mathbb{N}^*$, be a holomorphic map, all values of which are of degree n and monic.*

Then the set of splitting points of the zeros of P is a nowhere dense closed analytic subset of X.

Moreover, if X is irreducible and there is at least one splitting point of the zeros of P, then there exist holomorphic functions $h_1, \dots, h_\ell$ on X, where $\ell \le ((2n-1)!)^2$, each of which is a sum of not more than $(2n-1)!$ products of $(2n-1)n$ functions from $\{\pm p_0, \dots, \pm p_n\}$, where $p_0, \dots, p_n$ are the coefficients of P, i.e., the holomorphic functions on X functions with

$$P(\zeta)(\lambda) = p_0(\zeta) + p_1\zeta)\lambda + \ldots + p_n(\zeta)\lambda^n, \quad \zeta \in X,\ \lambda \in \mathbb{C},$$

such that the set of splitting points of the zeros of P is equal to

$$\big\{\zeta \in X \,\big|\, h_1(\zeta) = \ldots = h_\ell(\zeta) = 0\big\}, \tag{4.7}$$

and

$$|h_j(\zeta)| \le (2n)^{4n} \max_{0\le k\le n} |p_k(\zeta)|^{2n-1} \quad \textit{for all} \quad \zeta \in X \textit{ and } 1 \le j \le \ell. \tag{4.8}$$

Proof. If there is no splitting point of the zeros of P, the claim of the theorem is trivial (as also the empty set is called analytic). Therefore, we may assume that P has at least one splitting point of the zeros of P. Then $\deg P \ge 2$ and, hence, $n \ge 2$.

First, we moreover assume that X is irreducible.

Let $L\big(\mathcal{P}_{n-2} \oplus \mathcal{P}_{n-1}, \mathcal{P}_{2n-2}\big)$ be the space of complex linear maps from $\mathcal{P}_{n-2} \oplus \mathcal{P}_{n-1}$ to $\mathcal{P}_{2n-2}$, and let

$$F : X \to L\big(\mathcal{P}_{n-2} \oplus \mathcal{P}_{n-1}, \mathcal{P}_{2n-2}\big)$$

be the holomorphic map defined by $F(\zeta) = \Phi_{P(\zeta)}$, $\zeta \in X$ (see Def. 4.1), i.e.,

$$F(\zeta)(s,q) := P(\zeta)s - P(\zeta)'q, \qquad \zeta \in X, \quad (s,q) \in \mathcal{P}_{n-2} \oplus \mathcal{P}_{n-1}. \tag{4.9}$$

Let $u_0, \dots, u_{2n-2}$ and $v_0, \dots, v_{2n-2}$ be the polynomials defined as follows:

$$\begin{aligned} u_j(\lambda) &= \lambda^j, && \lambda \in \mathbb{C} \quad \text{and} \quad j = 0, \dots, n-2,\\ u_j(\lambda) &= \lambda^{j+1-n}, && \lambda \in \mathbb{C} \quad \text{and} \quad j = n-1, \dots, 2n-2,\\ v_j(\lambda) &= \lambda^j, && \lambda \in \mathbb{C} \quad \text{and} \quad j = 0, \dots, 2n-2. \end{aligned}$$

Then

$$(u_0, 0), \dots, (u_{n-2}, 0), (0, u_{n-1}), \dots, (0, u_{2n-1}) \tag{4.10}$$

is a complex linear basis of $\mathcal{P}_{n-2} \oplus \mathcal{P}_{n-1}$ and

$$v_0, \dots, v_{2n-1} \tag{4.11}$$

is a complex linear basis of $\mathcal{P}_{2n-2}$. Therefore, and since F is holomorphic, we have uniquely determined holomorphic functions $M_{ij} : X \to \mathbb{C}$, $i,j = 0, \dots, 2n-2$, such that, for all $\zeta \in X$,

$$\sum_{j=0}^{2n-2} M_{ij}(\zeta) v_j = \begin{cases} F(\zeta)(u_i, 0) & \text{for} \quad 0 \le i \le n-2,\\ F(\zeta)(0, u_i) & \text{for} \quad n-1 \le i \le 2n-2. \end{cases} \tag{4.12}$$

By (4.9), this implies that, for all $\zeta \in X$,

$$\sum_{j=0}^{2n-2} M_{ij}(\zeta) v_j = \begin{cases} P(\zeta) u_i & \text{for } 0 \le i \le n-2,\\ -P'(\zeta) u_i & \text{for } n-1 \le i \le 2n-2. \end{cases} \tag{4.13}$$

Now let $p_0, \dots, p_n$ be the holomorphic functions on X such that, for all $\zeta \in X$,

$$P(\zeta)(\lambda) = \sum_{k=0}^{n} p_k(\zeta) \lambda^k, \quad \lambda \in \mathbb{C}.$$

(Recall that P is of degree n and monic, so that $p_n \equiv 1$.) Then (4.13) takes the form

$$\sum_{j=0}^{2n-2} M_{ij}(\zeta) \lambda^j = \begin{cases} \Big(\sum_{k=0}^{n} p_k(\zeta) \lambda^k \Big) \lambda^i & \text{for } 0 \le i \le n-2,\\ -\Big(\sum_{k=1}^{n} k p_k(\zeta) \lambda^{k-1} \Big) \lambda^{i+1-n} & \text{for } n-1 \le i \le 2n-2, \end{cases}$$

i.e., for all $\zeta \in X$, we have

$$\sum_{j=0}^{2n-2} M_{ij}(\zeta) \lambda^j = \begin{cases} \sum_{k=0}^{n} p_k(\zeta) \lambda^{k+i} & \text{for } 0 \le i \le n-2,\\ -\sum_{k=1}^{n} k p_k(\zeta) \lambda^{k+i-n} & \text{for } n-1 \le i \le 2n-2, \end{cases}$$

Comparing the coefficients of λ^j, from this, we see:

(4.14) Each of the functions M_{ij} is one of the functions

$$p_0, p_1, \dots, p_n \text{ and } -2p_2, -3p_3, \dots, -np_n.$$

Let $M : X \to M_{2n-1}(\mathbb{C})$ be the holomorphic map such that $M_{i-1,j-1}$ is the element in row i and column j. Again using that (4.10) is a basis of $\mathcal{P}_{n-2} \oplus \mathcal{P}_{n-1}$ and (4.11) is a basis of $\mathcal{P}_{2n-2}$, then it follows from (4.12) that

$$\operatorname{rank} M(\zeta) = \operatorname{rank} F(\zeta) \quad \text{for all } \zeta \in X.$$

Since, by Lemma 4.2, for all $\zeta \in X$, the number of zeros of $P(\zeta)$ is equal to $\operatorname{rank} F(\zeta) - n + 1$, this implies:

(4.15)

$$\text{for all } \zeta \in X, \text{ the number of zeros of } P(\zeta) \text{ is equal to } \operatorname{rank} M(\zeta) - n + 1.$$

Hence:

(4.16) The set of splitting points of the zeros of P is equal to the set of jump points of M.

