PROJECTIVE REPRESENTATIONS OF GROUPS USING HILBERT RIGHT C*-MODULES

CORNELIU CONSTANTINESCU

Communicated by Lucian Beznea

The projective representation of groups was introduced in 1904 by Issai Schur in his paper [6]. It differs from the normal representation of groups (introduced by his tutor Ferdinand Georg Frobenius at the suggestion of Richard Dedekind) by a twisting factor, which we call Schur function in this paper and which is called sometimes multipliers or normalized factor set in the literature (other names are also used). It starts with a group T and a Schur function f for T. This is a scalar valued function on $T \times T$ satisfying the conditions $f(1,1) = 1$ and $|f(s,t)| = 1\,,\ f(r,s)f(rs,t) = f(r,st)f(s,t)$ for all $r,s,t \in T$. The projective representation of T twisted by f is a unital C*-subalgebra of the C*-algebra $\mathcal{L}(l^2(T))$ of operators on the Hilbert space $l^2(T)$. This representation can be used in order to construct many examples of C*-algebras (see e.g. [1, Chapter 7]). By replacing the scalars $\mathbb{R}$ or $\mathbb{C}$ with an arbitrary unital (real or complex) C*-algebra E, the field of applications is enhanced in an essential way. In this case, $l^2(T)$ is replaced by the Hilbert right E-module $\bigoplus\limits_{t\in T} E \approx E \otimes l^2(T)$ and $\mathcal{L}(l^2(T))$ is replaced by $\mathcal{L}_E(E \otimes l^2(T))$, the C*-algebra of adjointable operators of $\mathcal{L}(E \otimes l^2(T))$. The projective representation of groups, which we present in this paper, has some similarities with the construction of cross products with discrete groups. It opens the way to create many K-theories. In a first section, we introduce some results which are needed for this construction, which is developed in the second section. In the third section, we present examples of C*-algebras obtained by this method. Examples of a special kind (the Clifford algebras) are presented in the last section.

AMS 2020 Subject Classification: Primary 22D25; Secondary 20C25, 46L08.

Key words: Hilbert right C* modules, projective groups representations.

CONTENTS

0 Notation and Terminology **2**

1 Preliminaries **5**

1.1 Schur functions 5
1.2 E-C*-algebras 10
1.3 Some topologies 18

REV. ROUMAINE MATH. PURES APPL. **68** (2023), *3-4*, 199–328
doi: 10.59277/RRMPA.2023.199.328

2 Main Part **32**
2.1 The representations 32
2.2 Variation of the parameters 57
2.3 The functor S 79

3 Examples **84**
3.1 T:=Z_2 84
3.2 T:=Z_2xZ_2 97
3.3 T=(Z_2)^ n 103
3.4 T:=Z_n 107
3.5 T:=Z 110

4 Clifford Algebras **110**
4.1 The general case 110
4.2 A special case 119

0. NOTATION AND TERMINOLOGY

Throughout this paper, we use the following notation: T is a group, 1 is its neutral element, $M := l^2(T)$, $1_M := id_K :=$ identity map of M, E is a unital C*-algebra (resp. a W*-algebra), 1_E is its unit, $\breve{E}$ denotes the set E endowed with its canonical structure of a Hilbert right E-module ([1, Proposition 5.6.1.5]),

$$L := \breve{E} \otimes M \approx \bigoplus_{t \in T} \breve{E}\,, \quad (\text{resp.}\, L := \breve{E} \bar{\otimes} M \approx \overset{W}{\bigoplus_{t \in T}} \breve{E})$$

([3, Proposition 2.1], (resp. [3, Corollary 2.2])). In some examples, in which T is additive, 1 will be replaced by 0.

The map

$$\mathcal{L}_E(\breve{E}) \longrightarrow E, \quad u \longmapsto \langle\, u1_E \mid 1_E \,\rangle = u1_E$$

is an isomorphism of C*-algebras with inverse

$$E \longrightarrow \mathcal{L}_E(\breve{E}), \quad x \longmapsto x \cdot \ .$$

We identify E with $\mathcal{L}_E(\breve{E})$ using these isomorphisms.

In general, we use the notation of [1]. For tensor products of C*-algebras we use [8], for W*-tensor products of W*-algebras we use [7], for tensor products of Hilbert right C^*-modules we use [5], and for the exterior W*-tensor products of selfdual Hilbert right W^*-modules we use [2] and [3].

In the sequel, we give a list of notations used in this paper.

1) $\mathbb{K}$ denotes the field of real numbers ($:= \mathbb{R}$) or the field of complex numbers ($:= \mathbb{C}$). In general, the C*-algebras will be complex or real. $\mathbb{H}$ denotes the field of quaternions, $\mathbb{N}$ denotes the set of natural numbers ($0 \notin \mathbb{N}$), and for every $n \in \mathbb{N} \cup \{0\}$ we put

$$\mathbb{N}_n := \{\, m \in \mathbb{N} \mid m \leq n \,\}.$$

$\mathbb{Z}$ denotes the group of integers and for $n \in \mathbb{N}$ we put $\mathbb{Z}_n := \mathbb{Z}/(n\mathbb{Z})$.

2) For every set A, $\mathfrak{P}(A)$ denotes the set of subsets of A, $\mathfrak{P}_f(A)$ the set of finite subsets of A, and Card A denotes the cardinal number of A. If f is a function defined on A and B is a subset of A then $f|B$ denotes the restriction of f to B.

3) If A, B are sets then A^B denotes the set of maps of B in A.

4) For all i, j we denote by $\delta_{i,j}$ Kronecker's symbol:

$$\delta_{i,j} := \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} .$$

5) If A, B are topological spaces then $\mathcal{C}(A, B)$ denotes the set of continuous maps of A into B. If A is locally compact space and E is a C*-algebra then $\mathcal{C}(A, E)$ (resp. $\mathcal{C}_0(A, E)$) denotes the C*-algebra of continuous maps $A \to E$, which are bounded (resp. which converge to 0 at the infinity).

6) For every set I and for every $J \subset I$ we denote by $e_J := e_J^I$ the characteristic function of J, i.e. the function on I equal to 1 on J and equal to 0 on $I \setminus J$. For $i \in I$ we put $e_i := (\delta_{i,j})_{j \in I} \in l^2(I)$.

7) If F is an additive group and S is a set then

$$F^{(S)} := \left\{\, x \in F^S \;\middle|\; \{\, s \in S \mid x_s \neq 0 \,\} \text{ is finite} \,\right\} .$$

8) If E, F are vector spaces in duality then E_F denotes the vector space E endowed with the locally convex topology of pointwise convergence on F, i.e. with the weak topology $\sigma(E, F)$.

9) If E is a normed vector space then E' denotes its dual and $E^{\#}$ denotes its unit ball:

$$E^{\#} := \{\, x \in E \mid \|x\| \leq 1 \,\}.$$

Moreover, if E is an ordered Banach space then E_+ denotes the convex cone of its positive elements. If E has a unique predual (up to isomorphisms), then we denote by $\ddot{E}$ this predual and so by $E_{\ddot{E}}$ the vector space E endowed with the locally convex topology of pointwise convergence on $\ddot{E}$.

10) The expressions of the form "..C*-...(resp. ..W*-...)", which appear often in this paper, will be replaced by expressions of the form "...C**-...".

11) If F is a unital C*-algebra and A is a subset of F then we denote by 1_F the unit of F, by $Pr\,F$ the set of orthogonal projections of F, by

$$A^c := \{\, x \in F \mid y \in A \Rightarrow xy = yx \,\}\,, \quad Re\,F := \{\, x \in F \mid x = x^* \,\}\,,$$

and by $Un\,F$ the set of unitary elements of F. If F is a real C*-algebra then $\overset{\circ}{F}$ denotes its complexification.

12) If F is a C*-algebra then we denote for every $n \in \mathbb{N}$ by $F_{n,n}$ the C*-algebra of $n \times n$ matrices with entries in F. If T is finite then $F_{T,T}$ has a corresponding signification.

13) Let F be a C*-algebra and H, K Hilbert right F-modules. We denote by $\mathcal{L}_F(H,K)$ the Banach subspace of $\mathcal{L}(H,K)$ of adjointable operators, by 1_H the identity map $H \to H$ which belongs to

$$\mathcal{L}_F(H) := \mathcal{L}_F(H,H)\,.$$

For $(\xi, \eta) \in H \times K$ we put

$$\eta \langle \cdot \mid \xi \rangle : H \longrightarrow K\,, \quad \zeta \longmapsto \eta \langle \zeta \mid \xi \rangle$$

and denote by $\mathcal{K}_F(H)$ the closed vector subspace of $\mathcal{L}_F(H)$ generated by $\{\, \eta \langle \cdot \mid \xi \rangle \mid \xi, \eta \in H \,\}$.

14) Let F be a W*-algebra and H, K Hilbert right F-modules. We put for $a \in \ddot{F}$ and $(\xi, \eta) \in H \times K$,

$$\widetilde{(a,\xi)} : H \longrightarrow \mathbb{K}\,, \quad \zeta \longmapsto \langle \langle \zeta \mid \xi \rangle \,, a \rangle\,,$$

$$\widetilde{(a,\xi,\eta)} : \mathcal{L}_F(H,K) \longrightarrow \mathbb{K}\,, \quad u \longmapsto \langle \langle u\xi \mid \eta \rangle \,, a \rangle$$

and denote by $\ddot{H}$ the closed vector subspace of the dual H' of H generated by

$$\left\{ \widetilde{(a,\xi)} \,\middle|\, a \in \ddot{F},\ \xi \in H \right\}$$

and by $\dddot{H}$ the closed vector subspace of $\mathcal{L}_F(H,K)'$ generated by

$$\left\{ \widetilde{(a,\xi,\eta)} \,\middle|\, (a,\xi,\eta) \in \ddot{F} \times H \times K \right\}\,.$$

If H is selfdual then $\dddot{H}$ is the predual of $\mathcal{L}_F(H)$ ([1, Theorem 5.6.3.5 b)]) and $\ddot{H}$ is the predual of H ([1, Proposition 5.6.3.3]). Moreover, a map defined on F is called W*-continuous if it is continuous on $F_{\ddot{F}}$. If G is a W*-algebra a C*-homomorphism $\varphi : F \to G$ is called a W*-homomorphism if the map $\varphi : F_{\ddot{F}} \to G_{\ddot{G}}$ is continuous; in this case, $\ddot{\varphi}$ denotes the pretranspose of φ.

15) If F is a C**-algebra and $(H_i)_{i\in I}$ a family of Hilbert right F-modules then we put

$$\bigoplus_{i\in I} H_i := \left\{ \xi \in \prod_{i\in I} H_i \;\middle|\; \text{the family } \langle\, \xi_i \mid \xi_i \,\rangle_{i\in I} \text{ is summable in } F \right\}$$

respectively

$$\bigoplus_{i\in I}^{W} H_i := \left\{ \xi \in \prod_{i\in I} H_i \;\middle|\; \text{the family } \langle\, \xi_i \mid \xi_i \,\rangle_{i\in I} \text{ is summable in } F_{\ddot{F}} \right\} .$$

16) $\odot$ denotes the algebraic tensor product of vector spaces.

17) If F, G are W*-algebras and H (resp. K) is a selfdual Hilbert right F-module (resp. G-module) then we denote by $H\bar{\otimes}K$ the W*-tensor product of H and K, which is a selfdual Hilbert right $F\bar{\otimes}G$-module ([2, Definition 2.3]).

18) $\approx$ denotes isomorphic.

If T is finite then (by [1, Theorem 5.6.6.1 f)])

$$\mathcal{L}_E(H) = E_{T,T} = \mathbb{K}_{T,T} \otimes E = \mathcal{K}_E(H) .$$

1. PRELIMINARIES

1.1. Schur functions

Definition 1.1.1. A **Schur E-function for T** is a map

$$f : T \times T \longrightarrow Un\, E^c$$

such that $f(1,1) = 1_E$ and

$$f(r,s)f(rs,t) = f(r,st)f(s,t)$$

for all $r, s, t \in T$. We denote by $\mathcal{F}(T,E)$ the set of Schur E-functions for T and put

$$\tilde{f} : T \longrightarrow Un\, E^c\,, \quad t \longmapsto f(t,t^{-1})^*\,,$$
$$\hat{f} : T \times T \longrightarrow Un\, E^c\,, \quad (s,t) \longmapsto f(t^{-1}, s^{-1})$$

for every $f \in \mathcal{F}(T,E)$.

Schur functions are also called normalized factor set or multiplier or two-co-cycle (for T with values in $Un\ E^c$) in the literature. We present in this subsection only some elementary properties (which will be used in the sequel) in order to fix the notation and the terminology. By the way, $Un\ E^c$ can be replaced in this subsection by an arbitrary commutative multiplicative group.

PROPOSITION 1.1.2. *Let* $f \in \mathcal{F}(T, E)$.

a) *For every* $t \in T$,

$$f(t,1) = f(1,t) = 1_E\,, \quad f(t,t^{-1}) = f(t^{-1},t)\,, \quad \tilde{f}(t) = \tilde{f}(t^{-1})\,.$$

b) *For all* $s, t \in T$,

$$f(s,t)\tilde{f}(s) = f(s^{-1}, st)^*\,, \qquad f(s,t)\tilde{f}(t) = f(st, t^{-1})^*\,.$$

Proof. a) Putting $s = 1$ in the equation of f we obtain

$$f(r,1)f(r,t) = f(r,t)f(1,t)$$

so

$$f(r,1) = f(1,t)$$

for all $r, t \in T$. Hence

$$f(t,1) = f(1,t) = f(1,1) = 1_E\,.$$

Putting $r = t$ and $s = t^{-1}$ in the equation of f we get

$$f(t,t^{-1})f(1,t) = f(t,1)f(t^{-1},t)\,.$$

By the above,

$$f(t,t^{-1}) = f(t^{-1},t)\,, \quad \tilde{f}(t) = \tilde{f}(t^{-1}).$$

b) Putting $r = s^{-1}$ in the equation of f, by a),

$$f(s,t)f(s^{-1},st) = f(s^{-1},s)f(1,t) = \tilde{f}(s)^*,$$

$$f(s,t)\tilde{f}(s) = f(s^{-1},st)^*.$$

Putting now $t = s^{-1}$ in the equation of f, by a) again,

$$f(r,s)f(rs,s^{-1}) = f(r,1)f(s,s^{-1}) = \tilde{f}(s)^*,$$

$$f(r,s)\tilde{f}(s) = f(rs,s^{-1})^*\,, \quad f(s,t)\tilde{f}(t) = f(st,t^{-1})^*.$$ □

Definition 1.1.3. We put

$$\Lambda(T,E) := \{\, \lambda : T \longrightarrow Un\ E^c \mid \lambda(1) = 1_E \,\}$$

and

$$\hat{\lambda} : T \longrightarrow Un\ E^c\,, \quad t \longmapsto \lambda(t^{-1}),$$

$$\delta\lambda : T \times T \longrightarrow Un\ E^c\,, \quad (s,t) \longmapsto \lambda(s)\lambda(t)\lambda(st)^*$$

for every $\lambda \in \Lambda(T,E)$.

PROPOSITION 1.1.4. a) *$\mathcal{F}(T,E)$ is a subgroup of the commutative multiplicative group $(Un\ E^c)^{T\times T}$ such that f^* is the inverse of f for every $f\in\mathcal{F}(T,E)$.*

b) *$\hat{f}\in\mathcal{F}(T,E)$ for every $f\in\mathcal{F}(T,E)$ and the map*

$$\mathcal{F}(T,E)\longrightarrow\mathcal{F}(T,E),\quad f\longmapsto\hat{f}$$

is an involutive group automorphism.

c) *$\Lambda(T,E)$ is a subgroup of the commutative multiplicative group $(Un\ E^c)^T$, $\delta\lambda\in\mathcal{F}(T,E)$ for every $\lambda\in\Lambda(T,E)$, and the map*

$$\delta:\Lambda(T,E)\longrightarrow\mathcal{F}(T,E)\,,\quad \lambda\longmapsto\delta\lambda$$

is a group homomorphism with kernel

$$\{\,\lambda\in\Lambda(T,E)\mid\lambda\text{ is a group homomorphism}\,\}$$

such that $\widehat{\delta\lambda}=\delta\hat{\lambda}$ for every $\lambda\in\Lambda(T,E)$.

Proof. a) is obvious.

b) For $r,s,t\in T$,

$$\begin{aligned}\hat{f}(r,s)\hat{f}(rs,t)&=f(s^{-1},r^{-1})f(t^{-1},s^{-1}r^{-1})\\&=f(t^{-1},s^{-1})f(t^{-1}s^{-1},r^{-1})=\hat{f}(r,st)\hat{f}(s,t),\end{aligned}$$

so $\hat{f}\in\mathcal{F}(T,E)$.

For $f,g\in\mathcal{F}(T,E)$,

$$\begin{aligned}\widehat{fg}(s,t)=(fg)(t^{-1},s^{-1})&=f(t^{-1},s^{-1})g(t^{-1},s^{-1})\\&=\hat{f}(s,t)\hat{g}(s,t)=(\hat{f}\hat{g})(s,t).\end{aligned}$$

Hence $\widehat{fg}=\hat{f}\hat{g}$.

$$\hat{f}^*(s,t)=\hat{f}(s,t)^*=f(t^{-1},s^{-1})^*=f^*(t^{-1},s^{-1})=\widehat{f^*}(s,t),$$

and therefore $(\hat{f})^*=\widehat{f^*}$.

c) For $r,s,t\in T$, we have:

$$\delta\lambda(r,s)\delta\lambda(rs,t)=\lambda(r)\lambda(s)\lambda(rs)^*\lambda(rs)\lambda(t)\lambda(rst)^*=\lambda(r)\lambda(s)\lambda(t)\lambda(rst)^*,$$
$$\delta\lambda(r,st)\delta\lambda(s,t)=\lambda(r)\lambda(st)\lambda(rst)^*\lambda(s)\lambda(t)\lambda(st)^*=\lambda(r)\lambda(s)\lambda(t)\lambda(rst)^*$$

so $\delta\lambda\in\mathcal{F}(T,E)$.

For $\lambda,\mu\in\mathcal{F}(T,E)$ and $s,t\in T$, we have:

$$\delta\lambda(s,t)\delta\mu(s,t)=\lambda(s)\lambda(t)\lambda(st)^*\mu(s)\mu(t)\mu(st)^*$$

$$= (\lambda\mu)(s)(\lambda\mu)(t)(\lambda\mu)(st)^* = \delta(\lambda\mu)(s,t).$$

Therefore $(\delta\lambda)(\delta\mu) = \delta(\lambda\mu)$.

$$\delta\lambda^*(s,t) = \lambda^*(s)\lambda^*(t)\lambda(st) = (\delta\lambda(s,t))^* = (\delta\lambda)^*(s,t),$$

and hence $\delta\lambda^* = (\delta\lambda)^*$. Therefore δ is a group homomorphism. The other assertions are obvious. $\square$

PROPOSITION 1.1.5. *Let $t \in T$, $m, n \in \mathbb{Z}$, and $f \in \mathcal{F}(T,E)$.*

a) $f(t^m, t^n) = f(t^n, t^m)$.

b) $m \in \mathbb{N} \Longrightarrow f(t^m, t^n) = \left(\prod_{j=0}^{m-1} f(t^{n+j}, t)\right)\left(\prod_{k=1}^{m-1} f(t^k, t)^*\right)$.

c) *We define*

$$\lambda : \mathbb{Z} \longrightarrow Un\, E^c, \quad n \longmapsto \begin{cases} \prod_{j=1}^{n-1} f(t^j, t)^* & \text{if} \quad n \in \mathbb{N} \\ \prod_{j=1}^{-n} f(t^{-j}, t) & \text{if} \quad n \notin \mathbb{N} \end{cases}.$$

If $t^p \neq 1$ for every $p \in \mathbb{N}$ then

$$f(t^m, t^n) = \lambda(m)\lambda(n)\lambda(m+n)^*$$

for all $m, n \in \mathbb{Z}$.

Proof. a) We may assume $m \in \mathbb{N}$ because otherwise we can replace t by t^{-1}. Put

$$P(m,n) :\Longleftrightarrow f(t^m, t^n) = f(t^n, t^m),$$
$$Q(m) :\Longleftrightarrow P(m,n) \text{ holds for all } n \in \mathbb{Z}.$$

From

$$f(t^m, t^{n-m})f(t^n, t^m) = f(t^m, t^n)f(t^{n-m}, t^m)$$

it follows

$$P(m,n) \Longleftrightarrow P(m, n-m) \Longleftrightarrow P(m, n-km)$$

for all $k \in \mathbb{Z}$.

We prove the assertion by induction. $P(m,0)$ follows from Proposition 1.1.2 a). By the above

$$P(1,0) \Longleftrightarrow P(1,k)$$

for all $k \in \mathbb{Z}$. Thus $Q(1)$ holds.

Assume $Q(p)$ holds for all $p \in \mathbb{N}_{m-1}$. Then $P(m,p)$ holds for all $p \in \mathbb{N}_{m-1} \cup \{0\}$. Let $n \in \mathbb{Z}$. There is a $k \in \mathbb{Z}$ such that

$$p := n - km \in \mathbb{N}_{m-1} \cup \{0\}\,.$$

By the above $P(m,n)$ holds. Thus $Q(m)$ holds and this finishes the inductive proof.

b) We prove the formula by induction with respect to m. By a), the formula holds for $m=1$. Assume the formula holds for an $m\in\mathbb{N}$. Since

$$f(t^m,t)f(t^{m+1},t^n)=f(t^m,t^{n+1})f(t,t^n)$$

we get by a),

$$\begin{aligned} f(t^{m+1},t^n) &= f(t^m,t^{n+1})f(t,t^n)f(t^m,t)^* \\ &= \left(\prod_{j=0}^{m-1} f(t^{n+1+j},t)\right)\left(\prod_{k=1}^{m-1} f(t^k,t)^*\right) f(t^n,t)f(t^m,t)^* \\ &= \left(\prod_{j=0}^{m} f(t^{n+j},t)\right)\left(\prod_{k=1}^{m} f(t^k,t)^*\right). \end{aligned}$$

Thus the formula holds also for $m+1$.

c) If $m,n\in\mathbb{N}$ then by b),

$$\begin{aligned} \lambda(m)\lambda(n)\lambda(m+n)^* &= \left(\prod_{k=1}^{m-1} f(t^k,t)^*\right)\left(\prod_{j=1}^{n-1} f(t^j,t)^*\right)\left(\prod_{j=1}^{m+n-1} f(t^j,t)\right) \\ &= \left(\prod_{j=0}^{m-1} f(t^{n+j},t)\right)\left(\prod_{k=1}^{m-1} f(t^k,t)^*\right) = f(t^m,t^n). \end{aligned}$$

If $m,n\in\mathbb{N}, n\le m-1$ then by b),

$$\begin{aligned} \lambda(m)\lambda(-n)\lambda(m-n)^* &= \left(\prod_{j=1}^{m-1} f(t^j,t)^*\right)\left(\prod_{j=1}^{n} f(t^{-j},t)\right)\left(\prod_{j=1}^{m-n-1} f(t^j,t)\right) \\ &= \left(\prod_{j=0}^{m-1} f(t^{-n+j},t)\right)\left(\prod_{k=1}^{m-1} f(t^k,t)^*\right) = f(t^m,t^{-n}). \end{aligned}$$

If $m,n\in\mathbb{N}, n\ge m$ then by b),

$$\begin{aligned} \lambda(m)\lambda(-n)\lambda(m-n)^* &= \left(\prod_{k=1}^{m-1} f(t^k,t)^*\right)\left(\prod_{j=1}^{n} f(t^{-j},t)\right)\left(\prod_{j=1}^{n-m} f(t^{-j},t)^*\right) \\ &= \left(\prod_{j=n-m+1}^{n} f(t^{-j},t)\right)\left(\prod_{k=1}^{m-1} f(t^k,t)^*\right) = f(t^m,t^{-n}). \end{aligned}$$

For all $m, n \in \mathbb{N}$ put

$$R(m,n) :\Longleftrightarrow f(t^{-m}, t^{-n}) = \lambda(-m)\lambda(-n)\lambda(-m-n)^* \,.$$

By the above and by Proposition 1.1.2 a),b),

$$\lambda(-1)\lambda(-1)\lambda(-2)^* = f(t^{-1},t)f(t^{-2},t)^* = \tilde{f}(t^{-1})^* f(t,t^{-2})^* = f(t^{-1},t^{-1})\,,$$

so $R(1,1)$ holds. Let now $m, n \in \mathbb{N}$ and assume $R(m,n)$ holds. Then

$$\begin{aligned}\lambda(-m)\lambda(-n-1)\lambda(-m-n-1)^* \\ &= \left(\prod_{j=1}^{m} f(t^{-j},t)\right)\left(\prod_{j=1}^{n+1} f(t^{-j},t)\right)\left(\prod_{j=1}^{m+n+1} f(t^{-j},t)^*\right) \\ &= f(t^{-m},t^{-n})f(t^{-n-1},t)f(t^{-m-n-1},t)^* = f(t^{-m},t^{-n-1}),\end{aligned}$$

so $R(m,n) \Rightarrow R(m,n+1)$.

By symmetry and a), $R(m,n)$ holds for all $m, n \in \mathbb{N}$. □

COROLLARY 1.1.6. *The map*

$$\Lambda(\mathbb{Z}, E) \longrightarrow \mathcal{F}(\mathbb{Z}, E), \quad \lambda \longmapsto \delta\lambda$$

is a surjective group homomorphism with kernel

$$\{\, \lambda \in \Lambda(\mathbb{Z}, E) \mid n \in \mathbb{Z} \Longrightarrow \lambda(n) = \lambda(1)^n \,\} \,.$$

Proof. By Proposition 1.1.4 c), only the surjectivity of the above map has to be proved and this follows from Proposition 1.1.5 c). □

1.2. *E*-C*-algebras

By replacing the scalars with the unital C*-algebra E we restrict the category of C*-algebras to the subcategory of those C*-algebras which are connected in a certain way with E. The category of unital C*-algebras is replaced by the category of E-C*-algebras, while the general category of C*-algebras is replaced by the category of adapted E-modules.

Definition 1.2.1. We call in this paper an ***E*-module** a C*-algebra F endowed with the bilinear maps

$$E \times F \longrightarrow F, \quad (\alpha, x) \longmapsto \alpha x\,,$$

$$F \times E \longrightarrow F, \quad (x, \alpha) \longmapsto x\alpha$$

such that for all $\alpha, \beta \in E$ and $x, y \in F$,

$$(\alpha\beta)x = \alpha(\beta x), \quad \alpha(x\beta) = (\alpha x)\beta, \quad x(\alpha\beta) = (x\alpha)\beta,$$

$$\alpha(xy) = (\alpha x)y, \quad (xy)\alpha = x(y\alpha), \quad \alpha \in E^c \Longrightarrow \alpha x = x\alpha,$$
$$(\alpha x)^* = x^*\alpha^*, \quad (x\alpha)^* = \alpha^* x^*, \quad 1_E x = x 1_E = x.$$

If F, G are E-modules then a C*-homomorphism $\varphi : F \to G$ is called **E-linear** if for all $(\alpha, x) \in E \times F$,

$$\varphi(\alpha x) = \alpha(\varphi x), \quad \varphi(x\alpha) = (\varphi x)\alpha.$$

For all $(\alpha, x) \in E \times F$,

$$\|\alpha x\|^2 = \|x^*\alpha^*\alpha x\| \leq \|x\|^2 \|\alpha\|^2, \quad \|x\alpha\|^2 = \|\alpha^* x^* x\alpha\| \leq \|\alpha\|^2 \|x\|^2$$

so

$$\|\alpha x\| \leq \|\alpha\| \|x\|, \quad \|x\alpha\| \leq \|x\| \|\alpha\|.$$

Definition 1.2.2. An **E-C**-algebra** is a unital C**-algebra F for which E is a *canonical* unital C**-subalgebra such that E^c defined with respect to E coincides with E^c defined with respect to F, i.e. for every $x \in E$, if $xy = yx$ for all $y \in E$ then $xy = yx$ for all $y \in F$. Every closed ideal of an E-C*-algebra is canonically an E-module.

Let F, G be E-C**-algebras. A map $\varphi : F \longrightarrow G$ is called an **E-C**-homomorphism** if it is an E-linear C**-homomorphism. If in addition φ is a C*-isomorphism then we say that φ is an **E-C*-isomorphism** and we use in this case the notation $\approx_E$. A C**-subalgebra F_0 of F is called **E-C**-subalgebra** of F if $E \subset F_0$.

With the notation of the above Definition $(\alpha - \varphi\alpha)\varphi x = 0$ for all $\alpha \in E$ and $x \in F$. Thus φ is unital iff $\varphi\alpha = \alpha$ for every $\alpha \in E$. The example

$$\mathbb{K} \longrightarrow \mathbb{K} \times \mathbb{K}, \quad x \longmapsto (x, 0)$$

shows that an E-C*-homomorphism need not be unital.

If we put $\mathbf{T} := \{ z \in \mathbb{C} \mid |z| = 1 \}$, $E := \mathcal{C}(\mathbf{T}, \mathbb{C})$, and

$$x : \mathbf{T} \longrightarrow \mathbb{C}, \quad z \longmapsto z$$

and if we denote by λ the Lebesgue measure on $\mathbf{T}$ then $L^\infty(\lambda)$ is an E-C*-algebra, $x \in Un\ E$, and x is homotopic to 1_E in $Un\ L^\infty(\lambda)$ but not in $Un\, \mathcal{C}(\mathbf{T}, \mathbb{C})$.

Definition 1.2.3. We denote by $\mathfrak{C}_E$ (resp. by $\mathfrak{C}_E^1$) the category of E-C*-algebras for which the morphisms are the E-C*-homomorphisms (resp. the unital E-C*-homomorphisms).

Proposition 1.2.4. *Let F be an E-module.*

a) *We denote by $\check{F}$ the vector space $E \times F$ endowed with the bilinear map*

$$(E \times F) \times (E \times F) \longrightarrow E \times F, \quad ((\alpha, x), (\beta, y)) \longmapsto (\alpha\beta, \alpha y + x\beta + xy)$$

and with the conjugate linear map

$$E \times F \longrightarrow E \times F, \quad (\alpha, x) \longmapsto (\alpha^*, x^*).$$

$\check{F}$ is an involutive unital algebra with $(1_E, 0)$ as unit.

b) *The maps*

$$\pi : \check{F} \longrightarrow E\,, \quad (\alpha, x) \longmapsto \alpha,$$
$$\lambda : E \longrightarrow \check{F}\,, \quad \alpha \longmapsto (\alpha, 0),$$
$$\iota : F \longrightarrow \check{F}\,, \quad x \longmapsto (0, x)$$

are involutive algebra homomorphisms such that $\pi \circ \lambda$ is the identity map of E, λ and ι are injective, and λ and π are unital. If there is a norm on $\check{F}$ with respect to which it is a C^*-algebra (in which case such a norm is unique), then we call F **adapted**. *We denote by $\mathfrak{M}_E$ the category of adapted E-modules for which the morphism are the E-linear C^*-homomorphisms.*

c) *If F is adapted then $\check{F}$ is an E-C^*-algebra by using canonically the injection λ and for all $\alpha \in E$ and $x \in F$,*

$$\|\alpha\| \leq \|(\alpha, x)\| \leq \|\alpha\| + \|x\|\ , \quad \|(0, x)\| = \|x\| \leq 2\,\|(\alpha, x)\|\ ,$$
$$\|(\alpha, 0)(0, x)\| \leq \|\alpha\|\,\|x\|\ , \qquad \|(0, x)(\alpha, 0)\| \leq \|x\|\,\|\alpha\|\ .$$

In particular, F (identified with $\iota(F)$) is a closed ideal of $\check{F}$.

d) *If E and F are C^*-subalgebras of a C^*-algebra G in such a way that the structure of E-module of F is inherited from G then*

$$\varphi : \check{F} \longrightarrow E \times G\,, \quad (\alpha, x) \longmapsto (\alpha, \alpha + x)$$

is an injective involutive algebra homomorphism, $\varphi(\check{F})$ is closed, F is adapted, and for all $\alpha \in E$ and $x \in F$,

$$\|(\alpha, x)\|_{E \times F} = \sup\{\|\alpha\|\,, \|\alpha + x\|\}\ .$$

In particular, every closed ideal of an E-C^*-algebra is adapted and $\mathfrak{C}_E$ is a full subcategory of $\mathfrak{M}_E$.

e) *A closed ideal G of an adapted E-module F, which is at the same time an E-submodule of F, is adapted.*

f) *If F is unital then it is adapted and*

$$\check{F} \longrightarrow \mathbb{R}_+, \quad (\alpha, x) \longmapsto \sup\{\|\alpha\|\,, \|\alpha 1_F + x\|\}$$

is the C^*-norm of $\check{F}$.

g) *If*

$$\lim_{y,\mathfrak{F}} \|\alpha y - y\alpha\| = 0$$

for all $\alpha \in E_+$*, where* $\mathfrak{F}$ *denotes the canonical approximate unit of* F*, then* F *is adapted and*

$$\check{F} \longrightarrow \mathbb{R}_+, \quad (\alpha, x) \longmapsto \sup\left\{\|\alpha\|\,,\, \limsup_{y,\mathfrak{F}} \|\alpha y + x\|\right\}$$

is the C^**-norm of* $\check{F}$*. In particular* F *is adapted if* E *is commutative.*

h) *If* F *is an adapted* E*-module then (with the notation of b))*

$$0 \longrightarrow F \overset{\iota}{\longrightarrow} \check{F} \underset{\overleftarrow{\lambda}}{\overset{\pi}{\rightarrow}} E \longrightarrow 0$$

is a split exact sequence in the category $\mathfrak{M}_E$*.*

Proof. a) and b) are easy to see.

c) Since λ and ι are injective and

$$\pi(\alpha, x) = \alpha\,, \qquad (\alpha, x) = (\alpha, 0) + (0, x)\,,$$
$$(\alpha, 0)(0, x) = (0, \alpha x)\,, \qquad (0, x)(\alpha, 0) = (0, x\alpha)$$

we get the first and the last two inequalities as well as the identity $\|(0, x)\| = \|x\|$. It follows

$$\|(0, x)\| \le \|(\alpha, x)\| + \|(\alpha, 0)\| = \|(\alpha, x)\| + \|\lambda\pi(\alpha, x)\|$$
$$\le \|(\alpha, x)\| + \|(\alpha, x)\| = 2\,\|(\alpha, x)\|\ .$$

d) It is easy to see that φ is an injective involutive algebra homomorphism. Let $(\alpha, x) \in \overline{\varphi(\check{F})}$. There are sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ in E and F, respectively, such that

$$\lim_{n\to\infty} (\alpha_n, \alpha_n + x_n) = (\alpha, x)\ .$$

It follows

$$\alpha = \lim_{n\to\infty} \alpha_n \in E\,, \quad x - \alpha = \lim_{n\to\infty} x_n \in F\,, \quad (\alpha, x) = \varphi(\alpha, x - \alpha) \in \varphi(\check{F})\ .$$

Thus $\varphi(\check{F})$ is closed, which proves the assertion by pulling back the norm of $E \times G$.

e) By c), F is a closed ideal of $\check{F}$ so G is a closed ideal of $\check{F}$ (use an approximate unit of F). Since G is an E-submodule of F its structure of E-module is inherited from $\check{F}$. By d), G is adapted.

f) The map

$$\check{F} \longrightarrow E \times F, \quad (\alpha, x) \longmapsto (\alpha, \alpha 1_F + x)$$

is an isomorphism of involutive algebras and so we can pull back the norm of $E \times F$.

g) It is easy to see that the above map is a norm. Since

$$\sup\{\|\alpha\|, \frac{1}{2}\|x\|\} \le \|(\alpha, x)\| \le \|\alpha\| + \|x\|$$

for all $(\alpha, x) \in E \times F$, $\check{F}$ endowed with this norm is complete. For $(\alpha, x) \in E \times F$,

$$(\alpha, x)^*(\alpha, x) = (\alpha^*\alpha, \alpha^*x + x^*\alpha + x^*x),$$
$$\|(\alpha, x)^*(\alpha, x)\| = \sup\{\|\alpha\|^2, \limsup_{y, \mathfrak{F}} \|\alpha^*\alpha y + \alpha^*x + x^*\alpha + x^*x\|\}.$$

For $y \in F_+^{\#}$,

$$\left\|(\alpha y^{\frac{1}{2}} + x)^*(\alpha y^{\frac{1}{2}} + x) - (\alpha^*\alpha y + \alpha^*x + x^*\alpha + x^*x)\right\|$$
$$\le \left\|y^{\frac{1}{2}}\alpha^*\alpha - \alpha^*\alpha y^{\frac{1}{2}}\right\| + \left\|y^{\frac{1}{2}}\alpha^*x - \alpha^*x\right\| + \left\|x^*\alpha y^{\frac{1}{2}} - x^*\alpha\right\|$$

so

$$\lim_{y, \mathfrak{F}} \left\|(\alpha y^{\frac{1}{2}} + x)^*(\alpha y^{\frac{1}{2}} + x) - (\alpha^*\alpha y + \alpha^*x + x^*\alpha + x^*x)\right\| = 0.$$

Since the map $F_+ \to F_+$, $y \mapsto y^{\frac{1}{2}}$ maps $\mathfrak{F}$ into itself and

$$\|\alpha y + x\|^2 = \|y\alpha^*\alpha y + y\alpha^*x + x^*\alpha y + x^*x\|$$

we have by the above,

$$\|(\alpha, x)\|^2 = \sup\left\{\|\alpha\|^2, \limsup_{y, \mathfrak{F}} \left\|\alpha y^{\frac{1}{2}} + x\right\|^2\right\}$$
$$= \sup\left\{\|\alpha\|^2, \limsup_{y, \mathfrak{F}} \left\|(\alpha y^{\frac{1}{2}} + x)^*(\alpha y^{\frac{1}{2}} + x)\right\|\right\}$$
$$= \sup\{\|\alpha\|^2, \limsup_{y, \mathfrak{F}} \|\alpha^*\alpha y + \alpha^*x + x^*\alpha + x^*x\|\} = \|(\alpha, x)^*(\alpha, x)\|.$$

Thus the above norm is a C*-norm and F is adapted.

h) ι is an injective E-C*-homomorphism and its image is equal to $Ker\ \pi$.
□

Corollary 1.2.5. *Let F an E-module, G a C*-algebra, and $\otimes_\sigma$ the spatial tensor product.*

a) *$F \otimes_\sigma G$ is in a natural way an E-module the multiplication being given by*

$$\alpha(x \otimes y) = (\alpha x) \otimes y, \quad (x \otimes y)\alpha = (x\alpha) \otimes y$$

for all $\alpha \in E$, $x \in F$, and $y \in G$.

b) *If F is an E-C^*-algebra and G is unital then the map*

$$E \longrightarrow F \otimes_\sigma G, \quad \alpha \longmapsto \alpha \otimes 1_G$$

is an injective C^*-homomorphism. In particular, the E-module $F \otimes_\sigma G$ is an E-C^*-algebra.

c) *If F is an adapted E-module then the E-module $F \otimes_\sigma G$ is adapted and*

$$\|(\alpha, z)\| = \sup\{\|\alpha\|, \|\alpha + z\|\}$$

for all $(\alpha, z) \in E \times (F \otimes_\sigma G)$.

d) *If F is an adapted E-module and $G := \mathcal{C}_0(\Omega)$ for a locally compact space Ω then $\mathcal{C}_0(\Omega, F)$ is adapted and*

$$\|(\alpha, x)\| = \sup\{\|\alpha\|, \|\alpha e_\Omega + x\|\}$$

for all $(\alpha, x) \in E \times \mathcal{C}_0(\Omega, F)$.

Proof. a) and b) are easy to see.

c) If $\tilde{G}$ denotes the unitization of G then by b), $\check{F} \otimes_\sigma \tilde{G}$ is an E-C*-algebra and $F \otimes_\sigma G$ is a closed ideal of it, so the assertion follows from Proposition 1.2.4 d),e).

d) follows from c). □

PROPOSITION 1.2.6. a) *If F, G are E-modules and $\varphi : F \to G$ is an E-linear C^*-homomorphism then the map*

$$\check{\varphi} : \check{F} \longrightarrow \check{G}, \quad (\alpha, x) \longmapsto (\alpha, \varphi x)$$

is an involutive unital algebra homomorphism, injective or surjective if φ is so. If $F = G$ and if φ is the identity map then $\check{\varphi}$ is also the identity map.

b) *Let F_1, F_2, F_3 be E-modules and let $\varphi : F_1 \to F_2$ and $\psi : F_2 \to F_3$ be E-linear C^*-homomorphisms. Then $\overbrace{\psi \circ \varphi}^{\vee} = \check{\psi} \circ \check{\varphi}$.* □

PROPOSITION 1.2.7. *Let G be an E-module, F an E-submodule of G which is at the same time an ideal of G, and $\varphi : G \to G/F$ the quotient map.*

a) *G/F has a natural structure of E-module and φ is E-linear.*

b) *If G is adapted then G/F is also adapted. Moreover if $\psi : \check{G} \to \check{G}/F$ denotes the quotient map (where F is identified to $\{(0, x) | \, x \in F\}$) then there is an E-C^*-isomorphism $\theta : \overbrace{G/F}^{\vee} \to \check{G}/F$ such that $\psi = \theta \circ \check{\varphi}$.*

Proof. a) is easy to see.

b) Let $(\alpha, z) \in \overbrace{G/F}^{\vee}$ and let $x, y \in \overset{-1}{\varphi}(z)$. Then $\psi(\alpha, x) = \psi(\alpha, y)$ and we put $\theta(\alpha, z) := \psi(\alpha, x)$. It is straightforward to show that θ is an isomorphism of involutive algebras. By pulling back the norm of $\check{G}/F$ with respect to θ we see that G/F is adapted. $\square$

LEMMA 1.2.8. *Let $\{(F_i)_{i\in I}, (\varphi_{ij})_{i,j\in I})\}$ be an inductive system in the category of C^*-algebras, $\{F, (\varphi_i)_{i\in I}\}$ its inductive limit, G a C^*-algebra, for every $i \in I$, $\psi_i : F_i \to G$ a C^*-homomorphism such that $\psi_j \circ \varphi_{ji} = \psi_i$ for all $i, j \in I, i \leq j$, and $\psi : F \to G$ the resulting C^*-homomorphism. If $Ker\,\psi_i \subset Ker\,\varphi_i$ for every $i \in I$ then ψ is injective.*

Proof. Let $i \in I$. Since $Ker\,\varphi_i \subset Ker\,\psi_i$ is obvious, we have $Ker\,\varphi_i = Ker\,\psi_i$. Let $\rho : F_i \to F_i/Ker\,\psi_i$ be the quotient map and

$$\varphi_i' : F_i/Ker\,\psi_i \longrightarrow F\,, \qquad \psi_i' : F_i/Ker\,\psi_i \longrightarrow G$$

the injective C*-homomorphisms with

$$\varphi_i = \varphi_i' \circ \rho\,, \qquad \psi_i = \psi_i' \circ \rho\,.$$

Then

$$\psi_i' \circ \rho = \psi_i = \psi \circ \varphi_i = \psi \circ \varphi_i' \circ \rho\,.$$

For $x \in F_i$, since ψ_i' and φ_i' are norm-preserving,

$$\|\rho x\| = \|\psi_i' \rho x\| = \|\psi \varphi_i' \rho x\| \leq \|\varphi_i' \rho x\| = \|\rho x\|\,,$$

$$\|\psi \varphi_i x\| = \|\psi \varphi_i' \rho x\| = \|\varphi_i' \rho x\| = \|\varphi_i x\|\,.$$

Thus ψ preserves the norms on $\cup_{i\in I} \varphi_i(F_i)$. Since this set is dense in F, ψ is injective. $\square$

PROPOSITION 1.2.9. *Let $\{(F_i)_{i\in I}, (\varphi_{ij})_{i,j\in I}\}$ be an inductive system in the category $\mathfrak{M}_E$ and let $(F, (\varphi_i)_{i\in I})$ be its inductive limit in the category of E-modules* (Proposition 1.2.4 c)).

a) *F is adapted.*

b) *Let $(G, (\psi_i)_{i\in I})$ be the inductive limit in the category $\mathfrak{C}_E^1$ of the inductive system $\{(\check{F}_i)_{i\in I}, (\check{\varphi}_{ij})_{i,j\in I}\}$* (Proposition 1.2.6 a),b)) *and let $\psi : G \to \check{F}$ be the unital C^*-homomorphism such that $\psi \circ \psi_i = \check{\varphi}_i$ for every $i \in I$. Then ψ is an E-C^*-isomorphism.*

Proof. a) Put

$$F_0 := \left\{ (\alpha, x) \in \check{F} \;\middle|\; \alpha \in E,\; x \in \bigcup_{i\in I} \varphi_i(F_i) \right\},$$

$$p : F_0 \longrightarrow \mathbb{R}_+ , \quad (\alpha, x) \longmapsto \inf \{ \|(\alpha, x_i)\| \mid i \in I,\ x_i \in F_i,\ \varphi_i x_i = x \} \ .$$

F_0 is an involutive unital subalgebra of $\check{F}$. p is a norm and by Proposition 1.2.4 c),

$$q(\alpha, x) := \lim_{\substack{(\alpha,y)\in F_0 \\ y\to x}} p(\alpha, y)$$

exists and

$$\|\alpha\| \le q(\alpha, x) \le \|\alpha\| + \|x\| \ , \quad \|x\| \le 2q(\alpha, x)$$

for every $(\alpha, x) \in \check{F}$.