Set

$$r_{\max} := \max_{\zeta \in X} \operatorname{rank} M(\zeta),$$

and let $h_1, \dots, h_\ell$ be the minors of order $r_{\max}$ of M, $\ell = \binom{2n-1}{r_{\max}}^2$.

For all $\zeta \in X$, $P(\zeta)$ is of positive degree and, therefore, has at least one zero. By (4.15) this implies that $\operatorname{rank} M(\zeta) \geq n$ for all $\zeta \in X$. In particular, M is not identically zero. Therefore, we can apply Lemma 4.4 to M and obtain:

$$\left\{ \zeta \in X \,\middle|\, h_1(\zeta) = \ldots = h_\ell(\zeta) = 0 \right\}$$

is the set of jump points of the rank of M, which means, by (4.16), that (4.7) is the set of splitting points of the zeros of P.

By (4.14), each of the functions $h_l, \dots, h_\ell$ is the sum of $r_{\max}!$ products of $r_{\max}$ functions from $\{p_0, p_1, \dots, p_n, -2p_2, -3p_3, \dots, -np_n\}$. Therefore, for $j = 1, \dots, \ell$ and all $\zeta \in X$, we have

$$|h_j(\zeta)| \leq r_{\max}! \Big(n \max_{0 \leq k \leq n} |p_k(\zeta)| \Big)^{r_{\max}!} = r_{\max}! n^{r_{\max}!} \max_{0 \leq k \leq n} |p_k(\zeta)|^{r_{\max}!}.$$

Since $r_{\max} \leq 2n - 1$ and $p_n \equiv 1$ (P is monic), and, hence,

$$r_{\max}! n^{r_{\max}!} \max_{0 \leq k \leq n} |p_k(\zeta)|^{r_{\max}!} \leq (2n-1)! n^{2n-1} \max_{0 \leq k \leq n} |p_k(\zeta)|^{2n-1} \leq (2n)^{4n},$$

this implies estimate (4.8).

This completes the proof in the case when X is irreducible.

Now, we consider the general case. By the global decomposition theorem for complex spaces (see, e.g., [11, V.4.6] or [9, Ch. 9, §2.2]), there is a locally finite covering $\{X_i\}_{i\in I}$ of X such that each X_i is an irreducible closed analytic subset of X. Then, clearly, the set of splitting points of the zeros of P is the union of the sets of spitting points of the zeros of $P|_{X_i}$, $i \in I$. Since, as already proved, each of these sets is a nowhere dense analytic subset of X_i, and the covering $\{X_i\}_{i\in I}$ is locally finite, this proves that the set of splitting points of the zeros of P is a nowhere dense analytic subset of X. □

Remark 4.6. A disadvantage of our proof of Theorem 4.5 is that it does not show that the set of splitting points of the zeros of P is of codimension 1 in X (in distinction to the well-known proof outlined in Remark 3.6). An advantage is that it shows for in the irreducible case that the set of splitting points of the zeros of P can be defined by functions satisfyng estimate (4.8). This implies, for example:

– If P is bounded, then the set of splitting points of the zeros of P can be defined by bounded functions.

– If X is the unit disk and P is bounded, then the set of splitting points of the zeros of P satisfies the Blaschke condition.

– If $X = \mathbb{C}^N$, and the coefficients of P are holomorphic polynomials, then set of splitting points of the zeros of p can be defined by finitely many holomorphic polynomials. For $N = 1$, this means that P has only a finite number of splitting points of the zeros (which is well-known from the theory of algebraic functions).

5. SPLITTING POINTS OF THE EIGENVALUES OF A MATRIX FUNCTION

Definition 5.1. Let X be a topological space, and let $A : X \to \mathrm{Mat}_n(\mathbb{C})$ be continuous. A point $\xi \in X$ is called a **splitting point of the eigenvalues of** A if, for each neighborhood U of ξ, there exists $\zeta \in U$ such that $A(\zeta)$ has more eigenvalues than $A(\xi)$ (not counting multiplicities).

Since, for each $\Phi \in \mathrm{Mat}_n(\mathbb{C})$, the eigenvalues of Φ are the zeros of the characteristic polynomial $\det\big(\lambda - \Phi\big)$, $\lambda \in \mathbb{C}$, which is of degree n and monic, and since the algebraic multiplicity of an eigenvalue of Φ is the order of this eigenvalue as a zero of the characteristic polynomial, from Propositions 3.2 and 3.4, we immediately obtain the following two lemmas.

LEMMA 5.2. *Let X be a connected topological space, $n \in \mathbb{N}^*$, and let $A : X \to \mathrm{Mat}_n(\mathbb{C})$ be a continuous map such that the number of different eigenvalues of $A(\zeta)$, denoted by m, is the same for all $\zeta \in X$. Sup-*

pose $\lambda_1, \ldots, \lambda_m : X \to \mathbb{C}$ *are continuous functions such that, for all* $\zeta \in X$, $\lambda_1(\zeta), \ldots, \lambda_m(\zeta)$ *are the distinct eigenvalues of* $A(\zeta)$.[6] *Then:*

(i) *For each* $1 \leq j \leq m$, *the algebraic multiplicity of* $\lambda_j(\zeta)$ *as an eigenvalue of* $A(\zeta)$ *is the same for all* $\zeta \in X$.

(ii) *If there is a second collection of continuous functions* $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_m : X \to \mathbb{C}$ *such that also, for each* $\zeta \in X$, $\widetilde{\lambda}_1(\zeta), \ldots, \widetilde{\lambda}_m(\zeta)$ *are the distinct eigenvalues of* $A(\zeta)$, *and if, for at least one point* $\xi \in X$, $\widetilde{\lambda}_j(\xi) = \lambda_j(\xi)$ *for all* $1 \leq j \leq m$, *then* $\widetilde{\lambda}_j(\zeta) = \lambda_j(\zeta)$ *for all* $1 \leq j \leq m$ *and for all* $\zeta \in X$.

Lemma 5.3. *Let* X *be a topological space,* $n \in \mathbb{N}^*$, *and let* $A : X \to \mathrm{Mat}_n(\mathbb{C})$ *be a continuous map. Then:*

(i) $\xi \in X$ *is* not *a splitting point of the eigenvalues of* A *if and only if there exist a neighborhood* U *of* ξ *and continuous functions* $\lambda_1, \ldots, \lambda_m : U \to \mathbb{C}$ *such that, for each* $\zeta \in U$, $\lambda_1(\zeta), \ldots, \lambda_m(\zeta)$ *are the distinct eigenvalues of* $A(\zeta)$.[7]

(ii) *Assume that* X *is a complex space and* A *is holomorphic. Let* $Y \subseteq X$ *be an open set which does not contain splitting points of the eigenvalues of* A, *and let* $\gamma_1, \ldots, \gamma_m : Y \to \mathbb{C}$ *be continuous functions such that, for each* $\zeta \in Y$, $\gamma_1(\zeta), \ldots, \gamma_m(\zeta)$ *are the distinct eigenvalues of* $A(\zeta)$. *Then these functions are holomorphic on* Y.

Theorem 5.4. *Let* X *be a complex space, and let* $A : X \to \mathrm{Mat}_n(\mathbb{C})$ *be holomorphic. Denote by* $\operatorname{split} A$ *the set of splitting points of the eigenvalues of* A. *Then* $\operatorname{split} A$ *is a nowhere dense closed analytic subset of* X.

Moreover, if X *is irreducible and* $\operatorname{split} A \neq \emptyset$, *then there exist finitely many holomorphic functions* $h_1, \ldots, h_\ell : X \to \mathbb{C}$, *each of which is a finite sum of finite products of elements of* A, *such that*

$$\operatorname{split} A = \{h_1 = \ldots = h_\ell = 0\}, \tag{5.1}$$

and

$$|h_j(\zeta)| \leq (2n)^{6n^2}\Big(1 + \|A(\zeta)\|\Big)^{2n^2} \quad \textit{for all} \quad \zeta \in X \textit{ and } 1 \leq j \leq \ell. \tag{5.2}$$

Proof. Let $P(\zeta)(\lambda) := \det(\lambda - A(\zeta))$, for $\zeta \in X$ and $\lambda \in \mathbb{C}$, and let $\operatorname{split} P$ be the set of splitting points of the zeros of P. Since the eigenvalues of A are the zeros of P, then

$$\operatorname{split} A = \operatorname{split} P.$$

Therefore, by Theorem 4.5, $\operatorname{split} A$ is a nowhere dense analytic subset of X.

[6]By that we mean that $\{\lambda_1(\zeta), \ldots, \lambda_m(\zeta)\}$ is the set of all eigenvalues of $A(\zeta)$ and $\lambda_i(\zeta) \neq \lambda_j(\zeta)$ if $i \neq j$.

[7]By Lemma 5.2 (ii), up to the numbering, these functions are uniquely determined on each connected component of U.

Now, we assume that X is irreducible and split $A \neq \emptyset$. Let $p_1(\zeta), \ldots, p_n(\zeta)$ be the coefficients of $P(\zeta)$. Then, again by Theorem 4.5, there exist holomorphic functions $h_1, \ldots, h_\ell$ on X, where $\ell \leq ((2n-1)!)^2$, each of which is a sum of not more than $(2n-1)!$ products of $(2n-1)n$ functions from $\{\pm p_0, \ldots, \pm p_n\}$, such that

$$\text{(5.3)} \qquad \operatorname{split} P = \big\{\zeta \in X \,\big|\, h_1(\zeta) = \ldots = h_\ell(\zeta) = 0\big\},$$

and

$$\text{(5.4)} \qquad |h_j(\zeta)| \leq (2n)^{4n} \max_{0 \leq k \leq n} |p_k(\zeta)|^{2n} \quad \text{for all} \quad \zeta \in X \text{ and } 1 \leq j \leq \ell.$$

Since split $A =$ split P, then (5.1) follows from (4.7).

Since $\big|p_k(\zeta)\big| \leq n!(1 + \|A(\zeta)\|)^n$ for all $\zeta \in X$ and $0 \leq k \leq n$, it follows from (5.4) that, for all $\zeta \in X$ and $1 \leq j \leq \ell$,

$$|h_j(\zeta)| \leq (2n)^{4n}(n!)^{2n}(1 + \|A(\zeta)\|)^{2n^2} \leq (2n)^{6n^2} \max_{0 \leq k \leq n} (1 + \|A(\zeta)\|)^{2n^2},$$

i.e., we have (5.2). □

Remark 5.5. According to the end of Remark 3.6, the claim of Theorem 5.4 can be completed by the statement that, at each point of split A which is a smooth point of X, split A is of codimension 1 in X.

6. JORDAN STABLE POINTS

Definition 6.1. As usual, by a **Jordan block** we mean a matrix of the form $\lambda I_\ell + (\delta_{i,j-1})_{i,j=1}^{\ell}$, where δ_{ij} is the Kronecker symbol, $\lambda \in \mathbb{C}$ (the eigenvalue of the Jordan block) and $\ell \in \mathbb{N}^*$ (the size of the Jordan block).

If $\Phi \in \operatorname{Mat}_n(\mathbb{C})$ and $\lambda_1, \ldots, \lambda_m$ are the distinct eigenvalues of Φ, then, for $\ell \in \mathbb{N}^*$, we denote by $\vartheta_\ell\big(\Phi, \lambda_j\big)$ the number of Jordan blocks of size ℓ of the eigenvalue λ_j in the Jordan normal forms of Φ, and set

$$\vartheta_\ell\big(\Phi, \bullet\big) = \sum_{j=1}^{m} \vartheta_\ell\big(\Phi, \lambda_j\big).$$

Further, then we define

$$\Theta_\Phi = \big(\lambda_1 - \Phi\big) \cdot \ldots \cdot \big(\lambda_m - \Phi\big),$$

which is correct, for the matrices $\lambda_1 - \Phi, \ldots, \lambda_m - \Phi$ pairwise commute.

LEMMA 6.2. *Let $\Phi \in \operatorname{Mat}_n(\mathbb{C})$, let $\lambda_1, \ldots, \lambda_m$ be the different eigenvalues of Φ, and let n_j be the algebraic multiplicity of λ_j. Then*

$$\text{(6.1)} \qquad \operatorname{rank}(\lambda_j - \Phi)^k = n - n_j \quad \textit{for} \quad k \geq n_j \quad \textit{and} \quad 1 \leq j \leq m$$

(6.2) $$\Theta_\Phi^k = 0 \quad \textit{for} \quad k \geq n,$$

(6.3) $$\operatorname{rank} \Theta_\Phi^k = n - nm + \operatorname{rank}(\lambda_1 - \Phi)^k + \ldots + \operatorname{rank}(\lambda_m - \Phi)^k \quad \textit{if} \quad 1 \leq k \leq n-1,$$

(6.4) $$\operatorname{rank} \Theta_\Phi^k = n - \sum_{\ell=1}^{k} \ell \vartheta_\ell\big(\Phi, \bullet\big) - k \sum_{\ell=k+1}^{n} \vartheta_\ell\big(\Phi, \bullet\big) \quad \textit{if} \quad 1 \leq k \leq n-1,$$

(6.5) $$\vartheta_k\big(\Phi, \lambda_j\big) = \operatorname{rank}\,(\lambda_j - \Phi)^{k-1} + \operatorname{rank}(\lambda_j - \Phi)^{k+1} - 2 \operatorname{rank}\,(\lambda_j - \Phi)^k \quad \textit{if} \quad 1 \leq k \leq n \textit{ and } 1 \leq j \leq m,$$

where $(\lambda_j - \Phi)^0 := I_n$.