Let $(\alpha, x) \in F_0$. Let further $i \in I$, $x_i, y_i \in F_i$ with $\varphi_i x_i = x$, $\varphi_i y_i = \alpha^* x + x^* \alpha + x^* x$. Then

$$(0, \varphi_i(\alpha^* x_i + x_i^* \alpha + x_i^* x_i - y_i)) = \check{\varphi}_i((\alpha, x_i)^*(\alpha, x_i) - (\alpha^*\alpha, y_i)) = 0$$

so

$$\inf_{i\le j} \|\varphi_{ji}(\alpha^* x_i + x_i^* \alpha + x_i^* x_i - y_i)\| = 0 \ .$$

For $\epsilon > 0$ there is a $j \in I$, $i \le j$, with

$$\|\varphi_{ji}(\alpha^* x_i + x_i^* \alpha + x_i^* x_i - y_i)\| < \epsilon \ .$$

We get

$$\begin{aligned} p(\alpha, x)^2 &\le \|(\alpha, \varphi_{ji} x_i)\|^2 = \|(\alpha, \varphi_{ji} x_i)^*(\alpha, \varphi_{ji} x_i)\| \\ &= \|(\alpha^*\alpha, \alpha^*\varphi_{ji} x_i + (\varphi_{ji} x_i^*)\alpha + \varphi_{ji}(x_i^* x_i))\| = \|(\alpha^*\alpha, \varphi_{ji}(\alpha^* x_i + x_i^*\alpha + x_i^* x_i))\| \\ &\le \|(\alpha^*\alpha, \varphi_{ji} y_i)\| + \|(0, \varphi_{ji}(\alpha^* x_i + x_i^* \alpha + x_i^* x_i - y_i))\| < \|(\alpha^*\alpha, \varphi_{ji} y_i)\| + \epsilon. \end{aligned}$$

By taking the infimum on the right side it follows, since ϵ is arbitrary,

$$p(\alpha, x)^2 \le p(\alpha^*\alpha, \alpha^* x + x^*\alpha + x^* x) = p((\alpha, x)^*(\alpha, x))$$

and this shows that p is a C*-norm. It is easy to see that q is a C*-norms. By the above inequalities, $\check{F}$ endowed with the norm q is complete, i.e. $\check{F}$ is a C*-algebra and F is adapted.

b) Let $i \in I$ and let $(\alpha, x) \in Ker\, \check{\varphi}_i$. Then

$$0 = \check{\varphi}_i(\alpha, x) = (\alpha, \varphi_i x)$$

so

$$\alpha = 0, \quad \varphi_i x = 0, \quad \inf_{j\in I,\, j\ge i} \|\varphi_{ji} x\| = 0,$$

$$\|\check{\varphi}_{ji}(0, x)\| = \|(0, \varphi_{ji} x)\| = \|\varphi_{ji} x\| \, ,$$

$$\|\psi_i(\alpha, x)\| = \inf_{j\in I,\, j\ge i} \|\check{\varphi}_{ji}(0, x)\| = 0, \quad (\alpha, x) \in Ker\, \psi_i.$$

By Lemma 1.2.8, ψ is injective.

Let $(\beta, y) \in \check{F}$ and let $\varepsilon > 0$. There are $i \in I$ and $x \in F_i$ with $\|\varphi_i x - y\| < \varepsilon$. Then

$$\psi\psi_i(\beta, x) = \check{\varphi}_i(\beta, x) = (\beta, \varphi_i x),$$

$$\|\psi\psi_i(\beta,x)-(\beta,y)\| = \|\check{\varphi}_i(\beta,x)-(\beta,y)\| = \|\varphi_i x - y\| < \varepsilon.$$

Thus $\psi(G)$ is dense in $\check{F}$ and ψ is surjective. Hence ψ is a C*-isomorphism. □

COROLLARY 1.2.10. *We put* $\Phi_E(F) := \check{F}$ *for every* E*-module* F *and similarly* $\Phi_E(\varphi) := \check{\varphi}$ *for every* E*-linear C*-homomorphism* φ.

a) Φ_E *is a covariant functor from the category* $\mathfrak{M}_E$ *in the category* $\mathfrak{C}^1_E$.

b) *The categories* $\mathfrak{C}^1_E$ *and* $\mathfrak{M}_E$ *possess inductive limits and the functor* Φ_E *is continuous with respect to the inductive limits.*

Proof. a) follows from Proposition 1.2.6.
b) follows from Proposition 1.2.9. □

Remark. The category $\mathfrak{C}_E$ does not possess inductive limits in general. This happens for instance if $\varphi_{ij} = 0$ for all $i, j \in I$.

1.3. **Some topologies**

In this subsection, T is only a set.

If the group T is infinite then different topologies play a certain role in the construction of the projective representations of T. It will be shown that all these topologies conduct to the same construction, but the use of them simplifies the manipulations.

We introduce the following notation in order to unify the cases of C*-algebras and (resp. W*-algebras).

Definition 1.3.1.

$$\widetilde{\bigoplus} := \bigoplus \qquad (resp.\ \widetilde{\bigoplus} := \overset{W}{\bigoplus}),$$

$$\widetilde{\otimes} := \otimes \qquad (resp.\ \widetilde{\otimes} := \bar{\otimes}),$$

$$\widetilde{\sum} := \sum \qquad (resp.\ \widetilde{\sum} := \overset{\ddot{E}}{\sum}).$$

If $\mathfrak{T}$ is a Hausdorff topology on $\mathcal{L}_E(H)$ then for every $\mathcal{G} \subset \mathcal{L}_E(H)$, $\mathcal{G}_{\mathfrak{T}}$ denotes the set $\mathcal{G}$ endowed with the relative topology $\mathfrak{T}$ and $\overset{\mathfrak{T}}{\bar{\mathcal{G}}}$ denotes the closure of $\mathcal{G}$ in $\mathcal{L}_E(H)_{\mathfrak{T}}$. Moreover $\overset{\mathfrak{T}}{\sum}$ denotes the sum with respect to $\mathfrak{T}$.

Lemma 1.3.2. *For $x \in E$, by the above identification of E with $\mathcal{L}_E(\breve{E})$,*

$$x\widetilde{\otimes}1_K : H \longrightarrow H\,, \quad \xi \longmapsto (x\xi_t)_{t\in T}$$

is well-defined and belongs to $\mathcal{L}_E(H)$.

a) *The map*

$$\varphi : E \longrightarrow \mathcal{L}_E(H)\,, \quad x \longmapsto x\widetilde{\otimes}1_K$$

is an injective unital C^*-homomorphism.

b) *Assume E is a W^*-algebra. Then for every $(a,\xi,\eta) \in \ddot{E}\times H \times H$, the family $(\xi_t\, a\, \eta_t^*)_{t\in T}$ is summable in $\ddot{E}_E$ and for every $x \in E$,*

$$\left\langle \varphi x\,,\, \widetilde{(a,\xi,\eta)} \right\rangle = \left\langle x\,,\, \sum_{t\in T}^{E} \xi_t\, a\, \eta_t^* \right\rangle .$$

Thus φ is a W^*-homomorphism ([1, Theorem 5.6.3.5 d)]) *with*

$$\ddot{\varphi}\widetilde{(a,\xi,\eta)} = \sum_{t\in T}^{E} \xi_t\, a\, \eta_t^*\,,$$

where $\ddot{\varphi}$ denotes the pretranspose of φ.

c) *If we consider E as a canonical unital C^{**}-subalgebra of $\mathcal{L}_E(H)$ by using the embedding of a) then $\mathcal{L}_E(H)$ is an E-C^{**}-algebra.*

Proof. a) follows from [5, page 37] (resp. [3, Proposition 1.4]).

b) We have

$$\left\langle x\bar{\otimes}1_K\,,\, \widetilde{(a,\xi,\eta)} \right\rangle = \langle\, \langle\, (x\bar{\otimes}1_K)\xi \mid \eta \,\rangle\,,\, a \,\rangle = \left\langle \sum_{t\in T}^{\ddot{E}} \eta_t^*\, x\, \xi_t\,,\, a \right\rangle =$$

$$= \sum_{t\in T} \langle\, \eta_t^*\, x\, \xi_t\,,\, a \,\rangle = \sum_{t\in T} \langle\, x\,,\, \xi_t\, a\, \eta_t^* \,\rangle\ .$$

Thus the family $(\xi_t\, a\, \eta_t^*)_{t\in T}$ is summable in $\ddot{E}_E$ and

$$\left\langle \varphi x\,,\, \widetilde{(a,\xi,\eta)} \right\rangle = \left\langle x\,,\, \sum_{t\in T}^{E} \xi_t\, a\, \eta_t^* \right\rangle .$$

If $\varphi' : \mathcal{L}_E(H) \to E'$ denotes the transpose of φ then

$$\varphi'\widetilde{(a,\xi,\eta)} = \sum_{t\in T}^{E} \xi_t\, a\, \eta_t^* \in \ddot{E}\ .$$

By continuity $\varphi'\left(\overset{\cdot\cdot}{\overbrace{\mathcal{L}_E(H)}}\right) \subset \ddot{E}$ and φ is a unital W*-homomorphism.

c) Let $x \in E^c$ and $\xi, \eta \in \mathcal{L}_E(H)$. By [1, Proposition 3.17 d)],

$$\left\langle\, (x\widetilde{\otimes}1_K)\xi \,\middle|\, \eta \,\right\rangle = \widetilde{\sum_{t\in T}}\eta_t^*((x\widetilde{\otimes}1_K)\xi)_t = \widetilde{\sum_{t\in T}}\eta_t^* x\xi_t =$$

$$= \widetilde{\sum_{t\in T}} x\eta_t^*\xi_t = x\widetilde{\sum_{t\in T}}\eta_t^*\xi_t = x\left\langle\, \xi \,|\, \eta \,\right\rangle \ .$$

Thus for $u \in \mathcal{L}_E(H)$,

$$\left\langle\, u(x\widetilde{\otimes}1_K)\xi \,\middle|\, \eta \,\right\rangle = \left\langle\, (x\widetilde{\otimes}1_K)\xi \,\middle|\, u^*\eta \,\right\rangle = x\left\langle\, \xi \,|\, u^*\eta \,\right\rangle = x\left\langle\, u\xi \,|\, \eta \,\right\rangle \ ,$$

$$u(x\widetilde{\otimes}1_K) = (x\widetilde{\otimes}1_K)u\,,$$

and so $x\widetilde{\otimes}1_K \in \mathcal{L}_E(H)^c$. $\square$

Definition 1.3.3. We put for all $\xi, \eta \in H$ (resp. and $a \in \ddot{E}_+$)

$$p_{\xi,\eta} : \mathcal{L}_E(H) \longrightarrow \mathbb{R}_+\,, \quad X \longmapsto \|\langle\, X\xi \,|\, \eta \,\rangle\|\,,$$
$$(\textit{resp. } p_{\xi,\eta,a} : \mathcal{L}_E(H) \longrightarrow \mathbb{R}_+\,, \quad X \longmapsto |\langle\langle\, X\xi \,|\, \eta \,\rangle\ ,\, a\,\rangle|),$$
$$p_\xi : \mathcal{L}_E(H) \longrightarrow \mathbb{R}_+\,, \quad X \longmapsto \|X\xi\| = \|\langle\, X\xi \,|\, X\xi \,\rangle\|^{1/2}\,,$$
$$(\textit{resp. } p_{\xi,a} : \mathcal{L}_E(H) \longrightarrow \mathbb{R}_+\,, \quad X \longmapsto \langle\langle\, X\xi \,|\, X\xi \,\rangle\ ,\, a\,\rangle^{1/2}),$$
$$q_\xi : \mathcal{L}_E(H) \longrightarrow \mathbb{R}_+\,, \quad X \longmapsto p_\xi(X^*),$$
$$(\textit{resp. } q_{\xi,a} : \mathcal{L}_E(H) \longrightarrow \mathbb{R}_+\,, \quad X \longmapsto p_{\xi,a}(X^*)).$$

and denote, respectively, by $\mathfrak{T}_1\,, \mathfrak{T}_2\,, \mathfrak{T}_3$ the topologies on $\mathcal{L}_E(H)$ generated by the set of seminorms

$$\{\, p_{\xi,\eta} \mid \xi, \eta \in H\,\}\,, \quad \left(\textit{resp. } \left\{\, p_{\xi,\eta,a} \mid \xi, \eta \in H,\, a \in \ddot{E}_+ \,\right\}\right),$$
$$\{\, p_\xi \mid \xi \in H\,\}\,, \quad \left(\textit{resp. } \left\{\, p_{\xi,a} \mid \xi \in H,\, a \in \ddot{E}_+ \,\right\}\right),$$
$$\{\, p_\xi \mid \xi \in H\,\} \cup \{\, q_\xi \mid \xi \in H\,\}\,,$$
$$\left(\textit{resp. } \left\{\, p_{\xi,a} \mid \xi \in H,\, a \in \ddot{E}_+ \,\right\} \cup \left\{\, q_{\xi,a} \mid \xi \in H,\, a \in \ddot{E}_+ \,\right\}\right).$$

Moreover $\|\cdot\|$ denotes the norm topology on $\mathcal{L}_E(H)$.

Of course $\mathfrak{T}_2 \subset \mathfrak{T}_3$. In the C*-case, $\mathfrak{T}_2$ is the topology of pointwise convergence. If E is finite-dimensional then the C*-case and the W*-case coincide.

Proposition 1.3.4. *Let $X \in \mathcal{L}_E(H)$ and $\xi, \eta \in H$ (resp. and $a \in \ddot{E}$).*

a) $p_{\xi,\eta}(X) = p_{\eta,\xi}(X^*)$ (resp. $p_{\xi,\eta,|a|}(X) = p_{\eta,\xi,|a|}(X^*)$).

b) $p_{\xi,\eta}(X) \le p_\xi(X)\,\|\eta\|$.

c) *If E is a W^*-algebra and $a = x|a|$ is the polar representation of a then*

$$p_{\xi x,\eta,|a|}(X) = \left|\left\langle X \,,\, \widetilde{(a,\xi,\eta)} \right\rangle\right| \leq p_{\xi x,|a|}(X) \left\langle \langle \eta \mid \eta \rangle \,,\, |a| \right\rangle^{1/2}.$$

d) *If $Y, Z \in \mathcal{L}_E(H)$ then*

$$p_{\xi,\eta}(YXZ) = p_{Z\xi,Y^*\eta}(X) \quad (\text{resp. } p_{\xi,\eta,|a|}(YXZ) = p_{Z\xi,Y^*\eta,|a|}(X)),$$

$$p_\xi(YXZ) \leq \|Y\|\, p_{Z\xi}(X) \quad (\text{resp. } p_{\xi,|a|}(YXZ) \leq \|Y\|\, p_{Z\xi,|a|}(X)).$$

Proof. a) From

$$\langle X\xi \mid \eta \rangle = \langle \xi \mid X^*\eta \rangle = \langle X^*\eta \mid \xi \rangle^*$$

it follows

$$p_{\xi,\eta}(X) = \|\langle X\xi \mid \eta \rangle\| = \|\langle X^*\eta \mid \xi \rangle\| = p_{\eta,\xi}(X^*),$$
$$(\text{resp. } p_{\xi,\eta,|a|}(X) = |\langle \langle X^*\eta \mid \xi \rangle \,,\, |a| \rangle| = p_{\eta,\xi,|a|}(X^*)).$$

b) $p_{\xi,\eta}(X) = \|\langle X\xi \mid \eta \rangle\| \leq p_\xi(X)\, \|\eta\|$.

c) We have

$$p_{\xi x,\eta,|a|}(X) = |\langle \langle X(\xi x) \mid \eta \rangle \,,\, |a| \rangle| = |\langle \langle X\xi \mid \eta \rangle x \,,\, |a| \rangle| =$$

$$= |\langle \langle X\xi \mid \eta \rangle \,,\, x|a| \rangle| = |\langle \langle X\xi \mid \eta \rangle \,,\, a \rangle| = \left|\left\langle X \,,\, \widetilde{(a,\xi,\eta)} \right\rangle\right|.$$

By Schwarz' inequality ([1, Proposition 2.3.3.9]),

$$|\langle \langle X(\xi x) \mid \eta \rangle \,,\, |a| \rangle|^2 \leq \langle \langle X(\xi x) \mid X(\xi x) \rangle \,,\, |a| \rangle \langle \langle \eta \mid \eta \rangle \,,\, |a| \rangle,$$

so

$$p_{\xi x,\eta,|a|}(X) \leq p_{\xi x,|a|}(X) \langle \langle \eta \mid \eta \rangle \,,\, |a| \rangle^{1/2}.$$

d) The first equation follows from

$$p_{\xi,\eta}(YXZ) = \|\langle YXZ\xi \mid \eta \rangle\| = \|\langle XZ\xi \mid Y^*\eta \rangle\| = p_{Z\xi,Y^*\eta}(X)$$
$$(\text{resp. } p_{\xi,\eta,|a|}(YXZ) = |\langle \langle YXZ\xi \mid \eta \rangle \,,\, |a| \rangle|$$
$$= |\langle \langle XZ\xi \mid Y^*\eta \rangle \,,\, |a| \rangle| = p_{Z\xi,Y^*\eta,|a|}(X))$$

and the second from

$$p_\xi(YXZ) = \|YXZ\xi\| \leq \|Y\|\, \|XZ\xi\| = \|Y\|\, p_{Z\xi}(X)$$
$$(\text{resp. } p_{\xi,|a|}(YXZ) = \langle \langle YXZ\xi \mid YXZ\xi \rangle \,,\, |a| \rangle^{1/2}$$
$$\leq \|Y\| \langle \langle XZ\xi \mid XZ\xi \rangle \,,\, |a| \rangle^{1/2} = \|Y\|\, p_{Z\xi,|a|}(X)).$$

□

Lemma 1.3.5. *Let $n \in \mathbb{N}$ and $(x_i)_{i\in\mathbb{N}_n}$ a family in E. Then*

$$\left(\sum_{i\in\mathbb{N}_n} x_i\right)^* \left(\sum_{i\in\mathbb{N}_n} x_i\right) \le n \sum_{i\in\mathbb{N}_n} x_i^* x_i \,.$$

Proof. We prove the relation by induction with respect to n. By [1, Corollary 4.2.2.4] and by the hypothesis of the induction,

$$\left(\sum_{i\in\mathbb{N}_n} x_i\right)^* \left(\sum_{i\in\mathbb{N}_n} x_i\right) = \left(x_n^* + \sum_{i\in\mathbb{N}_{n-1}} x_i^*\right)\left(x_n + \sum_{i\in\mathbb{N}_{n-1}} x_i\right)$$

$$= x_n^* x_n + \sum_{i\in\mathbb{N}_{n-1}} (x_n^* x_i + x_i^* x_n) + \left(\sum_{i\in\mathbb{N}_{n-1}} x_i\right)^* \left(\sum_{i\in\mathbb{N}_{n-1}} x_i\right)$$

$$\le x_n^* x_n + \sum_{i\in\mathbb{N}_{n-1}} (x_n^* x_n + x_i^* x_i) + (n-1) \sum_{i\in\mathbb{N}_{n-1}} x_i^* x_i = n \sum_{i\in\mathbb{N}_n} x_i^* x_i .$$

□

Lemma 1.3.6. *Let $n \in \mathbb{N}$, $x \in E_{n,n}$, and for every $j \in \mathbb{N}_n$ put*

$$\eta_j := (\delta_{ji} 1_E)_{i\in\mathbb{N}_n} \in \bigoplus_{i\in\mathbb{N}_n} \breve{E}.$$

Then

$$\|x\| \le \sqrt{n} \sup_{j\in\mathbb{N}_n} \|x\eta_j\| \,.$$

Proof. For $\xi \in \left(\bigoplus_{i\in\mathbb{N}_n} \breve{E}\right)^{\#}$, by Lemma 1.3.5,

$$\langle\, x\xi \mid x\xi \,\rangle = \sum_{i\in\mathbb{N}_n} \langle\, (x\xi)_i \mid (x\xi)_i \,\rangle = \sum_{i\in\mathbb{N}_n} \left(\sum_{j\in\mathbb{N}_n} x_{ij}\xi_j\right)^* \left(\sum_{j\in\mathbb{N}_n} x_{ij\xi_j}\right)$$

$$\le n \sum_{i\in\mathbb{N}_n} \sum_{j\in\mathbb{N}_n} (x_{ij}\xi_j)^* (x_{ij}\xi_j) = n \sum_{i\in\mathbb{N}_n} \sum_{j\in\mathbb{N}_n} \xi_j^* x_{ij}^* x_{ij} \xi_j = n \sum_{j\in\mathbb{N}_n} \xi_j^* \left(\sum_{i\in\mathbb{N}_n} x_{ij}^* x_{ij}\right) \xi_j .$$

For $i, j \in \mathbb{N}_n$,

$$(x\eta_j)_i = \sum_{k\in\mathbb{N}_n} x_{ik}\eta_{jk} = x_{ij},$$

$$\langle\, x\eta_j \mid x\eta_j \,\rangle = \sum_{i\in\mathbb{N}_n} (x\eta_j)_i^* (x\eta_j)_i = \sum_{i\in\mathbb{N}_n} x_{ij}^* x_{ij},$$

so

$$\langle\, x\xi \mid x\xi \,\rangle \le n \sum_{j\in\mathbb{N}_n} \xi_j^* \,\langle\, x\eta_j \mid x\eta_j \,\rangle\, \xi_j \le n \sum_{j\in\mathbb{N}_n} \|x\eta_j\|^2 \,\xi_j^*\xi_j$$

$$\le n \sup_{j\in\mathbb{N}_n} \|x\eta_j\|^2 \sum_{j\in\mathbb{N}_n} \xi_j^*\xi_j \le n \sup_{j\in\mathbb{N}_n} \|x\eta_j\|^2 \, 1_E \,.$$

Hence $\|x\|^2 \le n \sup_{j\in\mathbb{N}_n} \|x\eta_j\|^2$. □

COROLLARY 1.3.7.

a) *The map*

$$\mathcal{L}_E(H)_{\mathfrak{T}_1} \longrightarrow \mathcal{L}_E(H)_{\mathfrak{T}_1}, \quad X \longmapsto X^*$$

is continuous. In particular, Re $\mathcal{L}_E(H)$ *is a closed set of* $\mathcal{L}_E(H)_{\mathfrak{T}_1}$.

b) $\mathfrak{T}_1 \subset \mathfrak{T}_2 \subset \mathfrak{T}_3 \subset$ *norm topology.*

c) *If E is a W^*-algebra then the identity map*

$$\mathcal{L}_E(H)_{\ddot{H}} \longrightarrow \mathcal{L}_E(H)_{\mathfrak{T}_1}$$

is continuous so

$$\mathcal{L}_E(H)^{\#}_{\mathfrak{T}_1} = \mathcal{L}_E(H)^{\#}_{\ddot{H}}$$

is compact.

d) *For $Y, Z \in \mathcal{L}_E(H)$ and $k \in \{1,2\}$, the map*

$$\mathcal{L}_E(H)_{\mathfrak{T}_k} \longrightarrow \mathcal{L}_E(H)_{\mathfrak{T}_k}, \quad X \longmapsto YXZ$$

is continuous.

e) $\mathcal{L}_E(H)_{\mathfrak{T}_3}$ *is complete in the C^*-case.*

f) *If T is finite then $\mathfrak{T}_2$ is the norm topology in the C^*-case.*

g) $\mathcal{K}_E(H)$ *is dense in* $\mathcal{L}_E(H)_{\mathfrak{T}_3}$.

Proof. a) follows from Proposition 1.3.4 a).

b) $\mathfrak{T}_1 \subset \mathfrak{T}_2$ follows from Proposition 1.3.4 b),c). $\mathfrak{T}_2 \subset \mathfrak{T}_3 \subset$ norm topology is trivial.

c) follows from Proposition 1.3.4 c) (and [1, Theorem 5.6.3.5 a)]).

d) follows from Proposition 1.3.4 d).

e) Let $\mathfrak{F}$ be a Cauchy filter on $\mathcal{L}_E(H)_{\mathfrak{T}_3}$. Put

$$Y : H \longrightarrow H\,, \quad \xi \longmapsto \lim_{X,\mathfrak{F}} (X\xi)\,,$$

$$Z : H \longrightarrow H\,, \quad \xi \longmapsto \lim_{X,\mathfrak{F}}(X^*\xi)\,,$$

where the limits are considered in the norm topology of H. For $\xi,\eta \in H$,

$$\langle\, Y\xi \mid \eta\,\rangle = \lim_{X,\mathfrak{F}} \langle\, X\xi \mid \eta\,\rangle = \lim_{X,\mathfrak{F}} \langle\, \xi \mid X^*\eta\,\rangle = \langle\, \xi \mid Z\eta\,\rangle\ ,$$

so $Y,Z \in \mathcal{L}_E(H)$ and $Z = Y^*$. Thus $\mathfrak{F}$ converges to Y in $\mathcal{L}_E(H)_{\mathfrak{T}_3}$ and $\mathcal{L}_E(H)_{\mathfrak{T}_3}$ is complete.

f) follows from b) and Lemma 1.3.6.

g) Let $X \in \mathcal{L}_E(H)$ and $\xi \in H$. For every $S \in \mathfrak{P}_f(T)$ put

$$P_S := \sum_{s\in S} e_s \,\langle\, \cdot \mid e_s\,\rangle \in Pr\,\mathcal{K}_E(H)$$

and let $\mathfrak{F}_T$ be the upper section filter or $\mathfrak{P}_f(T)$. Then $P_S X \in \mathcal{K}_E(H)$ for every $S \in \mathfrak{P}_f(T)$ and

$$\lim_{S,\mathfrak{F}_T} P_S X\xi = X\xi$$

in H (resp. in $H_{\ddot{H}}$) ([1, Proposition 5.6.4.1 e)] (resp. [1, Proposition 5.6.4.6 c)])). Thus

$$\lim_{S,\mathfrak{F}_T} P_S X = X$$

with respect to the topology $\mathfrak{T}_2$. Since the same holds for X^*, it follows that X belongs to the closure of $\mathcal{K}_E(H)$ in $\mathcal{L}_E(H)_{\mathfrak{T}_3}$. □

Remark. The inclusions in b) can be strict as it is known from the case $E := \mathbb{K}$.

LEMMA 1.3.8. *Let G be a W^*-algebra and F a C^*-subalgebra of G. Then the following are equivalent.*

a) *F generates G as a W^*-algebra.*

b) *$F^\#$ is dense in $G^\#_{\ddot{G}}$.*

c) *F is dense in $G_{\ddot{G}}$.*

Proof. $a \Longrightarrow b$ follows from [1, Corollary 6.3.8.7].
$b \Longrightarrow c$ is trivial.
$c \Longrightarrow a$ follows from [1, Corollary 4.4.4.12 a)]. □

PROPOSITION 1.3.9. *Let G be a W^*-algebra, F a C^*-subalgebra of G generating it as W^*-algebra, I a set, and*

$$L := \bigoplus_{i\in I} \breve{F}\,, \qquad M := \overset{W}{\bigoplus_{i\in I}} \breve{G}\,.$$

a) *M is the extension of L to a selfdual Hilbert right G-module ([2, Proposition 1.3 f)]) and $L^{\#}$ is dense in $M^{\#}_{\ddot{M}}$.*

b) *If we denote for every $X \in \mathcal{L}_F(L)$ by $\bar{X} \in \mathcal{L}_G(M)$ its unique extension ([3, Proposition 1.4 a)]) then the map*

$$\mathcal{L}_F(L) \longrightarrow \mathcal{L}_G(M), \quad X \longmapsto \bar{X}$$

is an injective C-homomorphism and its image is dense in $\mathcal{L}_G(M)_{\dddot{M}}$.*

c) *The map*

$$\mathcal{L}_F(L)^{\#}_{\mathfrak{T}_2} \longrightarrow \mathcal{L}_G(M)^{\#}_{\mathfrak{T}_1}, \quad X \longmapsto \bar{X}$$

is continuous.

Proof. a) By Lemma 1.3.8 $a \Rightarrow b$, $F^{\#}$ is dense in $G^{\#}_{\ddot{G}}$ so $\breve{F}^{\#}$ is dense in $\breve{G}^{\#}_{\ddot{G}}$ and $\breve{G}$ is the extension of $\breve{F}$ to a selfdual Hilbert right G-module ([3, Corollary 1.5 $a_2 \Rightarrow a_1$]). By [3, Proposition 1.8], M is the extension of L to a selfdual Hilbert right G-module. By [3, Corollary 1.5] $a_1 \Rightarrow a_2$, $L^{\#}$ is dense in $M^{\#}_{\ddot{M}}$.

b) By a) and [3, Proposition 1.4 e)], the map

$$\mathcal{L}_F(L) \longrightarrow \mathcal{L}_G(M), \quad X \longmapsto \bar{X}$$

is an injective C*-homomorphism. By [3, Proposition 1.9 b)], its image is dense in $\mathcal{L}_G(M)_{\dddot{M}}$.

c) Denote by N the vector subspace of $\dddot{M}$ generated by

$$\left\{ \widetilde{(a,\xi,\eta)} \;\middle|\; (a,\xi,\eta) \in \ddot{G} \times L \times L \right\} .$$

By a) and [3, Proposition 1.9 a)], N is dense in $\dddot{M}$ so by Corollary 1.3.7 c),

$$\mathcal{L}_G(M)^{\#}_{\mathfrak{T}_1} = \mathcal{L}_G(M)^{\#}_{N} .$$

For $(a,\xi,\eta) \in \ddot{G}_+ \times L \times L$ and $X \in \mathcal{L}_F(L)$, by Proposition 1.3.4 c),

$$p_{\xi,\eta,a}(\bar{X}) = |\langle \langle \, \bar{X}\xi \mid \eta \rangle \, , a \rangle| =$$

$$= |\langle \langle \, X\xi \mid \eta \rangle \, , a \rangle| \le p_{\xi x,|a|}(X) \langle \langle \, \eta \mid \eta \rangle \, , |a| \rangle^{\frac{1}{2}} \, ,$$

where $a = x|a|$ is the polar representation of a, so the map

$$\mathcal{L}_F(L)^{\#}_{\mathfrak{T}_2} \longrightarrow \mathcal{L}_G(M)^{\#}_{\mathfrak{T}_1}, \quad X \longmapsto \bar{X}$$

is continuous. □

LEMMA 1.3.10. *Let $n \in \mathbb{N}$, $\xi \in \bigoplus_{i\in\mathbb{N}_n} \breve{E}$, and*

$$x := [\xi_i \delta_{j,1}]_{i,j\in\mathbb{N}_n} \in E_{n,n} \,.$$

Then $\|x\| = \|\xi\|$.

Proof. For $\eta \in \bigoplus_{i\in\mathbb{N}_n} \breve{E}$ and $i \in \mathbb{N}_n$,

$$(x\eta)_i = \sum_{j\in\mathbb{N}_n} x_{ij}\eta_j = \sum_{j\in\mathbb{N}_n} \xi_i\delta_{j,1}\eta_j = \xi_i\eta_1,$$

$$\langle\, x\eta \mid x\eta \,\rangle = \sum_{i\in\mathbb{N}_n} \langle\, (x\eta)_i \mid (x\eta)_i \,\rangle = \sum_{i\in\mathbb{N}_n} \langle\, \xi_i\eta_1 \mid \xi_i\eta_1 \,\rangle = \sum_{i\in\mathbb{N}_n} \eta_1^*\xi_i^*\xi_i\eta_1$$

$$= \eta_1^* \left(\sum_{i\in\mathbb{N}_n} \xi_i^*\xi_i \right) \eta_1 = \eta_1^* \langle\, \xi \mid \xi \,\rangle\, \eta_1 \le \|\xi\|^2 \,\eta_1^*\eta_1.$$

Hence $\|x\eta\|^2 \le \|\xi\|^2 \|\eta_1\|^2 \le \|\xi\|^2 \|\eta\|^2$ and therefore $\|x\| \le \|\xi\|$.

On the other hand, if we put $\zeta := (\delta_{i,1} 1_E)_{i\in\mathbb{N}_n}$ then for $i \in \mathbb{N}_n$,

$$(x\zeta)_i = \sum_{j\in\mathbb{N}_n} x_{ij}\zeta_j = \sum_{j\in\mathbb{N}_n} \xi_i\delta_{j,1}1_E = \xi_i \,,$$

$$\langle\, x\zeta \mid x\zeta \,\rangle = \sum_{i\in\mathbb{N}_n} (x\zeta)_i^*(x\zeta)_i = \sum_{i\in\mathbb{N}_n} \xi_i^*\xi_i = \langle\, \xi \mid \xi \,\rangle \,.$$

We deduce that $\|x\| \ge \|x\zeta\| = \|\xi\|$, and hence $\|x\| = \|\xi\|$. □

LEMMA 1.3.11. *Let F, G be unital C^{**}-algebras, $\varphi : F \to G$ a surjective C^{**}-homomorphism, I a set,*

$$L := \widetilde{\bigoplus_{i\in I}} \breve{F} \approx \breve{F}\widetilde{\otimes} l^2(I)\,, \quad M := \widetilde{\bigoplus_{i\in I}} \breve{G} \approx \breve{G}\widetilde{\otimes} l^2(I)\,,$$

and for every $\xi \in L$ put $\tilde{\xi} := (\varphi\xi_i)_{i\in I}$.

a) *If $\xi, \eta \in L$ and $x \in F$ then*

$$\tilde{\xi} \in M\,, \quad \left\|\tilde{\xi}\right\| \le \|\xi\|\,, \quad \widetilde{(\xi x)} = (\tilde{\xi})\varphi x\,, \quad \left\langle\, \tilde{\xi} \,\middle|\, \tilde{\eta} \,\right\rangle = \varphi \langle\, \xi \mid \eta \,\rangle\,.$$

b) *For every $\eta \in M$ there is a $\xi \in L$ with $\tilde{\xi} = \eta$, $\|\xi\| = \|\eta\|$.*

c) *In the W^*-case, the map*

$$L_{\ddot{L}} \longrightarrow M_{\ddot{M}}, \quad \xi \longmapsto \tilde{\xi}$$

is continuous.

Proof. a) For $J \in \mathfrak{P}_f(I)$,

$$\sum_{i\in J} \langle\, \varphi\xi_i \mid \varphi\eta_i \,\rangle = \sum_{i\in J} (\varphi\eta_i)^*(\varphi\xi_i) = \varphi \sum_{i\in J} \eta_i^*\xi_i \,.$$

It follows $\tilde{\xi} \in M$, $\left\|\tilde{\xi}\right\| \le \|\xi\|\,,\ \left\langle\, \tilde{\xi} \,\middle|\, \tilde{\eta} \,\right\rangle = \varphi\,\langle\, \xi \mid \eta \rangle$. Moreover for $i \in I$,

$$(\widetilde{\xi x})_i = \varphi(\xi x)_i = \varphi(\xi_i x) = (\varphi\xi_i)(\varphi x) = \tilde{\xi}_i(\varphi x), \quad \widetilde{\xi x} = \tilde{\xi}(\varphi x).$$

b) Case 1. $\{\, i \in I \mid \eta_i \neq 0 \,\}$ is finite

For simplicity, we assume $\{\, i \in I \mid \eta_i \neq 0 \,\} = \mathbb{N}_n$ for some $n \in \mathbb{N}$. We put

$$\theta : F_{n,n} \longrightarrow G_{n,n}\,, \quad [x_{ij}]_{i,j\in\mathbb{N}_n} \longmapsto [\varphi x_{ij}]_{i,j\in\mathbb{N}_n}.$$

θ is obviously a surjective C*-homomorphism. So if we put

$$y := [\eta_i\delta_{j,1}]_{i,j\in\mathbb{N}_n} \in G_{n,n},$$

then there is an $x \in F_{n,n}$ with $\theta x = y$, $\|x\| = \|y\|$ ([4, Theorem 10.1.7]). If we put

$$\xi : I \longrightarrow \breve{F}\,, \quad i \longmapsto \begin{cases} x_{i1} & \text{if} \quad i \in \mathbb{N}_n \\ 0 & \text{if} \quad i \in I \setminus \mathbb{N}_n \end{cases}$$

and $z := [x_{ij}\delta_{j1}]_{i,j\in\mathbb{N}_n} \in F_{n,n}$ then

$$\theta z = [\varphi(x_{ij}\delta_{j1})]_{i,j\in\mathbb{N}_n} = [y_{ij}\delta_{j1}]_{i,j\in\mathbb{N}_n} = y$$

and by [1, Theorem 5.6.6.1 a)], $\|z\| \le \|x\|$. We get for $i \in \mathbb{N}_n$,

$$\tilde{\xi}_i = \varphi\xi_i = \varphi x_{i1} = y_{i1} = \eta_i.$$

By a) and Lemma 1.3.10, $\|\xi\| = \|z\| \le \|x\| = \|y\| = \|\eta\| = \left\|\tilde{\xi}\right\| \le \|\xi\|$, hence $\|\xi\| = \|\eta\|$.

Case 2. η arbitrary in the W*-case

We may assume $\|\eta\| = 1$. We put for every $J \in \mathfrak{P}_f(I)$,

$$\eta_J : I \longrightarrow G\,, \quad i \longmapsto \begin{cases} \eta_i & \text{if} \quad i \in J \\ 0 & \text{if} \quad i \in I \setminus J \end{cases}.$$

By Case 1, for every $J \in \mathfrak{P}_f(I)$ there is a $\xi_J \in L$ with $\tilde{\xi}_J = \eta_J$ and $\|\xi_J\| = \|\eta_J\| \le 1$. Let $\mathfrak{F}$ be an ultrafilter on $\mathfrak{P}_f(I)$ finer than the upper section filter of $\mathfrak{P}_f(I)$. By [1, Proposition 5.6.3.3] $a \Rightarrow b$,

$$\xi := \lim_{J,\mathfrak{F}} \xi_J$$

exists in $L^{\#}_{\ddot{L}}$. For $i \in I$,

$$\tilde{\xi}_i = \varphi\xi_i = \varphi \lim_{J,\mathfrak{F}} (\xi_J)_i = \lim_{J,\mathfrak{F}} \varphi(\xi_J)_i = \eta_i$$

so $\tilde{\xi} = \eta$. By a), $1 = \|\eta\| = \left\|\tilde{\xi}\right\| \leq \|\xi\| \leq 1$, so $\|\xi\| = \|\eta\|$.

CASE 3. η arbitrary in the C*-case

We put for every $J \in \mathfrak{P}_f(I)$ and every $\zeta \in M$,

$$\zeta_J : I \longrightarrow G\,, \quad i \longmapsto \begin{cases} \zeta_i & \text{if} \quad i \in J \\ 0 & \text{if} \quad i \in I \setminus J \end{cases}.$$

Moreover, we denote by $\mathfrak{F}_I$ the upper section filter of $\mathfrak{P}_f(I)$, set

$$M_0 := \{\, \zeta \in M \mid \{\, i \in I \mid \zeta_i \neq 0 \,\} \text{ is finite} \,\}\,,$$

and denote by $\mathcal{M}$ the vector subspace of $\mathcal{K}_G(M)$ generated by the set

$$\{\, \zeta_1 \langle \cdot | \, \zeta_2 \rangle \mid \zeta_1, \zeta_2 \in M_0 \,\}\,.$$

Let $\mathcal{G}$ be the vector subspace of $\mathcal{K}_F(L)$ generated by the set

$$\{\, \alpha \langle \cdot | \, \beta \rangle \mid \alpha, \beta \in L \,\}\,.$$

$\mathcal{G}$ is an involutive subalgebra of $\mathcal{K}_F(L)$. Let $(\alpha_q)_{q\in Q}$, $(\beta_q)_{q\in Q}$ be finite families in L such that

$$\sum_{q\in Q} \alpha_q \langle \cdot | \, \beta_q \rangle = 0.$$

Let further $\alpha', \beta' \in M_0$. By Case 1, there are $\alpha, \beta \in L$ with $\tilde{\alpha} = \alpha'\,, \tilde{\beta} = \beta'$ and we get by a),

$$\left\langle \sum_{q\in Q} \tilde{\alpha}_q \left\langle \beta' \,|\, \tilde{\beta}_q \right\rangle \middle| \, \alpha' \right\rangle = \sum_{q\in Q} \langle \tilde{\alpha}_q \,|\, \alpha' \rangle \left\langle \beta' \,|\, \tilde{\beta}_q \right\rangle = \sum_{q\in Q} \langle \tilde{\alpha}_q \,|\, \tilde{\alpha} \rangle \left\langle \tilde{\beta} \middle| \, \tilde{\beta}_q \right\rangle$$

$$= \varphi \left(\sum_{q\in Q} \langle \alpha_q \,|\, \alpha \rangle \langle \beta \,|\, \beta_q \rangle \right) = \varphi \left(\left\langle \left(\sum_{q\in Q} \alpha_q \langle \cdot |\, \beta_q \rangle \right) \beta \middle| \, \alpha \right\rangle \right) = 0.$$

It follows ([1, Proposition 5.6.4.1 e)])

$$\sum_{q\in Q} \tilde{\alpha}_q \left\langle \cdot | \, \tilde{\beta}_q \right\rangle = 0.$$

Thus the linear map

$$\psi : \mathcal{G} \longrightarrow \mathcal{K}_G(M)\,, \quad \sum_{q\in Q} \alpha_q \langle \cdot | \, \beta_q \rangle \longmapsto \sum_{q\in Q} \tilde{\alpha}_q \left\langle \cdot | \, \tilde{\beta}_q \right\rangle$$

is well-defined and it is easy to see (by a)) that ψ is an involutive algebra homomorphism.

STEP 1. $\|\psi\| \leq 1$.

We extend ψ by continuity to a map $\psi : \mathcal{K}_F(L) \to \mathcal{K}_G(M)$. Let

$$u := \sum_{q\in Q} \alpha_q \langle \cdot | \beta_q \rangle \in \mathcal{G}$$

and let $\zeta \in M_0^{\#}$. By Case 1, there is an $\alpha \in L^{\#}$ with $\tilde{\alpha} = \zeta$. By a),

$$(\psi u)\zeta = \sum_{q\in Q} \tilde{\alpha}_q \left\langle \tilde{\alpha} \,\middle|\, \tilde{\beta}_q \right\rangle = \sum_{q\in Q} \tilde{\alpha}_q \varphi \langle \alpha | \beta_q \rangle = \sum_{q\in Q} \widetilde{\overbrace{\alpha_q \langle \alpha | \beta_q \rangle}} = \widetilde{u\alpha}\,,$$

$$\|(\psi u)\zeta\| = \|\widetilde{u\alpha}\| \le \|u\alpha\| \le \|u\|\,.$$

Since M_0 is dense in M ([1, Proposition 5.6.4.1 e)]), it follows

$$\|\psi u\| \le \|u\|\,, \qquad \|\psi\| \le 1.$$

STEP 2. $\mathcal{M}$ is dense in $\mathcal{K}_G(M)$.

Let $\alpha, \beta \in M$. By [1, Proposition 5.6.4.1 e)],

$$\alpha = \lim_{J,\mathfrak{F}_I} \alpha_J, \qquad \beta = \lim_{J,\mathfrak{F}_I} \beta_J$$

so by [1, Proposition 5.6.5.2 a)],

$$\alpha \langle \cdot | \beta \rangle = \lim_{J,\mathfrak{F}_I} \alpha_J \langle \cdot | \beta_J \rangle\ ,$$

which proves the assertion.

STEP 3. ψ is a surjective C*-homomorphism.

By Step 1, ψ is a C*-homomorphism. Since its image contains $\mathcal{M}$ (by Case 1) it is surjective by Step 2.

STEP 4. The assertion.