For completeness, we give a proof of this lemma, although the relations collected there (and in its proof) are well-known, possibly, in somewhat different formulations, see, e.g., [2, Kap. II, §8.4] or [4, 2.9.4].

Proof. First recall that, if, for some $1 \leq j \leq m$, J is a Jordan block of size ℓ and with eigenvalue λ_j, then

(6.6) $$\begin{aligned} &\operatorname{rank}\,(\lambda_j - J)^k = \ell - k \quad \text{for} \quad 0 \leq k \leq \ell - 1, \\ &(\lambda_j - J)^\ell = 0, \\ &\lambda_i - J \in \mathrm{GL}(\ell, \mathbb{C}) \quad \text{for all} \quad 1 \leq i \leq m \text{ with } i \neq j. \end{aligned}$$

Denote by E_j the algebraic eigenspace of λ_j, i.e., $E_j := \operatorname{Ker}(\lambda_j - \Phi)^{n_j}$. Then each E_j is an invariant subspace of each $\lambda_i - \Phi$, and, since Φ is similar to a matrix in Jordan normal form, it follows from (6.6) that

(6.7) $$\mathbb{C}^n = E_1 \oplus \ldots \oplus E_m, \quad \text{and} \quad n_j = \dim E_j \quad \text{for} \quad 1 \leq j \leq m,$$

$\lambda_i - \Phi$ maps E_j isomorphically onto itself if $i \neq j$,

(6.8) $$\operatorname{Ker}(\lambda_j - \Phi)^k = E_j \quad \text{for} \quad k \geq n_j \quad \text{and} \quad 1 \leq j \leq m,$$

(6.9) $$\dim \operatorname{Ker}(\lambda_j - \Phi)^k = \sum_{\ell=1}^{k} \ell \vartheta_\ell\big(\Phi, \lambda_j\big) + k \sum_{\ell=k+1}^{n_j} \vartheta_\ell\big(\Phi, \lambda_j\big) \quad \text{for } 1 \leq j \leq m \text{ and } k \in \mathbb{N}^*,$$

and (taking into account that the matrices $\lambda_j - \Phi$ pairwise commute), for all $k \in \mathbb{N}^*$,

(6.10) $$\operatorname{Ker} \Theta_\Phi^k = \operatorname{Ker}(\lambda_1 - \Phi)^k \oplus \ldots \oplus \operatorname{Ker}(\lambda_m - \Phi)^k,$$

(6.11) $$\dim \operatorname{Ker} \Theta_\Phi^k = \dim \operatorname{Ker}(\lambda_1 - \Phi)^k + \ldots + \dim \operatorname{Ker}(\lambda_m - \Phi)^k.$$

From (6.9) and (6.11) together, we obtain

$$\dim \operatorname{Ker} \Theta_\Phi^k = \sum_{\ell=1}^{k} \ell \vartheta_\ell(\Phi, \bullet) + k \sum_{\ell=k+1}^{n} \vartheta_\ell(\Phi, \bullet), \quad k \in \mathbb{N}^*. \tag{6.12}$$

Now: (6.1) follows from (6.7) and (6.8); (6.2) follows from (6.7), (6.8) and (6.10); (6.3) follows from (6.11); (6.4) follows from (6.12).

To prove (6.5), we first note that (6.9) holds also for $k = 0$ – then both sides are zero. Hence, for $k \in \mathbb{N}^*$ and $1 \le j \le m$,

$$\begin{aligned}
&\dim \operatorname{Ker}(\lambda_j - \Phi)^k - \dim \operatorname{Ker}(\lambda_j - \Phi)^{k-1} \\
&\quad = \Bigg(\sum_{\ell=1}^{k} \ell \vartheta_\ell(\Phi, \lambda_j) - \sum_{\ell=1}^{k-1} \ell \vartheta_\ell(\Phi, \lambda_j) \Bigg) \\
&\qquad + \Bigg(k \sum_{\ell=k+1}^{n_j} \vartheta_\ell(\Phi, \lambda_j) - (k-1) \sum_{\ell=k}^{n_j} \vartheta_\ell(\Phi, \lambda_j) \Bigg) \\
&\quad = k \vartheta_k(\Phi, \lambda_j) + \sum_{\ell=k+1}^{n_j} \vartheta_\ell(\Phi, \lambda_j) - (k-1) \vartheta_k(\Phi, \lambda_j) = \sum_{\ell=k}^{n_j} \vartheta_\ell(\Phi, \lambda_j)
\end{aligned}$$

and, therefore,

$$\vartheta_k(\Phi, \lambda_j) = 2 \dim \operatorname{Ker}(\lambda_j - \Phi)^k - \dim \operatorname{Ker}(\lambda_j - \Phi)^{k-1} - \dim \operatorname{Ker}(\lambda_j - \Phi)^{k+1},$$

which implies (6.5). □

LEMMA 6.3. *Let X be a topological space, $A : X \to \operatorname{Mat}_n(\mathbb{C})$ a continuous map, and ξ a point in X which is not a splitting point of the eigenvalues of A. By Lemma 5.3, then we can find a neighborhood U of ξ and continuous functions $\lambda_1, \ldots, \lambda_m : U \to \mathbb{C}$ such that, for each $\zeta \in U$, $\lambda_1(\zeta), \ldots, \lambda_m(\zeta)$ are the distinct eigenvalues of $A(\zeta)$.*

Then (claim of the lemma) the following conditions are equivalent.

(i) *There exists a neighborhood $V \subseteq U$ of ξ such that, for all $1 \le j \le m$ and $1 \le \ell \le n$, the map*

$$V \ni \zeta \longmapsto \vartheta_\ell\big(A(\zeta), \lambda_j(\zeta)\big) \tag{6.13}$$

is constant.

(ii) *There exists a neighborhood $V \subseteq U$ of ξ such that, for all $1 \le \ell \le n$, the map*

$$V \ni \zeta \longmapsto \vartheta_\ell\big(A(\zeta), \bullet\big) \tag{6.14}$$

is constant.

(iii) *There exists a neighborhood $V \subseteq U$ of ξ such that, for all $1 \le k \le n-1$, the map*

$$V \ni \zeta \longmapsto \operatorname{rank}\big(\Theta_{A(\zeta)}\big)^k \tag{6.15}$$

is constant.

(iv) *There exists a neighborhood $V \subseteq U$ of ξ such that, for all $1 \le j \le n$ and $1 \le k \le n-1$, the map*

$$V \ni \zeta \longmapsto \operatorname{rank}\big(\lambda_j(\zeta) - A(\zeta)\big)^k \tag{6.16}$$

is constant.

(v) *There exists a neighborhood $V \subseteq U$ of ξ and a continuous map $T : V \to \mathrm{GL}(n, \mathbb{C})$, which is holomorphic if X is a complex space and A is holomorphic, such that $T(\zeta)^{-1}A(\zeta)T(\zeta)$ is in Jordan normal form for all $\zeta \in V$.*

If X is a domain in $\mathbb{C}$ and A is holomorphic, the equivalence of conditions (i), (ii) and (v) is due to G. P. A. Thiesse [13].

Proof. The equivalence of (i) - (iv) follows from Lemma 6.2. Indeed:

(i) ⇒ (ii) is trivial.

(ii) ⇒ (iii) follows from (6.4).

To prove that (iii) ⇒ (iv), we note that, by (6.3),

$$\operatorname{rank}\big(\Theta_{A(\zeta)}\big)^k = \text{const} + \operatorname{rank}\big(\lambda_1(\zeta) - A(\zeta)\big)^k + \ldots + \operatorname{rank}\big(\lambda_m(\zeta) - A(\zeta)\big)^k$$

for all $\zeta \in U$ and $1 \le k \le n-1$, and observe that the functions on the right hand side of this relation are lower semicontinuous in ζ (since the rank of a continuous matrix function is always lower semicontinuous). Therefore, constancy of the left hand side is possible only if all functions on the right hand side are constant.

(iv) ⇒ (i) follows from (6.5).

Moreover, it is clear that (v) ⇒ (i). To complete the proof of the lemma, therefore it is sufficient to prove that (i) ⇒ (v).

Assume (i) is satisfied.

Denote by n_j be the algebraic multiplicity of $\lambda_j(\xi)$. Then

$$n_j = \sum_{\ell} \vartheta_\ell\big(A(\xi), \lambda_j(\xi)\big).$$

Therefore, for each $1 \le j \le m$, we can choose a matrix $N_j \in \mathrm{Mat}_n(\mathbb{C})$, which is a block diagonal matrix with Jordan blocks on the diagonal, each of which has the eigenvalue 0, and such that, for each $\ell \in \mathbb{N}^*$, exactly $\vartheta_\ell\big(A(\xi), \lambda_j(\xi)\big)$ of them are of size ℓ. Let $J : U \to \mathrm{Mat}_n(\mathbb{C})$ (recall that $n_1 + \ldots + n_m = n$) be the map such that $J(\zeta)$, $\zeta \in U$, is the block diagonal matrix with the diagonal

$$\lambda_1(\zeta)I_{n_1} + N_1, \ldots, \lambda_m(\zeta)I_{n_m} + N_m.$$

Since the functions λ_j are continuous, J is continuous. If X is a complex space and A is holomorphic, then, by Lemma 5.3 (ii), the functions λ_j are even holomorphic. Therefore, in this case also, J is holomorphic.

Moreover, by condition (i), there is a neighborhood $V \subseteq U$ of ξ such that,

$$\vartheta_\ell\big(A(\zeta), \lambda_j(\zeta)\big) = \vartheta_\ell\big(A(\xi), \lambda_j(\xi)\big) \quad \text{for all} \quad \zeta \in V,$$

which means that, for each $\zeta \in V$, $J(\zeta)$ is a Jordan normal form of $A(\zeta)$. Therefore, for each $\zeta \in V$, we can choose a matrix $\Theta_\zeta \in \mathrm{GL}_n(\mathbb{C})$ with

$$\Theta_\zeta J(\zeta)\Theta_\zeta^{-1} = A(\zeta). \tag{6.17}$$

Now let $\mathrm{End}\big(\mathrm{Mat}_n(\mathbb{C})\big)$ be the space of linear endomorphisms of the complex vector space $\mathrm{Mat}_n(\mathbb{C})$. Following an idea of W. Wasow [15], we consider the continuous (and holomorphic if A is holomorphic) maps $\varphi, \psi : V \to \mathrm{End}\big(\mathrm{Mat}_n(\mathbb{C})\big)$ defined by

$$\begin{aligned} \varphi(\zeta)\Phi &= \Phi A(\zeta) - J(\zeta)\Phi, \qquad \zeta \in V, \quad \Phi \in \mathrm{Mat}_n(\mathbb{C}),\\ \psi(\zeta)\Phi &= \Phi J(\zeta) - J(\zeta)\Phi, \qquad \zeta \in V, \quad \Phi \in \mathrm{Mat}_n(\mathbb{C}). \end{aligned}$$

Further, for $\zeta \in V$, we denote by $\mathcal{T}_\zeta$ the linear automorphism of $\mathrm{Mat}_n(\mathbb{C})$ defined by

$$\mathcal{T}_\zeta\Phi = \Theta_\zeta\Phi, \quad \Phi \in \mathrm{Mat}_n(\mathbb{C}).$$

We claim that

$$\dim_{\mathbb{C}} \mathrm{Ker}\,\varphi(\zeta) = \dim_{\mathbb{C}} \mathrm{Ker}\,\psi(\zeta) \quad \text{for all } \zeta \in V. \tag{6.18}$$

Indeed, let $\Phi \in \mathrm{Ker}\,\varphi(\zeta)$, i.e., $\Phi A(\zeta) = J(\zeta)\Phi$. By (6.17), this implies $\Phi\Theta_\zeta J(\zeta)\Theta_\zeta^{-1} = J(\zeta)\Phi$ and, hence, $\Phi\Theta_\zeta J(\zeta) = J(\zeta)\Phi\Theta_\zeta$. By definition of $\mathcal{T}_\zeta$, this means that $(\mathcal{T}_\zeta\Phi)J(\zeta) = J(\zeta)(\mathcal{T}_\zeta\Phi)$, i.e., $\mathcal{T}_\zeta\Phi \in \mathrm{Ker}\,\psi(\zeta)$.

So, we have proved that $\mathcal{T}_\zeta\,\mathrm{Ker}\,\varphi(\zeta) \subseteq \mathrm{Ker}\,\psi(\zeta)$. Since $\mathcal{T}_\zeta$ is a linear automorphism of $\mathrm{Mat}_n(\mathbb{C})$, this shows that $\dim_{\mathbb{C}} \mathrm{Ker}\,\varphi(\zeta) \le \dim_{\mathbb{C}} \mathrm{Ker}\,\psi(\zeta)$.