Let $j \in I$. By Step 3 and [4, Theorem] 10.1.7 (and [1, Proposition 5.6.5.2 a)]), there is a $u \in \mathcal{K}_F(L)$ with

$$\psi u = \eta \langle \cdot | 1_G \otimes e_j \rangle\,, \qquad \|u\| = \|\eta \langle \cdot | 1_G \otimes e_j \rangle\| = \|\eta\|\ .$$

From

$$\begin{aligned}\psi(u((1_F \otimes e_j) \langle \cdot | 1_F \otimes e_j \rangle)) &= (\eta \langle \cdot | 1_G \otimes e_j \rangle)((1_G \otimes e_j) \langle \cdot | 1_G \otimes e_j \rangle)\\ &= \eta \langle \cdot | 1_G \otimes e_j \rangle\,,\end{aligned}$$

$$\begin{aligned}\|\eta\| = \|\eta \langle \cdot | 1_G \otimes e_j \rangle\| &\le \|u((1_F \otimes e_j) \langle \cdot | 1_F \otimes e_j \rangle)\|\\ &\le \|u\|\,\|(1_F \otimes e_j) \langle \cdot | 1_F \otimes e_j \rangle\| = \|u\| = \|\eta\|\,,\end{aligned}$$

$$\|u((1_F \otimes e_j) \langle \cdot | 1_F \otimes e_j \rangle)\| = \|\eta\|$$

we see that we may assume

$$u = u((1_F \otimes e_j) \langle \cdot | 1_F \otimes e_j \rangle)\ .$$

Then

$$u = (u(1_F \otimes e_j)) \langle \cdot \mid 1_F \otimes e_j \rangle \ .$$

If we put $\xi := u(1_F \otimes e_j) \in L$ then $u = \xi \langle \cdot \mid 1_F \otimes e_j \rangle$, $\|\eta\| = \|u\| = \|\xi\|$,

$$\eta \langle \cdot \mid 1_G \otimes e_j \rangle = \psi u = \tilde{\xi} \langle \cdot \mid 1_G \otimes e_j \rangle) \, ,$$

$$\eta = \eta \langle 1_G \otimes e_j \mid 1_G \otimes e_j \rangle = \tilde{\xi} \langle 1_G \otimes e_j \mid 1_G \otimes e_j \rangle = \tilde{\xi} \, .$$

c) Let $(a, \eta_0) \in \ddot{G} \times M$. By b), there is a $\xi_0 \in L$ with $\tilde{\xi}_0 = \eta_0$. By a), for $\xi \in L$,

$$\left\langle \tilde{\xi} \, , \widetilde{(a, \eta_0)} \right\rangle = \left\langle \left\langle \tilde{\xi} \,\middle|\, \eta_0 \right\rangle , a \right\rangle = \left\langle \left\langle \tilde{\xi} \,\middle|\, \tilde{\xi}_0 \right\rangle , a \right\rangle =$$

$$= \langle \varphi \langle \xi \mid \xi_0 \rangle \, , a \rangle = \langle \langle \xi \mid \xi_0 \rangle \, , \ddot{\varphi} a \rangle = \left\langle \xi \, , \widetilde{(\ddot{\varphi} a, \xi_0)} \right\rangle .$$

We put

$$\theta : L \longrightarrow M \, , \quad \xi \longmapsto \tilde{\xi}$$

and denote by $\theta' : M' \to L'$ its transpose. By the above, $\theta' \widetilde{(a, \eta_0)} \in \ddot{L}$. Since θ' is continuous, $\theta'(\ddot{M}) \subset \ddot{L}$ and this proves the assertion. □

Proposition 1.3.12. *We use the notation of* Lemma 1.3.11.

a) *If $X \in \mathcal{L}_F(L)$ and $\xi \in L$ with $\tilde{\xi} = 0$ then $\widetilde{X\xi} = 0$; we define*

$$\tilde{X} : M \longrightarrow M \, , \quad \eta \longmapsto \widetilde{X\xi} \, ,$$

where $\xi \in L$ with $\tilde{\xi} = \eta$ (Lemma 1.3.11 b)).

b) *For every $X \in \mathcal{L}_F(L)$, $\tilde{X}$ belongs to $\mathcal{L}_G(M)$ and the map*

$$\mathcal{L}_F(L) \longrightarrow \mathcal{L}_G(M), \quad X \longmapsto \tilde{X}$$

is a surjective C^{**}-homomorphism continuous with respect to the topologies $\mathfrak{T}_k$ with $k \in \{1, 2, 3\}$.

c) *For $\xi, \eta \in L$,*

$$\widetilde{\overbrace{\eta \langle \cdot \mid \xi \rangle}} = \tilde{\eta} \left\langle \cdot \mid \tilde{\xi} \right\rangle$$

and

$$\mathcal{K}_G(M) = \left\{ \tilde{X} \,\middle|\, X \in \mathcal{K}_F(L) \right\} \, .$$

Proof. a) For $i \in I$, $\varphi \xi_i = \tilde{\xi}_i = 0$ so by Lemma 1.3.11 a),

$$\widetilde{X(e_i \xi_i)} = \widetilde{(X e_i)} \xi_i = \widetilde{(X e_i)} \varphi \xi_i = 0 \, .$$

By [1, Proposition 5.6.4.1 e)] (resp. [1, Proposition 5.6.4.6 c)] and [1, Proposition 5.6.3.4 c)]),

$$X\xi = X\left(\sum_{i\in I} e_i\xi_i\right) = \sum_{i\in I} X(e_i\xi_i), \left(\text{resp. } X\xi = X\left(\sum_{i\in I}^{\ddot{L}} e_i\xi_i\right) = \sum_{i\in I}^{\ddot{L}} X(e_i\xi_i)\right)$$

so by Lemma 1.3.11 a) (resp. c)),

$$\widetilde{X\xi} = \widetilde{\overbrace{\sum_{i\in I} X(e_i\xi_i)}} = \sum_{i\in I} \widetilde{X(e_i\xi_i)} = 0$$

$$\left(\text{resp. } \widetilde{X\xi} = \widetilde{\overbrace{\sum_{i\in I}^{\ddot{L}} X(e_i\xi_i)}} = \sum_{i\in I}^{\ddot{M}} \widetilde{X(e_i\xi_i)} = 0\right).$$

b) For $X, Y \in \mathcal{L}_F(L)$ and $\xi, \eta \in L$, by Lemma 1.3.11 a),

$$\left\langle \tilde{X}\tilde{\xi} \,\middle|\, \tilde{\eta} \right\rangle = \left\langle \widetilde{X\xi} \,\middle|\, \tilde{\eta} \right\rangle = \varphi \langle X\xi \mid \eta \rangle$$

$$= \varphi \langle \xi \mid X^*\eta \rangle = \left\langle \tilde{\xi} \,\middle|\, \widetilde{X^*\eta} \right\rangle = \left\langle \tilde{\xi} \,\middle|\, \widetilde{X^*}\tilde{\eta} \right\rangle,$$

$$\tilde{X}\tilde{Y}\tilde{\xi} = \tilde{X}\widetilde{Y\xi} = \widetilde{X(Y\xi)} = \widetilde{(XY)\xi} = \widetilde{XY}\tilde{\xi}.$$

By Lemma 1.3.11 b), $\tilde{X} \in \mathcal{L}_G(M)$, $(\tilde{X})^* = \widetilde{X^*}$, and $\tilde{X}\tilde{Y} = \widetilde{XY}$, i.e. the map is a C*-homomorphism.

For $X \in \mathcal{L}_F(L)$ and $\xi, \eta \in L$ (resp. and $a \in \ddot{M}_+$), by Lemma 1.3.11 a),

$$p_{\tilde{\xi},\tilde{\eta}}(\tilde{X}) = \left\|\left\langle \tilde{X}\tilde{\xi} \,\middle|\, \tilde{\eta} \right\rangle\right\| = \left\|\left\langle \widetilde{X\xi} \,\middle|\, \tilde{\eta} \right\rangle\right\| = \|\varphi \langle X\xi \mid \eta \rangle\| \le p_{\xi,\eta}(X)$$

$$(\text{resp. } p_{\tilde{\xi},\tilde{\eta},a}(X) = \left|\left\langle \left\langle \tilde{X}\tilde{\xi} \,\middle|\, \tilde{\eta} \right\rangle, a \right\rangle\right| = |\langle \varphi \langle X\xi \mid \eta \rangle, a \rangle|$$

$$= |\langle \langle X\xi \mid \eta \rangle, \ddot{\varphi}a \rangle| = p_{\xi,\eta,\ddot{\varphi}a}(X)),$$

so by Lemma 1.3.11 b), the map is continuous with respect to the topology $\mathfrak{T}_1$. The proof for the other topologies is similar.

c) For $\zeta \in L$, by Lemma 1.3.11 a),

$$\widetilde{\overbrace{\eta \langle \cdot \mid \xi \rangle}}\tilde{\zeta} = \widetilde{\overbrace{(\eta \langle \cdot \mid \xi \rangle)\zeta}} = \widetilde{\overbrace{\eta \langle \zeta \mid \xi \rangle}}$$

$$= \tilde{\eta}\varphi \langle \zeta \mid \xi \rangle = \tilde{\eta}\left\langle \tilde{\zeta} \,\middle|\, \tilde{\xi} \right\rangle = \left(\tilde{\eta}\left\langle \cdot \,\middle|\, \tilde{\xi} \right\rangle\right)\tilde{\zeta}$$

so by Lemma 1.3.11 b),

$$\widetilde{\overbrace{\eta \langle \cdot \mid \xi \rangle}} = \tilde{\eta}\left\langle \cdot \,\middle|\, \tilde{\xi} \right\rangle.$$

The last assertion follows now from b). □

2. MAIN PART

Throughout this section, we fix $f \in \mathcal{F}(T, E)$.

2.1. The representations

We present here the projective representation of the groups and its main properties.

Definition 2.1.1. We put for every $t \in T$ and $\xi \in H$,

$$u_t : \breve{E} \longrightarrow H\,, \quad \zeta \longmapsto \zeta \otimes e_t\,,$$

$$V_t\xi : T \longrightarrow \breve{E}\,, \quad s \longmapsto f(t, t^{-1}s)\xi(t^{-1}s)\;.$$

If we want to emphasize the role of f then we put V_t^f instead of V_t. For $x \in E$,

$$(x\widetilde{\otimes}1_K)V_t\xi : T \longrightarrow \breve{E}\,, \quad s \longmapsto f(t, t^{-1}s)x\xi(t^{-1}s)\;.$$

Proposition 2.1.2. *Let $s, t \in T$, $x \in E$, $\zeta \in \breve{E}$, and $\xi \in H$.*

a) $V_t\xi \in H$.

b) $V_sV_t = (f(s,t)\widetilde{\otimes}1_K)V_{st}$.

c) $V_t(\zeta \otimes e_s) = (f(t,s)\zeta) \otimes e_{ts}$.

d) $V_t(x\widetilde{\otimes}1_K) = (x\widetilde{\otimes}1_K)V_t$.

e) $V_t \in Un\,\mathcal{L}_E(H)\,, \quad V_t^* = (\tilde{f}(t)\widetilde{\otimes}1_K)V_{t^{-1}}$.

f) $(x\widetilde{\otimes}1_K)V_t(\zeta \otimes e_s) = (f(t,s)x\zeta) \otimes e_{ts}$.

g) *If T is infinite and $\mathfrak{F}$ denotes the filter on T of cofinite subsets, i.e.*

$$\mathfrak{F} := \{\, S \mid S \in \mathfrak{P}(T)\,,\ T \setminus S \in \mathfrak{P}_f(T)\,\}\ ,$$

then

$$\lim_{t,\mathfrak{F}} V_t = 0$$

in $\mathcal{L}_E(H)_{\mathfrak{T}_1}$.

Proof. a) For $R \in \mathfrak{P}_f(T)$,

$$\sum_{r\in R} \langle\, (V_t\xi)_r \mid (V_t\xi)_r \,\rangle = \sum_{r\in R} \langle\, f(t,t^{-1}r)\xi_{t^{-1}r} \mid f(t,t^{-1}r)\xi_{t^{-1}r} \,\rangle =$$

$$= \sum_{r\in R} \langle\, \xi_{t^{-1}r} \mid \xi_{t^{-1}r} \,\rangle = \sum_{r\in R} \langle\, \xi_r \mid \xi_r \,\rangle \le \langle\, \xi \mid \xi \,\rangle$$

so $V_t\xi \in H$.

b) For $r \in T$,

$$(V_sV_t\xi)_r = f(s,s^{-1}r)(V_t\xi)_{s^{-1}r} = f(s,s^{-1}r)f(t,t^{-1}s^{-1}r)\xi_{t^{-1}s^{-1}r}$$
$$= f(s,t)f(st,t^{-1}s^{-1}r)\xi_{t^{-1}s^{-1}r} = f(s,t)(V_{st}\xi)_r = ((f(s,t)\widetilde{\otimes}1_K)V_{st}\xi)_r$$

so

$$V_sV_t = (f(s,t)\widetilde{\otimes}1_K)V_{st}.$$

c) For $r \in T$,

$$(V_t(\zeta\otimes e_s))_r = f(t,t^{-1}r)(\zeta\otimes e_s)_{t^{-1}r}$$
$$= \delta_{s,t^{-1}r}f(t,t^{-1}r)\zeta = \delta_{r,ts}f(t,s)\zeta = ((f(t,s)\zeta)\otimes e_{ts})_r$$

so

$$V_t(\zeta\otimes e_s) = (f(t,s)\zeta)\otimes e_{ts}\ .$$

d) We have

$$(V_t(x\widetilde{\otimes}1_K)\xi)_s = f(t,t^{-1}s)((x\widetilde{\otimes}1_K)\xi)_{t^{-1}s} = f(t,t^{-1}s)x\xi_{t^{-1}s} = ((x\widetilde{\otimes}1_K)V_t\xi)_s$$

so

$$V_t(x\widetilde{\otimes}1_K) = (x\widetilde{\otimes}1_K)V_t\ .$$

e) For $\eta \in H$, by Proposition 1.1.2 a),b),

$$\langle\, V_t\xi \mid \eta\,\rangle = \widetilde{\sum_{s\in T}}\langle\, (V_t\xi)_s \mid \eta_s\,\rangle = \widetilde{\sum_{s\in T}}\left\langle\, f(t,t^{-1}s)\xi_{t^{-1}s} \,\middle|\, \eta_s\,\right\rangle$$
$$= \widetilde{\sum_{r\in T}}\langle\, f(t,r)\xi_r \mid \eta_{tr}\,\rangle = \widetilde{\sum_{r\in T}}\left\langle\, \xi_r \mid \tilde{f}(t)f(t^{-1},tr)\eta_{tr}\,\right\rangle$$
$$= \widetilde{\sum_{r\in T}}\left\langle\, \xi_r \mid (((\tilde{f}(t)\widetilde{\otimes}1_K)V_{t^{-1}})\eta)_r\,\right\rangle = \left\langle\, \xi \mid ((\tilde{f}(t)\widetilde{\otimes}1_K)V_{t^{-1}})\eta\,\right\rangle$$

so $V_t \in \mathcal{L}_E(H)$ with $V_t^* = (\tilde{f}(t)\widetilde{\otimes}1_K)V_{t^{-1}}$. By b) and d),

$$V_t^*V_t = (\tilde{f}(t)\widetilde{\otimes}1_K)V_{t^{-1}}V_t = (\tilde{f}(t)\widetilde{\otimes}1_K)(f(t^{-1},t)\widetilde{\otimes}1_K)V_{t^{-1}t} = id_H\ ,$$

$$V_tV_t^* = V_t(\tilde{f}(t)\widetilde{\otimes}1_K)V_{t^{-1}} = (\tilde{f}(t)\widetilde{\otimes}1_K)V_tV_{t^{-1}}$$
$$= (\tilde{f}(t)\widetilde{\otimes}1_K)(f(t,t^{-1})\widetilde{\otimes}1_K)V_{tt^{-1}} = id_H.$$

f) follows from c).

g) Let us consider first the C*-case. Let $\xi,\eta \in H$, $t \in T$, and $\varepsilon > 0$. There is an $S \in \mathfrak{P}_f(T)$ such that $\left\|\eta e_{T\setminus S}\right\| < \varepsilon$. By e),

$$\left|\left\langle\, V_t\xi \mid \eta e_{T\setminus S}\,\right\rangle\right| \le \|V_t\xi\|\left\|\eta e_{T\setminus S}\right\| \le \varepsilon\,\|\xi\|$$

so

$$p_{\xi,\eta}(V_t) = |\langle V_t\xi \mid \eta \rangle| \le |\langle V_t\xi \mid \eta e_S \rangle| + \left|\left\langle V_t\xi \mid \eta e_{T\setminus S} \right\rangle\right| < |\langle V_t\xi \mid \eta e_S \rangle| + \varepsilon.$$

From

$$\langle V_t\xi \mid \eta e_S \rangle = \sum_{s\in S} \eta_s^* f(t, t^{-1}s)\xi_{t^{-1}s}$$

it follows

$$\lim_{t,\mathfrak{F}} \langle V_t\xi \mid \eta e_S \rangle = 0\,, \qquad \lim_{t,\mathfrak{F}} p_{\xi,\eta}(V_t) = 0.$$

The W*-case can be proved similarly. □

Remark. By e), $\mathfrak{T}_1$ cannot be replaced by $\mathfrak{T}_2$ in g).

Proposition 2.1.3. *Let $s, t \in T$.*

a) $u_t \in \mathcal{L}_E(\breve{E}, H)\,, \quad u_t^* = \langle \;\cdot\; \mid 1_E \otimes e_t \rangle.$

b) $u_s^* u_t = \delta_{s,t} 1_E.$

c) $u_s u_t^* = 1_E \widetilde{\otimes} (\langle \cdot \mid e_t \rangle\, e_s).$

d) $\sum\limits_{r\in T}^{\mathfrak{T}_2} u_r u_r^* = id_H.$

Proof. a) For $\zeta \in \breve{E}$ and $\xi \in H$,

$$\langle u_t\zeta \mid \xi \rangle = \langle \zeta \otimes e_t \mid \xi \rangle = \widetilde{\sum_{s\in T}} \xi_s^*(\zeta \otimes e_t)_s = \xi_t^*\zeta = \langle \zeta \mid \xi_t \rangle$$

so

$$u_t \in \mathcal{L}_E(\breve{E}, H)\,, \qquad u_t^*\xi = \xi_t = \langle \xi \mid 1_E \otimes e_t \rangle\ .$$

b) For $\zeta \in \breve{E}$, by a),

$$u_s^* u_t\zeta = u_s^*(\zeta \otimes e_t) = \langle \zeta \otimes e_t \mid 1_E \otimes e_s \rangle = \delta_{s,t}\zeta$$

so $u_s^* u_t = \delta_{s,t} 1_E$.

c) For $\zeta \in \breve{E}$ and $r \in T$, by a),

$$\begin{aligned} u_s u_t^*(\zeta \otimes e_r) &= u_s \delta_{r,t}\zeta = \delta_{r,t}(\zeta \otimes e_s) \\ &= \zeta \otimes \langle e_r \mid e_t \rangle\, e_s = (1_E \widetilde{\otimes} (\langle \cdot \mid e_t \rangle\, e_s))(\zeta \otimes e_r), \end{aligned}$$

so (by a) and [1, Proposition 5.6.4.1 e)] (resp. [1, Proposition 5.6.4.6 c) and Proposition 5.6.3.4 c)]) $u_s u_t^* = 1_E \widetilde{\otimes} (\langle \cdot \mid e_t \rangle\, e_s)$.

d) For $\xi \in H$ (resp. and $a \in \ddot{E}_+$) and $S \in \mathfrak{P}_f(T)$, by c),

$$p_\xi\left(\sum_{t\in S} u_t u_t^* - id_H\right) = \left\|\sum_{t\in T\setminus S} \langle \xi \mid \xi \rangle\right\|^{1/2}$$

$$\left(\text{resp. } p_{\xi,a}\left(\sum_{t\in S} u_t u_t^* - id_H\right) = \left\langle\left\langle \sum_{t\in S}(u_t u_t^* - id_H)\xi \,\middle|\, \sum_{t\in S}(u_t u_t^* - id_H)\xi \right\rangle, a\right\rangle^{\frac{1}{2}}\right.$$
$$\left. = \Big(\sum_{t\in T\setminus S} \langle \langle\, \xi \mid \xi \,\rangle \,,\, a \,\rangle\Big)^{\frac{1}{2}} \right)$$

and the assertion follows. □

Proposition 2.1.4. *Let $s, t \in T$ and $x \in E$.*

a) $V_s u_t = u_{st} f(s,t)$.

b) $u_s^* V_t = f(t, t^{-1}s) u_{t^{-1}s}^*$.

c) $(x \widetilde{\otimes} 1_K) u_t = u_t x$.

d) $x u_t^* = u_t^* (x \widetilde{\otimes} 1_K)$.

Proof. a) For $\zeta \in \breve{E}$, by Proposition 2.1.2 c),

$$V_s u_t \zeta = V_s(\zeta \otimes e_t) = (f(s,t)\zeta) \otimes e_{st} = u_{st} f(s,t)\zeta$$

so $V_s u_t = u_{st} f(s,t)$.

b) For $\zeta \in \breve{E}$ and $r \in T$, by Proposition 2.1.2 c) and Proposition 2.1.3 a),

$$u_s^* V_t(\zeta \otimes e_r) = u_s^*((f(t,r)\zeta) \otimes e_{tr}) = \delta_{s,tr} f(t,r)\zeta$$
$$= \delta_{t^{-1}s,r} f(t, t^{-1}s)\zeta = f(t, t^{-1}s) u_{t^{-1}s}^*(\zeta \otimes e_r)$$

so $u_s^* V_t = f(t, t^{-1}s) u_{t^{-1}s}^*$.

c) For $\zeta \in \breve{E}$,

$$(x \widetilde{\otimes} 1_K) u_t \zeta = (x \widetilde{\otimes} 1_K)(\zeta \otimes e_t) = (x\zeta) \otimes e_t = u_t x \zeta$$

so $(x \widetilde{\otimes} 1_K) u_t = u_t x$.

d) follows from c). □

Definition 2.1.5. We put for all $s, t \in T$ *(Proposition 2.1.3 a))*

$$\varphi_{s,t} : \mathcal{L}_E(H) \longrightarrow \mathcal{L}_E(\breve{E}) \approx E\,, \quad X \longmapsto u_s^* X u_t$$

and set $X_t := \varphi_{t,1} X$ for every $X \in \mathcal{L}_E(H)$.

Proposition 2.1.6. *Let $s, t \in T$.*

a) *$\varphi_{s,t}$ is linear with $\|\varphi_{s,t}\| = 1$.*

b) *For $X \in \mathcal{L}_E(H)$ and $x, y \in \breve{E}$,*

$$\langle\, (\varphi_{s,t} X)x \mid y \,\rangle = \langle\, X(x \otimes e_t) \mid y \otimes e_s \,\rangle \ .$$

c) *The map*

$$\varphi_{s,t} : \mathcal{L}_E(H)_{\mathfrak{T}_1} \longrightarrow E \text{ (resp. } E_{\ddot{E}})$$

is continuous.

d) $\varphi_{t,t}$ *is involutive and completely positive.*

e) *For* $r \in T$ *and* $x \in E$,

$$\varphi_{s,t}((x\widetilde{\otimes}1_K)V_r) = \delta_{s,rt} f(r,t)x \ .$$

f) *If* $(x_r)_{r\in T} \in E^{(T)}$ *and*

$$X := \sum_{r\in T}(x_r\widetilde{\otimes}1_K)V_r$$

then

$$\varphi_{s,t}X = f(st^{-1},t)x_{st^{-1}} \, , \qquad X_t = x_t \ .$$

g) *For* $X \in \mathcal{L}_E(H)$ *and* $x, y \in E$,

$$\varphi_{s,t}((x\widetilde{\otimes}1_K)X(y\widetilde{\otimes}1_K)) = x(\varphi_{s,t}X)y \, ,$$
$$((x\widetilde{\otimes}1_K)X(y\widetilde{\otimes}1_K))_t = xX_ty \ .$$

Proof. a) follows from Proposition 2.1.3 a),b).

b) We have

$$\langle\, (\varphi_{s,t}X)x \mid y \,\rangle = \langle\, u_s^*Xu_tx \mid y \,\rangle = \langle\, Xu_tx \mid u_sy \,\rangle = \langle\, X(x\otimes e_t) \mid y\otimes e_s \,\rangle \ .$$

c) *The* C^**-case.*

By b), for $X \in \mathcal{L}_E(H)$,

$$\|\varphi_{s,t}X\| = \|\langle\, (\varphi_{s,t}X)1_E \mid 1_E \,\rangle\| = \|\langle\, X(1_E\otimes e_t) \mid 1_E\otimes e_s \,\rangle\| = p_{1_E\otimes e_t, 1_E\otimes e_s}(X).$$

The W^**-case.*

Let $a \in \ddot{E}$ and let $a = x|a|$ be its polar representation. By b), for $X \in \mathcal{L}_E(H)$,

$$\begin{aligned} |\langle \varphi_{s,t}X \, , \, a \rangle| &= |\langle \langle\, (\varphi_{s,t}X)1_E \mid 1_E \,\rangle \, , \, x|a| \rangle| = |\langle \langle\, (\varphi_{s,t}X)x \mid 1_E \,\rangle \, , \, |a| \rangle| \\ &= |\langle \langle\, X(x\otimes e_t) \mid 1_E\otimes e_s \,\rangle \, , \, |a| \rangle| = p_{x\otimes e_t, 1_E\otimes e_s, |a|}(X). \end{aligned}$$

d) For $X \in \mathcal{L}_E(H)$,

$$(\varphi_{t,t}X)^* = (u_t^*Xu_t)^* = u_t^*X^*u_t = \varphi_{t,t}(X^*)$$

so $\varphi_{t,t}$ is involutive. For $n \in \mathbb{N}$, $X \in ((\mathcal{L}_E(H))_{n,n})_+$, and $\zeta \in \breve{E}^n$,

$$\sum_{i\in\mathbb{N}_n}\left\langle \sum_{j\in\mathbb{N}_n}((\varphi_{t,t}X_{ij})\zeta_j) \,\middle|\, \zeta_i \right\rangle = \sum_{i,j\in\mathbb{N}_n}\langle\, u_t^*X_{ij}u_t\zeta_j \mid \zeta_i \,\rangle =$$

$$= \sum_{i,j \in \mathbb{N}_n} \langle\, X_{ij} u_t \zeta_j \mid u_t \zeta_i \,\rangle \geq 0$$

([1, Theorem 5.6.6.1 f)] and [1, Theorem 5.6.1.11 $c_1 \Rightarrow c_2$]) so $\varphi_{t,t}$ is completely positive ([1, Theorem 5.6.6.1 f)] and [1, Theorem 5.6.1.11 $c_2 \Rightarrow c_1$]).

e) By Proposition 2.1.4 a),d) and Proposition 2.1.3 b),

$$\varphi_{s,t}((x\widetilde{\otimes}1_K)V_r) = u_s^*(x\widetilde{\otimes}1_K)V_r u_t = x u_s^* V_r u_t = x u_s^* u_{rt} f(r,t) = \delta_{s,rt} f(r,t) x .$$

f) By e) (and Proposition 1.1.2 a)),

$$\varphi_{s,t} X = \sum_{r \in T} \varphi_{s,t}((x_r \widetilde{\otimes} 1_K)V_r) = \sum_{r \in T} \delta_{s,rt} f(r,t) x_r = f(st^{-1},t) x_{st^{-1}} ,$$

$$X_t = \varphi_{t,1} X = f(t,1) x_t = x_t .$$

g) By Proposition 2.1.4 c),d),

$$\begin{aligned} \varphi_{s,t}((x\widetilde{\otimes}1_K)X(y\widetilde{\otimes}1_K)) &= u_s^*(x\widetilde{\otimes}1_K)X(y\widetilde{\otimes}1_K)u_t \\ &= x u_s^* X u_t y = x(\varphi_{s,t}X)y . \end{aligned}$$

□

Definition 2.1.7. We put

$$\mathcal{R}(f) := \left\{ \sum_{t \in T} (x_t \widetilde{\otimes} 1_K) V_t \;\middle|\; (x_t)_{t \in T} \in E^{(T)} \right\} ,$$

$$\mathcal{S}(f) := \overline{\mathcal{R}(f)}^{\mathfrak{T}_3} , \qquad \mathcal{S}_{\|\cdot\|}(f) := \overline{\mathcal{R}(f)}^{\|\cdot\|} .$$

Moreover, we put $\mathcal{S}_C(f) := \mathcal{S}(f)$ in the C*-case and $\mathcal{S}_W(f) := \mathcal{S}(f)$ in the W*-case. If F is a subset of E then we put

$$\mathcal{S}(f,F) := \{\, X \in \mathcal{S}(f) \mid t \in T \Longrightarrow X_t \in F \,\}$$

and use similar notation for the other $\mathcal{S}$.

By Proposition 2.1.2 b),d),e), $\mathcal{R}(f)$ is an involutive unital E-subalgebra of $\mathcal{L}_E(H)$ (with V_1 as unit). In particular, $\mathcal{S}_{\|\cdot\|}(f)$ is an E-C*-subalgebra of $\mathcal{L}_E(H)$. If T is finite then $\mathcal{R}(f) = \mathcal{S}(f)$. By Corollary 1.3.7 e), $\mathcal{S}_C(f)_{\mathfrak{T}_3}$ is complete.

Proposition 2.1.8. *For* $X \in \overline{\mathcal{R}(f)}^{\mathfrak{T}_1}$ *and* $s,t \in T$,

$$\varphi_{s,t} X = f(st^{-1},t) X_{st^{-1}} .$$

Proof. Let $\mathfrak{F}$ be a filter on $\mathcal{R}(f)$ converging to X in the $\mathfrak{T}_1$-topology. By Proposition 2.1.6 c),f) (and Corollary 1.3.7 d)),

$$\varphi_{s,t}X = \lim_{Y,\mathfrak{F}} \varphi_{s,t}Y = \lim_{Y,\mathfrak{F}} f(st^{-1},t)Y_{st^{-1}} = f(st^{-1},t)\lim_{Y,\mathfrak{F}} Y_{st^{-1}}$$
$$= f(st^{-1},t)\lim_{Y,\mathfrak{F}} \varphi_{st^{-1},1}Y = f(st^{-1},t)\varphi_{st^{-1},1}X = f(st^{-1},t)X_{st^{-1}}.$$

□

THEOREM 2.1.9. *Let* $X \in \overline{\mathcal{R}(f)}^{\mathfrak{T}_1}$.

a) *If* $(x_t)_{t\in T}$ *is a family in* E *such that*

$$X = \sum_{t\in T}^{\mathfrak{T}_1} (x_t \widetilde{\otimes} 1_K)V_t$$

then $X_t = x_t$ *for every* $t \in T$. *In particular, if* T *is finite then the map*

$$E^T \longrightarrow \mathcal{S}(f), \quad x \longmapsto \sum_{t\in T}(x_t \otimes 1_K)V_t$$

is bijective and E*-linear* (Proposition 2.1.2 d)).

b) *We have*

$$X = \sum_{t\in T}^{\mathfrak{T}_3} (X_t \widetilde{\otimes} 1_K)V_t \in \mathcal{S}(f)\ .$$

c) $(X^*)_t = \tilde{f}(t)(X_{t^{-1}})^*$ *for every* $t \in T$ *and*

$$X^* = \sum_{t\in T}^{\mathfrak{T}_3} ((X_t)^* \widetilde{\otimes} 1_K)V_t^* \in \overline{\mathcal{R}(f)}^{\mathfrak{T}_3}\ .$$

d) $\mathcal{S}(f) = \overline{\mathcal{R}(f)}^{\mathfrak{T}_1} = \overline{\mathcal{R}(f)}^{\mathfrak{T}_2}$.

e) *For* $\xi \in H$ *and* $t \in T$,

$$(X\xi)_t = \widetilde{\sum_{s\in T}} f(s, s^{-1}t)X_s\xi_{s^{-1}t}\ .$$

f) *If* T *is finite and if we identify* $\mathcal{L}_E(H)$ *with* $E_{T,T}$ *then* X *is identified with the matrix*

$$[f(st^{-1},t)X_{st^{-1}}]_{s,t\in T}\ ,$$

and for every $r \in T$, V_r *is identified with the matrix*

$$[f(st^{-1},t)\delta_{s,rt}]_{s,t\in T}\ .$$

g) *If* $X, Y \in \mathcal{S}(f)$ *and* $t \in T$ *then* $XY \in \mathcal{S}(f)$ *and*

$$(XY)_t = \widetilde{\sum_{s\in T}} f(s, s^{-1}t)X_s Y_{s^{-1}t}\,,$$

$$(X^*Y)_t = \widetilde{\sum_{s\in T}} f(s,t)^* X_s^* Y_{st}\,, \qquad (XY^*)_t = \widetilde{\sum_{s\in T}} f(t,s)^* X_{ts} Y_s^*\,,$$

$$(X^*Y)_1 = \widetilde{\sum_{s\in T}} X_s^* Y_s\,, \qquad (XY^*)_1 = \widetilde{\sum_{s\in T}} X_s Y_s^*\,.$$

h) *The map*

$$E \longrightarrow \mathcal{S}(f), \quad x \longmapsto x\widetilde{\otimes}1_K$$

is an injective unital C^{**}*-homomorphism and so* $\mathcal{S}(f)$ *is an* E-C^{**}*-subalgebra of* $\mathcal{L}_E(H)$ *and* $Re\,\mathcal{S}(f)$ *is closed in* $\mathcal{S}(f)_{\mathfrak{T}_1}$. *In the* W^**-case,* $\mathcal{S}_W(f)$ *is the* W^**-subalgebra of* $\mathcal{L}_E(H)$ *generated by* $\mathcal{R}(f)$ *and* $\mathcal{R}(f)^{\#}$ *is dense in* $\mathcal{S}_W(f)^{\#}_{\mathfrak{T}_1} = \mathcal{S}_W(f)^{\#}_{\ddot{H}}$*, which is compact.*

i) *If* E *is a* W^**-algebra then* $\mathcal{S}_C(f)$ *may be identified canonically with a unital* C^**-subalgebra of* $\mathcal{S}_W(f)$ *by using the map of Proposition 1.3.9 b). By this identification* $\mathcal{S}_C(f)$ *generates* $\mathcal{S}_W(f)$ *as* W^**-algebra.*

j) *If* F *is a closed ideal of* E *(resp. of* $E_{\ddot{E}}$ *) then* $\mathcal{S}(f,F)$ *is a closed ideal of* $\mathcal{S}(f)$ $\Big($*resp. of* $\mathcal{S}(f)_{\underbrace{\ddot{}}_{\mathcal{S}(f)}}\Big)$.

k) *If* F *is a unital* C^{**}*-subalgebra of* E *such that* $f(s,t) \in F$ *for all* $s, t \in T$ *then* $\mathcal{S}(f,F)$ *is a unital* C^{**}*-subalgebra of* $\mathcal{S}(f)$ *and the map*

$$\mathcal{S}(f,F) \longrightarrow \mathcal{S}(g), \quad X \longmapsto \sum_{t\in T}^{\mathfrak{T}_3} (X_t\widetilde{\otimes}1_K)V_t^g$$

is an injective C^{**}*-homomorphism, where*

$$g : T \times T \longrightarrow Un\; F^c\,, \quad (s,t) \longmapsto f(s,t)\,.$$

This map induces a C^**-isomorphism* $\mathcal{S}_{\|\cdot\|}(f,F) \to \mathcal{S}_{\|\cdot\|}(g)$.

l) $(X,Y) \in \overbrace{\mathcal{S}(f)}^{\circ}{}_{+} \Longrightarrow (X_1, Y_1) \in \overset{\circ}{E}_{+}$.

Proof. a) By Proposition 2.1.6 c),e),

$$X_t = \varphi_{t,1}X = \widetilde{\sum_{s\in T}} \varphi_{t,1}((x_s\widetilde{\otimes}1_K)V_s) = \widetilde{\sum_{s\in T}} \delta_{t,s} f(s,1)x_s = x_t\,.$$

b), c), and d)

STEP 1. $X = \sum_{t\in T}^{\mathfrak{T}_2} (X_t \widetilde{\otimes} 1_K) V_t$.

By Proposition 2.1.3 d), Corollary 1.3.7 d), Proposition 2.1.8, and Proposition 2.1.4 b),d),

$$X = \left(\sum_{s\in T}^{\mathfrak{T}_2} u_s u_s^*\right) X \left(\sum_{t\in T}^{\mathfrak{T}_2} u_t u_t^*\right) = \sum_{s\in T}^{\mathfrak{T}_2}\sum_{t\in T}^{\mathfrak{T}_2} u_s u_s^* X u_t u_t^*$$
$$= \sum_{s\in T}^{\mathfrak{T}_2}\sum_{t\in T}^{\mathfrak{T}_2} u_s (\varphi_{s,t} X) u_t^* = \sum_{s\in T}^{\mathfrak{T}_2}\sum_{t\in T}^{\mathfrak{T}_2} u_s f(st^{-1}, t) X_{st^{-1}} u_t^*$$
$$= \sum_{s\in T}^{\mathfrak{T}_2}\sum_{r\in T}^{\mathfrak{T}_2} u_s X_r f(r, r^{-1}s) u_{r^{-1}s}^* = \sum_{s\in T}^{\mathfrak{T}_2}\sum_{r\in T}^{\mathfrak{T}_2} u_s X_r u_s^* V_r$$
$$= \sum_{s\in T}^{\mathfrak{T}_2}\sum_{r\in T}^{\mathfrak{T}_2} u_s u_s^* (X_r \widetilde{\otimes} 1_K) V_r = \sum_{s\in T}^{\mathfrak{T}_2} u_s u_s^* \left(\sum_{t\in T}^{\mathfrak{T}_2} (X_t \widetilde{\otimes} 1_K) V_t\right) = \sum_{t\in T}^{\mathfrak{T}_2} (X_t \widetilde{\otimes} 1_K) V_t .$$

STEP 2.

By Step 1, Corollary 1.3.7 a), and Proposition 2.1.2 d),e) (and Proposition 1.1.2 a)),

$$X^* = \left(\sum_{s\in T}^{\mathfrak{T}_1} (X_s \widetilde{\otimes} 1_K) V_s\right)^* = \sum_{s\in T}^{\mathfrak{T}_1} (X_s^* \widetilde{\otimes} 1_K) V_s^*$$
$$= \sum_{s\in T}^{\mathfrak{T}_1} (X_s^* \widetilde{\otimes} 1_K)(\tilde{f}(s) \widetilde{\otimes} 1_K) V_{s^{-1}} = \sum_{r\in T}^{\mathfrak{T}_1} ((\tilde{f}(r) X_{r^{-1}}^*) \widetilde{\otimes} 1_K) V_r \in \overline{\mathcal{R}(f)}^{\mathfrak{T}_1} .$$

By a),

$$(X^*)_t = \tilde{f}(t)(X_{t^{-1}})^* .$$

By Step 1 and Proposition 2.1.2 e) (and Proposition 1.1.2 a)),

$$X^* = \sum_{t\in T}^{\mathfrak{T}_2} ((X^*)_t \widetilde{\otimes} 1_K) V_t = \sum_{t\in T}^{\mathfrak{T}_2} ((X_{t^{-1}})^* \widetilde{\otimes} 1_K)(\tilde{f}(t) \widetilde{\otimes} 1_K) V_t$$
$$= \sum_{t\in T}^{\mathfrak{T}_2} ((X_{t^{-1}})^* \widetilde{\otimes} 1_K) V_{t^{-1}}^* = \sum_{t\in T}^{\mathfrak{T}_2} ((X_t)^* \widetilde{\otimes} 1_K) V_t^* .$$

Together with Step 1 this proves

$$X = \sum_{t\in T}^{\mathfrak{T}_3} (X_t \widetilde{\otimes} 1_K) V_t \in \mathcal{S}(f) , \qquad X^* = \sum_{t\in T}^{\mathfrak{T}_3} ((X_t)^* \widetilde{\otimes} 1_K) V_t^* \in \mathcal{S}(f) .$$

In particular $\mathcal{S}(f) = \overline{\mathcal{R}(f)}^{\mathfrak{T}_1} = \overline{\mathcal{R}(f)}^{\mathfrak{T}_2}$.

e) By b) and Corollary 1.3.7 b), in the C*-case,

$$(X\xi)_t = \left\langle \left(\sum_{s\in T}^{\mathfrak{T}_1} (X_s \widetilde{\otimes} 1_K) V_s \right) \xi \,\middle|\, 1_E \otimes e_t \right\rangle = \sum_{s\in T} \left\langle \, (X_s \widetilde{\otimes} 1_K) V_s \xi \,\middle|\, 1_E \otimes e_t \, \right\rangle$$
$$= \sum_{s\in T} X_s f(s, s^{-1}t) \xi_{s^{-1}t} = \sum_{s\in T} f(s, s^{-1}t) X_s \xi_{s^{-1}t}.$$

The proof is similar in the W*-case.

f) For $\xi \in H$ and $s \in T$, by e),

$$(X\xi)_s = \sum_{t\in T} f(t, t^{-1}s) X_t \xi_{t^{-1}s} = \sum_{r\in T} f(sr^{-1}, r) X_{sr^{-1}} \xi_r \, .$$

g) By b), Corollary 1.3.7 b),d), and Proposition 2.1.2 b),d),

$$XY = \left(\sum_{s\in T}^{\mathfrak{T}_2} (X_s \widetilde{\otimes} 1_K) V_s \right) \left(\sum_{t\in T}^{\mathfrak{T}_2} (X_t \widetilde{\otimes} 1_K) V_t \right)$$
$$= \sum_{s\in T}^{\mathfrak{T}_2} \sum_{t\in T}^{\mathfrak{T}_2} (X_s \widetilde{\otimes} 1_K) V_s (Y_t \widetilde{\otimes} 1_K) V_t = \sum_{s\in T}^{\mathfrak{T}_2} \sum_{t\in T}^{\mathfrak{T}_2} (X_s \widetilde{\otimes} 1_K)(Y_t \widetilde{\otimes} 1_K) V_s V_t$$
$$= \sum_{s\in T}^{\mathfrak{T}_2} \sum_{t\in T}^{\mathfrak{T}_2} (X_s \widetilde{\otimes} 1_K)(Y_t \widetilde{\otimes} 1_K)(f(s,t) \widetilde{\otimes} 1_K) V_{st}$$
$$= \sum_{s\in T}^{\mathfrak{T}_2} \sum_{r\in T}^{\mathfrak{T}_2} ((f(s, s^{-1}r) X_s Y_{s^{-1}r}) \widetilde{\otimes} 1_K) V_r .$$

Since by d),

$$\sum_{r\in T}^{\mathfrak{T}_2} ((f(s, s^{-1}r) X_s Y_{s^{-1}r}) \widetilde{\otimes} 1_K) V_r \in \mathcal{S}(f)$$

for every $s \in T$ we get $XY \in \mathcal{S}(f)$, again by d). By Corollary 1.3.7 b) and Proposition 2.1.6 c),e),

$$(XY)_t = \varphi_{t,1}(XY) = \widetilde{\sum_{s\in T}} \widetilde{\sum_{r\in T}} \varphi_{t,1}((f(s, s^{-1}r) X_s Y_{s^{-1}r}) \widetilde{\otimes} 1_K) V_r$$
$$= \widetilde{\sum_{s\in T}} \widetilde{\sum_{r\in T}} \delta_{t,r} f(r,1) f(s, s^{-1}r) X_s Y_{s^{-1}r} = \widetilde{\sum_{s\in T}} f(s, s^{-1}t) X_s Y_{s^{-1}t} .$$

By the above, c), and Proposition 1.1.2 b),

$$(X^*Y)_t = \widetilde{\sum_{s\in T}} f(s, s^{-1}t)(X^*)_s Y_{s^{-1}t} = \widetilde{\sum_{s\in T}} f(s, s^{-1}t) \tilde{f}(s) (X_{s^{-1}})^* Y_{s^{-1}t}$$

$$= \widetilde{\sum_{s\in T}} f(s^{-1},t)^*(X_{s^{-1}})^*Y_{s^{-1}t} = \widetilde{\sum_{s\in T}} f(s,t)^*X_s^*Y_{st},$$

$$(XY^*)_t = \widetilde{\sum_{s\in T}} f(s,s^{-1}t)X_s(Y^*)_{s^{-1}t} = \widetilde{\sum_{s\in T}} f(s,s^{-1}t)X_s\tilde{f}(s^{-1}t)(Y_{t^{-1}s})^*$$

$$= \widetilde{\sum_{s\in T}} f(t,t^{-1}s)^*X_s(Y_{t^{-1}s})^* = \widetilde{\sum_{s\in T}} f(t,s)^*X_{ts}Y_s^*.$$

It follows by Proposition 1.1.2 a),

$$(X^*Y)_1 = \widetilde{\sum_{s\in T}} X_s^*Y_s\,, \qquad (XY^*)_1 = \widetilde{\sum_{s\in T}} X_sY_s^*\ .$$

h) By c) and g), $\mathcal{S}(f)$ is an involutive unital subalgebra of $\mathcal{L}_E(H)$. Being closed (resp. closed in $\mathcal{L}_E(H)_{\ddot{H}}$ (d) and Corollary 1.3.7 c))) it is a C**-subalgebra of $\mathcal{L}_E(H)$ (resp. generated by $\mathcal{R}(f)$ [1, Theorem 5.6.3.5 b)] and [1, Corollary 4.4.4.12 a)] and by [1, Corollary 6.3.8.7] $\mathcal{R}(f)^{\#}$ is dense in $\mathcal{S}_W(f)^{\#}_{\mathfrak{T}_1}$, which is compact by Corollary 1.3.7 c)). The assertion concerning E follows from Proposition 2.1.2 d) and Lemma 1.3.2 c). By Corollary 1.3.7 a), $Re\,\mathcal{S}(f)$ is a closed set of $\mathcal{S}(f)_{\mathfrak{T}_1}$.

i) The assertion follows from h), Proposition 1.3.9 b), and Lemma 1.3.8 *c)* $\Rightarrow$ *a)*.

j) For $X \in \mathcal{S}(f,F)$, $Y \in \mathcal{S}(f)$, and $t \in T$, by g), $(XY)_t, (YX)_t \in \mathcal{S}(f,F)$ so $\mathcal{S}(f,F)$ is an ideal of $\mathcal{S}(f)$. The closure properties follow from Proposition 2.1.6 c).

k) By c) and g), $\mathcal{S}(f,F)$ is a unital involutive subalgebra of $\mathcal{S}(f)$ and by Proposition 2.1.6 c), $\mathcal{S}(f,F)$ is a C**-subalgebra of $\mathcal{S}(f)$. The last assertion follows from the fact that the image of the map contains $\mathcal{R}(g)$.

l) There are $U,V \in \mathcal{S}(f)$ with

$$(X,Y) = (U,V)^*(U,V) = (U^*,-V^*)(U,V) = (U^*U+V^*V, U^*V-V^*U)\ .$$

For $t \in T$,

$$0 \le (U_t,V_t)^*(U_t,V_t) = (U_t^*,-V_t^*)(U_t,V_t) = (U_t^*U_t+V_t^*V_t, U_t^*V_t-V_t^*U_t)\ .$$

By g),

$$X_1 = (U^*U+V^*V)_1 = \widetilde{\sum_{t\in T}}(U_t^*U_t+V_t^*V_t)\,,$$

$$Y_1 = (U^*V-V^*U)_1 = \sum_{t\in T}(U_t^*V_t-V_t^*U_t)$$

so

$$(X_1,Y_1) = \widetilde{\sum_{t\in T}}(U_t^*U_t+V_t^*V_t, U_t^*V_t-V_t^*U_t) \in \overset{\circ}{E}_+\ .$$

□

Remark. It may happen that by the identification of i), $\mathcal{S}_C(f) \neq \mathcal{S}_W(f)$ (Remark of Proposition 2.1.23).

COROLLARY 2.1.10.

a) *If* $(x_t)_{t\in T}$ *is a family in* E *such that* $(\|x_t\|)_{t\in T}$ *is summable then*

$$((x_t\widetilde{\otimes}1_K)V_t)_{t\in T}$$

is norm summable in $\mathcal{L}_E(H)$ *and*

$$\left\|\sum_{t\in T}(x_t\widetilde{\otimes}1_K)V_t\right\| \leq \sum_{t\in T}\|x_t\| \ .$$

b) *The set*

$$\mathcal{A} := \left\{ X \in \mathcal{S}(f) \mid \sum_{t\in T}\|X_t\| < \infty \right\}$$

is a dense involutive unital subalgebra of $\mathcal{S}_{\|\cdot\|}(f)$ *with*

$$\sum_{t\in T}\|(X^*)_t\| = \sum_{t\in T}\|X_t\| \ ,$$

$$\sum_{t\in T}\|(XY)_t\| \leq \left(\sum_{t\in T}\|X_t\|\right)\left(\sum_{t\in T}\|Y_t\|\right)$$

for all $X, Y \in \mathcal{A}$*.*

c) $\mathcal{A}$ *endowed with the norm*

$$\mathcal{A} \longrightarrow \mathbb{R}_+, \quad X \longmapsto \sum_{t\in T}\|X_t\|$$

is an involutive Banach algebra and $\mathcal{S}_{\|\cdot\|}(f)$ *is its C*-hull.*

Proof. a) For $S \in \mathfrak{P}_f(T)$, by Proposition 2.1.2 e),

$$\left\|\sum_{t\in S}(x_t\widetilde{\otimes}1_K)V_t\right\| \leq \sum_{t\in S}\|x_t\widetilde{\otimes}1_K\|\,\|V_t\| = \sum_{t\in S}\|x_t\|$$

and the assertion follows.

b) By Theorem 2.1.9 c), $X^* \in \mathcal{S}(f)$ and

$$\|(X^*)_t\| = \|(X_{t^{-1}})^*\| = \|X_{t^{-1}}\|$$

for all $t \in T$ so

$$\sum_{t\in T}\|(X^*)_t\| = \sum_{t\in T}\|X_{t^{-1}}\| = \sum_{t\in T}\|X_t\| \ .$$

By Theorem 2.1.9 g), $XY \in \mathcal{S}(f)$ and

$$\|(XY)_t\| = \left\| \widetilde{\sum_{s\in T}} f(s, s^{-1}t) X_s Y_{s^{-1}t} \right\| \leq \sum_{s\in T} \|X_s\| \, \|Y_{s^{-1}t}\|$$

for every $t \in T$ so

$$\sum_{t\in T} \|(XY)_t\| \leq \sum_{t\in T} \sum_{s\in T} \|X_s\| \, \|Y_{s^{-1}t}\| = \sum_{s\in T} \|X_s\| \left(\sum_{t\in T} \|Y_{s^{-1}t}\| \right)$$
$$= \sum_{s\in T} \|X_s\| \left(\sum_{t\in T} \|Y_t\| \right) = \left(\sum_{t\in T} \|X_t\| \right) \left(\sum_{t\in T} \|Y_t\| \right).$$

c) is easy to see. □

Remark. There may exist $X \in \mathcal{S}_{\|\cdot\|}(f)$ for which $((X_t \widetilde{\otimes} 1_K)V_t)_{t\in T}$ is not norm summable, as it is known from the theory of trigonometric series (see Proposition 3.5.1). In particular, the inclusion $\mathcal{A} \subset \mathcal{S}_{\|\cdot\|}(f)$ may be strict.

COROLLARY 2.1.11. *Let F be a unital C^{**}-algebra and $\tau : E \to F$ a positive continuous (resp. W^*-continuous) unital trace.*

a) *$\tau \circ \varphi_{1,1}$ is a positive continuous (resp. W^*-continuous) unital trace.*

b) *If τ is faithful then $\tau \circ \varphi_{1,1}$ is faithful and V_1 is finite.*

c) *In the W^*-case, $\mathcal{S}_W(f)$ is finite iff E is finite.*

Proof. a) Let $X, Y \in \mathcal{S}(f)$. By Theorem 2.1.9 g) (and Proposition 1.1.2 a)),

$$\tau\varphi_{1,1}(XY) = \tau \left(\widetilde{\sum_{t\in T}} f(t, t^{-1}) X_t Y_{t^{-1}} \right) = \tau \left(\widetilde{\sum_{t\in T}} f(t, t^{-1}) X_{t^{-1}} Y_t \right)$$
$$= \widetilde{\sum_{t\in T}} \tau(f(t, t^{-1}) X_{t^{-1}} Y_t) = \widetilde{\sum_{t\in T}} \tau(f(t, t^{-1}) Y_t X_{t^{-1}})$$
$$= \tau \left(\widetilde{\sum_{t\in T}} f(t, t^{-1}) Y_t X_{t^{-1}} \right) = \tau\varphi_{1,1}(YX).$$

Thus $\tau \circ \varphi_{1,1}$ is a trace which is obviously positive, continuous (resp. W*-continuous), and unital (Proposition 2.1.6 c),d)).

b) By Theorem 2.1.9 g), $\varphi_{1,1}$ is faithful, so $\tau \circ \varphi$ is also faithful. Let $X \in \mathcal{S}(f)$ with $X^*X = V_1$. By a),

$$\tau\varphi_{1,1}(XX^*) = \tau \circ \varphi_{1,1}(X^*X) = \tau\varphi_{1,1} V_1 = 1_F$$

so

$$\tau\varphi_{1,1}(V_1 - XX^*) = 1_F - 1_F = 0\,, \qquad V_1 = XX^*\,,$$

and V_1 is finite.

c) By b), if E is finite then $\mathcal{S}_W(f)$ is also finite. The reverse implication follows from the fact that $E\bar{\otimes}1_K$ is a unital W*-subalgebra of $\mathcal{S}_W(f)$ (Theorem 2.1.9 h)). □

COROLLARY 2.1.12. *Assume T finite and for every $x' \in (E')^T$ put*

$$\widetilde{x'} : \mathcal{S}(f) \longrightarrow \mathbb{K}, \quad X \longmapsto \sum_{t\in T} \langle\, X_t\,,\, x'_t\,\rangle\ .$$

a) *$\widetilde{x'} \in \mathcal{S}(f)'$ and*

$$\sup_{t\in T} \|x'_t\| \leq \left\|\widetilde{x'}\right\| \leq \sum_{t\in T} \|x'_t\|$$

for every $x' \in (E')^T$ and the map

$$\varphi : (E')^T \longrightarrow \mathcal{S}(f)'\,, \quad x' \longmapsto \widetilde{x'}$$

is an isomorphism of involutive vector spaces such that

$$\varphi(xx') = (x \otimes 1_K)(\varphi x')\,, \qquad \varphi(x'x) = (\varphi x')(x \otimes 1_K)$$

([1, Proposition 2.2.7.2]) *for every $x \in E$ and $x' \in (E')^T$.*

b) *If E is a W*-algebra then the map*

$$\psi : (\ddot{E})^T \longrightarrow \ddot{\overbrace{\mathcal{S}(f)}}\,, \quad (a_t)_{t\in T} \longmapsto (\tilde{a}_t)_{t\in T}$$

is an isomorphism of involutive vector spaces such that

$$\psi(xa) = (x \otimes 1_K)(\psi a)\,, \qquad \psi(ax) = (\psi a)(x \otimes 1_K)$$

for every $x \in E$ and $a \in (\ddot{E})^T$.

COROLLARY 2.1.13. *Assume T finite and let M be a Hilbert right $\mathcal{S}(f)$-module. M endowed with the right multiplication*

$$M \times E \longrightarrow M, \quad (\xi, x) \longmapsto \xi(x\tilde{\otimes}1_K)$$

and with the inner-product

$$M \times M \longrightarrow E, \quad (\xi, \eta) \longmapsto \langle\, \xi \mid \eta\,\rangle_1$$

is a Hilbert right E-module denoted by $\widetilde{M}$, $\mathcal{L}_{\mathcal{S}(f)}(M)$ is a unital C-subalgebra of $\mathcal{L}_E(\widetilde{M})$, and M is selfdual if $\widetilde{M}$ is so.*

Proof. By Proposition 2.1.6 d),g) and Theorem 2.1.9 g),l), for $X, Y \in \mathcal{S}(f)$ and $x \in E$,

$$\varphi_{1,1}(X(x\tilde{\otimes}1_K)) = (\varphi_{1,1}X)x\,, \qquad X \geq 0 \Longrightarrow \varphi_{1,1}X \geq 0\,,$$

$$(X,Y) \in \overbrace{\mathcal{S}(f)_+}^{\circ} \Longrightarrow (\varphi_{1,1}X, \varphi_{1,1}Y) \in \overset{\circ}{E}_+\,,$$
$$\inf\{\, \|\varphi_{1,1}X\| \mid X \in \mathcal{S}(f)_+\,, \|X\| = 1\,\} > 0$$

and the assertion follows from Proposition 2.1.6 a),c),d) and [1, Proposition 5.6.2.5 a),c),d)]. □

COROLLARY 2.1.14. *Let $n \in \mathbb{N}$ and let $\varphi : \mathcal{S}(f) \to E_{n,n}$ be an E-C^*-homomorphism. Then $(\varphi V_t)_{i,j} \in E^c$ for all $t \in T$ and all $i, j \in \mathbb{N}_n$.*

Proof. For $x \in E$, by Proposition 2.1.2 d) and Theorem 2.1.9 h),

$$x(\varphi V_t) = \varphi(x\tilde{\otimes}1_K)(\varphi V_t) = \varphi((x\tilde{\otimes}1_K)V_t) =$$
$$= \varphi(V_t(x\tilde{\otimes}1_K)) = (\varphi V_t)\varphi(x\tilde{\otimes}1_K) = (\varphi V_t)x$$

so $(\varphi V_t)_{i,j} \in E^c$. □

COROLLARY 2.1.15. *Let S be a group and $g \in \mathcal{F}(S, \mathcal{S}(f))$. If we put*

$$h : (T\times S)\times(T\times S) \longrightarrow Un\,\mathcal{S}(f)^c\,, \quad ((t_1,s_1),(t_2,s_2)) \longmapsto$$
$$(f(t_1,t_2)\tilde{\otimes}1_K)g(s_1,s_2)$$

then $h \in \mathcal{F}(T\times S, \mathcal{S}(f))$.

Proof. The assertion follows from Theorem 2.1.9 h). □

COROLLARY 2.1.16. *Let $X \in \mathcal{S}(f)$ (resp. $X \in \mathcal{S}_{\|\cdot\|}(f)$).*

a) *For every $S \subset T$,*

$$\sum_{s\in S}^{\mathfrak{T}_3}(X_s\tilde{\otimes}1_K)V_s \in \mathcal{S}(f) \qquad (\text{resp. } \sum_{s\in S}^{\|\cdot\|}(X_s\tilde{\otimes}1_K)V_s \in \mathcal{S}_{\|\cdot\|}(f))$$

and

$$\gamma := \sup\left\{ \left\|\sum_{t\in S}(X_t\tilde{\otimes}1_K)V_t\right\| \;\middle|\; S \in \mathfrak{P}_f(T) \right\} < \infty\,.$$

b) *We put for every $\alpha \in l^\infty(T)$*

$$\alpha X : T \longrightarrow E\,, \quad t \longmapsto \alpha_t X_t\,.$$

Then $\alpha X \in \mathcal{S}(f)$ (resp. $\alpha X \in \mathcal{S}_{\|\cdot\|}(f)$) for every $\alpha \in l^\infty(T)$ and the map

$$l^\infty(T) \longrightarrow \mathcal{S}(f)\ (\text{resp. } \mathcal{S}_{\|\cdot\|}(f)), \quad \alpha \longmapsto \alpha X$$

is norm-continuous.

c) *Assume E is a W^*-algebra and let $l^\infty(T,E)$ be the C^*-direct product of the family $(E)_{t\in T}$, which is a W^*-algebra* ([1, Proposition 4.4.4.21 a)]). *We put for every $\alpha \in l^\infty(T,E)$,*

$$\alpha X : T \longrightarrow E\,, \quad t \longmapsto \alpha_t X_t\ .$$

Then $\alpha X \in \mathcal{S}_W(f)$ for every $\alpha \in l^\infty(T,E)$ and the map

$$l^\infty(T,E) \longrightarrow \mathcal{S}_W(f), \quad \alpha \longmapsto \alpha X$$

is continuous and W^*-continuous.

Proof. a) In the C*-case the family $((X_s \otimes 1_K)V_s)_{s\in S}$ is summable since $\mathcal{S}_C(f)_{\mathfrak{T}_3}$ is complete. By Banach-Steinhaus Theorem, γ is finite.

In the W*-case the summability follows now from Corollary 1.3.7 b),c) and Theorem 2.1.9 b).

b) Let G be the vector subspace $\{\, \alpha \in l^\infty(T) \mid \alpha(T) \text{ is finite} \,\}$ of $l^\infty(T)$. By a), the map

$$G \longrightarrow \mathcal{S}(f) \text{ (resp. } \mathcal{S}_{\|\cdot\|}(f)), \quad \alpha \longmapsto \alpha X$$

is well-defined, linear, and continuous. The assertion follows by continuity.

c) Let $x \in E$, $S \subset T$, and $\alpha := x e_S$. For $\xi, \eta \in H$ and $a \in \ddot{E}$, by a) and Lemma 1.3.2 b) (and Theorem 2.1.9 b)),

$$\left\langle \alpha X\,, \widetilde{\overbrace{(a,\xi,\eta)}} \right\rangle = \langle\, \langle\, \alpha X \xi \mid \eta \,\rangle\ , a\,\rangle = \left\langle \sum_{t\in T}^{\ddot{E}} \eta_t^* x((e_S X)\xi)_t\,, a \right\rangle$$

$$= \sum_{t\in T} \langle\, x\,,\, ((e_S X)\xi)_t a \eta_t^* \,\rangle = \left\langle x\,, \sum_{t\in T}^{E} ((e_S X)\xi)_t a \eta_t^* \right\rangle .$$

Let G be the involutive subalgebra $\{\, \alpha \in l^\infty(T,E) \mid \alpha(T) \text{ is finite} \,\}$ of $l^\infty(T,E)$ and let $\bar{G}$ be its norm-closure in $l^\infty(T,E)$, which is a C*-subalgebra of $l^\infty(T,E)$. By [1, Proposition 4.4.4.21 a)], G is dense in $l^\infty(T,E)_{\ddot{F}}$, where $F := l^\infty(T,E)$.

Let $\alpha \in l^\infty(T,E)^\#$ and let $\mathfrak{F}$ be a filter on $G^\#$ converging to α in $l^\infty(T,E)_{\ddot{F}}$ ([1, Corollary 6.3.8.7]). By the above (and by Theorem 2.1.9 h)),

$$\lim_{\beta,\mathfrak{F}} \beta X = \alpha X$$

in $\mathcal{S}_W(f)_{\underbrace{\ddot{\ }}_{\mathcal{S}_W(f)}}$ and so $\alpha X \in \mathcal{S}_W(f)$. The assertion follows. □

Corollary 2.1.17. *Let S be a subgroup of T. Put*

$$f_S := f|(S\times S)\,, \quad K_S := l^2(S)\,, \quad \mathcal{G} := \{\, X \in \mathcal{S}(f) \mid t \in T\setminus S \Longrightarrow X_t = 0 \,\}\ .$$

a) $f_S \in \mathcal{F}(S,E)$.

b) *$\mathcal{G}$ is an E-C**-subalgebra of $\mathcal{S}(f)$.*

c) *For every $X \in \mathcal{G}$, the family $((X_s \widetilde{\otimes} 1_{K_S}) V_s^{f_S})_{s \in S}$ is summable in $\mathcal{L}_E(K_S)_{\mathfrak{T}_3}$ and the map*

$$\varphi : \mathcal{G} \longrightarrow \mathcal{S}(f_S)\,, \quad X \longmapsto \sum_{s \in S}^{\mathfrak{T}_3} (X_s \widetilde{\otimes} 1_{K_S}) V_s^{f_S}$$

*is an injective E-C**-homomorphism.*

d) *If $X \in \mathcal{G} \cap \mathcal{S}_{\|\cdot\|}(f)$ then $\varphi X \in \mathcal{S}_{\|\cdot\|}(f_S)$ and the map*

$$\mathcal{G} \cap \mathcal{S}_{\|\cdot\|}(f) \longrightarrow \mathcal{S}_{\|\cdot\|}(f_S), \quad X \longmapsto \varphi X$$

is an E-C-isomorphism.*

e) *If S is finite then the map*

$$\mathcal{G} \longrightarrow \mathcal{S}(f_S), \quad X \longmapsto \sum_{t \in S} (X_t \otimes 1_{K_S}) V_t^{f_S}$$

is an E-C-isomorphism.*

Proof. a) is obvious.

b) By Theorem 2.1.9 c),g), $\mathcal{G}$ is an involutive unital subalgebra of $\mathcal{S}(f)$ and by Proposition 2.1.6 a) (resp. Proposition 2.1.6 c) and Corollary 1.3.7 c)) and Theorem 2.1.9 h), it is an E-C**-subalgebra of $\mathcal{S}(f)$.

c) follows from Theorem 2.1.9 b) and Corollary 2.1.16 a).

d) follows from c).

e) is contained in d). □

Definition 2.1.18. We denote by $\mathfrak{S}_T$ the set of finite subgroups of T and call T **locally finite** if $\mathfrak{S}_T$ is upward directed and

$$\bigcup_{S \in \mathfrak{S}_T} S = T\,.$$

T is locally finite iff the subgroups of T generated by finite subsets of T are finite.

Corollary 2.1.19. *Assume T locally finite. We put $f_S := f|(S \times S)$ for every $S \in \mathfrak{S}_T$ and identify $\mathcal{S}(f_S)$ with $\{\, X \in \mathcal{S}(f) \mid t \in T \setminus S \Rightarrow X_t = 0 \,\}$* (Corollary 2.1.17 e)).

a) *For every $X \in \mathcal{S}_{\|\cdot\|}(f)$ and $\varepsilon > 0$ there is an $S \in \mathfrak{S}_T$ such that*

$$\left\| \sum_{t \in R} (X_t \otimes 1_K) V_t - X \right\| < \varepsilon$$

for every $R \in \mathfrak{S}_T$ with $S \subset R$.

b) $\mathcal{S}_{\|\cdot\|}(f)$ *is the norm closure of* $\cup_{s\in\mathfrak{S}_T}\mathcal{S}(f_S)$ *and so it is canonically isomorphic to the inductive limit of the inductive system* $\{\,\mathcal{S}(f_S)\mid S\in\mathfrak{S}_T\,\}$ *and for every* $S\in\mathfrak{S}_T$ *the inclusion map* $\mathcal{S}(f_S)\to\mathcal{S}_{\|\cdot\|}(f)$ *is the associated canonical morphism.*

Proof. a) There is a $Y\in\mathcal{R}(f)$ with $\|X-Y\|<\frac{\varepsilon}{2}$. Let $S\in\mathfrak{S}_T$ with $Y\in\mathcal{S}(f_S)$. By Corollary 2.1.17 b), for $R\in\mathfrak{S}_T$ with $S\subset R$,

$$\left\|\sum_{t\in R}((X_t-Y_t)\widetilde{\otimes}1_K)V_t\right\|\le\|X-Y\|<\frac{\varepsilon}{2}$$

so

$$\left\|\sum_{t\in R}(X_t\widetilde{\otimes}1_K)V_t-X\right\|\le\left\|\sum_{t\in R}((X_t-Y_t)\widetilde{\otimes}1_K)V_t\right\|+\|Y-X\|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\,.$$

b) follows from a). □

Remark. The C*-algebras of the form $\mathcal{S}_{\|\cdot\|}(f)$ with T locally finite can be seen as a kind of AF-E-C*-algebras.

Proposition 2.1.20. *The following are equivalent for all* $t\in T$ *with* $t^2=1$ *and* $\alpha\in Un\,E$.

a) $\frac{1}{2}(V_1+(\alpha\widetilde{\otimes}1_K)V_t)\in Pr\,\mathcal{S}(f)$.

b) $\alpha^2=\tilde{f}(t)$.

Proof. By Proposition 2.1.2 b),d),e),

$$(V_t)^*=(\tilde{f}(t)\widetilde{\otimes}1_K)V_t\,,\qquad (V_t)^2=(\tilde{f}(t)^*\widetilde{\otimes}1_K)V_1$$

so

$$\frac{1}{2}(V_1+(\alpha\widetilde{\otimes}1_K)V_t)^*=\frac{1}{2}(V_1+((\alpha^*\tilde{f}(t))\widetilde{\otimes}1_K)V_t)\,,$$

$$\left(\frac{1}{2}(V_1+(\alpha\widetilde{\otimes}1_K)V_t)\right)^2=\frac{1}{4}((1_E+\alpha^2\tilde{f}(t)^*)\widetilde{\otimes}1_K)V_1+\frac{1}{2}(\alpha\widetilde{\otimes}1_K)V_t\,.$$

Thus a) is equivalent to $\alpha^*\tilde{f}(t)=\alpha$ and $\alpha^2\tilde{f}(t)^*=1_E$, which is equivalent to b). □

Corollary 2.1.21. *Let* $t\in T$ *such that* $t^2=1$ *and* $\tilde{f}(t)=1_E$. *Then*

$$\frac{1}{2}(V_1\pm V_t)\in Pr\,\mathcal{S}(f)\,,\qquad (V_1+V_t)(V_1-V_t)=0\,.$$

Proof. The assertion follows from Proposition 2.1.20. □

COROLLARY 2.1.22. *Let $\alpha, \beta \in Un\, E$, $s, t \in T$ with $s^2 = t^2 = 1$, $st = ts$,*

$$\gamma := \frac{1}{2}(\alpha^*\beta f(s,st)^* + \beta^*\alpha f(t,st)^*)\,, \quad \gamma' := \frac{1}{2}(\alpha\beta^* f(st,t)^* + \beta\alpha^* f(st,s)^*)\,,$$

and

$$X := \frac{1}{2}((\alpha\widetilde{\otimes}1_K)V_s + (\beta\widetilde{\otimes}1_K)V_t)\;.$$

a) $f(s,st)f(t,st) = f(st,t)f(st,s) = \tilde{f}(st)^*$.

b) $f(st,t)f(s,st) = f(st,s)f(t,st)$.

c) $X^*X = \frac{1}{2}(V_1 + (\gamma\widetilde{\otimes}1_K)V_{st})\,, \quad XX^* = \frac{1}{2}(V_1 + (\gamma'\widetilde{\otimes}1_K)V_{st})$.

d) *The following are equivalent.*

- d$_1$) $X^*X \in Pr\,\mathcal{S}(f)$.
- d$_2$) $XX^* \in Pr\,\mathcal{S}(f)$.
- d$_3$) $\alpha^*\beta f(t,st) = \beta^*\alpha f(s,st)$.
- d$_4$) $\alpha^*\beta f(st,t) = \beta^*\alpha f(st,s)$.

Proof. a) and b) follow from the equation of Schur functions (Definition1.1.1) and Proposition 1.1.2 a).

c) By Proposition 2.1.2 b),e) and Proposition 1.1.2 b),

$$X^* = \frac{1}{2}(((\alpha^*\tilde{f}(s))\widetilde{\otimes}1_K)V_s + ((\beta^*\tilde{f}(t))\widetilde{\otimes}1_K)V_t),$$

$$\begin{aligned} X^*X &= \frac{1}{2}V_1 + \frac{1}{4}((\alpha^*\beta\tilde{f}(s)f(s,t) + \beta^*\alpha\tilde{f}(t)f(t,s))\widetilde{\otimes}1_K)V_{st} \\ &= \frac{1}{2}V_1 + \frac{1}{4}((\alpha^*\beta f(s,st)^* + \beta^*\alpha f(t,st)^*)\widetilde{\otimes}1_K)V_{st} = \frac{1}{2}(V_1 + (\gamma\widetilde{\otimes}1_K)V_{st}), \end{aligned}$$

$$\begin{aligned} XX^* &= \frac{1}{2}V_1 + \frac{1}{4}((\alpha\beta^*\tilde{f}(t)f(s,t) + \beta\alpha^*\tilde{f}(s)f(t,s))\widetilde{\otimes}1_K)V_{st} \\ &= \frac{1}{2}V_1 + \frac{1}{4}((\alpha\beta^* f(st,t)^* + \beta\alpha^* f(st,s)^*)\widetilde{\otimes}1_K)V_{st} = \frac{1}{2}(V_1 + (\gamma'\widetilde{\otimes}1_K)V_{st}). \end{aligned}$$

$d_1 \Leftrightarrow d_2$ is known.

$d_1 \Leftrightarrow d_3$. By a), we have

$$\begin{aligned} \gamma^2 - \tilde{f}(st) &= \frac{1}{4}(\alpha^*\beta\alpha^*\beta f(s,st)^{*2} + \beta^*\alpha\beta^*\alpha f(t,st)^{*2} + 2f(s,st)^*f(t,st)^*) \\ &\quad - f(s,st)^*f(t,st)^* = \frac{1}{4}(\alpha^*\beta f(s,st)^* - \beta^*\alpha f(t,st)^*)^2. \end{aligned}$$

By Proposition 2.1.20 d_1) is equivalent to $\gamma^2 = \tilde{f}(st)$ so, by the above, since $\alpha^*\beta f(s,st)^* - \beta^*\alpha f(t,st)^*$ is normal, it is equivalent to

$$\alpha^*\beta f(s,st)^* = \beta^*\alpha f(t,st)^* \quad \text{or to} \quad \beta^*\alpha f(s,st) = \alpha^*\beta f(t,st)\,.$$

$d_3 \Leftrightarrow d_4$ follows from b). □

PROPOSITION 2.1.23. *Let $X \in \mathcal{S}(f)$.*

a) $\widetilde{\sum\limits_{t\in T}} X_t^* X_t = (X^*X)_1\,, \quad \widetilde{\sum\limits_{t\in T}} (X_t X_t^*) = (XX^*)_1.$

b) $(X_t)_{t\in T}\,, (X_t^*)_{t\in T} \in \widetilde{\bigoplus\limits_{t\in T}} \breve{E},$

$$\|(X_t)_{t\in T}\| \le \|X\|\ , \quad \|(X_t^*)_{t\in T}\| \le \|X\|\ .$$

c) *If T is finite and f is constant then there is an $X \in \mathcal{S}(f)$ with*

$$\|X\| \ge \sqrt{\text{Card T}}\,\|(X_t)_{t\in T}\|\ , \quad \|X\| \ge \sqrt{\text{Card T}}\,\|(X_t^*)_{t\in T}\|\ .$$

d) *If T is infinite and locally finite and f is constant then the map*

$$\mathcal{S}(f) \longrightarrow \widetilde{\bigoplus_{t\in T}} \breve{E}, \quad X \longmapsto (X_t)_{t\in T}$$

is not surjective.

Proof. a) follows from Theorem 2.1.9 g).
b) By a),

$$(X_t)_{t\in T}\,, (X_t^*)_{t\in T} \in \widetilde{\bigoplus_{t\in T}} \breve{E}$$

and by Proposition 2.1.6 a),

$$\|(X_t)_{t\in T}\|^2 = \|\varphi_{1,1}(X^*X)\| \le \|X^*X\| = \|X\|^2\,,$$

$$\|(X_t^*)_{t\in T}\|^2 = \|\varphi_{1,1}(XX^*)\| \le \|XX^*\| = \|X\|^2\,.$$

c) Let $n :=$ Card T and for every $t \in T$ put $X_t := 1_E$, $\xi_t := 1_E$. Then

$$\|(X_t)_{t\in T}\|^2 = \|(X_t^*)_{t\in T}\|^2 = n\,, \quad \|(\xi_t)_{t\in T}\|^2 = n\,.$$

For $t \in T$, by Theorem 2.1.9 e),

$$(X\xi)_t = \sum_{s\in T} f(s,s^{-1}t) X_s \xi_{s^{-1}t} = n1_E$$

so

$$\langle\, X\xi \mid X\xi\,\rangle = n^3 1_E\,, \quad n\,\|X\|^2 = \|X\|^2\,\|\xi\|^2 \ge \|X\xi\|^2 = n^3,$$

$$\|X\|^2 \ge n\,\|(X_t)_{t\in T}\|^2\ , \quad \|X\| \ge \sqrt{n}\,\|(X_t)_{t\in T}\|\,.$$

d) follows from c), Theorem 2.1.9 a), and the Principle of Inverse Operator. □

Remark. If E is a W*-algebra then it may exist a family $(x_t)_{t\in T}$ in E such that the family $((x_t\widetilde{\otimes}1_K)V_t)_{t\in T}$ is summable in $\mathcal{L}_E(H)_{\mathfrak{T}_2}$ in the W*-case but not in the C*-case as the following example shows. Take $T := \mathbb{Z}$, f constant, $E := l^\infty(\mathbb{Z})$, and $x_t := (\delta_{t,s})_{s\in T} \in E$ for every $t \in T$. By Proposition 2.1.23 b), $((x_t \otimes 1_K)V_t)_{t\in T}$ is not summable in $\mathcal{L}_E(H)_{\mathfrak{T}_2}$ in the C*-case. In the W*-case for $\xi \in H$ and $s,t \in T$,

$$\langle\, ((x_t\bar{\otimes}1_K)V_t\xi)_s \mid ((x_t\bar{\otimes}1_K)V_t\xi)_s \,\rangle = e_t|\xi_{s-t}|^2 ,$$

$$\langle\, (x_t\bar{\otimes}1_K)V_t\xi \mid (x_t\bar{\otimes}1_K)V_t\xi \,\rangle = e_t \left\|\xi\right\|^2 .$$

Thus

$$X := \sum_{t\in T}^{\mathfrak{T}_2}(x_t\bar{\otimes}1_K)V_t \in \mathcal{S}_W(f) .$$

Using the identification of Theorem 2.1.9 i), we get $X \in \mathcal{S}_W(f)\setminus\mathcal{S}_C(f)$.

Corollary 2.1.24. *Let $X \in \mathcal{S}(f)$.*

a) $X \in \left\{\, x\widetilde{\otimes}1_K \;\middle|\; x \in E \,\right\}^c$ *iff $X_t \in E^c$ for all $t \in T$.*

b) $X \in \{\, V_t \mid t \in T\,\}^c$ *iff*

$$X_{s^{-1}ts} = f(s,s^{-1}ts)^*f(t,s)X_t = f(s^{-1},ts)f(t,s)\tilde{f}(s)X_t$$

for all $s,t \in T$.

c) $X \in \mathcal{S}(f)^c$ *iff for all $s,t \in T$*

$$X_t \in E^c , \quad X_{s^{-1}ts} = f(s,s^{-1}ts)^*f(t,s)X_t = f(s^{-1},ts)f(t,s)\tilde{f}(s)X_t .$$

In particular if $f(s,t) = f(t,s)$ for all $s,t \in T$ then $X \in \mathcal{S}(f)^c$ iff $X_t \in E^c$ for all $t \in T$.

d) $\varphi_{1,1}(\mathcal{S}(f)^c) = E^c$.

e) *If the conjugacy class of $t \in T$ (i.e. the set $\left\{\, s^{-1}ts \;\middle|\; s \in T\,\right\}$) is infinite and $X \in \{\, V_t \mid t \in T\,\}^c$ then $X_t = 0$.*

f) *If the conjugacy class of every $t \in T\setminus\{1\}$ is infinite then*

$$\{\, V_t \mid t \in T\,\}^c = \left\{\, x\widetilde{\otimes}1_K \;\middle|\; x \in E \,\right\} , \quad \mathcal{S}(f)^c = \left\{\, x\widetilde{\otimes}1_K \;\middle|\; x \in E^c \,\right\} .$$

Thus in this case $\mathcal{S}(f)$ is a kind of E-factor.

g) *The following are equivalent:*

g_1) *$\mathcal{S}(f)$ is commutative.*

g_2) *T and E are commutative and $f(s,t) = f(t,s)$ for all $s,t \in T$.*

Proof. For $s,t\in T$, $x\in E$, and $Y:=(x\widetilde{\otimes}1_K)V_s$, by Theorem 2.1.9 g),

$$(XY)_t=\widetilde{\sum_{r\in T}}f(r,r^{-1}t)X_rY_{r^{-1}t}=\widetilde{\sum_{r\in T}}f(r,r^{-1}t)X_r\delta_{s,r^{-1}t}x=f(ts^{-1},s)X_{ts^{-1}}x\,,$$

$$(YX)_t=\widetilde{\sum_{r\in T}}f(r,r^{-1}t)Y_rX_{r^{-1}t}=\widetilde{\sum_{r\in T}}f(r,r^{-1}t)\delta_{r,s}xX_{r^{-1}t}=f(s,s^{-1}t)xX_{s^{-1}t}\,.$$

a) follows from the above by putting $s:=1$ (Proposition 1.1.2 a)).

b) follows from the above by putting $x:=1_E$ and $t:=rs$ (Proposition 1.1.2).

c) follows from a),b), and Corollary 1.3.7 d). The last assertion follows using Proposition 1.1.5 a).

d) follows from c) (and Proposition 1.1.2 a)).

e) follows from b) and Proposition 2.1.23 b).

f) follows from c), e), and Proposition 2.1.2 d).

$g_1\Rightarrow g_2$. By a), E is commutative. By Proposition 2.1.2 b),

$$f(s,t)V_{st}=V_sV_t=V_tV_s=f(t,s)V_tV_s=f(t,s)V_{ts}$$

and so by Theorem 2.1.9 a), $st=ts$ and $f(s,t)=f(t,s)$.

$g_2\Rightarrow g_1$ follows from c). □

COROLLARY 2.1.25. *If $\mathbb{K}=\mathbb{R}$ then the following are equivalent:*

a) $\mathcal{S}(f)^c=\mathcal{S}(f)=Re\,\mathcal{S}(f)$.

b) *T is commutative, $E^c=E=Re\,E$, and*

$$f(s,t)=f(t,s)\,,\quad \tilde{f}(t)=1_E\,,\quad t^2=1$$

for all $s,t\in T$.

Proof. $a\Rightarrow b$. By Corollary 2.1.24 $g_1\Rightarrow g_2$, T is commutative, $E=E^c$, and $f(s,t)=f(t,s)$ for all $s,t\in T$. Since E is isomorphic with a C*-subalgebra of $\mathcal{S}(f)$ (Theorem 2.1.9 h)), $E=Re\,E$. By Proposition 2.1.2 e),

$$V_t=V_t^*=(\tilde{f}(t)\widetilde{\otimes}1_K)V_{t^{-1}}$$

so by Theorem 2.1.9 a), $t=t^{-1}$, $\tilde{f}(t)=1_E$, so $t^2=1$.

$b\Rightarrow a$. By Corollary 2.1.24 $g_2\Rightarrow g_1$, $\mathcal{S}(f)^c=\mathcal{S}(f)$. For $X\in\mathcal{S}(f)$ and $t\in T$, by Theorem 2.1.9 c),

$$(X^*)_t=\tilde{f}(t)(X_{t^{-1}})^*=(X_t)^*=X_t$$

so $X^*=X$ (Theorem 2.1.9 a)). □

PROPOSITION 2.1.26. *Let* $(E_i)_{i\in I}$ *be a family of unital* C^{**}*-algebras such that* E *is the* C^**-direct product of this family. For every* $i \in I$*, we identify* E_i *with the corresponding closed ideal of* E *(resp. of* $E_{\ddot{E}}$*) and put*

$$f_i : T \times T \longrightarrow Un\, E_i^c\,, \quad (s,t) \longmapsto f(s,t)_i\ .$$

a) *For every* $i \in I$*,* $f_i \in \mathcal{F}(T, E_i)$*. We put* (by Theorem 2.1.9 b))

$$\varphi_i : \mathcal{S}(f) \longrightarrow \mathcal{S}(f_i)\,, \quad X \longmapsto \sum_{t\in T}^{\mathfrak{T}_2} ((X_t)_i \widetilde{\otimes} 1_K) V_t^{f_i}\ .$$

φ_i *is a surjective* C^{**}*-homomorphism.*

b) *In the* C^**-case, if* T *is finite then* $\mathcal{R}(f) = \mathcal{S}_{\|\cdot\|}(f) = \mathcal{S}_C(f)$ *is isomorphic to the* C^**-direct product of the family*

$$(\mathcal{R}(f_i) = \mathcal{S}_{\|\cdot\|}(f_i) = \mathcal{S}_C(f_i))_{i\in I}\ .$$

c) *In the* C^**-case, if* I *is finite then* $\mathcal{S}_C(f)$ *(resp.* $\mathcal{S}_{\|\cdot\|}(f)$*) is isomorphic to* $\prod_{i\in I} \mathcal{S}_C(f_i)$ *(resp.* $\prod_{i\in I} \mathcal{S}_{\|\cdot\|}(f_i)$*).*

d) *In the* W^**-case,* $\mathcal{S}_W(f)$ *is isomorphic to the* C^**-direct product of the family* $(\mathcal{S}_W(f_i))_{i\in I}$*.*

Remark. The C*-isomorphisms of b) and c) cease to be surjective in general if T and I are both infinite. Take $T := (\mathbb{Z}_2)^{\mathbb{N}}$, $I := \mathbb{N}$, $E_i := \mathbb{K}$ for every $i \in I$, and $E := l^\infty$ (i.e. E is the C*-direct product of the family $(E_i)_{i\in I}$). For every $n \in \mathbb{N}$ put $t_n := (\delta_{m,n})_{m\in\mathbb{N}} \in T$. Assume there is an $X \in \mathcal{S}_C(f)$ (resp. $X \in \mathcal{S}_{\|\cdot\|}(f)$) with $\psi X = (V_{t_i}^{f_i})_{i\in I}$ (resp. $\varphi X = (V_{t_i}^{f_i})_{i\in I}$), where ψ and φ are the maps of b) and c), respectively. Then $(X_{t_n})_i = \delta_{i,n}$ for all $i, n \in \mathbb{N}$ and this implies $(X_t)_{t\in T} \notin \bigoplus_{t\in T} \breve{E}$, which contradicts Proposition 2.1.23 b).

PROPOSITION 2.1.27. *Let* S *be a finite group,* $K' := l^2(S)$*,* $K'' := l^2(S \times T)$*, and* $g \in \mathcal{F}(S, \mathcal{S}(f))$ *such that* $g(s_1, s_2) \in Un\, E^c$ *(where* $Un\, E^c$ *is identified with* $(Un\, E^c)\widetilde{\otimes} 1_K \subset Un\, \mathcal{S}(f)^c$*) for all* $s_1, s_2 \in S$ *and put*

$$h : (S \times T) \times (S \times T) \longrightarrow Un\, E^c\,, \quad ((s_1,t_1),(s_2,t_2)) \longmapsto g(s_1,s_2) f(t_1,t_2)\ .$$

a) $h \in \mathcal{F}(S \times T, E)$*; for every* $X \in \mathcal{S}(g)$ *put*

$$\varphi X := \sum_{s\in S} \sum_{t\in T}^{\mathfrak{T}_3} ((X_s)_t \widetilde{\otimes} 1_{K''}) V_{(s,t)}^h \in \mathcal{S}(h)\ .$$

b) $\varphi : \mathcal{S}(g) \longrightarrow \mathcal{S}(h)$ *is an* E*-*C^**-isomorphism.*

Proof. a) is obvious.

b) For $X, Y \in \mathcal{S}(g)$ and $(s,t) \in S \times T$, by Theorem 2.1.9 c),g) and Proposition 2.1.6 g),

$$\begin{aligned}(\varphi X^*)_{(s,t)} &= ((X^*)_s)_t = \tilde{g}(s)((X_{s^{-1}})^*)_t \\ &= \tilde{g}(s)\tilde{f}(t)((X_{s^{-1}})_{t^{-1}})^* = \tilde{h}(s,t)(X_{(s,t)^{-1}})^* = ((\varphi X)^*)_{(s,t)},\end{aligned}$$

$$\begin{aligned}(\varphi(XY))_{(s,t)} &= ((XY)_s)_t = \sum_{r\in S} g(r, r^{-1}s)(X_r Y_{r^{-1}s})_t \\ &= \sum_{r\in S} g(r, r^{-1}s) \widetilde{\sum_{q\in T}} f(q, q^{-1}t)(X_r)_q (Y_{r^{-1}s})_{q^{-1}t} \\ &= \widetilde{\sum_{(r,q)\in S\times T}} h((r,q),(r,q)^{-1}(s,t)) X_{(r,q)} Y_{(r,q)^{-1}(s,t)} = ((\varphi X)(\varphi Y))_{(s,t)},\end{aligned}$$

so φ is a C*-homomorphism. If $\varphi X = 0$ then $X_{(s,t)} = 0$ for all $(s,t) \in S \times T$, so $X = 0$ and φ is injective. Let $Z \in \mathcal{S}(h)$. For every $s \in S$ put

$$X_s := \sum_{t\in T}^{\mathfrak{T}_3} (Z_{(s,t)} \tilde{\otimes} 1_K) V_t^f \in \mathcal{S}(f)\,,$$

$$X := \sum_{s\in S} (X_s \otimes 1_{K'}) V_s^g \in \mathcal{S}(g)\,.$$

Then $\varphi X = Z$ and φ is surjective. $\square$

PROPOSITION 2.1.28. *If T is infinite and $X \in \mathcal{S}(f) \setminus \{0\}$ then $X(H^{\#})$ is not precompact.*

Proof. Let $t \in T$ with $X_t \neq 0$. There is an $x' \in E'_+$ (resp. $x' \in \ddot{E}_+$) with $\langle X_t^* X_t\,,\, x' \rangle > 0$. We put $t_1 := 1$ and construct a sequence $(t_n)_{n\in\mathbb{N}}$ recursively in T such that for all $m, n \in \mathbb{N}$, $m < n$,

$$\left|\left\langle f(t,t_m)^* f(t t_m t_n^{-1}, t_n) X_t^* X_{t t_m t_n^{-1}}\,,\, x' \right\rangle\right| < \frac{1}{2} \left\langle X_t^* X_t\,,\, x' \right\rangle\,.$$

Let $n \in \mathbb{N} \setminus \{1\}$ and assume the sequence was constructed up to $n-1$. Since (Proposition 2.1.23 a))

$$\sum_{s\in T} \left\langle X_{t t_m s^{-1}}^* X_{t t_m s^{-1}}\,,\, x' \right\rangle < \infty$$

for all $m \in \mathbb{N}_{n-1}$ there is a $t_n \in T$ with

$$\left\langle X_{t t_m t_n^{-1}}^* X_{t t_m t_n^{-1}}\,,\, x' \right\rangle < \frac{1}{4} \left\langle X_t^* X_t\,,\, x' \right\rangle$$

for all $m \in \mathbb{N}_{n-1}$. By Schwarz' inequality ([1, Proposition 2.3.4.6 c)]) for $m \in \mathbb{N}_{n-1}$,

$$\left|\left\langle f(t,t_m)^* f(tt_m t_n^{-1}, t_n) X_t^* X_{tt_m t_n^{-1}}\, ,\, x' \right\rangle\right|^2$$
$$\leq \left\langle X_t^* X_t\, ,\, x' \right\rangle \left\langle X^*_{tt_m t_n^{-1}} X_{tt_m t_n^{-1}}\, ,\, x' \right\rangle < \frac{1}{4} \left\langle X_t^* X_t\, ,\, x' \right\rangle^2 .$$

This finishes the recursive construction.

For $r, s \in T$, by Theorem 2.1.9 e),

$$(X(1_E \otimes e_r))_s = \widetilde{\sum_{q \in T}} f(q, q^{-1}s) X_q \delta_{r, q^{-1}s} = f(sr^{-1}, r) X_{sr^{-1}}\, ,$$

$$\langle\, X(1_E \otimes e_r) \mid X_t \otimes e_s \,\rangle = f(sr^{-1}, r) X_t^* X_{sr^{-1}}\ .$$

For $m, n \in \mathbb{N}$, $m < n$, it follows

$$\langle\, X(1_E \otimes e_{t_m}) \mid X_t \otimes e_{tt_m} \,\rangle = f(t, t_m) X_t^* X_t,$$

$$\left\langle \langle\, X(1_E \otimes e_{t_m}) \mid X_t \otimes e_{tt_m} \,\rangle\, ,\, x' f(t,t_m)^* \right\rangle = \left\langle X_t^* X_t\, ,\, x' \right\rangle ,$$

$$\langle\, X(1_E \otimes e_{t_n}) \mid X_t \otimes e_{tt_m} \,\rangle = f(tt_m t_n^{-1}, t_n) X_t^* X_{tt_m t_n^{-1}},$$

$$\left| \left\langle \langle\, X(1_E \otimes e_{t_n}) \mid X_t \otimes e_{tt_m} \,\rangle\, ,\, x' f(t,t_m)^* \right\rangle \right|$$
$$= \left| \left\langle f(t,t_m)^* f(tt_m t_n^{-1}, t_n) X_t^* X_{tt_m t_n^{-1}}\, ,\, x' \right\rangle \right| < \frac{1}{2} \left\langle X_t^* X_t\, ,\, x' \right\rangle ,$$

$$\begin{aligned}
&\|x'\|\, \|X(1_E \otimes e_{t_m}) - X(1_E \otimes e_{t_n})\|\, \|X_t\| \\
&\quad \geq \left| \left\langle \langle\, X(1_E \otimes e_{t_m}) - X(1_E \otimes e_{t_n}) \mid X_t \otimes e_{tt_m} \,\rangle\, ,\, x' f(t,t_m)^* \right\rangle \right| \\
&\quad \geq \left| \left\langle \langle\, X(1_E \otimes e_{t_m}) \mid X_t \otimes e_{tt_m} \,\rangle\, ,\, x' f(t,t_m)^* \right\rangle \right| - \\
&\qquad - \left| \left\langle \langle\, X(1_E \otimes e_{t_n}) \mid X_t \otimes e_{tt_m} \,\rangle\, ,\, x' f(t,t_m)^* \right\rangle \right| \\
&\quad > \left\langle X_t^* X_t\, ,\, x' \right\rangle - \frac{1}{2} \left\langle X_t^* X_t\, ,\, x' \right\rangle = \frac{1}{2} \left\langle X_T^* X_T\, ,\, x' \right\rangle\ .
\end{aligned}$$

Thus the sequence $(X(1_E \otimes e_{t_n}))_{n \in \mathbb{N}}$ has no Cauchy subsequence and therefore $X(H^\#)$ is not precompact. □

Proposition 2.1.29. *Assume T finite and let Ω be a compact space, $\omega_0 \in \Omega$,*

$$g : T \times T \longrightarrow Un\, \mathcal{C}(\Omega, E)\, , \quad (s,t) \longmapsto f(s,t) 1_\Omega,$$

$$A := \{\, X \in \mathcal{S}(g) \mid t \in T, t \neq 1 \Longrightarrow X_t(\omega_0) = 0 \,\}\, ,$$

$$B := \{\, Y \in \mathcal{C}(\Omega, \mathcal{S}(f)) \mid t \in T, t \neq 1 \Longrightarrow Y(\omega_0)_t = 0 \,\}\, .$$

Then $g \in \mathcal{F}(T, \mathcal{C}(\Omega, E))$ *and we define for every* $X \in A$ *and* $Y \in B$*,*

$$\varphi X : \Omega \longrightarrow \mathcal{S}(f)\,, \quad \omega \longmapsto \sum_{t\in T} (X_t(\omega) \otimes 1_K) V_t^f\,,$$

$$\psi Y := \sum_{t\in T} (Y(\cdot)_t \otimes 1_K) V_t^g\,.$$

Then A *(resp.* B*) is a unital* C^**-subalgebra of* $\mathcal{S}(g)$ *(resp. of* $\mathcal{C}(\Omega, \mathcal{S}(f))$*)*

$$\varphi : A \longrightarrow B\,, \qquad \psi : B \longrightarrow A$$

are C^**-isomorphisms, and* $\varphi = \psi^{-1}$*.*

Proof. It is easy to see that A (resp. B) is a unital C*-subalgebra of $\mathcal{S}(g)$ (resp. of $\mathcal{C}(\Omega, \mathcal{S}(f))$) and that φ and ψ are well-defined. For $X, X' \in A$, $t \in T$, and $\omega \in \Omega$, by Theorem 2.1.9 c),g) and Proposition 2.1.2 e),

$$\begin{aligned}
(((\varphi X)(\varphi X'))(\omega))_t &= \sum_{s\in T} f(s, s^{-1}t)((\varphi X)(\omega))_s((\varphi X')(\omega))_{s^{-1}t} \\
&= \sum_{s\in T} f(s, s^{-1}t) X_s(\omega) X'_{s^{-1}t}(\omega) = \sum_{s\in T} (f(s, s^{-1}t) X_s X'_{s^{-1}t})(\omega) \\
&= (XX')_t(\omega) = (\varphi(XX')(\omega))_t,
\end{aligned}$$

$$\begin{aligned}
(\varphi X^*)(\omega) &= \sum_{s\in T} (((X^*)_s(\omega)) \otimes 1_K) V_s^f = \sum_{s\in T} ((\tilde{f}(s)((X_{s^{-1}})^*(\omega))) \otimes 1_K) V_s^f \\
&= \sum_{s\in T} ((X_{s^{-1}})(\omega)^* \otimes 1_K)(V_{s^{-1}}^f)^* = \sum_{s\in T} (X_s(\omega)^* \otimes 1_K)(V_s^f)^* = (\varphi X)^*(\omega)
\end{aligned}$$

so φ is a C*-homomorphism and we have

$$(\psi\varphi X)_t = (\varphi X)_t = X_t\,.$$

Moreover for $Y \in B$,

$$(\varphi\psi Y)_t(\omega) = ((\psi Y)(\omega))_t = Y_t(\omega)$$

which proves the assertion. □

2.2. **Variation of the parameters**

In this subsection, we examine the changes produced by the replacement of the groups and of the Schur functions.