Conversely, let $\Phi \in \mathrm{Ker}\,\psi(\zeta)$, i.e., $\Phi J(\zeta) = J(\zeta)\Phi$. By (6.17), this implies $\Phi J(\zeta) = \Theta_\zeta^{-1}A(\zeta)\Theta_\zeta\Phi$ and, hence, $\Theta_\zeta\Phi J(\zeta) = A(\zeta)\Theta_\zeta\Phi$. By definition of $\mathcal{T}_\zeta$, this means that $(\mathcal{T}_\zeta\Phi)J(\zeta) = A(\zeta)(\mathcal{T}_\zeta\Phi)$, i.e., $\mathcal{T}_\zeta\Phi \in \mathrm{Ker}\,\varphi(\zeta)$.

So, we have proved that also $\mathcal{T}_\zeta\,\mathrm{Ker}\,\psi(\zeta) \subseteq \mathrm{Ker}\,\varphi(\zeta)$, which shows the opposite inequality $\dim_{\mathbb{C}} \mathrm{Ker}\,\psi(\zeta) \le \dim_{\mathbb{C}} \mathrm{Ker}\,\varphi(\zeta)$.

(6.18) is proved.

Next, we claim that the map

$$V \ni \zeta \longmapsto \dim \mathrm{Ker}\,\psi(\zeta) = \Big\{\Phi \in \mathrm{Mat}_n(\mathbb{C}) \,\Big|\, \Phi J(\zeta) = J(\zeta)\Phi\Big\} \tag{6.19}$$

is constant. Indeed, since $\lambda_i(\zeta) \neq \lambda_j(\zeta)$ if $i \neq j$, it follows from [5, Ch. VIII, §1] that, for all $\zeta \in V$, a matrix $\Phi \in \mathrm{Mat}_n(\mathbb{C})$ satisfies $\Phi J(\zeta) = J(\zeta)\Phi$ if and only if it is a block diagonal matrix with a diagonal of the form $\Lambda_1, \ldots, \Lambda_m$, where Λ_j belongs to the space

$$\Phi J(\zeta) = J(\zeta)\Phi\Big\{\Phi \in \mathrm{Mat}_{n_j}(\mathbb{C}) \;\Big|\; \Phi\big(\lambda_j(\zeta) + N_j\big) = \big(\lambda_j(\zeta) + N_j\big)\Phi\Big\}$$
$$= \Big\{\Phi \in \mathrm{Mat}_{n_j}(\mathbb{C}) \;\Big|\; \Phi N_j = N_j\Phi\Big\}.$$

Since the latter space is independent of ζ, this means that (6.19) is constant.

Since φ is continuous, and holomorphic if A is holomorphic, the constancy of (6.19) means that the family $\{\mathrm{Ker}\,\varphi(\zeta)\}_{\zeta\in V}$ is a sub-vector bundle of the product bundle $V \times \mathrm{Mat}_n(\mathbb{C})$, which is holomorphic if A is holomorphic (see, e.g., [15, Lemma 1] or [12, Corollary 2]).

Therefore, through each point in this sub-vector bundle goes a local continuous (resp. holomorphic) section. Since, by (6.17), (ξ, Θ_ξ^{-1}) is such a point, it follows that there is a neighborhood V of ξ and a continuous (resp. holomorphic) map $S : V \to \mathrm{Mat}_n(\mathbb{C})$ with $S(\xi) = \Theta_\xi^{-1}$ and $S(\zeta)A(\zeta) = J(\zeta)S(\zeta)$ for all $\zeta \in V$. Since Θ_ξ^{-1} is invertible, shrinking V, we may achieve that moreover $S(\zeta) \in \mathrm{GL}(n, \mathbb{C})$ for all $\zeta \in V$. It remains to set $T(\zeta) = S(\zeta)^{-1}$ for $\zeta \in V$. □

Definition 6.4. Let X be a topological space, and $A : X \to \mathrm{Mat}_n(\mathbb{C})$ a continuous map. A point $\xi \in X$ is called **Jordan stable** for A if ξ is not a splitting point of the eigenvalues of A and and the equivalent conditions (i) - (v) in Lemma 6.3 are satisfied.

If G is a domain in some $\mathbb{C}^N$ and $A : G \to \mathrm{Mat}_n(\mathbb{C})$ is holomorphic, H. Baumgärtel proved that there exists a nowhere dense analytic subset B of G, which contains the splitting points of A, such that all points of $G \setminus B$ are Jordan stable for A (he proved that condition (v) in Lemma 6.3 is satisfied), see [1], [2, Kap. V, §7] and [4, 5.7] if $N = 1$, and [3] and [4, S 3.4] for arbitrary N.

In the present section, we give a new proof of Baumgärtel's theorem, which gives the following more precise and more general

Theorem 6.5. *Let X be a complex space, and let $A : X \to \mathrm{Mat}_n(\mathbb{C})$ be holomorphic. Denote by* $\mathrm{Jst}\,A$ *the set of Jordan stable points of A.*

Then $X \setminus \mathrm{Jst}\,A$ is a nowhere dense closed analytic subset of X.

Moreover, if X is irreducible and normal[8]*, and if* $\mathrm{Jst}\,A \neq X$*, then there*

[8]For the definition of a *normal* complex space, see, e.g., [11, Ch. VI, §2]. For example, each complex manifold is normal.

exist finitely many holomorphic functions $h_1, \ldots, h_\ell : X \to \mathbb{C}$ *such that*

$$X \setminus \operatorname{Jst} A = \big\{h_1 = \ldots = h_\ell = 0\big\} \tag{6.20}$$

and

$$|h_j(\zeta)| \le (2n)^{7n^2} 2^{n^3} (1 + \|A(\zeta)\|)^{3n^3} \quad \textit{for all} \quad \zeta \in X \ \textit{and}\ 1 \le j \le \ell. \tag{6.21}$$

Proof. For $\operatorname{Jst} A = \emptyset$, the claim of the theorem is trivial. Therefore, we may assume that $\operatorname{Jst} A \neq \emptyset$.

We first consider the case when X is normal and irreducible.

Let $\operatorname{split} A$ be the set of splitting points of the eigenvalues of A, and let X^0 be the manifold of smooth points of X. Since X^0 is connected (X is irreducible) and dense in X, and $\operatorname{split} A$ is a nowhere dense analytic subset of X (Theorem 5.4), $X \setminus \operatorname{split} A$ is connected. Therefore, for all $\zeta \in X \setminus \operatorname{split} A$, the number of distinct eigenvalues of $A(\zeta)$ is the same, we denote it by m.