Definition 2.2.1. We put for every $\lambda \in \Lambda(T, E)$ *(Definition 1.1.3)*

$$U_\lambda : H \longrightarrow H\,, \quad \xi \longmapsto (\lambda(t)\xi_t)_{t\in T}\,.$$

It is easy to see that U_λ is well-defined, $U_\lambda \in Un\,\mathcal{L}_E(H)$, and the map

$$\Lambda(T,E) \longrightarrow Un\,\mathcal{L}_E(H), \quad \lambda \longmapsto U_\lambda$$

is an injective group homomorphism with $U_\lambda^* = U_{\lambda^*}$ (Proposition 1.1.4 c)). Moreover

$$\|U_\lambda - U_\mu\| \le \|\lambda - \mu\|_\infty$$

for all $\lambda, \mu \in \Lambda(T,E)$.

PROPOSITION 2.2.2. *Let $f, g \in \mathcal{F}(T,E)$ and $\lambda \in \Lambda(T,E)$.*

a) *The following are equivalent:*

a$_1$) $g = f\delta\lambda$.

a$_2$) *There is a (unique) E-C^*-isomorphism*

$$\varphi : \mathcal{S}(f) \longrightarrow \mathcal{S}(g)$$

continuous with respect to the $\mathfrak{T}_2$-topologies such that for all $t \in T$ and $x \in E$,

$$\varphi V_t^f = (\lambda(t)^* \widetilde{\otimes} 1_K) V_t^g$$

(we call such an isomorphism an **$\mathcal{S}$-isomorphism** *and denote it by $\approx_{\mathcal{S}}$)*

b) *If the above equivalent assertions are fulfilled then for $X \in \mathcal{S}(f)$ and $t \in T$,*

$$\varphi X = U_\lambda^* X U_\lambda\,, \quad (\varphi X)_t = \lambda(t)^* X_t\,.$$

c) *There is a natural bijection*

$$\{\, \mathcal{S}(f) \mid f \in \mathcal{F}(T,E)\,\} / \approx_{\mathcal{S}} \longrightarrow \mathcal{F}(T,E)/\{\, \delta\lambda \mid \lambda \in \Lambda(T,E)\,\}\,.$$

Proof. By Proposition 1.1.4 c), $\delta\lambda \in \mathcal{F}(T,E)$ for every $\lambda \in \Lambda(T,E)$.

a$_1$) $\Rightarrow$ a$_2$) and b).

For $s, t \in T$ and $\zeta \in \breve{E}$, by Proposition 2.1.2 c),

$$\begin{aligned} U_\lambda^* V_t^f U_\lambda(\zeta \otimes e_s) &= U_\lambda^* V_t^f((\lambda(s)\zeta) \otimes e_s) = U_\lambda^*((f(t,s)\lambda(s)\zeta) \otimes e_{ts}) \\ &= (\lambda(ts)^* f(t,s)\lambda(s)\zeta) \otimes e_{ts} = (\lambda(t)^* g(t,s)\zeta) \otimes e_{ts} = (\lambda(t)^* \widetilde{\otimes} 1_K) V_t^g(\zeta \otimes e_s) \end{aligned}$$

so (by Proposition 2.1.2 e))

$$U_\lambda^* V_t^f U_\lambda = (\lambda(t)^* \widetilde{\otimes} 1_K) V_t^g\,.$$

Thus the map

$$\varphi : \mathcal{S}(f) \longrightarrow \mathcal{S}(g)\,, \quad X \longmapsto U_\lambda^* X U_\lambda$$

is well-defined. It is obvious that it has the properties described in a_2). The uniqueness follows from Theorem 2.1.9 b).

We have

$$\varphi((X_t\tilde{\otimes}1_K)V_t^f) = (X_t\tilde{\otimes}1_K)(\lambda(t)^*\tilde{\otimes}1_K)V_t^g = ((\lambda(t)^*X_t)\tilde{\otimes}1_K)V_t^g$$

so $(\varphi X)_t = \lambda(t)^*X_t$.

a$_2$) $\Rightarrow$ a$_1$). Put $h := f\delta\lambda$. By the above, for $t \in T$,

$$(\lambda(t)^*\tilde{\otimes}1_K)V_t^g = \varphi V_t^f = (\lambda(t)^*\tilde{\otimes}1_K)V_t^h$$

so $V_t^g = V_t^h$ and this implies $g = h$.

c) follows from a). $\square$

Remark. Not every E-C*-isomorphism $\mathcal{S}(f) \to \mathcal{S}(g)$ is an $\mathcal{S}$ isomorphism (see Remark of Proposition 3.2.3).

COROLLARY 2.2.3. *Let*

$$\Lambda_0(T,E) := \{\, \lambda \in \Lambda(T,E) \mid \lambda \textit{ is a group homomorphism}\}$$

and for every $\lambda \in \Lambda_0(T,E)$ put

$$\varphi_\lambda : \mathcal{S}(f) \longrightarrow \mathcal{S}(f)\,, \quad X \longmapsto U_\lambda^* X U_\lambda\,.$$

Then the map $\lambda \mapsto \varphi_\lambda$ is an injective group homomorphism.

Proof. By Proposition 1.1.4 c), $\Lambda_0(T,E)$ is the kernel of the map

$$\Lambda(T,E) \longrightarrow \mathcal{F}(T,E), \quad \lambda \longmapsto \delta\lambda$$

so by Proposition 2.2.2, φ_λ is well-defined. Thus only the injectivity of the map has to be proved. For $t \in T$ and $\zeta \in \breve{E}$, by Proposition 2.1.2 c),

$$\begin{aligned} U_\lambda^* V_t U_\lambda(\zeta \otimes e_1) &= U_\lambda^* V_t(\zeta \otimes e_1) = U_\lambda^*(\zeta \otimes e_t) \\ &= (\lambda(t)^*\zeta) \otimes e_t = (\lambda(t)^*\tilde{\otimes}1_K)V_t(\zeta \otimes e_1). \end{aligned}$$

So if φ_λ is the identity map then $\lambda(t) = 1_E$ for every $t \in T$. $\square$

PROPOSITION 2.2.4. *Let F be a unital C^{**}-algebra, $\varphi : E \to F$ a surjective C^{**}-homomorphism, $g := \varphi \circ f \in \mathcal{F}(T,F)$, and $L := \widetilde{\bigoplus_{t\in T}} \breve{F}$. We put for all $\xi \in H$, $\eta \in L$, and $X \in \mathcal{L}_E(H)$,*

$$\tilde{\xi} := (\varphi\xi_i)_{i\in I} \in L\,, \qquad \tilde{X}\eta := \widetilde{X\zeta} \in L\,,$$

where $\zeta \in H$ with $\tilde{\zeta} = \eta$ (Lemma 1.3.11 a),b) and Proposition 1.3.12 a)). *Then*

$$\tilde{X} = \sum_{t\in T}^{\mathfrak{T}_3} ((\varphi X_t)\tilde{\otimes}1_K)V_t^g \in \mathcal{S}(g)$$

for every $X \in \mathcal{S}(f)$ and the map

$$\tilde{\varphi} : \mathcal{S}(f) \longrightarrow \mathcal{S}(g)\,, \quad X \longmapsto \tilde{X}$$

is a surjective C^{**}-homomorphism, continuous with respect to the topologies $\mathfrak{T}_k$, $k \in \{1,2,3\}$ such that

$$Ker\,\tilde{\varphi} = \{\ X \in \mathcal{S}(f) \mid t \in T \Longrightarrow X_t \in Ker\,\varphi\ \}\ .$$

Proof. For $s, t \in T$ and $\xi \in H$,

$$\left(\widetilde{\overbrace{(X_t\widetilde{\otimes}1_K)V_t^f}}\tilde{\xi}\right)_s = (\widetilde{\overbrace{(X_t\widetilde{\otimes}1_K)V_t^f\xi}})_s = \varphi((X_t\widetilde{\otimes}1_K)V_t^f\xi)_s$$

$$= \varphi(f(t,t^{-1}s)X_t\xi_{t^{-1}s}) = g(t,t^{-1}s)(\varphi X_t)\tilde{\xi}_{t^{-1}s} = (((\varphi X_t)\widetilde{\otimes}1_K)V_t^g\tilde{\xi})_s$$

so by Lemma 1.3.11 b),

$$\widetilde{\overbrace{(X_t\widetilde{\otimes}1_K)V_t^f}} = ((\varphi X_t)\widetilde{\otimes}1_K)V_t^g\ .$$

By Theorem 2.1.9 b),

$$X = \sum_{t\in T}^{\mathfrak{T}_3}(X_t\widetilde{\otimes}1_K)V_t^f$$

so by the above and by Proposition 1.3.12 b),

$$\tilde{X} = \sum_{t\in T}^{\mathfrak{T}_3}((\varphi X_t)\widetilde{\otimes}1_K)V_t^g \in \mathcal{S}(g)\ .$$

By Proposition 1.3.12 b), $\tilde{\varphi}$ is a surjective C**-homomorphism, continuous with respect to the topologies $\mathfrak{T}_k$ ($k \in \{1,2,3\}$). The last assertion is easy to see. □

Corollary 2.2.5. *Let F be a unital C^*-algebra, $\varphi : E \to F$ a unital C^*-homomorphism such that $\varphi(Un\ E^c) \subset F^c$, $g := \varphi \circ f \in \mathcal{F}(T,F)$, and $L := \bigoplus_{t\in T} \breve{F}$. Then the map*

$$\tilde{\varphi} : \mathcal{S}_{\|\cdot\|}(f) \longrightarrow \mathcal{S}_{\|\cdot\|}(g)\,, \quad X \longmapsto \sum_{t\in T}^{\|\cdot\|}((\varphi X_t)\otimes 1_L)V_t^g$$

is C^*-homomorphism.

Proof. Put $G := E/Ker\,\varphi$ and denote by $\varphi_1 : E \to G$ the quotient map and by $\varphi_2 : G \to F$ the corresponding injective C*-homomorphism. By Proposition 2.2.4, the corresponding map

$$\tilde{\varphi}_1 : \mathcal{S}_{\|\cdot\|}(f) \longrightarrow \mathcal{S}_{\|\cdot\|}(\varphi_1 \circ f)$$

is a C*-homomorphism and by Theorem 2.1.9 k), the corresponding map

$$\tilde{\varphi}_2 : \mathcal{S}_{\|\cdot\|}(\varphi_1 \circ f) \longrightarrow \mathcal{S}_{\|\cdot\|}(g)$$

is also a C*-homomorphism. The assertion follows from $\tilde{\varphi} = \tilde{\varphi}_2 \circ \tilde{\varphi}_1$. □

PROPOSITION 2.2.6. *Let T' be a group, $K' := l^2(T')$, $H' := \breve{E}\widetilde{\otimes}K'$, $\psi : T \to T'$ a surjective group homomorphism such that*

$$\sup_{t'\in T'} \operatorname{Card} \overset{-1}{\psi}(\mathrm{t}') \in \mathbb{N}\,,$$

and $f' \in \mathcal{F}(T', E)$ such that $f' \circ (\psi \times \psi) = f$. If we put

$$X'_{t'} := \sum_{t\in \overset{-1}{\psi}(t')} X_t$$

for every $X \in \mathcal{S}(f)$ and $t' \in T'$ then the family $((X'_{t'}\widetilde{\otimes}1_{K'})V_{t'}^{f'})_{t'\in T'}$ is summable in $\mathcal{L}_E(H')_{\mathfrak{T}_2}$ for every $X \in \mathcal{S}(f)$ and the map

$$\tilde{\psi} : \mathcal{S}(f) \longrightarrow \mathcal{S}(f')\,, \quad X \longmapsto X' := \sum_{t'\in T'}^{\mathfrak{T}_1} (X'_{t'}\widetilde{\otimes}1_{K'})V_{t'}^{f'}$$

*is a surjective E-C**-homomorphism.*

We may drop the hypothesis that ψ is surjective if we replace $\mathcal{S}$ by $\mathcal{S}_{\|\cdot\|}$.

Proof. Let $X \in \mathcal{S}(f)$. By Corollary 2.1.16 a), since ψ is surjective and

$$\sup_{t'\in T'} \operatorname{Card} \overset{-1}{\psi}(\mathrm{t}') \in \mathbb{N}\,,$$

it follows that the family $((X'_{t'}\widetilde{\otimes}1_{K'})V_{t'}^{f'})_{t'\in T'}$ is summable in $\mathcal{L}_E(H')_{\mathfrak{T}_2}$ and therefore $X' \in \mathcal{S}(f')$.

Let $X, Y \in \mathcal{S}(f)$. By Theorem 2.1.9 c),g), for $t' \in T'$,

$$(X'^*)_{t'} = \widetilde{f'}(t')(X_{t'^{-1}})^* = \widetilde{f'}(t')\left(\sum_{t\in \overset{-1}{\psi}(t'^{-1})} X_t\right)^* = \widetilde{f'}(t') \sum_{s\in \overset{-1}{\psi}(t')} (X_{s^{-1}})^*$$

$$= \sum_{s\in \overset{-1}{\psi}(t')} \tilde{f}(s)(X_{s^{-1}})^* = \sum_{s\in \overset{-1}{\psi}(t')} (X^*)_s = (X^*)'_{t'},$$

$$(X'Y')_{t'} = \widetilde{\sum_{s'\in T'}} f'(s', s'^{-1}t')X'_{s'}Y'_{s'^{-1}t'}$$

$$= \widetilde{\sum_{s'\in T'}} f'(s',s'^{-1}t') \left(\sum_{s\in \overset{-1}{\psi}(s')} X_s\right)\left(\sum_{r\in \overset{-1}{\psi}(s'^{-1}t')} Y_r\right)$$

$$= \widetilde{\sum_{s'\in T'}} f'(s',s'^{-1}t') \left(\sum_{s\in \overset{-1}{\psi}(s')} \sum_{t\in \overset{-1}{\psi}(t')} X_s Y_{s^{-1}t}\right)$$

$$= \widetilde{\sum_{s'\in T'}} \left(\sum_{s\in \overset{-1}{\psi}(s')} \sum_{t\in \overset{-1}{\psi}(t')} f(s,s^{-1}t) X_s Y_{s^{-1}t}\right)$$

$$= \sum_{t\in \overset{-1}{\psi}(t')} \widetilde{\sum_{s\in T}} f(s,s^{-1}t) X_s Y_{s^{-1}t} = \sum_{t\in \overset{-1}{\psi}(t')} (XY)_t = (XY)'_{t'}.$$

Thus ψ is a C*-homomorphism. The other assertions are easy to see.

The last assertion follows from Corollary 2.1.17 d). $\square$

COROLLARY 2.2.7. *If we use the notation of* Proposition 2.2.6 *and* Corollary 2.2.5 *and define* $\widetilde{\varphi}'$ *and* $\widetilde{\psi}'$ *in an obvious way then* $\widetilde{\varphi}' \circ \tilde{\psi} = \widetilde{\psi}' \circ \tilde{\varphi}$.

Proof. For $X \in \mathcal{S}(f)$ and $t' \in T'$,

$$(\widetilde{\varphi}'\tilde{\psi}X)_{t'} = \varphi((\tilde{\psi}X)_{t'}) = \varphi \sum_{t\in \overset{-1}{\psi}(t')} X_t = \sum_{t\in \overset{-1}{\psi}(t')} \varphi X_t\,,$$

$$(\widetilde{\psi}'\tilde{\varphi}X)_{t'} = \sum_{t\in \overset{-1}{\psi}(t')} (\tilde{\varphi}X)_t = \sum_{t\in \overset{-1}{\psi}(t')} \varphi X_t\,,$$

so

$$\widetilde{\varphi}' \circ \tilde{\psi} = \widetilde{\psi}' \circ \tilde{\varphi}.$$

$\square$

PROPOSITION 2.2.8. *Let F be a unital C*-subalgebra of E such that $f(s,t) \in F$ for all $s,t \in T$. We denote by $\psi : F \to E$ the inclusion map and put*

$$f^F : T\times T \longrightarrow Un\, F^c\,, \quad (s,t) \longmapsto f(s,t),$$

$$H^F := \bigoplus_{t\in T} \breve{F} \approx \breve{F} \otimes K,$$

$$\tilde{\psi} : H^F \longrightarrow H\,, \quad \xi \longmapsto (\psi\xi_t)_{t\in T}.$$

Moreover, we denote for all $s,t\in T$ *by* u_t^F, V_t^F, *and* $\varphi_{s,t}^F$ *the corresponding operators associated with* F ($f^F\in\mathcal{F}(T,F)$). *Let* $X\in\mathcal{S}_C(f)$ *such that* $X(\tilde{\psi}\xi)\in\tilde{\psi}(H^F)$ *for every* $\xi\in H^F$ *and put*

$$X^F : H^F \longrightarrow H^F\,,\quad \xi\longmapsto\xi'\,,$$

where $\xi'\in H^F$ *with* $\tilde{\psi}\xi'=X(\tilde{\psi}\xi)$, *and* $X_t^F:=(u_1^F)^*X^Fu_t^F\in F$ *(by the canonical identification of* F *with* $\mathcal{L}_F(\breve{F})$*) for every* $t\in T$.

a) $\xi,\eta\in H^F\Rightarrow\left\langle\,\tilde{\psi}\xi\,\middle|\,\tilde{\psi}\eta\,\right\rangle=\psi\langle\,\xi\mid\eta\,\rangle.$

b) $\tilde{\psi}$ *is linear and continuous with* $\left\|\tilde{\psi}\right\|=1.$

c) X^F *is linear and continuous with* $\left\|X^F\right\|=\|X\|.$

d) *For* $s,t\in T$,

$$\psi\varphi_{s,t}^FX^F=\varphi_{s,t}X\,,\qquad \psi X_t^F=X_t\,,\qquad \varphi_{s,t}^FX^F=f^F(st^{-1},t)X_{st^{-1}}^F\,.$$

e) $X^F\in\mathcal{S}(f^F).$

f) $\xi\in H^F\Rightarrow X(\tilde{\psi}\xi)=\sum\limits_{t\in T}^{\|\cdot\|}(X_t\otimes 1_K)V_t(\tilde{\psi}\xi).$

Proof. a), b), and c) are easy to see.
d) By a) and Proposition 2.1.6 b),

$$\varphi_{s,t}^FX^F=\left\langle\,X^F(1_F\otimes e_t)\,\middle|\,1_F\otimes e_s\,\right\rangle\,,$$

$$\begin{aligned}\psi\varphi_{s,t}^FX^F&=\psi\left\langle\,X^F(1_F\otimes e_t)\,\middle|\,1_F\otimes e_s\,\right\rangle=\left\langle\,\tilde{\psi}(X^F(1_F\otimes e_t))\,\middle|\,\tilde{\psi}(1_F\otimes e_s)\,\right\rangle\\&=\langle\,X(1_E\otimes e_t)\mid 1_E\otimes e_s\,\rangle=\varphi_{s,t}X.\end{aligned}$$

In particular,

$$\psi X_t^F=\psi\varphi_{1,t}^FX^F=\varphi_{1,t}X=X_t$$

and by Proposition 2.1.8,

$$\psi\varphi_{s,t}^FX^F=\varphi_{s,t}X=f(st^{-1},t)X_{st^{-1}}=\psi(f^F(st^{-1},t)X_{st^{-1}}^F)\,,$$

$$\varphi_{s,t}^FX^F=f^F(st^{-1},t)X_{st^{-1}}^F\,.$$

e) By c) and Proposition 2.1.3 d), for $\xi\in H^F$,

$$\sum_{t\in T}^{\|\cdot\|}u_t^F(u_t^F)^*\xi=\xi\,,$$

$$X^F\xi = X^F \sum_{t\in T}^{\|\cdot\|} u_t^F (u_t^F)^*\xi = \sum_{t\in T}^{\|\cdot\|} X^F u_t^F (u_t^F)^*\xi\,,$$

$$X^F\xi = \sum_{s\in T}^{\|\cdot\|} u_s^F (u_s^F)^* X^F \xi = \sum_{s\in T}^{\|\cdot\|}\sum_{t\in T}^{\|\cdot\|} u_s^F ((u_s^F)^* X^F u_t^F)(u_t^F)^*\xi\,.$$

By d) and Proposition 2.1.4 b),d),

$$X^F\xi = \sum_{s\in T}^{\|\cdot\|}\sum_{t\in T}^{\|\cdot\|} u_s^F f^F(st^{-1},t) X_{st^{-1}}^F (u_t^F)^*\xi = \sum_{s\in T}^{\|\cdot\|}\sum_{t\in T}^{\|\cdot\|} u_s^F X_{st^{-1}}^F (u_s^F)^* V_{st^{-1}}^F \xi$$

$$= \sum_{s\in T}^{\|\cdot\|}\sum_{r\in T}^{\|\cdot\|} u_s^F X_r^F (u_s^F)^* V_r^F\xi = \sum_{s\in T}^{\|\cdot\|}\sum_{r\in T}^{\|\cdot\|} u_s^F (u_s^F)^* (X_r^F\otimes 1_F) V_r^F \xi$$

$$= \sum_{s\in T}^{\|\cdot\|} u_s^F (u_s^F)^* \sum_{t\in T}^{\|\cdot\|} (X_t^F\otimes 1_K) V_t^F\xi = \sum_{t\in T}^{\|\cdot\|} (X_t^F\otimes 1_K)V_t^F\xi$$

by Proposition 2.1.3 d), again. Thus

$$X^F = \sum_{t\in T}^{\mathfrak{T}_2} (X_t^F\otimes 1_K)V_t^F \in \mathcal{S}_C(f^F)\,.$$

f) For $s,t\in T$, by d),

$$(\tilde\psi((X_t^F\otimes 1_K)V_t^F\xi))_s = \psi((X_t^F\otimes 1_K)V_t^F\xi)_s = \psi(f^F(t,t^{-1}s)X_t^F\xi_{t^{-1}s})$$
$$= f(t,t^{-1}s)X_t(\tilde\psi\xi)_{t^{-1}s} = ((X_t\otimes 1_K)V_t\tilde\psi\xi)_s,$$

$$\tilde\psi((X_t^F\otimes 1_K)V_t^F\xi) = (X_t\otimes 1_K)V_t\tilde\psi\xi$$

so by b) and e),

$$X(\tilde\psi\xi) = \tilde\psi(X^F\xi) = \tilde\psi\left(\sum_{t\in T}^{\|\cdot\|}(X_t^F\otimes 1_K)V_t^F\xi\right)$$

$$= \sum_{t\in T}^{\|\cdot\|} \tilde\psi((X_t^F\otimes 1_K)V_t^F\xi) = \sum_{t\in T}^{\|\cdot\|}(X_t\otimes 1_K)V_t(\tilde\psi\xi).$$

□

Proposition 2.2.9. *Let F be a W^*-algebra such that E is a unital C^*-subalgebra of F generating it as W^*-algebra, $\varphi : E\to F$ the inclusion map, and $\tilde\xi := (\varphi\xi_t)_{t\in T}\in L$ for every $\xi\in H$, where*

$$L := \bigoplus_{t\in T}^{W} \breve F \approx \breve F\bar\otimes K\,.$$

a) $\varphi(Un\ E^c) \subset Un\ F^c$ *and* $g := \varphi \circ f \in \mathcal{F}(T, F)$.

b) *If*

$$\psi : \mathcal{L}_E(H) \longrightarrow \mathcal{L}_F(L)\,, \quad X \longmapsto \bar{X}$$

is the injective C-homomorphism defined in Proposition 1.3.9 b), then* $\psi(\mathcal{S}_C(f)) \subset \mathcal{S}_W(g)$, $\psi(\mathcal{S}_C(f))$ *generates* $\mathcal{S}_W(g)$ *as* W^**-algebra, and for every* $X \in \mathcal{S}_C(f)$ *and* $t \in T$ *we have* $(\bar{X})_t = \varphi X_t$.

c) *The following are equivalent for every* $Y \in \mathcal{S}_W(g)$*:*

c$_1$) $Y \in \psi(\mathcal{S}_C(f))$.

c$_2$) $\xi \in H \Rightarrow Y\tilde{\xi} \in H$.

If these conditions are fulfilled then

c$_3$) $(Y_t)_{t\in T} \in H$.

c$_4$) $(Y_t^*)_{t\in T} \in H$.

c$_5$) $\xi \in H \Rightarrow Y\tilde{\xi} = \sum\limits_{t\in T}^{\|\cdot\|} (Y_t \bar{\otimes} 1_K) V_t^g \tilde{\xi} \in H$.

Proof. a) follows from the density of $\varphi(E)$ in $F_{\ddot{F}}$ (Lemma 1.3.8 $a \Rightarrow c$).
b) For $x \in E$, $t \in T$, and $\xi \in H$,

$$\begin{aligned}(((\varphi x)\bar{\otimes}1_K)V_t^g\tilde{\xi})_s &= g(t,t^{-1}s)(\varphi x)\tilde{\xi}_{t^{-1}s} \\ &= \varphi(f(t,t^{-1}s)x\xi_{t^{-1}s}) = \varphi((x\otimes 1_K)V_t\xi_s)\end{aligned}$$

so

$$((\varphi x)\bar{\otimes}1_K)V_t^g = \overline{(x\otimes 1_K)V_t^f}\,.$$

Let now $X \in \mathcal{S}(f)$. By Theorem 2.1.9 b),

$$X = \sum_{t\in T}^{\mathfrak{T}_2} (X_t \otimes 1_K) V_t^f$$

so by the above and by Proposition 1.3.9 c) (and Theorem 2.1.9 d)),

$$\bar{X} = \sum_{t\in T}^{\mathfrak{T}_1} \overline{(X_t \otimes 1_K)V_t^f} = \sum_{t\in T}^{\mathfrak{T}_1} ((\varphi X_t)\bar{\otimes}1_K)V_t^g \in \mathcal{S}_W(g)$$

so $\psi(\mathcal{S}_C(f)) \subset \mathcal{S}_W(f)$. By Theorem 2.1.9 a), $(\bar{X})_t = \varphi X_t$ for every $t \in T$.
Since $\varphi(E)$ is dense in $F_{\ddot{F}}$ (Lemma 1.3.8 $a) \Rightarrow c)$) it follows that

$$\mathcal{R}(g) \subset \overline{\varphi(\mathcal{R}(f))}^{\mathfrak{T}_1}$$

so $\psi(\mathcal{S}(f))$ is dense in $\mathcal{S}(g)_{\underbrace{\ddot{}}_{\mathcal{S}(g)}}$ and therefore generates $\mathcal{S}(g)$ as W*-algebra (Lemma 1.3.8 $c \Rightarrow a$).

c$_1$) $\Rightarrow$ c$_2$) follows from the definition of ψ.

c$_2$) $\Rightarrow$ c$_1$) follows from Proposition 2.2.8 e).

c$_2$) $\Rightarrow$ c$_3$) and c$_4$) follows from Proposition 2.1.23 b).

c$_2$) $\Rightarrow$ c$_5$) follows from Proposition 2.2.8 f). □

LEMMA 2.2.10. *Let E, F be W^*-algebras, $G := E\bar{\otimes}F$, and*

$$L := \bigoplus_{t\in T}^{W} \breve{G} \approx \breve{G}\bar{\otimes}K.$$

a) *If $z \in G^{\#}$ then $z\bar{\otimes}1_K$ belongs to the closure of*

$$\{\, w\bar{\otimes}1_K \mid w \in E \odot F,\ \|w\| \le 1 \,\}$$

in $\mathcal{L}_G(L)_{\ddot{L}}$.

b) *For every $y \in F$, the map*

$$E^{\#}_{\ddot{E}} \longrightarrow G_{\ddot{G}}, \quad x \longmapsto x \otimes y$$

is continuous.

Proof. a) By [1, Corollary 6.3.8.7], there is a filter $\mathfrak{F}$ on

$$\{\, w \in E \odot F \mid \|w\| \le 1 \,\}$$

converging to z in $G^{\#}_{\ddot{G}}$. By Lemma 1.3.2 b), for $(a,\xi,\eta) \in \ddot{G} \times L \times L$,

$$\left\langle z\bar{\otimes}1_K \,,\, \widetilde{(a,\xi,\eta)} \right\rangle = \left\langle z \,,\, \sum_{t\in T}^{G} \xi_t\, a\, \eta_t^* \right\rangle$$

$$= \lim_{w,\mathfrak{F}} \left\langle w \,,\, \sum_{t\in T}^{G} \xi_t\, a\, \eta_t^* \right\rangle = \lim_{w,\mathfrak{F}} \left\langle w\bar{\otimes}1_K \,,\, \widetilde{(a,\xi,\eta)} \right\rangle$$

which proves the assertion.

b) Let $(a_i, b_i)_{i\in I}$ be a finite family in $\ddot{E} \times \ddot{F}$. For $x \in E$,

$$\left\langle x \otimes y \,,\, \sum_{i\in I} a_i \otimes b_i \right\rangle = \sum_{i\in I} \langle x \,,\, a_i \rangle \langle y \,,\, b_i \rangle = \left\langle x \,,\, \sum_{i\in I} \langle y \,,\, b_i \rangle\, a_i \right\rangle .$$

Since $\{\, x \otimes y \mid x \in E^{\#} \,\}$ is a bounded set of G, the above identity proves the continuity. □

Proposition 2.2.11. *Let F be a unital C^{**}-algebra, S a group, and $g \in \mathcal{F}(S,F)$. We denote by $\otimes_\sigma$ the spatial tensor product and put*

$$G := E \otimes_\sigma F \qquad (resp.\ G := E \bar{\otimes} F)\,,$$

$$L := \widetilde{\bigoplus_{s\in S}} \breve{F} \approx \breve{F}\widetilde{\otimes} l^2(S), \qquad M := \widetilde{\bigoplus_{(t,s)\in T\times S}} \breve{G} \approx \breve{G}\widetilde{\otimes} l^2(T\times S)\,,$$

$$h : (T\times S)\times(T\times S) \longrightarrow Un\, G^c\,, \quad ((t_1,s_1),(t_2,s_2)) \longmapsto f(t_1,t_2)\otimes g(s_1,s_2)\,.$$

a) $h \in \mathcal{F}(T\times S, G), \qquad M \approx H\widetilde{\otimes} L,$
$\mathcal{L}_E(H)\otimes_\sigma \mathcal{L}_F(L) \subset \mathcal{L}_G(M)$ *in the* C^**-case,*
$\mathcal{L}_E(H)\bar{\otimes}\mathcal{L}_F(L) \approx \mathcal{L}_G(M)$ *in the* W^**-case .*

b) *For* $t\in T$, $s\in S$, $x\in E$, $y\in F$,

$$((x\widetilde{\otimes}1_{l^2(T)})V_t^f)\otimes((y\widetilde{\otimes}1_{l^2(S)})V_s^g) = ((x\otimes y)\widetilde{\otimes}1_{l^2(T\times S)})V_{(t,s)}^h\,.$$

c) *In the* C^**-case,* $\mathcal{S}_{\|\cdot\|}(f)\otimes_\sigma\mathcal{S}_{\|\cdot\|}(g) \approx \mathcal{S}_{\|\cdot\|}(h)$ *and* $\mathcal{S}_C(f)\otimes_\sigma\mathcal{S}_C(g)\approx\mathcal{S}_C(h)$.

d) *In the* W^**-case, if* $z\in G^\#$ *and* $(t,s)\in T\times S$ *then* $(z\bar{\otimes}1_{l^2(T\times S)})V_{(t,s)}^h$ *belongs to the closure of* $\left\{(w\bar{\otimes}1_{l^2(T\times S)})V_{(t,s)}^h \,\middle|\, w\in(E\odot F)^\#\right\}$ *in* $\mathcal{L}_G(M)_{\ddot{M}}$

e) *In the* W^**-case,* $\mathcal{S}_W(f)\bar{\otimes}\mathcal{S}_W(g)\approx\mathcal{S}_W(h)$.

Proof. a) $h\in\mathcal{F}(T\times S,G)$ is obvious.

Let us treat the C*-case first. For $\xi,\xi'\in H$ and $\eta,\eta'\in L$,

$$\begin{aligned}\langle\, \xi'\otimes\eta' \mid \xi\otimes\eta\,\rangle &= \langle\, \xi' \mid \xi\,\rangle\otimes\langle\,\eta'\mid\eta\,\rangle\\ &= \left(\sum_{t\in T}\xi_t^*\xi_t'\right)\otimes\left(\sum_{s\in S}\eta_s^*\eta_s'\right) = \sum_{(t,s)\in T\times S}((\xi_t^*\xi_t')\otimes(\eta_s^*\eta_s'))\\ &= \sum_{(t,s)\in T\times S}(\xi_t^*\otimes\eta_s^*)(\xi_t'\otimes\eta_s') = \sum_{(t,s)\in T\times S}(\xi_t\otimes\eta_s)^*(\xi_t'\otimes\eta_s'),\end{aligned}$$

so the linear map

$$H\odot L\longrightarrow M,\quad \xi\otimes\eta\longmapsto(\xi_t\otimes\eta_s)_{(t,s)\in T\times S}$$

preserves the scalar products and it may be extended to a linear map $\varphi : H\otimes L\to M$ preserving the scalar products.

Let $z\in G$, $(t,s)\in T\times S$, and $\varepsilon>0$. There is a finite family $(x_i,y_i)_{i\in I}$ in $E\times F$ such that

$$\left\|\sum_{i\in I}x_i\otimes y_i - z\right\| < \varepsilon\,.$$

Then

$$\left\| \sum_{i\in I} (x_i \otimes e_t) \otimes (y_i \otimes e_s) - z \otimes e_{(t,s)} \right\| < \varepsilon$$

so $z \otimes e_{(t,s)} \in \overline{\varphi(H \otimes L)} = \varphi(H \otimes L)$. It follows that φ is surjective and so $H \otimes L \approx M$.

The proof for the inclusion $\mathcal{L}_E(H) \otimes_\sigma \mathcal{L}_F(L) \subset \mathcal{L}_G(M)$ can be found in [5, page 37].

Let us now discus the W*-case. $\breve{E}\bar{\otimes}\breve{F} \approx \breve{G}$ follows from [2, Proposition 1.3 e)], $M \approx H\bar{\otimes}L$ follows from [3, Corollary 2.2], and $\mathcal{L}_E(H)\bar{\otimes}\mathcal{L}_F(L) \approx \mathcal{L}_G(M)$ follows from [2, Theorem 2.4 d)] or [3, Theorem 2.4].

b) For $t_1, t_2 \in T$, $s_1, s_2 \in S$, $\xi \in \breve{E}$, and $\eta \in \breve{F}$, by Proposition 2.1.2 f) and [3, Corollary 2.11],

$$\begin{aligned}&(((x\widetilde{\otimes}1_{l^2(T)})V^f_{t_1})\widetilde{\otimes}((y\widetilde{\otimes}1_{l^2(S)})V^g_{s_1}))((\xi \otimes e_{t_2}) \otimes (\eta \otimes e_{s_2}))\\ &\qquad = (((x\widetilde{\otimes}1_{l^2(T)})V^f_{t_1})(\xi \otimes e_{t_2}))\widetilde{\otimes}(((y\widetilde{\otimes}1_{l^2(S)})V^g_{s_1})(\eta \otimes e_{s_2})),\end{aligned}$$

$$\begin{aligned}&((((x \otimes y)\widetilde{\otimes}1_{l^2(T\times S)}))V^h_{(t_1,s_1)})((\xi \otimes \eta) \otimes e_{(t_2,s_2)})\\ &\qquad = (h((t_1, s_1), (t_2, s_2))(x \otimes y)(\xi \otimes \eta)) \otimes e_{(t_1t_2,s_1s_2)}\\ &\qquad = ((f(t_1, t_2)x\xi) \otimes (g(s_1, s_2)y\eta)) \otimes e_{t_1t_2} \otimes e_{s_1s_2}\\ &\qquad = (((x\widetilde{\otimes}1_{l^2(T)})V^f_{t_1})(\xi \otimes e_{t_2}))\widetilde{\otimes}(((y\widetilde{\otimes}1_{l^2(S)})V^g_{s_1})(\eta \otimes e_{s_2})).\end{aligned}$$

We put

$$u := ((x\widetilde{\otimes}1_{l^2(T)})V^f_t)\widetilde{\otimes}((y\widetilde{\otimes}1_{l^2(S)})V^g_s) - ((x \otimes y)\widetilde{\otimes}1_{l^2(T\times S)})V^h_{t,s} \in \mathcal{L}_G(M)\ .$$

By the above, $u(\zeta \otimes e_r) = 0$ for all $\zeta \in \breve{E} \odot \breve{F}$ and $r \in T \times S$.

Let us consider the C*-case first. Since $\breve{E} \odot \breve{F}$ is dense in $\breve{G}$, we get $u(z \otimes e_r) = 0$ for all $z \in \breve{G}$ and $r \in T \times S$. For $\zeta \in M$, by [1, Proposition 5.6.4.1 e)],

$$u\zeta = u\left(\sum_{r\in T\times S} (\zeta_r \otimes e_r)\right) = \sum_{r\in T\times S} u(\zeta_r \otimes e_r) = 0\,,$$

which proves the assertion in this case.

Let us consider now the W*-case. Let $z \in G^\#$ and $r \in T \times S$ and let $\mathfrak{F}$ be a filter on $(E \odot F)^\#$ converging to z in $G_{\ddot{G}}$ ([1, Corollary 6.3.8.7]). For $\eta \in M$, $a \in \ddot{G}$, and $r \in T \times S$,

$$\begin{aligned}\left\langle\, z \otimes e_r\,,\, \widetilde{(a, \eta)}\,\right\rangle &= \langle\, \langle\, z \otimes e_r \mid \eta\,\rangle\,,\, a\,\rangle = \langle\, \eta^*_r z\,,\, a\,\rangle = \langle\, z\,,\, a\eta^*_r\,\rangle\\ &= \lim_{w,\mathfrak{F}} \langle\, w\,,\, a\eta^*_r\,\rangle = \lim_{w,\mathfrak{F}} \left\langle\, w \otimes e_r\,,\, \widetilde{(a, \eta)}\,\right\rangle,\end{aligned}$$

Hence

$$\lim_{w,\mathfrak{F}} w\otimes e_r = z\otimes e_r$$

in $M_{\ddot{M}}$. Since $u : M_{\ddot{M}} \to M_{\ddot{M}}$ is continuous ([1, Proposition 5.6.3.4 c)]), we get by the above $u(z\otimes e_r)=0$. For $\zeta\in M$ it follows by [1, Proposition 5.6.4.6 c)] that

$$u\zeta = u\left(\sum_{r\in T\times S}^{\ddot{M}}(\zeta_r\otimes e_r)\right) = \sum_{r\in T\times S}^{\ddot{M}} u(\zeta_r\otimes e_r)=0$$

which proves the assertion in the W*-case.

c) By b), $\mathcal{R}(f)\odot\mathcal{R}(g)\subset\mathcal{R}(h)$ so by a),

$$\mathcal{S}_{\|\cdot\|}(f)\odot\mathcal{S}_{\|\cdot\|}(g)\subset\mathcal{S}_{\|\cdot\|}(h),\qquad \mathcal{S}_C(f)\odot\mathcal{S}_C(g)\subset\mathcal{S}_C(h),$$
$$\mathcal{S}_{\|\cdot\|}(f)\otimes_\sigma\mathcal{S}_{\|\cdot\|}(g)\subset\mathcal{S}_{\|\cdot\|}(h),\qquad \mathcal{S}_C(f)\otimes_\sigma\mathcal{S}_C(g)\subset\mathcal{S}_C(h).$$

Let $z\in G^{\#}$, $(t,s)\in T\times S$, and $\varepsilon>0$. There is a finite family $(x_i,y_i)_{i\in I}$ in $E\times F$ such that

$$\left\|\sum_{i\in I}(x_i\otimes y_i)\right\|<1\,,\qquad \left\|\sum_{i\in I}(x_i\otimes y_i)-z\right\|<\varepsilon\,.$$

By b),

$$\left\|\sum_{i\in I}(((x_i\otimes 1_{l^2(T)})V_t^f)\otimes((y_i\otimes 1_{l^2(S)})V_s^g))-(z\otimes 1_{l^2(T\times S)})V_{(t,s)}^h\right\|<\varepsilon$$

and so by a),

$$\mathcal{R}(h)\subset\overline{\mathcal{R}(f)\odot\mathcal{R}(g)}^{\|\cdot\|}\subset\overline{\mathcal{R}(f)\odot\mathcal{R}(g)}^{\mathfrak{T}_2}\,,$$
$$\mathcal{S}_{\|\cdot\|}(h)\subset\mathcal{S}_{\|\cdot\|}(f)\otimes_\sigma\mathcal{S}_{\|\cdot\|}(g)\,,\quad \mathcal{S}_C(h)\subset\mathcal{S}_C(f)\otimes_\sigma\mathcal{S}_C(g).$$

d) By a) and Lemma 2.2.10 a), there is a filter $\mathfrak{F}$ on

$$\left\{\,w\bar{\otimes}1_{l^2(T\times S)}\;\middle|\;w\in(E\odot F)^{\#}\,\right\}$$

converging to $z\bar{\otimes}1_{l^2(T\times S)}$ in $\mathcal{L}_G(M)_{\dddot{M}}$. For $\xi,\eta\in M$ and $a\in\ddot{G}$,

$$\left\langle\,(z\bar{\otimes}1_{l^2(T\times S)})V_{(t,s)}^h\,,\,\widetilde{(a,\xi,\eta)}\,\right\rangle=\left\langle\,z\bar{\otimes}1_{l^2(T\times S)}\,,\,V_{(t,s)}^h\widetilde{(a,\xi,\eta)}\,\right\rangle$$
$$=\lim_{w,\mathfrak{F}}\left\langle\,w\bar{\otimes}1_{l^2(T\times S)}V_{(t,s)}^h\,,\,\widetilde{(a,\xi,\eta)}\,\right\rangle=\lim_{w,\mathfrak{F}}\left\langle\,(w\bar{\otimes}1_{l^2(T\times S)})V_{(t,s)}^h\,,\,\widetilde{(a,\xi,\eta)}\,\right\rangle,$$

which proves the assertion.

e) By Theorem 2.1.9 h),

$$\left(\frac{\dddot{H}}{\mathcal{R}(f)}\right)^{\#}=\mathcal{S}_W(f)^{\#}\subset\mathcal{L}_E(H),\qquad \left(\frac{\dddot{L}}{\mathcal{R}(g)}\right)^{\#}=\mathcal{S}_W(g)^{\#}\subset\mathcal{L}_F(L)\,.$$

By b), $\mathcal{R}(f)\odot\mathcal{R}(g)\subset\mathcal{R}(h)$, so by Lemma 2.2.10 b),

$$\mathcal{S}_W(f)^{\#}\odot\mathcal{R}(g)^{\#}\subset\mathcal{S}_W(h)^{\#},\qquad \mathcal{S}_W(f)^{\#}\otimes\mathcal{S}_W(g)^{\#}\subset\mathcal{S}_W(h)^{\#}\,.$$

$$\mathcal{S}_W(f)\otimes\mathcal{S}_W(g)\subset\mathcal{S}_W(h)$$

By [3, Proposition 2.5],

$$\mathcal{S}_W(f)\bar{\otimes}\mathcal{S}_W(g)\approx\overline{\mathcal{S}_W(f)\otimes\mathcal{S}_W(g)}^{\dddot{M}}\subset\mathcal{S}_W(h)\,.$$

For $x\in E$, $y\in F$, and $(t,s)\in T\times S$, by b),

$$((x\otimes y)\bar{\otimes}1_{l^2(T\times S)})V^h_{(t,s)}=((x\bar{\otimes}1_{l^2(T)})V^f_t)\bar{\otimes}((y\bar{\otimes}1_{l^2(S)})V^g_s)\in\mathcal{S}_W(f)\bar{\otimes}\mathcal{S}_W(g)\,.$$

Let $z\in G^{\#}$. By d), there is a filter $\mathfrak{F}$ on

$$\left\{\,(w\bar{\otimes}1_{l^2(T\times S)})V^h_{(t,s)}\;\middle|\;w\in(E\odot F)^{\#}\,\right\}$$

converging to $(z\bar{\otimes}1_{l^2(T\times S)})V^h_{(t,s)}$ in $\mathcal{L}_G(M)_{\dddot{M}}$, so by the above

$$(z\bar{\otimes}1_{l^2(T\times S)})V^h_{(t,s)}\in\mathcal{S}_W(f)\bar{\otimes}\mathcal{S}_W(g)\,.$$

We get

$$\mathcal{R}(h)\subset\mathcal{S}_W(f)\bar{\otimes}\mathcal{S}_W(g),\qquad \mathcal{S}_W(h)\subset\mathcal{S}_W(f)\bar{\otimes}\mathcal{S}_W(g)\,,$$

$$\mathcal{S}_W(h)=\mathcal{S}_W(f)\bar{\otimes}\mathcal{S}_W(g).$$

□

COROLLARY 2.2.12. *Let* $n\in\mathbb{N}$ *and*

$$g:T\times T\longrightarrow Un\,(E_{n,n})^c\,,\quad (s,t)\longmapsto[\delta_{i,j}f(s,t)]_{i,j\in\mathbb{N}_n}\,.$$

a) $(\mathcal{S}(f))_{n,n}\approx\mathcal{S}(g),\quad(\mathcal{S}_{\|\cdot\|}(f))_{n,n}\approx\mathcal{S}_{\|\cdot\|}(g)\,.$

b) *Let us denote by* $\rho:\mathcal{S}(g)\to(\mathcal{S}(f))_{n,n}$ *the isomorphism of a). For* $X\in\mathcal{S}(g)$, $t\in T$, *and* $i,j\in\mathbb{N}_n$,

$$((\rho X)_{i,j})_t=(X_t)_{i,j}\,.$$

Proof. a) Take $F:=\mathbb{K}_{n,n}$ and $S:=\{1\}$ in Proposition 2.2.11. Then $G\approx E_{n,n}$ and

$$g:T\times T\longrightarrow Un\,G^c\,,\quad (s,t)\longmapsto f(s,t)\otimes 1_F\,.$$

By Proposition 2.2.11 c),e),

$$\mathcal{S}(g)\approx\mathcal{S}(f)\otimes\mathbb{K}_{n,n}\approx(\mathcal{S}(f))_{n,n}$$

$$\mathcal{S}_{\|\cdot\|}(g)\approx\mathcal{S}_{\|\cdot\|}(f)\otimes\mathbb{K}_{n,n}\approx(\mathcal{S}_{\|\cdot\|}(f))_{n,n}\,.$$

b) By Theorem 2.1.9 b),

$$X = \sum_{s\in T}^{\mathfrak{T}_3} (X_s \widetilde{\otimes} 1_K) V_s^g$$

so

$$(\rho X)_{i,j} = \sum_{s\in t}^{\mathfrak{T}_3} ((X_s)_{i,j} \widetilde{\otimes} 1_K) V_s^f \,,$$

$$((\rho X)_{i,j})_t = (X_t)_{i,j}$$

by Theorem 2.1.9 a). □

Corollary 2.2.13. *Let $n \in \mathbb{N}$. If $\mathbb{K} = \mathbb{C}$ (resp. if $n = 4^m$ for some $m \in \mathbb{N}$) then there is an $f \in \mathcal{F}(\,\mathbb{Z}_n \times \mathbb{Z}_n, E)$ (resp. $f \in \mathcal{F}(\,(\mathbb{Z}_2)^{2m}, E)$) such that*

$$\mathcal{R}(f) = \mathcal{S}(f) \approx E_{n,n} \,.$$

Proof. By [1, Proposition 7.1.4.9 b),d)] (resp. [1, Theorem 7.2.2.7 i),k)]) there is a $g \in \mathcal{F}(\,\mathbb{Z}_n \times \mathbb{Z}_n, \mathbb{C})$ (resp. $g \in \mathcal{F}(\,(\mathbb{Z}_2)^{2m}, \mathbb{K})$) such that

$$\mathcal{S}(g) \approx \mathbb{C}_{n,n} \qquad \text{(resp. } \mathcal{S}(g) \approx \mathbb{K}_{n,n}).$$

If we put

$$f : (\,\mathbb{Z}_n \times \mathbb{Z}_n) \times (\,\mathbb{Z}_n \times \mathbb{Z}_n) \longrightarrow Un\; E^c\,, \quad (s,t) \longmapsto g(s,t) \otimes 1_E$$

$$\text{(resp. } f : (\mathbb{Z}_2)^{2m} \times (\mathbb{Z}_2)^{2m} \longrightarrow Un\; E^c\,, \quad (s,t) \longmapsto g(s,t) \otimes 1_E)$$

then by Proposition 2.2.11 a),e), $f \in \mathcal{F}(\,\mathbb{Z}_n \times \mathbb{Z}_n, E)$ (resp. $f \in \mathcal{F}(\,(\mathbb{Z}_2)^{2m}, E)$) and

$$\mathcal{S}(f) \approx \mathcal{S}(g) \otimes E \approx \mathbb{K}_{n,n} \otimes E \approx E_{n,n}.$$

□

Corollary 2.2.14. *Let F be a unital C^{**}-algebra, $G := E \widetilde{\otimes} F$, and*

$$h : T \times T \longrightarrow Un\; G^c\,, \quad (s,t) \longmapsto f(s,t) \otimes 1_F.$$

Then $h \in \mathcal{F}(T, G)$ and

$$\mathcal{S}_{\|\cdot\|}(h) \approx \mathcal{S}_{\|\cdot\|}(f) \otimes F\,, \qquad \mathcal{S}(h) \approx \mathcal{S}(f) \widetilde{\otimes} F.$$

Corollary 2.2.15. *If E is a W^*-algebra then the following are equivalent:*

a) *E is semifinite.*

b) *$\mathcal{S}_W(f)$ is semifinite.*

Proof. a) ⇒ b). Assume first that there are a finite W*-algebra F and a Hilbert space L such that $E \approx F\bar{\otimes}\mathcal{L}(L)$. Put

$$g : T \times T \longrightarrow Un\ F^c\,, \quad (s,t) \longmapsto f(s,t).$$

By Corollary 2.2.14,

$$\mathcal{S}_W(f) \approx \mathcal{S}_W(g)\bar{\otimes}\mathcal{L}(L).$$

By Corollary 2.1.11 c), $\mathcal{S}_W(g)$ is finite and so $\mathcal{S}_W(f)$ is semifinite.

The general case follows from the fact that E is the C*-direct product of W*-algebras of the above form ([7, Proposition V.1.40]).

b) ⇒ a). E is isomorphic to a W*-subalgebra of $\mathcal{S}_W(f)$ (Theorem 2.1.9 h)) and the assertion follows from [7, Theorem V.2.15]. □

Proposition 2.2.16. *Let S, T be finite groups and $g \in \mathcal{F}(S, \mathcal{S}(f))$ and put $L := l^2(S)$, $M := l^2(S \times T)$, and*

$$h : (S \times T) \times (S \times T) \longrightarrow Un\ \mathcal{S}(f)^c\,, \quad ((s_1,t_1),(s_2,t_2)) \longmapsto f(t_1,t_2)g(s_1,s_2)\,.$$

Then $h \in \mathcal{F}(S \times T, \mathcal{S}(f))$ and the map

$$\varphi : \mathcal{S}(g) \longrightarrow \mathcal{S}(h)\,, \quad X \longmapsto \sum_{(s,t)\in S\times T} ((X_s)_t \otimes 1_M)V^h_{(s,t)}$$

is an $\mathcal{S}(f)$-C-isomorphism.*

Proof. For $X, Y \in \mathcal{S}(g)$, $Z \in \mathcal{S}(f)$, and $(s,t) \in S \times T$, by Theorem 2.1.9 c), g),

$$\begin{aligned}(\varphi(X^*))_{(s,t)} &= ((X^*)_s)_t = (\tilde{g}(s)(X_{s^{-1}})^*)_t = ((\tilde{g}(s)^* X_{s^{-1}})^*)_t \\ &= \tilde{f}(t)((\tilde{g}(s)^* X_{s^{-1}})_{t^{-1}})^* = \tilde{f}(t)\tilde{g}(s)((X_{s^{-1}})_{t^{-1}})^* \\ &= \tilde{h}(s,t)((\varphi X)_{(s^{-1},t^{-1})})^* = \tilde{h}(s,t)((\varphi X)_{(s,t)^{-1}})^* = ((\varphi X)^*)_{(s,t)},\end{aligned}$$

$$((\varphi X)(\varphi Y))_{(s,t)} = \sum_{(r,u)\in S\times T} h((r,u),(r,u)^{-1}(s,t))(\varphi X)_{(r,u)}(\varphi Y)_{(r,u)^{-1}(s,t)}$$

$$= \sum_{(r,u)\in S\times T} g(r,r^{-1}s)f(u,u^{-1}t)(X_r)_u(Y_{r^{-1}s})_{u^{-1}t} = \sum_{r\in S} g(r,r^{-1}s)(X_r Y_{r^{-1}s})_t$$

$$= \left(\sum_{r\in S} g(r,r^{-1}s)X_r Y_{r^{-1}s}\right)_t = ((XY)_s)_t = (\varphi(XY))_{(s,t)},$$

$$(\varphi(ZX))_{(s,t)} = ((ZX)_s)_t = ((ZX)_s)_t = (ZX_s)_t = Z(X_s)_t = Z(\varphi X)_{(s,t)}$$

so

$$\varphi(X^*) = (\varphi X)^*, \quad \varphi(XY) = (\varphi X)(\varphi Y), \quad \varphi(ZX) = Z\varphi(X)$$

and φ is an $\mathcal{S}(f)$-C*-homomorphism.

If $X\in\mathcal{S}(g)$ with $\varphi X=0$ then for $(s,t)\in S\times T$,

$$(X_s)_t=(\varphi X)_{(s,t)}=0,\quad X_s=0,\quad X=0,$$

so φ is injective.

Let $x\in E$ and $(s,t)\in S\times T$. Put

$$Z:=(x\otimes 1_K)V_t^f\in\mathcal{S}(f),\qquad X:=(Z\otimes 1_L)V_s^g\in\mathcal{S}(g).$$

Then for $(r,u)\in S\times T$,

$$(\varphi X)_{(r,u)}=(X_r)_u=\delta_{r,s}Z_u=\delta_{r,s}\delta_{u,t}x$$

so

$$\varphi X=(x\otimes 1_M)V^h_{(s,t)}$$

and φ is surjective. □

PROPOSITION 2.2.17. *Let S be a finite subgroup of T and $g:=f|(S\times S)$. We identify $\mathcal{S}(g)$ with the E-C^{**}-subalgebra $\{\,Z\in\mathcal{S}(f)\mid t\in T\setminus S\Rightarrow Z_t=0\,\}$ of $\mathcal{S}(f)$* (Corollary 2.1.17 e)). *Let $X\in\mathcal{S}(f)\cap\mathcal{S}(g)^c$, $P_+:=X^*X$, and $P_-:=XX^*$ and assume $P_\pm\in Pr\,\mathcal{S}(f)$.*

a) $P_\pm\in\mathcal{S}(g)^c$.

b) *The map*

$$\varphi_\pm:\mathcal{S}(g)\longrightarrow P_\pm\mathcal{S}(f)P_\pm\,,\quad Y\longmapsto P_\pm YP_\pm$$

is a unital C^{**}-homomorphism.

c) *For every $Z\in\varphi_+(\mathcal{S}(g))$, $XZX^*\in\varphi_-(\mathcal{S}(g))$ and the map*

$$\psi:\varphi_+(\mathcal{S}(g))\longrightarrow\varphi_-(\mathcal{S}(g))\,,\quad Z\longmapsto XZX^*$$

is a C^*-isomorphism with inverse

$$\varphi_-(\mathcal{S}(g))\longrightarrow\varphi_+(\mathcal{S}(g)),\quad Z\longmapsto X^*ZX$$

such that $\varphi_-=\psi\circ\varphi_+$.

d) *If $p\in Pr\,\mathcal{S}(g)$ then*

$$(X(\varphi_+p))^*(X(\varphi_+p))=\varphi_+p\,,\qquad (X((\varphi_+p))(X(\varphi_+p))^*=\varphi_-p\,.$$

e) *If φ_+ is injective then φ_- is also injective, the map*

$$E\longrightarrow P_\pm\mathcal{S}(f)P_\pm,\quad x\longmapsto P_\pm(x\widetilde{\otimes}1_K)P_\pm$$

is an injective unital C^{**}-homomorphism, $P_\pm\mathcal{S}(f)P_\pm$ is an E-C^{**}-algebra, $\varphi_\pm(\mathcal{S}(g))$ is an E-C^{**}-subalgebra of it, and $\varphi_\pm$ and ψ are E-C^{**}-homomorphisms.

f) *The above results still hold for an arbitrary subgroup S of T if we replace $\mathcal{S}$ by $\mathcal{S}_{\|\cdot\|}$.*

Proof. a) follows from the hypothesis on X.

b) follows from a).

c) Let $Y \in \mathcal{S}(g)$ with $Z = P_+ Y P_+$. By the hypotheses of the Proposition,

$$\begin{aligned} XZX^* &= XP_+YP_+X^* = XX^*XYX^*XX^* \\ &= XX^*YXX^*XX^* = P_-YP_- \in \varphi_-(\mathcal{S}(g)) \end{aligned}$$

and ψ is a C*-homomorphism. The other assertions follow from

$$X^*(XZX^*)X = P_+ZP_+ = P_+YP_+ \,.$$

d) By b) and c),

$$(X(\varphi_+ p))^*(X(\varphi_+ p)) = (\varphi_+ p)X^*X(\varphi_+ p) = (\varphi_+ p)P_+(\varphi_+ p) = \varphi_+ p\,,$$

$$(X(\varphi_+ p))(X(\varphi_+ p))^* = X(\varphi_+ p)(\varphi_+ p)^*X^* = X(\varphi_+ p)X^* = \psi\varphi_+ p = \varphi_- p\,.$$

e) follows from b), c), and Lemma 1.3.2.

f) follows from Corollary 2.1.17 d). □

Remark. Even if $\varphi_\pm$ is injective $P_\pm \mathcal{S}(f) P_\pm$ is not an E-C*-subalgebra of $\mathcal{S}(f)$.

THEOREM 2.2.18. *Let S be a finite subgroup of T, $L := l^2(S)$, $g := f|(S\times S)$, $\omega : \mathbb{Z}_2 \times \mathbb{Z}_2 \to T$ an injective group homomorphism such that $S \cap \omega(\mathbb{Z}_2\times\mathbb{Z}_2) = \{1\}$,*

$$a := \omega(1,0)\,, \quad b := \omega(0,1)\,, \quad c := \omega(1,1)\,, \quad \alpha_1 := f(a,a)\,, \quad \alpha_2 := f(b,b)\,,$$

$\beta_1, \beta_2 \in Un\, E^c$ such that $\alpha_1\beta_1^2 + \alpha_2\beta_2^2 = 0$,

$$\gamma := \frac{1}{2}(\alpha_1^*\beta_1^*\beta_2 - \alpha_2^*\beta_1\beta_2^*) = \alpha_1^*\beta_1^*\beta_2 = -\alpha_2^*\beta_1\beta_2^*\,,$$

$$X := \frac{1}{2}((\beta_1 \widetilde{\otimes} 1_K)V_a^f + (\beta_2 \widetilde{\otimes} 1_K)V_b^f)\,, \qquad P_+ := X^*X\,, \quad P_- := XX^*\,.$$

*We assume $f(s,c) = f(c,s)$ and $cs = sc$ for every $s \in S$, and $f(a,b) = -f(b,a) = 1_E$. Moreover, we consider $\mathcal{S}(g)$ as an E-C**-subalgebra of $\mathcal{S}(f)$* (Corollary 2.1.17 e)).

a) *We have*

$$f(a,c) = -f(c,a) = \alpha_1\,, \quad f(b,c) = -f(c,b) = -\alpha_2\,, \quad f(c,c) = -\alpha_1\alpha_2\,,$$

$$\gamma^2 = -\alpha_1^*\alpha_2^*\,, \qquad V_c^f \in \mathcal{S}(g)^c\,.$$

b) *We have*

$$P_{\pm}=\frac{1}{2}(V_1^f\pm(\gamma\widetilde{\otimes}1_K)V_c^f)\in\mathcal{S}(g)^c\cap Pr\,\mathcal{S}(f),\ P_++P_-=V_1^f,\ P_+P_-=0\,,$$

$$X^2=0,\ XP_+=X,\ P_-X=X,\ P_+X=XP_-=0,\ X+X^*\in Un\,\mathcal{S}(f)\,,$$
$$Y\in\mathcal{S}(g)\Longrightarrow XYX=0\,.$$

c) *The map*

$$E\longrightarrow P_\pm\mathcal{S}(f)P_\pm,\quad x\longmapsto(x\widetilde{\otimes}1_K)P_\pm$$

is a unital injective C^{**}*-homomorphism; we shall consider* $P_\pm\mathcal{S}(f)P_\pm$ *as an* E-C^{**}*-algebra using this map.*

d) *The maps*

$$\varphi_+:\mathcal{S}(g)\longrightarrow P_+\mathcal{S}(f)P_+\,,\quad Y\longmapsto P_+YP_+\,,$$
$$\varphi_-:\mathcal{S}(g)\longrightarrow P_-\mathcal{S}(f)P_-\,,\quad Y\longmapsto XYX^*$$

are orthogonal injective E-C^{**}*-homomorphisms and* $\varphi_++\varphi_-$ *is an injective* E-C^**-homomorphism. If* $Y_1,Y_2\in Un\,\mathcal{S}(g)$ *(resp.* $Y_1,Y_2\in Pr\,\mathcal{S}(g)$*) then* $\varphi_+Y_1+\varphi_-Y_2\in Un\,\mathcal{S}(f)$ *(resp.* $\varphi_+Y_1+\varphi_-Y_2\in Pr\,\mathcal{S}(f)$*). Moreover, the map*

$$\psi:\mathcal{S}(f)\longrightarrow\mathcal{S}(f)\,,\quad Z\longmapsto(X+X^*)Z(X+X^*)$$

is an E-C^{**}*-isomorphism such that*

$$\psi^{-1}=\psi\,,\qquad\psi(P_+\mathcal{S}(f)P_+)=P_-\mathcal{S}(f)P_-\,,\qquad\psi\circ\varphi_+=\varphi_-\,.$$

If $\mathbb{K}=\mathbb{C}$ *then* $X+X^*$ *is homotopic to* V_1^f *in* $Un\,\mathcal{S}(f)$ *and* ψ *is homotopic to the identity map of* $\mathcal{S}(f)$*. Using this homotopy we find that* φ_+Y *is homotopic in the above sense to* φ_-Y *for every* $Y\in\mathcal{S}(g)$ *and* $\varphi_+Y_1+\varphi_-Y_2$, $\varphi_-Y_1+\varphi_+Y_2$, $\varphi_+(Y_1Y_2)+P_-$*, and* $\varphi_+(Y_2Y_1+P_-$ *are homotopic in the above sense for all* $Y_1,Y_2\in\mathcal{S}(g)$.

e) *Let* $s\in S$ *such that* $sa=as$*. Then*

$$sb=bs\,,\qquad f(sc,c)f(s,c)=-\alpha_1\alpha_2\,,$$
$$f(sa,c)f(c,sa)^*=-1_E\,,\qquad f(a,s)f(s,a)^*=f(b,s)f(s,b)^*\,.$$

f) *If* $sa=as$ *for every* $s\in S$ *then the map*

$$S\times(\mathbb{Z}_2\times\mathbb{Z}_2)\longrightarrow T,\quad(s,r)\longmapsto s(\omega r)$$

is an injective group homomorphism.

g) *If T is generated by $S \cup \omega(\mathbb{Z}_2 \times \mathbb{Z}_2)$ and $sa = as$ for every $s \in S$ then φ_+ and ψ_- are E-C^*-isomorphisms with inverse*

$$P_\pm \mathcal{S}(f) P_\pm \longrightarrow \mathcal{S}(g), \quad Z \longmapsto 2\sum_{s\in S}(Z_s \widetilde{\otimes} 1_L) V_s^g \,,$$

where

$$\psi_- : \mathcal{S}(g) \longrightarrow P_- \mathcal{S}(f) P_- \,, \quad Y \longmapsto P_- Y P_- \,.$$

h) *If $sa = as$ and $f(a,s) = f(s,a)$ for every $s \in S$ then $X \in \mathcal{S}(g)^c$, $\varphi_- Y = P_- Y$ for every $Y \in \mathcal{S}(g)$, and there is a unique $\mathcal{S}(g)$-C^{**}-homomorphism $\phi : \mathcal{S}(g)_{2,2} \to \mathcal{S}(f)$ such that*

$$\phi \begin{bmatrix} 0 & 0 \\ (\alpha_1 \beta_1^2) \otimes 1_L & 0 \end{bmatrix} = X \,.$$

ϕ is injective and

$$\phi \begin{bmatrix} V_1^g & 0 \\ 0 & 0 \end{bmatrix} = P_+ \,, \qquad \phi \begin{bmatrix} 0 & 0 \\ 0 & V_1^g \end{bmatrix} = P_- \,.$$

i) *If $sa = as$ and $f(a,s) = f(s,a)$ for all $s \in S$ and if T is generated by $S \cup \omega(\mathbb{Z}_2 \times \mathbb{Z}_2)$ then ϕ is an $\mathcal{S}(g)$-C^*-isomorphism and*

$$\phi^{-1} V_1^f = \begin{bmatrix} 1_E \otimes 1_L & 0 \\ 0 & 1_E \otimes 1_L \end{bmatrix}, \; \phi^{-1} V_c^f = \begin{bmatrix} \gamma^* \otimes 1_L & 0 \\ 0 & -\gamma^* \otimes 1_L \end{bmatrix},$$

$$\phi^{-1} V_a^f = \begin{bmatrix} 0 & -\beta_1^* \otimes 1_L \\ (\beta_2 \gamma^*) \otimes 1_L & 0 \end{bmatrix},$$

$$\phi^{-1} V_b^f = \begin{bmatrix} 0 & -\beta_2^* \otimes 1_L \\ (\beta_1 \gamma^*) \otimes 1_L & 0 \end{bmatrix},$$

$$\phi^{-1} P_+ = \begin{bmatrix} V_1^g & 0 \\ 0 & 0 \end{bmatrix}, \qquad \phi^{-1} P_- = \begin{bmatrix} 0 & 0 \\ 0 & V_1^g \end{bmatrix},$$

and for every $s \in S$

$$\phi^{-1} V_s^f = \begin{bmatrix} V_s^g & 0 \\ 0 & V_s^g \end{bmatrix}.$$

j) *The above results still hold for an arbitrary subgroup S of T if we replace $\mathcal{S}$ with $\mathcal{S}_{\|\cdot\|}$.*

Proof. a) By the equation of the Schur functions,

$f(a,a) = f(a,c)f(a,b)$, $f(a,b)f(c,a) = f(a,c)f(b,a)$, $f(a,b)f(c,b) = f(b,b)$, $f(b,a)f(c,b) = f(b,c)f(a,b)$, $f(a,b)f(c,c) = f(a,a)f(b,c)$, and so

$$\alpha_1 = f(a,c), \quad f(c,a) = -f(a,c) = -\alpha_1, \quad f(c,b) = \alpha_2,$$
$$-\alpha_2 = -f(c,b) = f(b,c), \quad f(c,c) = \alpha_1 f(b,c) = -\alpha_1\alpha_2.$$

For $s \in S$, by Proposition 2.1.2 b),

$$V_c^f V_s^f = (f(c,s)\widetilde{\otimes}1_K)V_{cs}^f = (f(s,c)\widetilde{\otimes}1_K)V_{sc}^f = V_s^f V_c^f$$

and so $V_c^f \in \mathcal{S}(g)^c$ (by Proposition 2.1.2 d)).

b) By Proposition 2.1.2 b),d),e) (and Corollary 2.1.22 c)),

$$X^* = \frac{1}{2}(((\alpha_1^*\beta_1^*)\widetilde{\otimes}1_K)V_a^f + ((\alpha_2^*\beta_2^*)\widetilde{\otimes}1_K)V_b^f),$$

$$P_+ = \frac{1}{4}(2V_1^f + ((\alpha_1^*\beta_1^*\beta_2)\widetilde{\otimes}1_K)V_c^f - ((\alpha_2^*\beta_2^*\beta_1)\widetilde{\otimes}1_K)V_c^f) = \frac{1}{2}(V_1^f + (\gamma\widetilde{\otimes}1_K)V_c^f),$$
$$P_- = \frac{1}{4}(2V_1^f + ((\beta_1\alpha_2^*\beta_2^*)\widetilde{\otimes}1_K)V_c^f - ((\beta_2\alpha_1^*\beta_1^*)\widetilde{\otimes}1_K)V_c^f) = \frac{1}{2}(V_1^f - (\gamma\widetilde{\otimes}1_K)V_c^f).$$

By a),

$$P_\pm^* = \frac{1}{2}(V_1^f \pm (\gamma^*\widetilde{\otimes}1_K)((-\alpha_1^*\alpha_2^*)\widetilde{\otimes}1_K)V_c^f) = P_\pm,$$
$$P_\pm^2 = \frac{1}{4}(V_1^f \pm 2(\gamma\widetilde{\otimes}1_K)V_c^f + (\gamma^2\widetilde{\otimes}1_K)((-\alpha_1\alpha_2)\widetilde{\otimes}1_K)V_1^f) =$$
$$= \frac{1}{2}(V_1^f \pm (\gamma\widetilde{\otimes}1_K)V_c^f) = P_\pm,$$

so, by a) again, $P_\pm \in \mathcal{S}(g)^c \cap Pr\,\mathcal{S}(f)$. By Proposition 2.1.2 b),d),

$$X^2 = \frac{1}{4}(((\beta_1^2\alpha_1 + \beta_2^2\alpha_2)\widetilde{\otimes}1_K)V_1^f + ((\beta_1\beta_2)\widetilde{\otimes}1_K)(V_a^fV_b^f + V_b^fV_a^f)) = 0,$$

$$(X + X^*)^2 = X^2 + XX^* + X^*X + X^{*2} = P_+ + P_- = V_1^f.$$

For the last relation we remark that by the above,

$$XYX = X(P_+ + P_-)YX = XP_+YX = XYP_+X = 0.$$

c) follows from b) and Lemma 1.3.2.

d) By b) and c), the map $\varphi_\pm$ is an E-C**-homomorphism. Let $Y \in \mathcal{S}(g)$ with $\varphi_\pm Y = 0$. By b), $Y = \mp Y(\gamma\widetilde{\otimes}1_K)V_c^f$ so by Proposition 2.1.2 b),d) and Theorem 2.1.9 b),

$$\sum_{s\in S}(Y_s\widetilde{\otimes}1_K)V_s^f = \mp Y(\gamma\widetilde{\otimes}1_K)V_c^f = \mp\sum_{s\in S}((Y_s\gamma f(s,c))\widetilde{\otimes}1_K)V_{sc}^f,$$

which implies $Y_s = 0$ for every $s \in S$ (Theorem 2.1.9 a)). Thus $\varphi_\pm$ is injective. It follows that $\varphi_+ + \varphi_-$ is also injective.

Assume first $Y_1, Y_2 \in Un\,\mathcal{S}(g)$. By b),

$$(\varphi_+Y_1 + \varphi_-Y_2)^*(\varphi_+Y_1 + \varphi_-Y_2) = (\varphi_+Y_1^* + \varphi_-Y_2^*)(\varphi_+Y_1 + \varphi_-Y_2)$$

$$= \varphi_+(Y_1^*Y_1) + \varphi_-(Y_2^*Y_2) = P_+ + P_- = V_1^f .$$

Similarly $(\varphi_+Y_1 + \varphi_-Y_2)(\varphi_+Y_1 + \varphi_-Y_2)^* = V_1^f$. The case $Y_1, Y_2 \in Pr\,\mathcal{S}(g)$ is easy to see.

By b), ψ is an E-C**-isomorphism with

$$\psi^{-1} = \psi\,, \qquad \psi P_+ = (X+X^*)X^*X(X+X^*) = XX^*XX^* = P_-\ .$$

Moreover for $Y \in \mathcal{S}(g)$,

$$\psi\varphi_+Y = (X+X^*)P_+YP_+(X+X^*) = XYX^* = \varphi_-Y\ .$$

Assume now $\mathbb{K} = \mathbb{C}$. By b), $X + X^* \in Un\,\mathcal{S}(f)$. Being selfadjoint its spectrum is contained in $\{-1, +1\}$ and so it is homotopic to V_1^f in $Un\,\mathcal{S}(f)$.

e) We have $sb = sac = asc = acs = bs$. By a),

$$\begin{aligned}
&f(s,c)f(sc,c) = f(s,1)f(c,c) = -\alpha_1\alpha_2,\\
&f(s,a)f(sa,c) = f(s,b)f(a,c) = \alpha_1 f(s,b),\\
&f(c,as)f(a,s) = f(c,a)f(b,s) = -\alpha_1 f(b,s),\\
&f(c,bs)f(b,s) = f(c,b)f(a,s) = \alpha_2 f(a,s),\\
&f(s,c)f(sc,b) = f(s,a)f(c,b) = \alpha_2 f(s,a),\\
&f(c,s)f(cs,b) = f(c,sb)f(s,b)
\end{aligned}$$

so

$$\begin{aligned}
f(sa,c)f(c,as)^* &= -f(s,b)f(s,a)^*f(b,s)^*f(a,s)\\
&= -f(c,s)f(cs,b)f(c,sb)^*\alpha_2 f(s,c)^*f(sc,b)^*\alpha_2^* f(c,bs) = -1_E.
\end{aligned}$$

From

$$f(s,c)f(sc,a) = f(s,b)f(c,a),\ \ f(c,a)f(b,s) = f(c,as)f(a,s)\,,$$
$$f(c,s)f(cs,a) = f(c,sa)f(s,a)$$

we get

$$f(a,s)f(s,a)^* = f(b,s)f(s,b)^*.$$

f) Since S and $\omega(\mathbb{Z}_2 \times \mathbb{Z}_2)$ commute, the map is a group homomorphism. If $s(\omega r) = 1$ for $(s,r) \in S \times (\mathbb{Z}_2 \times \mathbb{Z}_2)$ then $\omega r = s^{-1} \in S \cap \omega(\mathbb{Z}_2 \times \mathbb{Z}_2)$, which implies $s = 1$ and $r = (0,0)$. Thus this group homomorphism is injective.

g) By e) and the hypothesis of f), for every $t \in T$ there are uniquely $s \in S$ and $d \in \{1,a,b,c\}$ with $t = sd$. Let $Z \in P_\pm\mathcal{S}(f)P_\pm$. By b) and Theorem 2.1.9 b) (and Corollary 1.3.7 d)),

$$Z = \pm(\gamma\widetilde{\otimes}1_K)ZV_c^f = \pm(\gamma\widetilde{\otimes}1_K)V_c^f Z$$

By Proposition 2.1.2 b),

$$ZV_c^f = \sum_{s\in S}((Z_s f(s,c))\widetilde{\otimes}1_K)V_{sc}^f + \sum_{s\in S}((Z_{sa}f(sa,c))\widetilde{\otimes}1_K)V_{sb}^f$$

$$+\sum_{s\in S}((Z_{sb}f(sb,c))\widetilde{\otimes}1_K)V^f_{sa}+\sum_{s\in S}((Z_{sc}f(sc,c))\widetilde{\otimes}1_K)V^f_s,$$

$$V^f_c Z=\sum_{s\in S}((f(c,s)Z_s)\widetilde{\otimes}1_K)V^f_{sc}+\sum_{s\in S}((f(c,sa)Z_{sa})\widetilde{\otimes}1_K)V^f_{sb}$$

$$+\sum_{s\in S}((f(c,sb)Z_{sb})\widetilde{\otimes}1_K)V^f_{sa}+\sum_{s\in S}((f(c,sc)Z_{sc})\widetilde{\otimes}1_K)V^f_s$$

and so by Theorem 2.1.9 a),

$$\begin{aligned}
Z_s &= \pm\gamma f(sc,c)Z_{sc}=\pm\gamma f(c,sc)Z_{sc},\\
Z_{sc} &= \pm\gamma f(s,c)Z_s=\pm\gamma f(c,s)Z_s,\\
Z_{sa} &= \pm\gamma f(sb,c)Z_{sb}=\pm\gamma f(c,sb)Z_{sb},\\
Z_{sb} &= \pm\gamma f(sa,c)Z_{sa}=\pm\gamma f(c,sa)Z_{sa}.
\end{aligned}$$

By e), $Z_{sa}=Z_{sb}=0$ for every $s\in S$. We get (by a), d), and Proposition 2.1.2 b))

$$\varphi_\pm(2\sum_{s\in S}(Z_s\widetilde{\otimes}1_L)V^g_s)=\sum_{s\in S}(Z_s\widetilde{\otimes}1_K)V^f_s\pm(\gamma\widetilde{\otimes}1_K)V^f_c\sum_{s\in S}(Z_s\widetilde{\otimes}1_K)V^f_s=$$

$$=\sum_{s\in S}(Z_s\widetilde{\otimes}1_K)V^f_s\pm\sum_{s\in S}((\gamma f(c,s)Z_s)\widetilde{\otimes}1_K)V^f_{sc}=$$

$$=\sum_{s\in S}(Z_s\widetilde{\otimes}1_K)V^f_s+\sum_{s\in S}(Z_{sc}\widetilde{\otimes}1_K)V^f_{sc}=Z\,.$$

Thus $\varphi_\pm$ is an E-C*-isomorphism with the mentioned inverse.

h) is a long calculation using e).

i) follows from h).

j) follows from Corollary 2.1.17 d). □

Remark. An example in which the above hypotheses are fulfilled is given in Theorem 4.1.7.

2.3. **The functor $\boldsymbol{\mathcal{S}}$**

Throughout this subsection, we assume T finite.

In this subsection, we present the construction in the frame of category theory. Some of the results still hold for T locally finite.

Definition 2.3.1. The above construction of $\mathcal{S}(f)$ can be done for an arbitrary E-module F, in which case we shall denote the result by $\mathcal{S}(F)$. Moreover, we shall write V^F_t instead of V^f_t in this case.

If F is an E-module then $\mathcal{S}(F)$ is canonically an E-module. If, in addition, F is adapted then $\mathcal{S}(F)$ is adapted and isomorphic to $\mathcal{S}(\check{F}, F)$. If F is an E-C*-algebra then $\mathcal{S}(F)$ is also an E-C*-algebra.

Proposition 2.3.2. *If F, G are E-modules and $\varphi : F \to G$ is an E-linear C*-homomorphism then the map*

$$\mathcal{S}(\varphi) : \mathcal{S}(F) \longrightarrow \mathcal{S}(G)\,, \quad X \longmapsto \sum_{t \in S} ((\varphi X_t) \otimes 1_K) V_t^G$$

is an E-linear C-homomorphism, injective or surjective if φ is so.*

Proof. The assertion follows from Theorem 2.1.9 a),c),g). □

Corollary 2.3.3. *Let F_1, F_2, F_3 be E-modules and let $\varphi : F_1 \to F_2$, $\psi : F_2 \to F_3$ be E-linear C*-homomorphisms.*

a) $\mathcal{S}(\psi) \circ \mathcal{S}(\varphi) = \mathcal{S}(\psi \circ \varphi)$.

b) *If the sequence*

$$0 \longrightarrow F_1 \xrightarrow{\varphi} F_2 \xrightarrow{\psi} F_3$$

is exact then the sequence

$$0 \longrightarrow \mathcal{S}(F_1) \xrightarrow{\mathcal{S}(\varphi)} \mathcal{S}(F_2) \xrightarrow{\mathcal{S}(\psi)} \mathcal{S}(F_3)$$

is also exact.

c) *The covariant functor $\mathcal{S} : \mathfrak{M}_E \to \mathfrak{M}_E$ is exact.*

Proof. a) is obvious.

b) Let $Y \in Ker\,\mathcal{S}(\psi)$. For every $t \in T$, $Y_t \in Ker\,\psi = Im\,\varphi$. If we identify F_1 with $Im\,\varphi$ then $Y_t \in F_1$. It follows $Y \in Im\,\mathcal{S}(\varphi)$, $Ker\,\mathcal{S}(\psi) = Im\,\mathcal{S}(\varphi)$.

c) follows from b) and Proposition 2.3.2. □

Corollary 2.3.4. *Let F be an adapted E-module and put*

$$\iota : F \longrightarrow \check{F}\,, \quad x \longmapsto (0, x),$$

$$\pi : \check{F} \longrightarrow E\,, \quad (\alpha, x) \longmapsto \alpha,$$

$$\lambda : E \longrightarrow \check{F}\,, \quad \alpha \longmapsto (\alpha, 0).$$

Then the sequence

$$0 \longrightarrow \mathcal{S}(F) \xrightarrow{\mathcal{S}(\iota)} \mathcal{S}(\check{F}) \underset{\underset{\mathcal{S}(\lambda)}{\longleftarrow}}{\overset{\mathcal{S}(\pi)}{\longrightarrow}} \mathcal{S}(E) \longrightarrow 0$$

is split exact.

PROPOSITION 2.3.5. *The covariant functor* $\mathcal{S} : \mathfrak{M}_E \to \mathfrak{M}_E$ *(resp.* $\mathcal{S} : \mathfrak{C}^1_E \to \mathfrak{C}^1_E$*)* (Proposition 2.3.2, Corollary 2.3.3 a)) *is continuous with respect to the inductive limits* (Proposition 1.2.9 a),b)).

Proof. Let $\{(F_i)_{i\in I},\ (\varphi_{ij})_{i,j\in I}\}$ be an inductive system in the category $\mathfrak{M}_E$ (resp. $\mathfrak{C}^1_E$) and let $\{F,\ (\varphi_i)_{i\in I}\}$ be its limit in the category $\mathfrak{M}_E$ (resp. $\mathfrak{C}^1_E$). Then $\{(\mathcal{S}(F_i))_{i\in I},\ (\mathcal{S}(\varphi_{ij})_{i,j\in I})\}$ is an inductive system in the category $\mathfrak{M}_E$ (resp. $\mathfrak{C}^1_E$). Let $\{G,\ (\psi_i)_{i\in I}\}$ be its limit in this category and let $\psi : G \to \mathcal{S}(F)$ be the E-linear C*-homomorphism such that $\psi\circ\psi_i = \mathcal{S}(\varphi_i)$ for every $i\in I$. In the $\mathfrak{C}^1_E$ case, for $\alpha\in E$ and $i\in I$,

$$\psi(\alpha\otimes 1_K) = \psi\circ\psi_i(\alpha\otimes 1_K) = (\mathcal{S}(\varphi_i))(\alpha\otimes 1_K) = \alpha\otimes 1_K$$

so that ψ is an E-C*-homomorphism.

Let $i\in I$ and let $X\in Ker\,\mathcal{S}(\varphi_i)$. Then $\varphi_i X_t = 0$ for every $t\in T$. Since T is finite, for every $\varepsilon>0$ there is a $j\in I$, $j\geq i$, with

$$\|\varphi_{ji}X_t\| < \frac{\varepsilon}{\text{Card T}}$$

for every $t\in T$. Then

$$\|(\mathcal{S}(\varphi_{ji}))X\| = \left\|\sum_{t\in T}((\varphi_{ji}X_t)\otimes 1_K)V_t^{F_j}\right\| < \varepsilon\ .$$

It follows

$$\|\psi_i X\| = \inf_{j\in I, j\geq i}\|(\mathcal{S}(\varphi_{ji}))X\| = 0,$$

$$\psi_i X = 0,\quad X\in Ker\,\psi_i,\quad Ker\,\mathcal{S}(\varphi_i)\subset Ker\,\psi_i.$$

By Lemma 1.2.8, ψ is injective. Since

$$\bigcup_{i\in I} Im\,\mathcal{S}(\varphi_i)\subset Im\,\psi\,,$$

$Im\,\psi$ is dense in $\mathcal{S}(F)$. Thus ψ is surjective and so an E-C*-isomorphism. □

PROPOSITION 2.3.6. *Let* $\theta : F\to G$ *be a surjective morphism in the category* $\mathfrak{C}^1_E$*. We use the notation of* Theorem 2.2.18 *and mark with an exponent if this notation is used with respect to* F *or to* G*. For every* $Y\in Un\ \mathcal{S}(g^G)$*, there is a* $Z\in\mathcal{S}(g^F)$ *such that*

$$Z^*Z = P^F_+,\quad \mathcal{S}(\theta)Z = \varphi^G_+Y.$$

Proof. By Proposition 2.3.2 c), $\mathcal{S}(\theta)$ is surjective and so there is a $Z_0\in\mathcal{S}(g^F)$ with $\|Z_0\| = 1$ and $\mathcal{S}(\theta)Z_0 = Y$. Put

$$Z := P^F_+Z_0 + X^F(1 - Z_0^*Z_0)^{\frac{1}{2}}.$$

By Theorem 2.2.18 b),

$$Z^*Z = P_+^F Z_0^* Z_0 + (1 - Z_0^* Z_0)^{\frac{1}{2}} (X^F)^* X^F (1 - Z_0^* Z_0)^{\frac{1}{2}}$$
$$= P_+^F Z_0^* Z_0 + P_+^F (1 - Z_0^* Z_0) = P_+^F .$$

Since

$$\mathcal{S}(\theta)(1 - Z_0^* Z_0) = 1 - Y^* Y = 0$$

we get

$$\mathcal{S}(\theta)(1 - Z_0^* Z_0)^{\frac{1}{2}} = 0, \quad \mathcal{S}(\theta) Z = P_+^G Y = \varphi_+^G Y.$$

□

PROPOSITION 2.3.7. *Let F be an adapted E-module and Ω a locally compact space. We define for $X \in \mathcal{S}(\mathcal{C}_0(\Omega, F))$* (see Corollary 1.2.5 d)) *and $Y \in \mathcal{C}_0(\Omega, \mathcal{S}(F))$,*

$$\varphi X : \Omega \longrightarrow \mathcal{S}(F), \quad \omega \longmapsto \sum_{t \in T} (X_t(\omega) \otimes 1_K) V_t^F ,$$

$$\psi Y := \sum_{t \in T} (Y(\cdot)_t \otimes 1_K) V_t^{\mathcal{C}_0(\Omega, F)} .$$

Then

$$\varphi : \mathcal{S}(\mathcal{C}_0(\Omega, F)) \longrightarrow \mathcal{C}_0(\Omega, \mathcal{S}(F)),$$
$$\psi : \mathcal{C}_0(\Omega, \mathcal{S}(F)) \longrightarrow \mathcal{S}(\mathcal{C}_0(\Omega, F))$$

are E-linear C^*-isomorphisms and $\varphi = \psi^{-1}$.

Let $\omega_0 \in \Omega$ and assume F is an E-C^*-algebra. Then the above maps φ and ψ induce the following E-C^*-isomorphisms

$$\mathcal{S}(\{\, X \in \mathcal{C}_0(\Omega, F) \mid X(\omega_0) \in E \,\}) \underset{\longleftarrow}{\overset{\longrightarrow}{}} \{\, Y \in \mathcal{C}_0(\Omega, \mathcal{S}(F)) \mid Y(\omega_0) \in \mathcal{S}(E) \,\} .$$

Proof. Let $X, X' \in \mathcal{S}(\mathcal{C}_0(\Omega, F))$ and $Y, Y' \in \mathcal{C}_0(\Omega, \mathcal{S}(F))$. By Proposition 2.1.23 b) and Corollary 2.1.10 a),

$$\varphi X \in \mathcal{C}_0(\Omega, \mathcal{S}(F)) , \qquad \psi Y \in \mathcal{S}(\mathcal{C}_0(\Omega, F))$$

and it is easy to see that φ and ψ are E-linear. By Theorem 2.1.9 c),g), for $t \in T$ and $\omega \in \Omega$,

$$((\varphi X)^*(\omega))_t = \tilde{f}(t)(((\varphi X)(\omega)_{t^{-1}}))^*$$
$$= \tilde{f}(t) X_{t^{-1}}(\omega)^* = (X^*(\omega))_t = ((\varphi X^*)(\omega))_t ,$$

$$(((\varphi X)(\varphi X'))(\omega))_t = \sum_{s \in T} f(s, s^{-1}t)((\varphi X)(\omega))_s ((\varphi X')(\omega))_{s^{-1}t}$$

$$= \sum_{s \in T} f(s, s^{-1}t) X_s(\omega) X'_{s^{-1}t}(\omega) = \left(\sum_{s \in T} f(s, s^{-1}t) X_s X'_{s^{-1}t} \right)(\omega)$$

$$= (XX')_t(\omega) = ((\varphi(XX'))(\omega))_t,$$

so

$$(\varphi X)^* = \varphi X^*, \qquad (\varphi X)(\varphi X') = \varphi(XX')$$

and φ is a C*-homomorphism. Similarly

$$(\psi Y^*)_t(\omega) = (Y^*(\omega))_t = \tilde{f}(t)(Y(\omega)_{t^{-1}})^* = \tilde{f}(t)((\psi Y)_{t^{-1}}(\omega))^* = ((\psi Y)^*)_t(\omega),$$

$$((\psi Y)(\psi Y'))_t(\omega) = \left(\sum_{s\in T} f(s,s^{-1}t)(\psi Y)_s(\psi Y')_{s^{-1}t}\right)(\omega)$$
$$= \sum_{s\in T} f(s,s^{-1}t)(\psi Y)_s(\omega)(\psi Y')_{s^{-1}t}(\omega) = \sum_{s\in T} f(s,s^{-1}t)Y(\omega)_s Y'(\omega)_{s^{-1}t}$$
$$= (Y(\omega)Y'(\omega))_t = (\psi(YY')_t)(\omega)$$

so

$$\psi Y^* = (\psi Y)^*, \qquad (\psi Y)(\psi Y') = \psi(YY')$$

and ψ is a C*-homomorphism. Moreover

$$(\psi\varphi X)_t(\omega) = ((\varphi X)(\omega))_t = X_t(\omega), \quad ((\varphi\psi Y)(\omega))_t = (\psi Y)_t(\omega) = (Y(\omega))_t,$$

so $\psi\varphi X = X$ and $\varphi\psi Y = Y$ which proves the assertion.