Consider the map

$$X \setminus \operatorname{split} A \ni \zeta \longmapsto \Theta_{A(\zeta)}. \tag{6.22}$$

By Lemma 5.3, for each $\xi \in X \setminus \operatorname{split} A$, we have an open neighborhood $U_\xi \subseteq X \setminus \operatorname{split} A$ of ξ and holomorphic functions $\lambda_1^{(\xi)}, \ldots, \lambda_m^{(\xi)} : U_\xi \to \mathbb{C}$ such that, for all $\zeta \in U_\xi$, $\lambda_1^{(\xi)}(\zeta), \ldots, \lambda_m^{(\xi)}(\zeta)$ are the distinct eigenvalues of $A(\zeta)$ and, hence,

$$\Theta_{A(\zeta)} = \big(\lambda_1^{(\xi)}(\zeta) - A(\zeta)\big) \cdot \ldots \cdot \big(\lambda_m^{(\xi)}(\zeta) - A(\zeta)\big). \tag{6.23}$$

In particular, this shows that (6.22) is holomorphic on $X \setminus \operatorname{split} A$.

Moreover, as $|\lambda_j(\zeta)| \le \|A(\zeta)\|$, from (6.23) it follows that

$$\big\|\Theta_{A(\zeta)}\big\| \le 2^m \|A(\zeta)\|^m \quad \text{for all} \quad \zeta \in X \setminus \operatorname{split} A. \tag{6.24}$$

Since $X \cap \operatorname{split} A$ is a nowhere dense analytic subset of X, and X is normal, this implies that (6.22) extends holomorphically to X. We denote this extended map by Θ. By (6.24), then

$$\big\|\Theta(\zeta)^k\big\| \le 2^{mk} \|A(\zeta)\|^{mk} \quad \text{for all} \quad \zeta \in X \text{ and } 1 \le k \le n. \tag{6.25}$$

Set

$$r_k = \max_{\zeta \in X} \operatorname{rank} \Theta(\zeta)^k \quad \text{for} \quad 1 \le k \le n.$$

First case: $r_1 = 0$. Then $\big(\Theta_{A(\zeta)}\big)^k = 0$ for all $\zeta \in X \setminus \operatorname{split} A$ and $k \in \mathbb{N}^*$. In particular, each $\xi \in X \setminus \operatorname{split} A$ satisfies condition (iii) in Lemma 6.3. Hence, $X \setminus \operatorname{Jst} A = \operatorname{split} A$, and the claim of the theorem follows from Theorem 5.4.

Second case: $r_1 > 0$. Then, by (6.2), $n \ge 2$ and there is an integer $1 \le k_0 \le n-1$ with $r_{k_0} > 0$ and $r_{k_0+1} = 0$. For $1 \le k \le k_0$, let $f_1^{(k)}, \ldots, f_{s_k}^{(k)}$

be the minors of order r_k of Θ^k which do not vanish identically on X. Since X is irreducible (i.e., the manifold of smooth points of X is connected), and the functions $f_j^{(k)}$ are holomorphic and $\not\equiv 0$, none of them can vanish identically on an open subset of X. Hence,

$$Z := \bigcup_{k=1}^{k_0} \{f_1^{(k)} = \ldots = f_{s_k}^{(k)} = 0\} \tag{6.26}$$

is a nowhere dense analytic subset of X, and $\xi \in Z$ if and only if ξ is a jump point (Def. 4.3) for at least one of the maps $\Theta^1, \ldots, \Theta^{k_0}$. Since $\Theta^k \equiv 0$ if $k_0 + 1 \le k \le n-1$, the latter means that $\xi \in Z$ if and only if ξ is a jump point for at least one of the maps $\Theta^1, \ldots, \Theta^{n-1}$. In particular, $\xi \in Z \cap (X \setminus \operatorname{split} A)$ if and only if $\xi \in (X \setminus \operatorname{split})A$ and ξ is a jump point of at least one of the maps

$$X \setminus \operatorname{split} A \longmapsto \big(\Theta_{A(\zeta)}\big)^1, \quad \ldots \quad , X \setminus \operatorname{split} A \longmapsto \big(\Theta_{A(\zeta)}\big)^{n-1},$$

i.e., $\xi \in Z \cap (X \setminus \operatorname{split} A)$ if and only if $\xi \in X \setminus \operatorname{split} A$ and ξ violates condition (iii) in Lemma 6.3. Hence

$$(X \setminus \operatorname{Jst} A) \cap (X \setminus \operatorname{split} A) = Z \cap (X \setminus \operatorname{split} A).$$

Since $\operatorname{split} A \subseteq X \setminus \operatorname{Jst} A$, it follows that

$$X \setminus \operatorname{Jst} A = Z \cup \operatorname{split} A. \tag{6.27}$$

By Theorem 5.4, we have finitely many holomorphic functions $g_1, \ldots, g_p : X \to \mathbb{C}$, each of which is a finite sum of finite products of elements of A, such that

$$\operatorname{split} A = \big\{g_1 = \ldots = g_p = 0\big\}, \tag{6.28}$$

and

$$|g_j(\zeta)| \le (2n)^{6n^2} (1 + \|A(\zeta)\|)^{2n^2} \quad \text{for all } \zeta \in X \text{ and } 1 \le j \le p. \tag{6.29}$$

Now let $\{h_1, \ldots, h_\ell\}$ be the set of all functions of the form

$$g_\mu \cdot \prod_{k=1}^{k_0} f_{\kappa_k}^{(k)}$$

with $1 \le \mu \le p$ and $1 \le \kappa_k \le s_k$ for $1 \le k \le k_0$. Then (6.20) follows from (6.26), (6.27) and (6.28), and

$$|h_j(\zeta)| \le \max_\mu |g_\mu(\zeta)| \max_k |f_{\kappa_k}^{(k)}(\zeta)|^n \quad \text{for all } \zeta \in X \text{ and } j = 1, \ldots, \ell. \tag{6.30}$$

To prove estimate (6.21), we first recall that each $f_{\kappa_k}^{(k)}$ is a minor of Θ^k, which implies that

$$\big|f_{\kappa_k}^{(k)}(\zeta)\big| \le n! \|\Theta^k(\zeta)\|^n \quad \text{for all } \zeta \in X,$$

and further, by (6.25),

$$\begin{aligned}\left|f^{(k)}_{\kappa_k}(\zeta)\right| &\le n!\|\Theta^k(\zeta)\|^n \le n!\Big(2^{mk}\|A(\zeta)\|^{mk}\Big)^n \le n!2^{mkn}\|A(\zeta)\|^{mkn}\\ &\le n^n 2^{n^3}\|A(\zeta)\|^{mkn} \le n^n 2^{n^3}\Big(1+\|A(\zeta)\|\Big)^{n^3} \quad \text{for all } \zeta\in X.\end{aligned}$$

Together with (6.29) and (6.30), this yields (6.21):

$$\begin{aligned}|h_j(\zeta)| &\le (2n)^{6n^2}(1+\|A(\zeta)\|)^{2n^2} n^n 2^{n^3}(1+\|A(\zeta)\|)^{n^3}\\ &\le (2n)^{7n^2}2^{n^3}(1+\|A(\zeta)\|)^{3n^3}.\end{aligned}$$

Next, we consider the case when X is irreducible, but, possibly, not normal.