The last assertion is easy to see. $\square$

PROPOSITION 2.3.8. *Let F be an adapted E-module,*

$$0 \longrightarrow F \overset{\iota}{\longrightarrow} \check{F} \overset{\pi}{\longrightarrow} E \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{S}(F) \overset{\iota_0}{\longrightarrow} \overbrace{\mathcal{S}(F)}^{\vee} \overset{\pi_0}{\longrightarrow} E \longrightarrow 0$$

the associated exact sequences (Proposition 1.2.4 h)), *and*

$$j : E \longrightarrow \mathcal{S}(E), \quad \alpha \longmapsto (\alpha\otimes 1_K)V_1^E,$$

$$\varphi : \overbrace{\mathcal{S}(F)}^{\vee} \longrightarrow \mathcal{S}(\check{F}), \quad (\alpha, X) \longmapsto \mathcal{S}(\iota)X + (\alpha\otimes 1_K)V_1^{\check{F}}.$$

Then φ is an injective E-C^*-homomorphism and $\mathcal{S}(\pi)\circ\varphi = j\circ\pi_0$.

PROPOSITION 2.3.9. *If E is commutative and F is an E-module then the map*

$$\varphi : \mathcal{S}(E)\otimes F \longrightarrow \mathcal{S}(F), \quad X\otimes x \longmapsto \sum_{t\in T}((X_t x)\otimes 1_K)V_t^F$$

is a surjective C^*-homomorphism. If in addition $E = \mathbb{K}$ then φ is a C^*-isomorphism with inverse

$$\psi : \mathcal{S}(F) \longrightarrow \mathcal{S}(E)\otimes F, \quad Y \longmapsto \sum_{t\in T}(V_t^E\otimes Y_t).$$

Proof. It is obvious that φ is surjective. For $X, Y \in \mathcal{S}(E)$ and $x, y \in F$, by Theorem 2.1.9 c),g) and Proposition 2.1.2 b),d),e),

$$\begin{aligned}\varphi((X\otimes x)^*) &= \varphi(X^*\otimes x^*) = \sum_{t\in T}(((X^*)_t x^*)\otimes 1_K)V_t^F \\ &= \sum_{t\in T}((\tilde{f}(t)(X_{t^{-1}})^* x^*)\otimes 1_K)V_t^F = \sum_{t\in T}(((X_{t^{-1}})^* x^*)\otimes 1_K)(V_{t^{-1}}^F)^* \\ &= \sum_{t\in T}((x^*(X_t)^*)\otimes 1_K)(V_t^F)^* = (\varphi(X\otimes x))^*,\end{aligned}$$

$$\begin{aligned}\varphi(X\otimes x)\varphi(Y\otimes y) &= \sum_{s,t\in T}((X_s x Y_t y)\otimes 1_K)V_s^F V_t^F \\ &= \sum_{s,t\in T}((f(s,t)X_s x Y_t y)\otimes 1_K)V_{st}^F \\ &= \sum_{r\in T}\sum_{s\in T}((f(s,s^{-1}r)X_s Y_{s^{-1}r} xy)\otimes 1_K)V_r^F \\ &= \sum_{r\in T}(((XY)_r xy)\otimes 1_K)V_r^F = \varphi((X\otimes x)(Y\otimes y))\end{aligned}$$

so φ is a C*-homomorphism.

Assume now $E = \mathbb{K}$ and let $X \in \mathcal{S}(E)$ and $x \in F$. Then

$$\begin{aligned}\psi\varphi(X\otimes x) &= \psi\sum_{t\in T}((X_t x)\otimes 1_K)V_t^F = \sum_{t\in T}V_t^E\otimes(X_t x) = \\ &= \left(\sum_{t\in T}X_t V_t^E\right)\otimes x = X\otimes x\end{aligned}$$

which proves the last assertion (by using the first assertion). □

3. EXAMPLES

We draw the reader's attention to the fact that in additive groups the neutral element is denoted by 0 and not by 1.

3.1. $T := \mathbb{Z}_2$

PROPOSITION 3.1.1. a) *The map*

$$\psi : \mathcal{F}(\mathbb{Z}_2, E) \longrightarrow Un\, E^c, \quad f \longmapsto f(1,1)$$

is a group isomorphism.

b) $\psi(\{\, \delta\lambda \mid \lambda \in \Lambda(\mathbb{Z}_2, E)\,\}) = \left\{\, x^2 \;\middle|\; x \in Un\ E^c \,\right\}$.

c) *If there is an* $x \in E^c$ *with* $x^2 = f(1,1)$ *(in which case* $x \in Un\ E^c$*) then the map*

$$\varphi : \mathcal{S}(f) \longrightarrow E \times E\,, \quad X \longmapsto (X_0 + xX_1, X_0 - xX_1)$$

is an E*-*C^**-isomorphism.*

d) *If* $\mathbb{K} = \mathbb{C}$ *and if* A *is a connected and simply connected compact space or a totally disconnected compact space then for every* $x \in Un\,\mathcal{C}(A)$ *there is a* $y \in \mathcal{C}(A, \mathbb{C})$ *with* $x = e^y$.

e) *Assume* $\mathbb{K} = \mathbb{R}$.

e$_1$) *There are uniquely* $p, q \in \Pr\ E^c$ *with*

$$p + q = 1_E, \qquad pf(1,1) = p, \qquad qf(1,1) = -q\,.$$

e$_2$) *The map*

$$\varphi : \mathcal{S}(f) \longrightarrow (pE) \times (pE) \times \overset{\circ}{\overbrace{qE}}\,, \quad X \longmapsto \tilde{X}\,,$$

where $\overset{\circ}{\overbrace{qE}}$ *denotes the complexification of the* C^**-algebra* qE *and*

$$\tilde{X} := (p(X_0 + X_1), p(X_0 - X_1), (qX_0, qX_1))$$

for every $X \in \mathcal{S}(f)$, *is an* E*-*C^**-isomorphism. In particular, if* $f(1,1) = -1_E$ *then* $\mathcal{S}(f)$ *is isomorphic to the complexification of* E.

f) *Assume* $\mathbb{K} = \mathbb{C}$, *let* $\sigma(E^c)$ *be the spectrum of* E^c, *and let* $\widehat{f_{11}}$ *be the function of* $\mathcal{C}(\sigma(E^c), \mathbb{C})$ *corresponding to* f_{11} *by the Gelfand transform. Then*

$$\left\{\, e^{i\theta} \;\middle|\; \theta \in \mathbb{R},\ e^{2i\theta} \in \widehat{f_{11}}(\sigma(E^c)) \,\right\}$$

is the spectrum of V_1.

Proof. a) follows from Proposition 1.1.2 a) (and Proposition 1.1.4 a)).
b) follows from Definition 1.1.3.
c) For $X, Y \in \mathcal{S}(f)$, by Theorem 2.1.9 c),g) (and Proposition 1.1.2 a)),

$$(X^*)_0 = (X_0)^*\,, \quad (X^*)_1 = (x^*)^2 (X_1)^*,$$

$$(XY)_0 = X_0Y_0 + x^2X_1Y_1, \quad (XY)_1 = X_0Y_1 + X_1Y_0,$$

so

$$\varphi(X^*) = ((X_0)^* + x(x^*)^2(X_1)^*, (X_0)^* - x(x^*)^2(X_1)^*)$$

$$= ((X_0)^* + x^*(X_1)^* \, , (X_0)^* - x^*(X_1)^*) = (\varphi X)^*,$$

$$\begin{aligned}(\varphi X)(\varphi Y) &= ((X_0 + xX_1)(Y_0 + xY_1), (X_0 - xX_1)(Y_0 - xY_1)) \\ &= (X_0Y_0 + xX_0Y_1 + xX_1Y_0 + x^2X_1Y_1, X_0Y_0 - xX_0Y_1 - xX_1Y_0 + x^2X_1Y_1) \\ &= ((XY)_0 + x(XY)_1, (XY)_0 - x(XY)_1 = \varphi(XY)\end{aligned}$$

i.e. φ is an E-C*-homomorphism. φ is obviously injective.

Let $(y, z) \in E \times E$. If we take $X \in \mathcal{S}(f)$ with

$$X_0 := \frac{1}{2}(y + z), \qquad X_1 := \frac{1}{2}x^*(y - z)$$

then $\varphi X = (y, z)$, i.e. φ is surjective.

d) is known.

e_1) follows by using the spectrum of E^c.

e_2) Put

$$\psi : \mathcal{S}(f) \longrightarrow \overbrace{qE}^{\circ}, \quad X \longmapsto (qX_0, qX_1) \; .$$

For $X, Y \in \mathcal{S}(f)$, by Theorem 2.1.9 c),g),

$$\begin{aligned}\psi(X^*) &= (q(X^*)_0, q(X^*)_1) = (q(X_0)^*, qf(1,1)^*(X_1)^*) \\ &= ((qX_0)^*, -(qX_1)^*) = (\psi X)^*,\end{aligned}$$

$$\begin{aligned}(\psi X)(\psi Y) &= (qX_0, qX_1)(qY_0, qY_1) \\ &= (q(X_0Y_0 - X_1Y_1), (q(X_0Y_1 + X_1Y_0))) = \psi(XY)\end{aligned}$$

so ψ is an E-C*-homomorphism. Thus by c), φ is an E-C*-homomorphism. The bijectivity of φ is easy to see.

f) By Proposition 2.1.2 e), V_1 is unitary so its spectrum is contained in $\left\{ e^{i\theta} \;\middle|\; \theta \in \mathbb{R} \right\}$. For $\theta \in \mathbb{R}$ and $X \in \mathcal{S}(f)$,

$$\begin{aligned}(e^{i\theta}V_0 - V_1)X &= X(e^{i\theta} - V_1) \\ &= ((e^{i\theta}X_0) \otimes 1_K)V_0 + ((e^{i\theta}X_1) \otimes 1_K)V_1 - (X_0 \otimes 1_K)V_1 - ((f_{11}X_1) \otimes 1_K)V_1 \\ &= ((e^{i\theta}X_0 - f_{11}X_1) \otimes 1_K)V_0 + ((e^{i\theta}X_1 - X_0) \otimes 1_K)V_1.\end{aligned}$$

Thus X is the inverse of $e^{i\theta}V_0 - V_1$ iff $X_0 = e^{i\theta}X_1$ and $e^{i\theta}X_0 - f_{11}X_1 = 1_E$, i.e. $(e^{2i\theta} - f_{11})X_1 = 1_E$. Therefore $e^{i\theta}V_0 - V_1$ is invertible iff $e^{2i\theta} - \widehat{f_{11}}$ does not vanish on $\sigma(E^c)$. $\square$

Corollary 3.1.2. *Assume $\mathbb{K} := \mathbb{R}$ and let S be a group, F a unital C^*-algebra, $g \in \mathcal{F}(S, F)$, and*

$$h : (S \times \mathbb{Z}_2) \times (S \times \mathbb{Z}_2) \longrightarrow Un\ F^c,$$

$$((s_1, t_1), (s_2, t_2)) \mapsto \begin{cases} -g(s_1, s_2) & \textit{if} \quad (t_1, t_2) = (1, 1) \\ g(s_1, s_2) & \textit{if} \quad (t_1, t_2) \neq (1, 1) \end{cases} .$$

a) $h \in \mathcal{F}(S \times \mathbb{Z}_2, F)$.

b) $\mathcal{S}(h) \approx \overbrace{\mathcal{S}(g)}^{\circ}, \qquad \mathcal{S}_{\|\cdot\|}(h) \approx \overbrace{\mathcal{S}_{\|\cdot\|}(g)}^{\circ}$.

Proof. Put $E := \mathbb{R}$ in the above Proposition and define $f \in \mathcal{F}(\mathbb{Z}_2, \mathbb{R})$ by $f(1,1) = -1$ (Proposition 3.1.1 a)). By this Proposition e_2), $\mathcal{S}(f) \approx \mathbb{C}$. Thus by Proposition 2.2.11 c),e),

$$\mathcal{S}(h) \approx \mathcal{S}(g) \otimes \mathcal{S}(f) \approx \overbrace{\mathcal{S}(g)}^{\circ}, \quad \mathcal{S}_{\|\cdot\|}(h) \approx \mathcal{S}_{\|\cdot\|}(g) \otimes \mathcal{S}_{\|\cdot\|}(f) \approx \overbrace{\mathcal{S}_{\|\cdot\|}(g)}^{\circ}.$$

□

Definition 3.1.3. We put

$$\mathbf{T} := \{\, z \in \mathbb{C} \mid |z| = 1 \,\} .$$

Example 3.1.4. Let $E := \mathcal{C}(\mathbf{T}, \mathbb{C})$ and $f \in \mathcal{F}(\mathbb{Z}_2, E)$ with

$$f(1,1) : \mathbf{T} \longrightarrow Un\,\mathbb{C}, \quad z \longmapsto z .$$

If we put

$$\tilde{X} : \mathbf{T} \longrightarrow \mathbb{C}, \quad z \longmapsto X_0(z^2) + zX_1(z^2)$$

for every $X \in \mathcal{S}(f)$ then the map

$$\varphi : \mathcal{S}(f) \longrightarrow E, \quad X \longmapsto \tilde{X}$$

is an isomorphism of C*-algebras (but not an E-C*-isomorphism).

Proof. For $X, Y \in \mathcal{S}(f)$, by Theorem 2.1.9 c),g),

$$(X^*)_0 = (X_0)^*, \quad (X^*)_1 = \overline{f(1,1)}(X_1)^*,$$

$$(XY)_0 = X_0Y_0 + f(1,1)X_1Y_1, \quad (XY)_1 = X_0Y_1 + X_1Y_0$$

so for $z \in \mathbf{T}$,

$$\widetilde{X^*}(z) = X_0^*(z^2) + z\bar{z}^2 X_1^*(z^2) = \overline{X_0(z^2) + zX_1(z^2)} = \tilde{X}^*(z),$$

$$\begin{aligned}(\tilde{X}(z))(\tilde{Y}(z)) &= (X_0(z^2) + zX_1(z^2))(Y_0(z^2) + zY_1(z^2)) \\ &= X_0(z^2)Y_0(z^2) + zX_0(z^2)Y_1(z^2) + zX_1(z^2)Y_0(z^2) + z^2X_1(z^2)Y_1(z^2) \\ &= (XY)_0(z^2) + z(XY)_1(z^2) = \widetilde{XY}(z),\end{aligned}$$

$$\widetilde{X^*} = \tilde{X}^*, \qquad \tilde{X}\tilde{Y} = \widetilde{XY},$$

i.e. φ is a C*-homomorphism. If $\varphi X = 0$ then for $z \in \mathbf{T}$,

$$X_0(z^2) + zX_1(z^2) = 0$$

so, successively,

$$X_0(z^2) - zX_1(z^2) = 0, \; X_0(z^2) = X_1(z^2) = 0, \; X_0 = X_1 = 0, \; X = 0$$

and φ is injective.

Put

$$\mathcal{G} := \left\{ \sum_{k\in\mathbb{Z}} c_k z^k \;\middle|\; (c_k)_{k\in\mathbb{Z}} \in \mathbb{C}^{(\mathbb{Z})} \right\} \subset E \,.$$

Let

$$x := \sum_{k\in\mathbb{Z}} c_k z^k \in \mathcal{G}$$

and take $X \in \mathcal{S}(f)$ with

$$X_0 := \sum_{k\in\mathbb{Z}} c_{2k} z^k, \qquad X_1 := \sum_{k\in\mathbb{Z}} c_{2k+1} z^k \,.$$

Then

$$\tilde{X} = \sum_{k\in\mathbb{Z}} c_{2k} z^{2k} + z \sum_{k\in\mathbb{Z}} c_{2k+1} z^{2k} = x$$

so $\mathcal{G} \subset \varphi(\mathcal{S}(f))$. Since $\mathcal{G}$ is dense in E, $\varphi(\mathcal{S}(f)) = E$ and φ is surjective. □

Definition 3.1.5. For every $x \in \mathcal{C}(\mathbf{T}, \mathbb{C})$ which does not take the value 0 we put

$$w(x) := \textbf{winding number of x} := \frac{1}{2\pi i} \int_x \frac{\mathrm{d}z}{z} = \frac{1}{2\pi i} [\log x(e^{i\theta})]_{\theta=0}^{\theta=2\pi} \in \mathbb{Z} \,.$$

If A is a connected compact space and γ is a cycle in A (i.e. a continuous map of $\mathbf{T}$ in A), which is homologous to 0 (or more generally, if a multiple of γ is homologous to 0), then for every $x \in \mathcal{C}(A, Un\ \mathbb{C})$ we have $w(x \circ \gamma) = 0$. If A is a compact space and $x \in \mathcal{C}(A, Un\ \mathbb{C})$ such that $w(x \circ \gamma) = 0$ for every cycle γ in A then there is a $y \in \mathcal{C}(A, \mathbb{C})$ with $x = e^y$.

Example 3.1.6. Let $E := \mathcal{C}(\mathbf{T}, \mathbb{C})$, $f \in \mathcal{F}(\mathbb{Z}_2, E)$, and $n := w(f(1,1))$.

a) If n is even then there is an $x \in Un\ E$ with winding number equal to $\frac{n}{2}$ such that the map

$$\mathcal{S}(f) \longrightarrow E \times E, \quad X \longmapsto (X_0 + xX_1, X_0 - xX_1)$$

is an E-C*-isomorphism.

b) If n is odd then $\mathcal{S}(f)$ is isomorphic to E.

c) The group $\mathcal{F}(\mathbb{Z}_2, E)/\Lambda(\mathbb{Z}_2, E)$ is isomorphic to $\mathbb{Z}_2$ and

$$\mathrm{Card}\,(\{\ \mathcal{S}(\mathrm{g}) \mid \mathrm{g} \in \mathcal{F}(\mathbb{Z}_2, \mathrm{E})\ \}/{\approx_{\mathcal{S}}}) = 2 \,.$$

d) There is a complex unital C*-algebra E and a family $(f_\beta)_{\beta\in\mathfrak{P}(\mathbb{N})}$ in $\mathcal{F}(\mathbb{Z}_2, E)$ such that for distinct $\beta, \gamma \in \mathfrak{P}(\mathbb{N})$, $\mathcal{S}(f_\beta) \not\approx \mathcal{S}(f_\gamma)$.

Proof. Put

$$\alpha : \mathbf{T} \longrightarrow Un\,\mathbb{C}\,, \quad z \longmapsto z\,.$$

Since $w(f(1,1)\alpha^{-n}) = 0$, there is a $y \in Un\ E$ with $w(y) = 0$ and $f(1,1)\alpha^{-n} = y^2$.

a) If we put $x := y\alpha^{\frac{n}{2}}$ then $w(x) = \frac{n}{2}$ and $f(1,1) = x^2$ and the assertion follows from Proposition 3.1.1 c).

b) We put $x := y\alpha^{\frac{n-1}{2}}$. Then $f(1,1) = \alpha x^2$. Take $g \in \mathcal{F}(\mathbb{Z}_2, E)$ with $g(1,1) = \alpha$ and $\lambda \in \Lambda(\mathbb{Z}_2, E)$ with $(\delta\lambda)(1,1) = x^2$ (Proposition 3.1.1 a),b)). Then $f = g\delta\lambda$. By Example 3.1.4, $\mathcal{S}(g)$ is isomorphic to E and by Proposition 2.2.2 $a_1 \Rightarrow a_2$, $\mathcal{S}(f)$ is also isomorphic to E.

c) follows from Proposition 3.1.1 b) and Proposition 2.2.2 a),c).

d) Denote by E the C*-direct product of the sequence $(\mathcal{C}(\mathbf{T}, \mathbb{C}_{n,n}))_{n\in\mathbb{N}}$ and for every $\beta \in \{0,1\}^{\mathbb{N}}$ define $f_\beta \in \mathcal{F}(\mathbb{Z}_2, E)$ by

$$f_\beta(1,1) : \mathbb{N} \longrightarrow Un\ E^c\,, \quad n \longmapsto \alpha^{\beta(n)} 1_{\mathbb{C}_{n,n}}\,.$$

By a) and b), for distinct $\beta, \gamma \in \{0,1\}^{\mathbb{N}}$, $\mathcal{S}(f_\beta) \not\approx \mathcal{S}(f_\gamma)$ (Proposition 2.1.26 a)). □

Example 3.1.7. Let I, J be finite disjoint sets and for all $i \in I \cup J$ and $j \in J$ put $A_i := B_j := \mathbf{T}$. We define the compact spaces A and B in the following way. For A we take first the disjoint union of the spaces A_i for all $i \in I \cup J$ and identify then the points $1 \in A_i$ for all $i \in I \cup J$. For B we take first the disjoint union of all the spaces A_i for all $i \in I \cup J$ and of the spaces B_j for all $j \in J$ and identify first the points $1 \in A_i$ for all $i \in I \cup J$ and identify then also the points $-1 \in A_i$ for all $i \in I$ and $1 \in B_j$ for all $j \in J$.

Let $E := \mathcal{C}(A, \mathbb{C})$ and $f \in \mathcal{F}(\mathbb{Z}_2, E)$ with

$$f(1,1) : A \longrightarrow Un\,\mathbb{C}\,, \quad z \longmapsto \begin{cases} z & \text{if} \quad z \in A_i \text{ with } i \in I \\ 1 & \text{if} \quad z \in A_i \textit{ with } i \in J \end{cases}.$$

For every $X \in \mathcal{S}(f)$ define $\tilde{X} \in \mathcal{C}(B, \mathbb{C})$ by

$$\tilde{X} : B \longrightarrow \mathbb{C}\,, \quad z \longmapsto \begin{cases} X_0(z^2) + zX_1(z^2) & \text{if} \quad z \in A_i \textit{ with } i \in I \\ X_0(z) + X_1(z) & \text{if} \quad z \in A_i \textit{ with } i \in J \\ X_0(z) - X_1(z) & \text{if} \quad z \in B_j \textit{ with } j \in J \end{cases}.$$

Then the map

$$\varphi : \mathcal{S}(f) \longrightarrow \mathcal{C}(B, \mathbb{C})\,, \quad X \longmapsto \tilde{X}$$

is an isomorphism of C*-algebras.

Proof. Let $X, Y \in \mathcal{S}(f)$. By Theorem 2.1.9 c),g),

$$(X^*)_0 = (X_0)^*, \quad (X^*)_1 = \overline{f(1,1)}(X_1)^*,$$

$$(XY)_0 = X_0Y_0 + f(1,1)X_1Y_1\,, \quad (XY)_1 = X_0Y_1 + X_1Y_0.$$

For $z \in A_i$ with $i \in I$,

$$\begin{aligned}\widetilde{X^*}(z) &= (X^*)_0(z^2) + z(X^*)_1(z^2) = \overline{X_0(z^2)} + z\bar{z}^2\overline{X_1(z^2)}\\ &= \overline{X_0(z^2) + zX_1(z^2)} = (\tilde{X})^*(z),\end{aligned}$$

$$\begin{aligned}\tilde{X}(z)\tilde{Y}(z) &= (X_0(z^2) + zX_1(z^2))(Y_0(z^2) + zY_1(z^2))\\ &= X_0(z^2)Y_0(z^2) + zX_0(z^2)Y_1(z^2) + zX_1(z^2)Y_0(z^2) + z^2X_1(z^2)Y_1(z^2)\\ &= (XY)_0(z^2) + z(XY)_1(z^2) = \widetilde{XY}(z).\end{aligned}$$

For $z \in A_j$ or $z \in B_j$ with $j \in J$,

$$\widetilde{X^*}(z) = (X^*)_0(z) \pm (X^*)_1(z) = \overline{X_0(z)} \pm \overline{X_1(z)} = (\tilde{X})^*(z),$$

$$\begin{aligned}\tilde{X}(z)\tilde{Y}(z) &= (X_0(z) \pm X_1(z))(Y_0(z) \pm Y_1(z))\\ &= X_0(z)Y_0(z) \pm X_0(z)Y_1(z) \pm X_1(z)Y_0(z) + X_1(z)Y_1(z)\\ &= (XY)_0(z) \pm (XY)_1(z) = \widetilde{XY}(z).\end{aligned}$$

Thus φ is a C*-homomorphism. Assume $\tilde{X} = 0$. For $z \in A_i$ with $i \in I$,

$$X_0(z^2) + zX_1(z^2) = 0$$

so, successively,

$$X_0(z^2) - zX_1(z^2) = 0\,, \quad X_0(z^2) = X_1(z^2) = 0\,, \quad X(z) = 0.$$

For $z \in A_j$ with $j \in J$,

$$\left\{\begin{aligned} X_0(z) + X_1(z) &= 0\\ X_0(z) - X_1(z) &= 0\end{aligned}\right.,$$

so

$$X_0(z) = X_1(z) = 0\,, \qquad X(z) = 0.$$

Thus φ is injective.

Let $x \in \mathcal{C}(B, \mathbb{C})$ such that for every $i \in I$ there is a family $(c_{i,k})_{k\in\mathbb{Z}} \in \mathbb{C}^{(\mathbb{Z})}$ with

$$x(z) = \sum_{k\in\mathbb{Z}} c_{i,k}z^k$$

for all $z \in A_i$. Define $X_0, X_1 \in E$ in the following way. If $z \in A_i$ with $i \in I$ we put

$$X_0(z) := \sum_{k\in\mathbb{Z}} c_{i,2k}z^k\,, \quad X_1(z) := \sum_{k\in\mathbb{Z}} c_{i,2k+1}z^k.$$

If $z \in A_j$ with $j \in J$ then we put $z' := z \in B_j$,

$$X_0(z) := \frac{1}{2}(x(z) + x(z'))\,, \qquad X_1(z) := \frac{1}{2}(x(z) - x(z')).$$

It is easy to see that X_0 and X_1 are well defined. Then

$$\tilde{X}(z) = \sum_{k\in\mathbb{Z}} c_{i,2k} z^{2k} + z \sum_{k\in\mathbb{Z}} c_{i,2k+1} z^{2k} = x(z)$$

for all $z \in A_i$ with $i \in I$ and $\tilde{X}(z) = x(z)$ for all $z \in A_j \cup B_j$ with $j \in J$. Since the elements x of the above form are dense in $\mathcal{C}(B, \mathbb{C})$, φ is surjective. □

Example 3.1.8. Let $E := \mathcal{C}(\mathbf{T}^2, \mathbb{C})$ and $f, g \in \mathcal{F}(\mathbb{Z}_2, E)$ with

$$\begin{cases} f(1,1) : \mathbf{T}^2 \longrightarrow Un\,\mathbb{C}\,, & (z_1, z_2) \longmapsto z_1 \\ g(1,1) : \mathbf{T}^2 \longrightarrow Un\,\mathbb{C}\,, & (z_1, z_2) \longmapsto z_2 \end{cases} .$$

Then the maps

$$\begin{cases} \mathcal{S}(f) \longrightarrow E, & X \longmapsto X_0(z_1^2, z_2) + z_1 X_1(z_1^2, z_2) \\ \mathcal{S}(g) \longrightarrow E, & X \longmapsto X_0(z_1, z_2^2) + z_2 X_1(z_1, z_2^2) \end{cases}$$

are isomorphisms of C*-algebras.

Remark. $\mathcal{S}(f)$ and $\mathcal{S}(g)$ are isomorphic but not E-C*-isomorphic.

Example 3.1.9. Let $E := \mathcal{C}(\mathbf{T}^2, \mathbb{C})$ and $f \in \mathcal{F}(\mathbb{Z}_2, E)$ with

$$f(1,1) : \mathbf{T}^2 \longrightarrow Un\,\mathbb{C}\,, \quad (z_1, z_2) \longmapsto z_1 z_2\,.$$

If we put

$$\tilde{X} : \mathbf{T}^2 \longrightarrow \mathbb{C}\,, \quad (z_1, z_2) \longmapsto X_0(z_1^2, z_2^2) + z_1 z_2 X_1(z_1^2, z_2^2)$$

for every $X \in \mathcal{S}(f)$ then the map

$$\varphi : \mathcal{S}(f) \longrightarrow E\,, \quad X \longmapsto \tilde{X}$$

is an injective unital C*-homomorphism with

$$\varphi(\mathcal{S}(f)) = \mathcal{G} := \left\{\, x \in E \mid (z_1, z_2) \in \mathbf{T}^2 \Longrightarrow x(z_1, z_2) = x(-z_1, -z_2) \,\right\} .$$

In particular $\mathcal{S}(f)$ is isomorphic to E.

Proof. Let $X, Y \in \mathcal{S}(f)$. By Theorem 2.1.9 c),g),

$$(X^*)_0 = (X_0)^*\,, \quad (X^*)_1 = \overline{f(1,1)}(X_1)^*\,,$$

$$(XY)_0 = X_0 Y_0 + f(1,1) X_1 Y_1\,, \quad (XY)_1 = X_0 Y_1 + X_1 Y_0$$

so for $(z_1, z_2) \in \mathbf{T}^2$,

$$\widetilde{X^*}(z_1, z_2) = X_0^*(z_1^2, z_2^2) + z_1 z_2 \bar{z}_1^2 \bar{z}_2^2 X_1^*(z_1^2, z_2^2)$$

$$= \overline{X_0(z_1^2, z_2^2) + z_1 z_2 X_1(z_1^2, z_2^2)} = \overline{\tilde{X}(z_1, z_2)},$$

$$\begin{aligned}
&(\tilde{X}(z_1, z_2))(\tilde{Y}(z_1, z_2)) \\
&\quad = (X_0(z_1^2, z_2^2) + z_1 z_2 X_1(z_1^2, z_2^2))(Y_0(z_1^2, z_2^2) + z_1 z_2 Y_1(z_1^2, z_2^2)) \\
&\quad = X_0(z_1^2, z_2^2) Y_0(z_1^2, z_2^2) + z_1 z_2 X_0(z_1^2, z_2^2) Y_1(z_1^2, z_2^2) \\
&\quad + z_1 z_2 X_1(z_1^2, z_2^2) Y_0(z_1^2, z_2^2) + z_1^2 z_2^2 X_1(z_1^2, z_2^2) Y_1(z_1^2, z_2^2) \\
&\quad = (XY)_0(z_1^2, z_2^2) + z_1 z_2 (XY)_1(z_1^2, z_2^2) = \widetilde{XY}(z_1, z_2),
\end{aligned}$$

i.e. φ is a unital C*-homomorphism. If $\tilde{X} = 0$ then for $(z_1, z_2) \in \mathbf{T}^2$,

$$X_0(z_1^2, z_2^2) + z_1 z_2 X_1(z_1^2, z_2^2) = 0$$

so, successively,

$$X_0(z_1^2, z_2^2) - z_1 z_2 X_1(z_1^2, z_2^2) = 0\,, \quad X_0(z_1^2, z_2^2) = X_1(z_1^2, z_2^2) = 0\,,$$
$$X_0 = X_1 = 0\,, \quad X = 0$$

and φ is injective.

The inclusion $\mathcal{S}(f) \subset \mathcal{G}$ is obvious. Let $(a_{j,k})_{j,k\in\mathbb{Z}},\ (b_{j,k})_{j,k\in\mathbb{Z}} \in \mathbb{C}^{(\mathbb{Z}\times\mathbb{Z})}$ and

$$x = \sum_{j,k\in\mathbb{Z}} a_{j,k} z_1^{2j} z_2^{2k} + \sum_{j,k\in\mathbb{Z}} b_{j,k} z_1^{2j+1} z_2^{2k+1} \in \mathcal{G}\,.$$

Define

$$X_0 := \sum_{j,k\in\mathbb{Z}} a_{j,k} z_1^j z_2^k\,, \qquad X_1 := \sum_{j,k\in\mathbb{Z}} b_{j,k} z_1^j z_2^k\,.$$

Then $\tilde{X} = x$. Since the elements of the above form are dense in $\mathcal{G}$, $\varphi(\mathcal{S}(f)) = \mathcal{G}$.

If we consider the equivalence relation $\sim$ on $\mathbf{T}^2$ defined by

$$(z_1, z_2) \sim (w_1, w_2) :\Longleftrightarrow z_1 = -w_1,\ z_2 = -w_2$$

then the quotient space $\mathbf{T}^2/\sim$ is homeomorphic to $\mathbf{T}^2$. Thus $\mathcal{S}(f)$ is isomorphic to E. □

Example 3.1.10. Let $E := \mathcal{C}(\mathbf{T}^2, \mathbb{C})$.

a) For $x \in Un\ E$ and $z \in \mathbf{T}$, $w(x(\,\cdot\,, z))$ and $w(x(z, \,\cdot\,))$ do not depend on z, where w denotes the winding number (Definition 3.1.5).

b) If $x \in Un\ E$ and if

$$w(x(\,\cdot\,, 1)) = w(x(1, \,\cdot\,)) = 0$$

then there is a $y \in Un\ E$ with $x = y^2$.

c) Let $f \in \mathcal{F}(\mathbb{Z}_2, E)$ and put

$$\alpha : \mathbf{T} \longrightarrow \mathbf{T}^2\,, \quad z \longmapsto (z, 1)\,, \quad \beta : \mathbf{T} \longrightarrow \mathbf{T}^2\,, \quad z \longmapsto (1, z)\,,$$
$$m := w(f(1,1) \circ \alpha)\,, \quad n := w(f(1,1) \circ \beta)\,.$$

c_1) If $m+n$ is odd then $\mathcal{S}(f)$ is isomorphic to E.

c_2) If m and n are even then $\mathcal{S}(f)$ is isomorphic to $E \times E$.

c_3) If m and n are odd then $\mathcal{S}(f)$ is isomorphic to E.

d) The group $\mathcal{F}(\mathbb{Z}_2, E)/\Lambda(\mathbb{Z}_2, E)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and

$$\mathrm{Card}\,(\{\, \mathcal{S}(\mathrm{f}) \mid \mathrm{f} \in \mathcal{F}(\mathbb{Z}_2, \mathrm{E}) \,\} / \approx_{\mathcal{S}}) = 4\,.$$

Proof. a) follows by continuity.
b) follows from a).
c) Let $g \in \mathcal{F}(\mathbb{Z}_2, E)$ with

$$g(1,1) : \mathbf{T}^2 \longrightarrow Un\,\mathbb{C}\,, \quad (z_1, z_2) \longmapsto z_1^m z_2^n\,.$$

Then

$$w(g(1,1)\circ\alpha) = m\,, \qquad w(g(1,1)\circ\beta) = n\,.$$

By b), there is an $x \in Un\, E$ with $f(1,1) = x^2 g(1,1)$. By Proposition 3.1.1 b) and Proposition 2.2.2 $a_1 \Rightarrow a_2$, $\mathcal{S}(f) \approx \mathcal{S}(g)$.

c_1) Assume m even and put

$$y : \mathbf{T}^2 \longrightarrow Un\,\mathbb{C}\,, \quad (z_1, z_2) \longmapsto z_1^{\frac{m}{2}} z_2^{\frac{n-1}{2}}\,.$$

If $h \in \mathcal{F}(\mathbb{Z}_2, E)$ with

$$h(1,1) : \mathbf{T}^2 \longrightarrow Un\,\mathbb{C}\,, \quad (z_1, z_2) \longmapsto z_2$$

then $g(1,1) = y^2 h(1,1)$. By Proposition 3.1.1 b) and Proposition 2.2.2 $a_1 \Rightarrow a_2$, $\mathcal{S}(g) \approx \mathcal{S}(h)$ and by Example 3.1.8 $a_1 \Rightarrow a_2$, $\mathcal{S}(h) \approx E$. Thus $\mathcal{S}(f) \approx E$.

c_2) If we put

$$y : \mathbf{T}^2 \longrightarrow Un\,\mathbb{C}\,, \quad (z_1, z_2) \longmapsto z_1^{\frac{m}{2}} z_2^{\frac{n}{2}}$$

then $g(1,1) = y^2$ and the assertion follows from Proposition 3.1.1 c).

c_3) We put

$$y : \mathbf{T}^2 \longrightarrow Un\,\mathbb{C}\,, \quad (z_1, z_2) \longmapsto z_1^{\frac{m-1}{2}} z_2^{\frac{n-1}{2}}$$

and take $h \in \mathcal{F}(\mathbb{Z}_2, E)$ with

$$h(1,1) : \mathbf{T}^2 \longrightarrow Un\,\mathbb{C}\,, \quad (z_1, z_2) \longmapsto z_1 z_2$$

then $g(1,1) = y^2 h(1,1)$ so by Proposition 3.1.1 b) and Proposition 2.2.2 $a_1 \Rightarrow a_2$, $\mathcal{S}(g) \approx \mathcal{S}(h)$. By Example 3.1.9 $\mathcal{S}(h) \approx E$, so $\mathcal{S}(f) \approx E$.

d) follows from b), Proposition 3.1.1 b), and Proposition 2.2.2 a),c). □

Remark. In a similar way, it is possible to show that for every $n \in \mathbb{N}$, $\mathcal{F}(\mathbb{Z}_2, \mathbf{T}^n)/\Lambda(\mathbb{Z}_2, \mathbf{T}^n)$ is isomorphic to $(\mathbb{Z}_2)^n$ and

$$\mathrm{Card}\,(\{\, \mathcal{S}(\mathrm{f}) \mid \mathrm{f} \in \mathcal{F}(\mathbb{Z}_2, \mathbf{T}^{\mathrm{n}}) \,\} / \approx_{\mathcal{S}}) = 2^{\mathrm{n}}\,.$$

Example 3.1.11. Let I, J, K be finite pairwise disjoint sets and for every $i \in I \cup J \cup K$ and $k \in K$ put $A_i := B_k := \mathbf{T}^2$. We define the compact spaces A and B in the following way. For A we take first the disjoint union of the spaces A_i with $i \in I \cup J \cup K$ and then identify the points $(1,1) \in A_i$ for all $i \in I \cup J \cup K$. For B we take first the disjoint union of the spaces A_i with $i \in I \cup J \cup K$ and of the spaces B_k with $k \in K$. Then we identify the points $(1,1) \in A_i$ for all $i \in I \cup J \cup K$ and then we identify for every $j \in J$ the points $(z_1, z_2) \in A_j$ with the points $(-z_1, -z_2) \in A_j$ and finally we identify the points $(-1,1) \in A_i$ for all $i \in I \cup J$ with the points $(1,1) \in B_k$ for all $k \in K$.

Let $E := \mathcal{C}(A, \mathbb{C})$ and $f \in \mathcal{F}(\mathbb{Z}_2, A)$ such that

$$f(1,1) : A \longrightarrow Un\,\mathbb{C}, \quad (z_1, z_2) \longmapsto \begin{cases} z_1 & \text{if} \quad (z_1, z_2) \in A_i \text{ with } i \in I \\ z_1 z_2 & \text{if} \quad (z_1, z_2) \in A_i \text{ with } i \in J \\ 1 & \text{if} \quad (z_1, z_2) \in A_i \text{ with } i \in K \end{cases} .$$

We define for every $X \in \mathcal{S}(f)$ a map $\tilde{X} : B \to \mathbb{C}$ by

$$(z_1, z_2) \mapsto \begin{cases} X_0(z_1^2, z_2) + z_1 X_1(z_1^2, z_2) & \text{if} \quad (z_1, z_2) \in A_i \text{ with i} \in \text{I} \\ X_0(z_1^2, z_2^2) + z_1 z_2 X_1(z_1^2, z_2^2) & \text{if} \quad (z_1, z_2) \in A_i \text{ with } i \in J \\ X_0(z_1, z_2) + X_1(z_1, z_2) & \text{if} \quad (z_1, z_2) \in A_i \text{ with } i \in K \\ X_0(z_1, z_2) - X_1(z_1, z_2) & \text{if} \quad (z_1, z_2) \in B_k \text{ with } k \in K \end{cases} .$$

Then the map

$$\mathcal{S}(f) \longrightarrow \mathcal{C}(B, \mathbb{C}), \quad X \longmapsto \tilde{X}$$

is an isomorphism of C*-algebras.

The proof is similar to the proof of Example 3.1.7.

Example 3.1.12. If $n \in \mathbb{N}$, $E := \mathcal{C}(\mathbf{T}^n, \mathbb{C})$, and $f \in \mathcal{F}(\mathbb{Z}_2, \mathcal{C}(\mathbf{T}^n, \mathbb{C}))$ then $\mathcal{S}(f)$ is isomorphic either to $\mathcal{C}(\mathbf{T}^n, \mathbb{C})$ or to $\mathcal{C}(\mathbf{T}^n, \mathbb{C}) \times \mathcal{C}(\mathbf{T}^n, \mathbb{C})$.

Example 3.1.13. Assume $E := \mathcal{C}(A, \mathbb{C})$, where A denotes Moebius's band (resp. Klein's bottle), i.e. the topological space obtained from $[0, 2\pi] \times [-\pi, \pi]$ by identifying the points $(0, \alpha)$ and $(2\pi, -\alpha)$ for all $\alpha \in [-\pi, \pi]$ (resp. and the points $(\theta, -\pi)$ and (θ, π) for all $\theta \in [0, 2\pi]$). We put $B := \mathbf{T} \times [-\pi, \pi]$ (resp. $B := \mathbf{T}^2$) and

$$\tilde{x} : [0, 2\pi] \times [-\pi, \pi] \longrightarrow \mathbb{C}, \quad (\theta, \alpha) \longmapsto \begin{cases} x(2\theta, \alpha) & \text{if} \quad \theta \in [0, \pi] \\ x(2(\theta - \pi), -\alpha) & \text{if} \quad \theta \in [\pi, 2\pi] \end{cases}$$

for every $x \in E$.

a) $\tilde{x}$ is well-defined and belongs to $\mathcal{C}(B, \mathbb{C})$ for every $x \in E$.

b) If $f_{1,1}(\theta, \alpha) = e^{i\theta}$ for all $(\theta, \alpha) \in [0, 2\pi] \times [-\pi, \pi]$ then the map

$$\varphi : \mathcal{S}(f) \longrightarrow \mathcal{C}(B, \mathbb{C}), \quad X \longmapsto \widetilde{X_0} + e^{i\theta} \widetilde{X_1}$$

is a C*-isomorphism.

c) Let $x \in Un\, E$. If $w(x(\cdot\,,0)) = 0$ (where w denotes the winding number) then there is a $y \in E$ with $e^y = x$.

d) Let $x \in Un\, E$ and put $n := w(x(\cdot\,,0))$. Then there is a $y \in E$ with $e^y = e^{-in\theta}x$.

e) The group $\mathcal{F}(\mathbb{Z}_2, A)/\Lambda(\mathbb{Z}_2, A)$ is isomorphic to $\mathbb{Z}_2$.

f) If $w(f_{1,1}(\cdot\,,0))$ is even (resp. odd) then $\mathcal{S}(f)$ is isomorphic to $E \times E$ (resp. to $\mathcal{C}(B, \mathbb{C})$).

Proof. a) For $\alpha \in [-\pi, \pi]$,

$$\tilde{x}(\pi, \alpha) = x(2\pi, \alpha) = x(0, -\alpha) = \tilde{x}(\pi, \alpha)$$

so $\tilde{x}$ is well-defined. Moreover

$$\tilde{x}(0, \alpha) = x(0, \alpha) = x(2\pi, -\alpha) = \tilde{x}(2\pi, \alpha)$$

and in the case of Klein's bottle

$$\begin{cases} \tilde{x}(\theta, -\pi) = x(2\theta, -\pi) = x(2\theta, \pi) = \tilde{x}(\theta, \pi) & \text{if} \quad \theta \in [0, \pi] \\ \tilde{x}(\theta, -\pi) = x(2(\theta - \pi), \pi) = x(2(\theta - \pi), -\pi) = \tilde{x}(\theta, \pi) & \text{if} \quad \theta \in [\pi, 2\pi] \end{cases}$$

i.e. $\tilde{x} \in \mathcal{C}(B, \mathbb{C})$.

b) For $X, Y \in \mathcal{S}(f)$ and $(\theta, \alpha) \in [0, 2\pi] \times [-\pi, \pi]$, by Theorem 2.1.9 c),g),

$$(\varphi X^*)(\theta, \alpha) = \widetilde{(X^*)_0}(\theta, \alpha) + e^{i\theta}\widetilde{(X^*)_1}(\theta, \alpha)$$

$$= \widetilde{(X_0)^*}(\theta, \alpha) + e^{i\theta}\widetilde{\overbrace{(e^{-i\theta}(X_1)^*)}}(\theta, \alpha)$$

$$= \begin{cases} \overline{X_0(2\theta, \alpha)} + e^{i\theta}(e^{-2i\theta}\overline{X_1(2\theta, \alpha)}) & \text{if} \quad \theta \in [0, \pi] \\ \overline{X_0(2(\theta - \pi), -\alpha)} + e^{i\theta}(e^{-2i(\theta-\pi)}\overline{X_1(2(\theta - \pi), -\alpha)}) & \text{if} \quad \theta \in [\pi, 2\pi] \end{cases}$$

$$= \begin{cases} \overline{X_0(2\theta, \alpha) + e^{i\theta}X_1(2\theta, \alpha)} & \text{if} \quad \theta \in [0, \pi] \\ \overline{X_0(2(\theta - \pi), -\alpha) + e^{i\theta}X_1(2(\theta - \pi), -\alpha)} & \text{if} \quad \theta \in [\pi, 2\pi] \end{cases} = \overline{\varphi X}(\theta, \alpha),$$

$$(\varphi X)(\varphi Y) = (\widetilde{X_0} + e^{i\theta}\widetilde{X_1})(\widetilde{Y_0} + e^{i\theta}\widetilde{Y_1}) = \widetilde{X_0}\widetilde{Y_0} + e^{i\theta}\widetilde{X_0}\widetilde{Y_1} + e^{i\theta}\widetilde{X_1}\widetilde{Y_0} + e^{2i\theta}\widetilde{X_1}\widetilde{Y_1}\,,$$

$$\begin{aligned} \varphi(XY) &= \widetilde{(XY)_0} + e^{i\theta}\widetilde{(XY)_1} \\ &= \widetilde{X_0}\widetilde{Y_0} + e^{2i\theta}\widetilde{X_1}\widetilde{Y_1} + e^{i\theta}(\widetilde{X_0}\widetilde{Y_1} + \widetilde{X_1}\widetilde{Y_0}) = (\varphi X)(\varphi Y)\,, \end{aligned}$$

i.e. φ is a C*-homomorphism. If $\varphi X = 0$ then for $\alpha \in [-\pi, \pi]$,

$$\begin{cases} X_0(2\theta, \alpha) + e^{i\theta}X_1(2\theta, \alpha) = 0 & \text{if} \quad \theta \in [0, \pi] \\ X_0(2(\theta - \pi), -\alpha) + e^{i\theta}X_1(2(\theta - \pi), -\alpha) = 0 & \text{if} \quad \theta \in [\pi, 2\pi] \end{cases}$$

so for $\theta \in [0, \pi]$, replacing θ by $\theta + \pi$ and α by $-\alpha$ in the second relation,

$$X_0(2\theta, \alpha) - e^{i\theta}X_1(2\theta, \alpha) = 0\,.$$

It follows successively

$$X_0(2\theta,\alpha)=X_1(2\theta,\alpha)=0\,,$$
$$X_0=X_1=0\,,\quad X=0.$$

Thus φ is injective.