Let $\pi : \widetilde X \to X$ be the normalization of X (see, e.g., [11, Ch. VI, §4]) and $\widetilde A := A\circ\pi$. Then $\widetilde X$ is normal and irreducible. Therefore, by part (i) of the theorem, $\widetilde X \setminus \operatorname{Jst}\widetilde A$ is a nowhere dense closed analytic subset of X. Since, clearly,

$$\pi\big(\widetilde X \setminus \operatorname{Jst}\widetilde A\big) = X\setminus \operatorname{Jst} A, \tag{6.31}$$

this implies, by Remmert's proper mapping theorem (see, e.g., [11, Ch. V, §5.1])), that $X\setminus\operatorname{Jst}A$ is a closed analytic subset of X.

To prove that $X\setminus\operatorname{Jst}A$ is nowhere dense in X, let X^0 be the manifold of smooth points of X. Then π is biholomorphic between $\pi^{-1}(X^0)$ and X^0, and, by (6.31),

$$\pi\big(\pi^{-1}(X^0)\setminus\operatorname{Jst}\widetilde A\big) = X^0\setminus\operatorname{Jst}A.$$

Since $\pi^{-1}(X^0)\setminus\operatorname{Jst}\widetilde A$ is nowhere dense in $\pi^{-1}(X^0)$, this implies that $X^0\setminus\operatorname{Jst}A$ is nowhere dense in X^0. Since $X\setminus X^0$ is nowhere dense in X, it follows that $X\setminus\operatorname{Jst}A$ is nowhere dense in X.

Finally, we consider the general case.

By the global decomposition theorem for complex spaces (see, e.g., [11, V.4.6] or [9, Ch. 9, §2.2]), there is a locally finite covering $\{X_i\}_{i\in I}$ of X such that each X_i is an irreducible closed analytic subset of X. Then, as already proved, each $X_i\setminus\operatorname{Jst}(A|_{X_i})$ is a nowhere dense analytic subset of X_i. Since the covering $\{X_i\}_{i\in I}$ is locally finite and, clearly,

$$X\setminus\operatorname{Jst}A = \bigcup_{i\in I}\Big(X_i\setminus\operatorname{Jst}(A|_{X_i})\Big),$$

this proves that $X\setminus\operatorname{Jst}A$ is a nowhere dense analytic subset of X. □

Remark 6.6. Estimate (6.21) shows that the claim of Theorem 6.5 can be completed. For example:

– If A is bounded, then $X \setminus \operatorname{Jst} A$ can be defined by bounded holomorphic functions. In the case of the disk $X = \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}$, this implies that $X \setminus \operatorname{Jst} A$ satisfies the Blaschke condition.

– If $X = \mathbb{C}^N$ and the elements of A are holomorphic polynomials, then $\mathbb{C}^N \setminus \operatorname{Jst} A$ is the common zero set of finitely many holomorphic polynomials, i.e., it is affine algebraic. For $N = 1$ this means that $\mathbb{C} \setminus \operatorname{Jst} A$ is finite.

Remark 6.7. It is possible (in contrast to Remark 5.5) that the set of points which are not Jordan stable is of codimension > 1, also at smooth points. Here is an example. Let

$$A(z,w) := \begin{pmatrix} zw & -z^2 \\ w^2 & -zw \end{pmatrix} \quad \text{for} \quad (z,w) \in \mathbb{C}^2.$$

Then $A(z,w)^2 = 0$ for all $(z,w) \in \mathbb{C}^2$, and $A(z,w) = 0$ if and only if $(z,w) = 0$. This means that $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is the Jordan normal form of $A(0,0)$, whereas, for all $(z,w) \in \mathbb{C}^2 \setminus \{(0,0)\}$, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is the Jordan normal form of $A(z,w)$. Hence, $(0,0)$ is the only point in $\mathbb{C}^2$ which is not Jordan stable for A.

REFERENCES

[1] H. Baumgärtel, *Jordansche Normalform holomorpher Matizen.* Monatsber. Deutsch. Akad. Wiss. Berlin **11** (1969), 23–24.

[2] H. Baumgärtel, *Endlichdimensionale analytische Störungstheorie.* Mathematische Lehrbücher und Monographien. II. Abteilung: Mathematische Monographien. Akademie-Verlag, Berlin, 1972.

[3] H. Baumgärtel, *Analytic perturbation theory for linear operators depending on several complex variables* (in Russian). Mat. Issled. **9** (1974), *1*, 17–39.

[4] H. Baumgärtel, *Analytic perturbation theory for matrices and operators.* Oper. Theory Adv. Appl. **15**, Birkhäuser Verlag, Basel-Boston-Stuttgart, 1985.

[5] F.R. Gantmacher, *The Theory of Matrices. Volume one.* AMS Chelsea Publishing, Providence, 1959.

[6] K. Fritzsche and H. Grauert, *From holomorphic functions to complex manifolds.* Grad. Texts in Math. **213**, Springer, New York, 2002.

[7] H. Grauert and K. Fritzsche, *Einführung in die Funktionentheorie mehrerer Veränderlicher.* Springer-Verlag, Berlin-Heidelberg-New York, 1974.

[8] I.Z. Gohberg and G. Heinig, *The resultant matrix and its generalizations. I: The resultant operator for matrix polynomials* (in Russian). Acta Sci. Math. **37** (1975), 41–61.

[9] H. Grauert and R. Remmert, *Coherent analytic sheaves.* Grundlehren Math. Wiss. **265**, Springer-Verlag, Berlin, 1984.

[10] M.G. Krein and A. Naimark, *The method of symmetric and Hermitian forms in the theory of the separation of the roots of algebraic equations.* Linear and Multilinear Algebra **10** (1981), 265–308.

[11] S. Łojasiewicz, *Introduction to complex analytic geometry*. Birkhäuser, 1991.

[12] M.A. Shubin, *Holomorphic families of subspaces of a Banach space*. Integral Equations and Operator Theory **2** (1979), *3*, 407–420. Translation from Mat. Issled. **5** (1970), *4*, 153–165 (Russian).

[13] G.P.A. Thijsse, *Global holomorphic similarity to a Jordan form*. Results Math. **8** (1985), 78–87.

[14] B.L. van der Waerden, *Algebra I*. Springer-Verlag, Berlin-Heidelberg-New York, 1971.

[15] W. Wasow, *On holomorphically similar matrices*. J. Math. Anal. Appl. **4** (1962), 202–206.

Institut für Mathematik
Humboldt-Universität zu Berlin
Rudower Chaussee 25
D-12489 Berlin, Germany
`juergen.leiterer@hu-berlin.de`