Let $y\in\mathcal{C}(B,\mathbb{C})$. Put

$$\left\{\begin{array}{l} X_0:[0,2\pi]\times[-\pi,\pi]\longrightarrow\mathbb{C}\,,\quad(\theta,\alpha)\longmapsto\frac{1}{2}(y(\frac{\theta}{2},\alpha)+y(\frac{\theta}{2}+\pi,-\alpha))\\ X_1:[0,2\pi]\times[-\pi,\pi]\longrightarrow\mathbb{C}\,,\quad(\theta,\alpha)\longmapsto\frac{1}{2}e^{-i\frac{\theta}{2}}(y(\frac{\theta}{2},\alpha)-y(\frac{\theta}{2}+\pi,-\alpha))\end{array}\right..$$

For $\alpha\in[-\pi,\pi]$,

$$\left\{\begin{array}{l} X_0(0,\alpha)=\frac{1}{2}(y(0,\alpha)+y(\pi,-\alpha))\\ X_0(2\pi,-\alpha)=\frac{1}{2}(y(\pi,-\alpha)+y(2\pi,\alpha))\end{array}\right.$$

$$\left\{\begin{array}{l} X_1(0,\alpha)=\frac{1}{2}(y(0,\alpha)-y(\pi,-\alpha))\\ X_1(2\pi,-\alpha)=-\frac{1}{2}(y(\pi,-\alpha)-y(2\pi,\alpha))\end{array}\right.$$

so $X_0,X_1\in E$. Moreover for $(\theta,\alpha)\in[0,2\pi]\times[-\pi,\pi]$,

$$\begin{aligned}&\widetilde{X_0}(\theta,\alpha)+e^{i\theta}\widetilde{X_1}(\theta,\alpha)\\ &=\left\{\begin{array}{lll} X_0(2\theta,\alpha)+e^{i\theta}X_1(2\theta,\alpha) & \text{if} & \theta\in[0,\pi]\\ X_0(2(\theta-\pi),-\alpha)+e^{i\theta}X_1(2(\theta-\pi),-\alpha) & \text{if} & \theta\in[\pi,2\pi]\end{array}\right.\\ &=\left\{\begin{array}{lll} \frac{1}{2}(y(\theta,\alpha)+y(\theta+\pi,-\alpha)+y(\theta,\alpha)-y(\theta+\pi,-\alpha)) & \text{if} & \theta\in[0,\pi]\\ \frac{1}{2}(y(\theta-\pi,-\alpha)+y(\theta,\alpha)-y(\theta-\pi,-\alpha)+y(\theta,\alpha)) & \text{if} & \theta\in[\pi,2\pi]\end{array}\right.\\ &=y(\theta,\alpha)\end{aligned}$$

i.e. φ is surjective.

c) If A is Moebius's band then the assertion is obvious so assume A is Klein's bottle. The winding numbers of

$$\left\{\begin{array}{l} [0,2\pi]\longrightarrow\mathbb{C},\quad\alpha\longmapsto x(0,\alpha)\\ [0,2\pi]\longrightarrow\mathbb{C},\quad\alpha\longmapsto x(2\pi,\alpha)\end{array}\right.$$

are equal by homotopy, but their sum is equal to 0. Thus these winding numbers are equal to 0. The paths θ and α on A generate the homotopy group of A. Thus the winding number of x on any path of A is 0 and the assertion follows.

d) The winding number of

$$[0,2\pi]\longrightarrow\mathbb{C},\quad\theta\longmapsto e^{-in\theta}x(\theta,0)$$

is 0 and the assertion follows from c).

e) The assertion follows from d) and Proposition 3.1.1 b).

f) The assertion follows from b), d), Proposition 2.2.2 $a_1\Rightarrow a_2$, and Proposition 3.1.1 c). □

3.2. $\boldsymbol{T} := \mathbb{Z}_2 \times \mathbb{Z}_2$

PROPOSITION 3.2.1. *Let E be a unital C^*-algebra and let a, b, c be the three elements of $(\mathbb{Z}_2 \times \mathbb{Z}_2) \setminus \{(0,0)\}$. Put*

$$A := \left\{ (\alpha, \beta, \gamma, \varepsilon) \in (Un\ E^c)^4 \;\middle|\; \varepsilon^2 = 1_E \right\}$$

and for every $\varrho \in A$ and $\sigma \in (Un\ E^c)^3$ denote by f_ϱ and g_σ the functions defined by the following tables:

f_ϱ	a	b	c
a	$\beta\gamma$	γ	β
b	$\varepsilon\gamma$	$\varepsilon\alpha\gamma$	α
c	$\varepsilon\beta$	$\varepsilon\alpha$	$\alpha\beta$

g_σ	a	b	c
a	α^2	$\alpha\beta\gamma^*$	$\alpha\gamma\beta^*$
b	$\alpha\beta\gamma^*$	β^2	$\beta\gamma\alpha^*$
c	$\alpha\gamma\beta^*$	$\beta\gamma\alpha^*$	γ^2

a) *$f_\varrho \in \mathcal{F}(\mathbb{Z}_2 \times \mathbb{Z}_2, E)$ for every $\varrho \in A$ and the map*

$$A \longrightarrow \mathcal{F}(\mathbb{Z}_2 \times \mathbb{Z}_2, E), \quad \varrho \longmapsto f_\varrho$$

is bijective.

b) *$g_\sigma \in \{\, \delta\lambda \mid \lambda \in \Lambda(\mathbb{Z}_2 \times \mathbb{Z}_2, E) \,\}$ for every $\sigma \in (Un\ E^c)^3$ and the map*

$$(Un\ E^c)^3 \longrightarrow \{\, \delta\lambda \mid \lambda \in \Lambda(\mathbb{Z}_2 \times \mathbb{Z}_2, E) \,\}, \quad \sigma \longmapsto g_\sigma$$

is bijective.

c) *The following are equivalent for all $\varrho := (\alpha, \beta, \gamma, \epsilon) \in A$ and $\varrho' := (\alpha', \beta', \gamma', \epsilon') \in A$:*

c$_1$) $\mathcal{S}(f_\varrho) \approx_{\mathcal{S}} \mathcal{S}(f_{\varrho'})$.

c$_2$) *$\varepsilon = \varepsilon'$ and there are $x, y, z \in Un\ E^c$ with*

$$x^2 = \beta\beta'^*\gamma\gamma'^*, \qquad y^2 = \alpha\alpha'^*\gamma\gamma'^*, \qquad z^2 = \alpha\alpha'^*\beta\beta'^*.$$

c$_3$) *$\varepsilon = \varepsilon'$ and there are $x, y \in Un\ E^c$ with*

$$x^2 = \beta\beta'^*\gamma\gamma'^*, \qquad y^2 = \alpha\alpha'^*\gamma\gamma'^*.$$

d) *The following are equivalent for all $\varrho := (\alpha, \beta, \gamma, \varepsilon \in A)$ and $X \in \mathcal{S}(f_\varrho)$:*

d$_1$) $X \in \left\{ V_t^{f_\varrho} \;\middle|\; t \in \mathbb{Z}_2 \times \mathbb{Z}_2 \right\}^c$.

d$_2$) $t \in \mathbb{Z}_2 \times \mathbb{Z}_2 \Longrightarrow \varepsilon X_t = X_t$.

e) *The following are equivalent for all $\varrho := (\alpha, \beta, \gamma, \varepsilon \in A)$ and $X \in \mathcal{S}(f_\varrho)$:*

e$_1$) $X \in \mathcal{S}(f_\varrho)^c$.

e$_2$) $t \in \mathbb{Z}_2 \times \mathbb{Z}_2 \Longrightarrow \varepsilon X_t = X_t \in E^c$.

f) *For* $\varrho := (\alpha, \beta, \gamma, \varepsilon) \in A$ *and* $X, Y \in \mathcal{S}(f_\varrho)$,

$$(X^*)_0 = X_0^*\,,\ (X^*)_a = \beta^*\gamma^* X_a^*\,,\ (X^*)_b = \varepsilon\alpha^*\gamma^* X_b^*\,,\ (X^*)_c = \alpha^*\beta^* X_c^*\,,$$

$$\begin{aligned}(XY)_0 &= X_0Y_0 + \beta\gamma X_aY_a + \varepsilon\alpha\gamma X_bY_b + \alpha\beta X_cY_c\,,\\ (XY)_a &= X_0Y_a + X_aY_0 + \alpha X_bY_c + \varepsilon\alpha X_cY_b\,,\\ (XY)_b &= X_0Y_b + \beta X_aY_c + X_bY_0 + \varepsilon\beta X_cY_a\,,\\ (XY)_c &= X_0Y_c + \gamma X_aY_b + \varepsilon\gamma X_bY_a + X_cY_0\,.\end{aligned}$$

g) *Assume* $\mathbb{K} = \mathbb{C}$, *let* $\sigma(E^c)$ *be the spectrum of* E^c, *and for every* $\delta \in E^c$ *let* $\hat{\delta}$ *be its Gelfand transform. Then*

$$\sigma(V_a) = \left\{ e^{i\theta} \;\middle|\; \theta \in \mathbb{R},\ e^{2i\theta} \in \widehat{\beta\gamma}(\sigma(E^c)) \right\},$$

$$\sigma(V_b) = \left\{ e^{i\theta} \;\middle|\; \theta \in \mathbb{R},\ e^{2i\theta} \in \widehat{\alpha\gamma}(\sigma(E^c)) \right\},$$

$$\sigma(V_c) = \left\{ e^{i\theta} \;\middle|\; \theta \in \mathbb{R},\ e^{2i\theta} \in \widehat{\alpha\beta}(\sigma(E^c)) \right\}.$$

Proof. a) is a long calculation.

b) is easy to verify.

$c_1 \Rightarrow c_2$ By Proposition 2.2.2 $a_2 \Rightarrow a_1$ there is a $\lambda \in \Lambda(\mathbb{Z}_2 \times \mathbb{Z}_2, E)$ with $f_\varrho = f_{\varrho'}\delta\lambda$. By b), there is a $\sigma := (x, y, z) \in (Un\ E^c)^3$ with $f_\varrho = f_{\varrho'}g_\sigma$. We get $\varepsilon = \varepsilon'$ and

$$\alpha\alpha'^* = x^*yz\,,\quad \beta\beta'^* = xy^*z\,,\quad \gamma\gamma'^* = xyz^*.$$

It follows $xyz = \alpha\alpha'^*\beta\beta'^*\gamma\gamma'^*$ so

$$x^2 = \beta\beta'^*\gamma\gamma'^*\,,\quad y^2 = \alpha\alpha'^*\gamma\gamma'^*\,,\quad z^2 = \alpha\alpha'^*\beta\beta'^*.$$

$c_2 \Rightarrow c_3$ is trivial.

$c_3 \Rightarrow c_2$ If we put $z := xy\gamma^*\gamma'$ then

$$z^2 = \beta\beta'^*\gamma\gamma'^*\alpha\alpha'^*\gamma\gamma'^*\gamma^{*2}\gamma'^2 = \alpha\alpha'^*\beta\beta'^*.$$

$c_2 \Rightarrow c_1$ follows from b) and Proposition 2.2.2 $a_1 \Rightarrow a_2$.

d) follows from Corollary 2.1.24 b).

e) follows from Corollary 2.1.24 c).

f) follows from Theorem 2.1.9 c),g).

g) follows from f). $\square$

Corollary 3.2.2. *We use the notation of* Proposition 3.2.1 *and take* $\varrho := (\alpha, \beta, \gamma, \varepsilon) \in A$.

a) *Assume* $\varepsilon = 1_E$ *and there are* $x, y \in Un\ E$ *with* $x^2 = \beta\gamma\,, y^2 = \alpha\gamma$. *Put* $z := xy\gamma^*$.

a$_1$) $x, y, z \in Un\ E^c\,,\ z^2 = \alpha\beta$.

a$_2$) *For every* $\lambda, \mu \in \{-1, 1\}$ *the map*

$$\varphi_{\lambda,\mu} : \mathcal{S}(f_\varrho) \longrightarrow E\,, \quad X \longmapsto X_0 + \lambda x X_a + \mu y X_b + \lambda\mu z X_c$$

is an E-C^*-*homomorphism.*

a$_3$) *The map*

$$\mathcal{S}(f_\varrho) \longrightarrow E^4, \quad X \longmapsto (\varphi_{1,1}X, \varphi_{1,-1}X, \varphi_{-1,1}X.\varphi_{-1,-1}X)$$

is an E-C^*-*isomorphism.*

b) *Assume* $\mathbb{K} := \mathbb{R}\,, \varepsilon = 1_E$, *and there are* $x, y \in Un\ E$ *with*

$$x^2 = -\beta\gamma\,, \quad y^2 = \alpha\gamma\,, \quad (resp.\ y^2 = -\alpha\gamma)\,.$$

Put $z := xy\gamma^*$. *Then* $x, y, z \in Un\ E^c$, $z^2 = -\alpha\beta$ *(resp.* $z^2 = \alpha\beta$*), and the maps*

$$\mathcal{S}(f_\varrho) \longrightarrow (\overset{\circ}{E})^2, \quad X \longmapsto (X_0{+}ixX_a{+}yX_b{+}izX_c, X_0{+}ixX_a{-}yX_b{-}izX_c)$$

$$\mathcal{S}(f_\varrho) \longrightarrow (\overset{\circ}{E})^2, \quad X \longmapsto (X_0{+}ixX_a{+}iyX_b{-}zX_c, X_0{+}ixX_a{-}iyX_b{+}zX_c)$$

are respectively E-C^*-*isomorphisms (where* $\overset{\circ}{E}$ *denotes the complexification of* E*).*

c) *Assume* $\mathbb{K} := \mathbb{R}$, $\varepsilon = -1_E$, *and there are* $x, y \in E^c$ *with* $x^2 = -\beta\gamma\,,\ y^2 = \alpha\gamma$. *Put* $z := xy\gamma^*$. *Then* $x, y, z \in Un\ E^c$, $z^2 = -\alpha\beta$, *and the map*

$$\mathcal{S}(f_\varrho) \longrightarrow \mathbb{H} \otimes E, \quad X \longmapsto X_0 + ixX_a + jyX_b + kzX_c\,,$$

where i, j, k *are the canonical units of* $\mathbb{H}$, *is an* E-C^*-*isomorphism.*

d) *If* $\varepsilon = -1_E$ *and there is an* $x \in Un\ E^c$ *with* $x^2 = \alpha\beta$ *then for every* $\delta \in Un\ E^c$ *the map*

$$\mathcal{S}(f_\varrho) \longrightarrow E_{2,2}, \quad X \longmapsto \begin{bmatrix} X_0 + xX_c & \gamma\delta^*(\beta X_a - xX_b) \\ \delta(X_a + x\beta^* X_b) & X_0 - xX_c \end{bmatrix}$$

is an E-C^*-*isomorphism.*

The proof is a long calculation using Proposition 3.2.1 f).

Remarks. d) is contained in Proposition 3.2.3 c). An example with $\varepsilon = 1_E$ but different from a) is presented in Proposition 3.3.2.

Proposition 3.2.3. *We use the notation of* Proposition 3.2.1 *and take* $\varrho := (\alpha, \beta, \gamma, \varepsilon) \in A$.

a) *Let* $\varphi : \mathcal{S}(f_\varrho) \to E_{2,2}$ *be an* E-C^**-isomorphism and put*

$$\begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix} := \varphi V_t$$

for every $t \in \mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{(0,0)\}$. *Then* $\varepsilon = -1_E$, $A_t, B_t, C_t, D_t \in E^c$ *and* $A_t + D_t = 0$ *for every* $t \in \mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{(0,0)\}$, *and*

$$\begin{aligned}
&A_a^* = \beta^*\gamma^* A_a\,, \quad A_b^* = -\alpha^*\gamma^* A_b\,, \quad A_c^* = \alpha^*\beta^* A_c\,,\\
&B_a^* = \beta^*\gamma^* C_a\,, \quad B_b^* = -\alpha^*\gamma^* C_b\,, \quad B_c^* = \alpha^*\beta^* C_c\,,\\
&A_a^2 + B_a C_a = \beta\gamma\,, \quad A_b^2 + B_b C_b = -\alpha\gamma\,, \quad A_c^2 + B_c C_c = \alpha\beta\,,\\
&A_a^2 = \beta\gamma(1_E - |B_a|^2)\,, \; A_b^2 = -\alpha\gamma(1_E - |B_b|^2)\,, \; A_c^2 = \alpha\beta(1_E - |B_c|^2)\,,\\
&2A_aA_b + B_aC_b + B_bC_a = 0\,, \quad 2A_bA_c + B_bC_c + B_cC_b = 0\,,\\
&2A_cA_a + B_cC_a + B_aC_c = 0\,,\\
&\alpha A_a = A_bA_c + B_bC_c\,, \quad \alpha B_a = A_bB_c - A_cB_b\,, \quad \alpha C_a = A_cC_b - A_bC_c\,,\\
&\beta A_b = A_aA_c + B_aC_c\,, \quad \beta B_b = A_aB_c - A_cB_a\,, \quad \beta C_b = A_cC_a - A_aC_c\,,\\
&\gamma A_c = A_aA_b + B_aC_b\,, \quad \gamma B_c = A_aB_b - A_bB_a\,, \quad \gamma C_c = A_bC_a - A_aC_b\,,\\
&|A_a| + |A_b| + |A_c| \neq 0\,, \quad |B_a| + |B_b| + |B_c| \neq 3.1_E.
\end{aligned}$$

b) *Let* $(A_t)_{t\in T}$, $(B_t)_{t\in T}$, $(C_t)_{t\in T}$, $(D_t)_{t\in T}$ *be families in* E^c *satisfying the above conditions and put*

$$X' := A_aX_a + A_bX_b + A_cX_c, \quad X'' := B_aX_a + B_bX_b + B_cX_c,$$

$$X''' := C_aX_a + C_bX_b + C_cX_c$$

for every $X \in \mathcal{S}(f_\varrho)$. *If* $\varepsilon = -1_E$ *then the map*

$$\mathcal{S}(f_\varrho) \longrightarrow E_{2,2}, \quad X \longmapsto \begin{bmatrix} X_0 + X' & X'' \\ X''' & X_0 - X' \end{bmatrix}$$

is an E-C^**-isomorphism.*

c) *Let* $\varepsilon = -1_E$ *and assume there is an* $x \in E^c$ *with* $x^2 = \beta\gamma$. *Let* $y \in Un\, E^c$ *and put* $z := \gamma^* xy$. *Then* $x, y, z \in Un\, E^c$ *and the map*

$$\varphi : \mathcal{S}(f_\varrho) \longrightarrow E_{2,2}\,, \quad X \longmapsto \begin{bmatrix} X_0 + xX_a & \alpha(yX_b + zX_c) \\ -\gamma y^* X_b + \beta z^* X_c & X_0 - xX_a \end{bmatrix}$$

is an E-C^**-isomorphism such that*

$$\varphi(\frac{1}{2}(V_0 + (x^* \otimes 1_K)V_a)) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

In particular (by the symmetry of a,b,c), if $\varepsilon = -1_E$ *and if there is an* $x \in E^c$ *with* $x^2 = \beta\gamma$, *or* $x^2 = -\alpha\gamma$, *or* $x^2 = \alpha\beta$ *then* $\mathcal{S}(f_\varrho) \approx_E E_{2,2}$.

Remark. Take $\varrho := (1_E, 1_E, 1_E, -1_E)$, $\varrho' := (1_E, 1_E, \gamma', -1_E)$. By c), $\mathcal{S}(f_\varrho) \approx_E \mathcal{S}(f_{\varrho'})$ and by Proposition 3.2.1 $c_1 \Rightarrow c_2$, $\mathcal{S}(f_\varrho) \approx_{\mathcal{S}} \mathcal{S}(f_{\varrho'})$ implies the existence of an $x \in Un\, E^c$ with $x^2 = \gamma'$.

COROLLARY 3.2.4. *We use the notation of* Proposition 3.2.3 *and take* $E := \mathbb{K}$, $\alpha = 1$, *and* $\beta = \gamma = \varepsilon = -1$. *Let* S *be a group,* F *a unital* C^**-algebra,* $g \in \mathcal{F}(S, F)$, *and*

$$h : ((S \times (\mathbb{Z}_2)^2) \times (S \times (\mathbb{Z}_2)^2)) \longrightarrow Un\, F^c$$

$$((s_1, t_1), (s_2, t_2)) \longmapsto f_\varrho(t_1, t_2) g(s_1, s_2).$$

a) $h \in \mathcal{F}(S \times (\mathbb{Z}_2)^2, F)$.

b) $\mathcal{S}(h) \approx \mathcal{S}(g)_{2,2}\,, \qquad \mathcal{S}_{\|\cdot\|}(h) \approx \mathcal{S}_{\|\cdot\|}(g)_{2,2}$.

Proof. By Proposition 3.2.3 c), $\mathcal{S}(f) \approx \mathbb{K}_{2,2}$, so by Proposition 2.2.11 c),e),

$$\mathcal{S}(h) \approx \mathbb{K}_{2,2} \otimes \mathcal{S}(g) \approx \mathcal{S}(g)_{2,2}, \quad \mathcal{S}_{\|\cdot\|}(h) \approx \mathbb{K}_{2,2} \otimes \mathcal{S}_{\|\cdot\|}(g) \approx \mathcal{S}_{\|\cdot\|}(g)_{2,2}.$$

□

Example 3.2.5. Let $\mathbb{K} := \mathbb{C}$ and $E := \mathcal{C}(\mathbf{T}, \mathbb{C})$.

a) With the notation of *Proposition 3.2.1*, if $\varrho := (\alpha, \beta, \gamma, -1) \in A$ then $\mathcal{S}(f_\varrho) \approx_E E_{2,2}$.

b) $\mathrm{Card}\,(\{\, \mathcal{S}(\mathrm{f}) \mid \mathrm{f} \in \mathcal{F}(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathrm{E}) \,\} / \approx_{\mathcal{S}}) = 16$.

Proof. Put

$$m := w(\alpha), \quad n := w(\beta), \quad p := w(\gamma),$$

where w denotes the winding number. By Proposition 2.2.2 $a_1 \Rightarrow a_2$, we may assume $\alpha = z^m$, $\beta = z^n$, $\gamma = z^p$.

a) If $n + p$ is even then the assertion follows from Proposition 3.2.3 c). If $n + p$ is odd then either $m + p$ or $m + n$ is even and the assertion follows again from Proposition 3.2.3 c).

b) follows from Proposition 2.2.2 a),c). □

Remark. Assume $\mathbb{K} := \mathbb{R}$ and let E be the real C*-algebra $\mathcal{C}(\mathbf{T}, \mathbb{C})$ ([1, Theorem 4.1.1.8 a)]), $\varepsilon = -1_E$,

$$\alpha : \mathbf{T} \longrightarrow \mathbb{C}\,, \quad z \longmapsto z,$$

$$\beta : \mathbf{T} \longrightarrow \mathbb{C}\,, \quad z \longmapsto -z,$$
$$\gamma : \mathbf{T} \longrightarrow \mathbb{C}\,, \quad z \longmapsto \bar{z},$$

and $\varrho := (\alpha, \beta, \gamma, \varepsilon)$. Then by Corollary 3.2.2 c), $\mathcal{S}(f_\varrho) \approx \mathbb{H} \otimes E$.

Example 3.2.6. We put $E := \mathcal{C}(\mathbf{T}^2, \mathbb{C})$, $\gamma := 1_E$,

$\alpha : \mathbf{T}^2 \longrightarrow \mathbb{C}\,, \quad (z_1, z_2) \longmapsto z_1\,, \qquad \beta : \mathbf{T}^2 \longrightarrow \mathbb{C}\,, \quad (z_1, z_2) \longmapsto z_2\,,$

and (with the notation of *Proposition 3.2.1*) $\varrho := (\alpha, \beta, \gamma, -1_E) \in A$.

a) $\mathcal{S}(f_\varrho)$ is not commutative and not E-C*-isomorphic to $E_{2,2}$.

b) If we put
$$\tilde{x} : \mathbf{T}^2 \longrightarrow \mathbb{C}\,, \quad (z_1, z_2) \longmapsto x(z_1^2, z_2^2)$$
for every $x \in E$ then the map
$$\mathcal{S}(f_\varrho) \longrightarrow E_{2,2}, \quad X \longmapsto \begin{bmatrix} \tilde{X}_0 + \alpha\beta\tilde{X}_c & \beta\tilde{X}_a - \alpha\tilde{X}_b \\ \beta\tilde{X}_a + \alpha\tilde{X}_b & \tilde{X}_0 - \alpha\beta\tilde{X}_c \end{bmatrix}$$
is a C*-isomorphism.

c) $E_{2,2} \approx \mathcal{S}(f_\varrho) \not\approx_E E_{2,2}$.

Proof. a) By Proposition 3.2.1 d), $\mathcal{S}(f_\varrho)$ is not commutative. Assume $\mathcal{S}(f_\varrho) \approx_E E_{2,2}$ and let us use the notation of Proposition 3.2.3 a).

Step 1. $\{A_a \neq 0\} \subset \{A_b = 0\}$.
Assume $\{A_a \neq 0\} \cap \{A_b \neq 0\} \neq \emptyset$. By Proposition 3.2.3 a),
$$2A_aA_b + B_aC_b + B_bC_a = 0, \quad B_a^* = \beta^* C_a, \quad B_b^* = -\alpha^* C_b$$
so $B_a \neq 0$ and $B_b \neq 0$ on this set. We put
$$A_a =: |A_a| e^{i\tilde{A}_a},\ A_b =: |A_b| e^{i\tilde{A}_b},\ B_a =: |B_a| e^{i\tilde{B}_a},\ B_b =: |B_b| e^{i\tilde{B}_b},$$
$$z_1 =: e^{i\theta_1},\ z_2 =: e^{i\theta_2},$$
with $\tilde{A}_a$, $\tilde{A}_b$, $\tilde{B}_a$, $\tilde{B}_b \in \mathbb{R}$. By Proposition 3.2.3 a), $2\tilde{A}_a = \theta_2$, $2\tilde{A}_b = \theta_1 + \pi$,
$$B_aC_b + B_bC_a = -\alpha\gamma B_a B_b^* + \beta\gamma B_b B_a^* = |B_a||B_b|(e^{i(\theta_2 + \tilde{B}_b - \tilde{B}_a)} - e^{i(\theta_1 + \tilde{B}_a - \tilde{B}_b)})$$
$$= |B_a||B_b| e^{i\frac{\theta_1+\theta_2}{2}} (e^{i(\frac{\theta_2-\theta_1}{2} + \tilde{B}_b - \tilde{B}_a)} - e^{i(\frac{\theta_1-\theta_2}{2} + \tilde{B}_a - \tilde{B}_b)})$$
$$= 2|B_a||B_b| \sin(\frac{\theta_2 - \theta_1}{2} + \tilde{B}_b - \tilde{B}_a) e^{i\frac{\theta_1+\theta_2+\pi}{2}}.$$
Since $2A_aA_b = -(B_aC_b + B_bC_a)$ there is a $k \in \mathbb{Z}$ with
$$\frac{\theta_2}{2} + \frac{\theta_1 + \pi}{2} = \frac{\theta_1 + \theta_2 + \pi}{2} + (2k+1)\pi$$

which is a contradiction.

Step 2. $\{A_a \neq 0\} \subset \{A_c = 0\}$.
The assertion follows from Step 1 by symmetry.

Step 3. $\{A_a \neq 0\} = \{A_b = A_c = 0\}$.
The assertion follows from Steps 1 and 2 and from $|A_a| + |A_b| + |A_c| \neq 0$.
Step 4. The contradiction.
By Step 3 and by the symmetry, the sets $\{A_a \neq 0\}$, $\{A_b \neq 0\}$, and $\{A_c \neq 0\}$ are clopen and by $|A_a| + |A_b| + |A_c| \neq 0$ their union is equal to $\mathbf{T}^2$. So there is exactly one of these sets equal to $\mathbf{T}^2$ which implies

$$A_a^2 = z_2\,, \quad \text{or} \quad A_b^2 = -z_1 \quad \text{or} \quad A_c^2 = z_1 z_2$$

and no one of these identities can hold.
b) is a direct verification.
c) follows from a) and b). □

3.3. $T := (\mathbb{Z}_2)^n$ with $n \in \mathbb{N}$

Example 3.3.1. Assume f constant and put

$$\langle\, s \,|\, t \,\rangle := \prod_{i=1}^{n} (-1)^{s(i)t(i)}$$

for all $s, t \in T$ (where $\mathbb{Z}_2$ is identified with $\{0, 1\}$) and

$$\varphi_t : \mathcal{S}(f) \longrightarrow E\,, \quad X \longmapsto \sum_{s \in T} \langle\, t \,|\, s \,\rangle\, X_s$$

for all $t \in T$. Then the map

$$\varphi : \mathcal{S}(f) \longrightarrow E^{2^n}\,, \quad X \longmapsto (\varphi_t X)_{t \in T}$$

is an E-C*-isomorphism.

Proof. For $r, s, t \in T$,

$$t + t = 0,\ \langle\, s \,|\, t \,\rangle = \langle\, t \,|\, s \,\rangle\,,\ \langle\, r + s \,|\, t \,\rangle = \langle\, r \,|\, t \,\rangle \langle\, s \,|\, t \,\rangle\,,$$

$$\langle\, r \,|\, s + t \,\rangle = \langle\, r \,|\, s \,\rangle \langle\, r \,|\, t \,\rangle\,.$$

For $t \in T$ and $X, Y \in \mathcal{S}(f)$, by Theorem 2.1.9 c),g),

$$\varphi_t(X^*) = \sum_{s \in T} \langle\, t \,|\, s \,\rangle\, (X^*)_s = \sum_{s \in T} \langle\, t \,|\, s \,\rangle\, (X_s)^* = (\varphi_t X)^*,$$

$$(\varphi_t X)(\varphi_t Y) = \sum_{r,s \in T} \langle\, t \,|\, r \,\rangle \langle\, t \,|\, s \,\rangle\, X_r Y_s = \sum_{q,r \in T} \langle\, t \,|\, r \,\rangle \langle\, t \,|\, q - r \,\rangle\, X_r Y_{q-r}$$

$$= \sum_{q,r\in T} \langle\, t \,|\, q \,\rangle\, X_r Y_{q-r} = \sum_{q\in T} \langle\, t \,|\, q \,\rangle\, (XY)_q = \varphi_t(XY)$$

so φ_t and φ are E-C*-homomorphisms.

We have

$$\sum_{t\in T} \langle\, 0 \,|\, t \,\rangle = 2^n\ .$$

We want to prove

$$\sum_{t\in T} \langle\, s \,|\, t \,\rangle = 0$$

for all $s \in T$, $s \neq 0$, by induction with respect to $\mathrm{Card}\,\{\, \mathrm{i} \in \mathbb{N}_\mathrm{n} \mid \mathrm{s(i)} \neq 0 \,\}$. Let $i \in \mathbb{N}_n$ with $s(i) \neq 0$ and put $r := s + e_i$,

$$T_0 := \{\, t \in T \mid t(i) = 0 \,\}\ , \qquad T_1 := \{\, t \in T \mid t(i) = 1 \,\}\ .$$

Then

$$\sum_{t\in T_0} \langle\, s \,|\, t \,\rangle = \sum_{t\in T_0} \langle\, r \,|\, t \,\rangle\ , \qquad \sum_{t\in T_1} \langle\, s \,|\, t \,\rangle = -\sum_{t\in T_1} \langle\, r \,|\, t \,\rangle\ .$$

But

$$\sum_{t\in T_0} \langle\, r \,|\, t \,\rangle = \sum_{t\in T_1} \langle\, r \,|\, t \,\rangle = 2^{n-1}$$

if $r = 0$. By the hypothesis of the induction

$$\sum_{t\in T_0} \langle\, r \,|\, t \,\rangle = \sum_{t\in T_1} \langle\, r \,|\, t \,\rangle = 0$$

if $r \neq 0$ (with $\mathbb{N}_n$ replaced by $\mathbb{N}_n \setminus \{i\}$, since $r(i) = 0$). This finishes the proof by induction.

For $r \in T$ and $X \in \mathcal{S}(f)$, by the above,

$$\sum_{t\in T} \langle\, r \,|\, t \,\rangle\, \varphi_t X = \sum_{s,t\in T} \langle\, r \,|\, t \,\rangle \langle\, t \,|\, s \,\rangle\, X_s = \sum_{s,t\in T} \langle\, r+s \,|\, t \,\rangle\, X_s$$

$$= \sum_{s\in T\setminus\{r\}} \sum_{t\in T} \langle\, r+s \,|\, t \,\rangle\, X_s + \sum_{t\in T} \langle\, 0 \,|\, t \,\rangle\, X_r = 2^n X_r.$$

Hence φ is bijective. $\square$

Example 3.3.2. Let $E := \mathcal{C}(\mathbf{T}^n, \mathbb{C})$, denote by $z := (z_1, z_2, \cdots, z_n)$ the points of $\mathbf{T}^n$, and put $z^2 := (z_1^2, z_2^2, \cdots, z_n^2)$ for every $z \in \mathbf{T}^n$. We identify $(\mathbb{Z}_2)^n$ with $\mathfrak{P}(\mathbb{N}_n)$ by using the bijection

$$\mathfrak{P}(\mathbb{N}_n) \longrightarrow (\mathbb{Z}_2)^n, \quad I \longmapsto e_I$$

and denote by

$$I \triangle J := (I \setminus J) \cup (J \setminus I)$$

the addition on $\mathfrak{P}(\mathbb{N}_n)$ corresponding to this identification. We put $\lambda_I := \prod\limits_{i\in I} z_i$ for every $I \subset \mathbb{N}_n$ and

$$f : \mathfrak{P}(\mathbb{N}_n) \times \mathfrak{P}(\mathbb{N}_n) \longrightarrow Un\ E^c\,, \quad (I, J) \longmapsto \lambda_{I\cap J}\ .$$

Then $f \in \mathcal{F}((\mathbb{Z}_2)^n, E)$ and, if we put

$$\tilde{X} := \sum_{I\subset \mathbb{N}_n} \lambda_I(z) X_I(z^2) \in E$$

for every $X \in \mathcal{S}(f)$, the map

$$\varphi : \mathcal{S}(f) \longrightarrow E\,, \quad X \longmapsto \tilde{X}$$

is an isomorphism of C*-algebras.

Proof. Let $X, Y \in \mathcal{S}(f)$. By Theorem 2.1.9 c),g),

$$\widetilde{X^*} = \sum_{I\subset \mathbb{N}_n} \lambda_I\,(X^*)_I(z^2) = \sum_{I\subset \mathbb{N}_n} \lambda_I\,\overline{\lambda_I^2}\,X_I^* = \overline{\tilde{X}},$$

$$\begin{aligned}\widetilde{XY} &= \sum_{I\subset \mathbb{N}_n} \lambda_I\,(XY)_I(z^2) = \sum_{I\subset \mathbb{N}_n} \lambda_I \sum_{J\subset \mathbb{N}_n} f(J, J\triangle I)^2 X_J Y_{J\triangle I}\\ &= \sum_{J,K\subset \mathbb{N}_n} \lambda_{J\triangle K}\lambda_{J\cap K}^2 X_J Y_K = \sum_{J,K\subset \mathbb{N}_n} \lambda_J\lambda_K X_J Y_K = \tilde{X}\tilde{Y}\end{aligned}$$

so φ is a C*-homomorphism.

We put for $k \in \mathbb{N}_n$, $i \in \mathbb{Z}^n$, and $I \subset \mathbb{N}_n$,

$$i_k^I := \left\{ \begin{array}{lll} 2i_k + 1 & \text{if} & k \in I \\ 2i_k & \text{if} & k \in \mathbb{N}_n \setminus I \end{array}\right. , \qquad i^I := (i_1^I, i_2^I, \cdots, i_n^I) \in \mathbb{Z}^n$$

and

$$\mathcal{G} := \left\{ \sum_{i\in\mathbb{Z}^n} a_i z_1^{i_1} z_2^{i_2} \cdots z_n^{i_n} \;\middle|\; (a_i)_{i\in\mathbb{Z}^n} \in \mathbb{C}^{(\mathbb{Z}^n)} \right\} .$$

Let

$$x := \sum_{i\in\mathbb{Z}^n} a_i z_1^{i_1} z_2^{i_2} \cdots z_n^{i_n} \in \mathcal{G}$$

and for every $I \subset \mathbb{N}_n$ put

$$X_I := \sum_{i\in\mathbb{Z}^n} a_{i^I} z_1^{i_1} z_2^{i_2} \cdots z_n^{i_n}\,, \qquad X := \sum_{I\subset \mathbb{N}_n} (X_I \otimes 1_K) V_I\ .$$

Then $\varphi X = x$ and so $\mathcal{G} \subset \varphi(\mathcal{S}(f))$. Since $\mathcal{G}$ is dense in E, it follows that φ is surjective.

We prove that φ is injective by induction with respect to $n \in \mathbb{N}$. The case $n = 1$ was proved in Example 3.1.4. Assume the assertion holds for $n-1$. Let $X \in Ker\, \varphi$. Then

$$\sum_{I \subset \mathbb{N}_n} \lambda_I(z) X_I(z^2) = 0 \,.$$

By replacing z_n by $-z_n$ in the above relation, we get

$$\sum_{I \subset \mathbb{N}_{n-1}} \lambda_I(z) X_I(z^2) - \sum_{n \in I \subset \mathbb{N}_n} \lambda_I(z) X_I(z^2) = 0$$

and so

$$\sum_{I \subset \mathbb{N}_{n-1}} \lambda_I(z) X_I(z^2) = \sum_{n \in I \subset \mathbb{N}_n} \lambda_I(z) X_I(z^2) = 0 \,.$$

By the induction hypothesis, we get $X_I = 0$ for all $I \subset \mathbb{N}_n$ and so $X = 0$. Thus φ is injective and a C*-isomorphism. □

Example 3.3.3. Let $f \in \mathcal{F}((\mathbb{Z}_2)^3, E)$, put

$$a := (0,0,1)\,, \quad b := (0,1,0)\,, \quad c := (0,1,1)\,, \quad s := (1,0,0)\,,$$

and denote by g the element of $\mathcal{F}(\mathbb{Z}_2, E)$ defined by $g(1,1) := f(s,s)$ *Proposition 3.1.1 a).*

a) There is a family $(\alpha_i\,, \beta_i\,, \gamma_i\,, \varepsilon_i)_{i \in \mathbb{N}_7}$ in $(Un\; E^c)^4$ such that f is given by the attached table and such that $\varepsilon_i^2 = 1_E$ for every $i \in \mathbb{N}_7$ and

$$\varepsilon_3 = \varepsilon_1\varepsilon_2, \quad \varepsilon_5 = \varepsilon_1\varepsilon_4, \quad \varepsilon_6 = \varepsilon_2\varepsilon_4, \quad \varepsilon_7 = \varepsilon_1\varepsilon_2\varepsilon_4,$$
$$\alpha_3 = \varepsilon_2\varepsilon_4\alpha_1\alpha_2^*\alpha_4\alpha_6\gamma_2^*, \quad \alpha_5 = \alpha_6\beta_1\gamma_2^*, \quad \alpha_7 = \alpha_4\gamma_1\gamma_2^*,$$
$$\beta_2 = \beta_1\gamma_1\gamma_2^*, \quad \beta_3 = \varepsilon_2\alpha_4^*\alpha_6\beta_1\,, \quad \beta_4 = \varepsilon_1\varepsilon_2\varepsilon_4\alpha_1\alpha_2^*\alpha_4\gamma_1\gamma_2^*,$$
$$\beta_5 = \varepsilon_4\alpha_1\alpha_2^*\alpha_6, \quad \beta_6 = \varepsilon_4\alpha_1\alpha_2^*\alpha_6\beta_1\gamma_2^*\,, \quad \beta_7 = \varepsilon_1\varepsilon_2\varepsilon_4\alpha_1\alpha_2^*\alpha_6,$$
$$\gamma_3 = \varepsilon_2\alpha_4\alpha_6^*\gamma_1, \quad \gamma_4 = \varepsilon_2\varepsilon_4\alpha_2\alpha_4^*\gamma_1^*\gamma_2, \quad \gamma_5 = \varepsilon_1\varepsilon_4\alpha_2\alpha_6^*\gamma_1,$$
$$\gamma_6 = \varepsilon_4\alpha_2\alpha_6^*\gamma_2, \quad \gamma_7 = \varepsilon_1\varepsilon_2\varepsilon_4\alpha_2\alpha_4^*\beta_1.$$

f	a	b	c	s	$a+s$	$b+s$	$c+s$
a	$\beta_1\gamma_1$	γ_1	β_1	γ_2	β_2	γ_3	β_3
b	$\varepsilon_1\gamma_1$	$\varepsilon_1\alpha_1\gamma_1$	α_1	γ_4	γ_5	β_4	β_5
c	$\varepsilon_1\beta_1$	$\varepsilon_1\alpha_1$	$\alpha_1\beta_1$	γ_6	γ_7	β_7	β_6
s	$\varepsilon_2\gamma_2$	$\varepsilon_4\gamma_4$	$\varepsilon_6\gamma_6$	$\varepsilon_2\alpha_2\gamma_2$	α_2	α_4	α_6
$a+s$	$\varepsilon_2\beta_2$	$\varepsilon_5\gamma_5$	$\varepsilon_7\gamma_7$	$\varepsilon_2\alpha_2$	$\alpha_2\beta_2$	α_7	α_5
$b+s$	$\varepsilon_3\gamma_3$	$\varepsilon_4\gamma_4$	$\varepsilon_7\gamma_7$	$\varepsilon_4\alpha_4$	$\varepsilon_7\alpha_7$	$\varepsilon_3\alpha_3\gamma_3$	α_3
$c+s$	$\varepsilon_3\beta_3$	$\varepsilon_5\beta_5$	$\varepsilon_6\beta_6$	$\varepsilon_6\beta_6$	$\varepsilon_5\alpha_5$	$\varepsilon_3\alpha_3$	$\alpha_3\beta_3$

b) If $\varepsilon_1 = -1_E$, $\varepsilon_2 = \varepsilon_4$, $\gamma_1 = 1_E$, and there is an $x \in E^c$ with $x^2 = \alpha_1\beta_1^*$ then there are $P_\pm \in (E\widetilde{\otimes}1_K)^c \cap Pr\,\mathcal{S}(f)$ with $P_+ + P_- = V_1^f$ and (Theorem 2.2.18 b))

$$P_+\mathcal{S}(f)P_+ \approx_E \mathcal{S}(g) \approx_E P_-\mathcal{S}(f)P_- \,.$$

c) If $\varepsilon_1 = -1_E$, $\varepsilon_2 = \varepsilon_4 = \gamma_1 = 1_E$, and there is an $x \in E^c$ with $x^2 = \alpha_1\beta_1^*$ then $\mathcal{S}(f) \approx_E \mathcal{S}(g)_{2,2}$.

d) Assume $\varepsilon_1 = -1_E$, $\varepsilon_2 = \varepsilon_4 = \alpha_1 = \beta_1 = \gamma_1 = 1_E$, $\gamma_2 = \alpha_2^*$, and $\alpha_2^4 = \alpha_4^4 = \alpha_6 = 1_E$ and put $\varphi_\pm : \mathcal{S}(f) \longrightarrow E_{2,2}$

$$X \mapsto \begin{bmatrix} X_0 + X_c \pm X_s \pm X_{c+s} & X_a - X_b \pm \alpha_2^* X_{a+s} \mp \alpha_4^* X_{b+s} \\ X_a + X_b \pm \alpha_2^* X_{a+s} \pm \alpha_4^* X_{b+s} & X_0 - X_c \pm X_s \mp X_{c+s} \end{bmatrix}.$$

Then the map

$$\mathcal{S}(f) \longrightarrow E_{2,2} \times E_{2,2}, \quad X \longmapsto (\varphi_+ X, \varphi_- X)$$

is an E-C*-isomorphism.

Proof. a) is a long calculation.
b) and c) follow from a) and Theorem 2.2.18 e).
d) is a long calculation using a). □

3.4. $T := \mathbb{Z}_n$ with $n \in \mathbb{N}$

PROPOSITION 3.4.1. *Put $A := Un\,E^c$ and for every $\alpha \in A^{n-1}$ put*

$$f_\alpha : \mathbb{Z}_n \times \mathbb{Z}_n \longrightarrow A\,, \quad (p,q) \longmapsto \left(\prod_{j=p}^{p+q-1} \alpha_j\right)\left(\prod_{k=1}^{q-1} \alpha_k^*\right),$$

where $\mathbb{Z}_n$ and $\mathbb{N}_n$ are canonically identified and $\alpha_n := 1_E$.

a) *For every $f \in \mathcal{F}(\mathbb{Z}_n, E)$ and $X \in \mathcal{S}(f)$, $X \in \mathcal{S}(f)^c$ iff $X_t \in E^c$ for all $t \in T$. In particular, $\mathcal{S}(f)$ is commutative if E is commutative.*

b) *$f_\alpha \in \mathcal{F}(\mathbb{Z}_n, E)$ for every $\alpha \in A^{n-1}$ and the map*

$$A^{n-1} \longrightarrow \mathcal{F}(\mathbb{Z}_n, E), \quad \alpha \longmapsto f_\alpha$$

is a group isomorphism.

c) *The following are equivalent for all $\alpha, \beta \in A^{n-1}$.*

c$_1$) $\mathcal{S}(f_\alpha) \approx_{\mathcal{S}} \mathcal{S}(f_\beta)$.

c$_2$) *There is a* $\gamma \in A$ *such that*

$$\gamma^n = \prod_{j=1}^{n-1} (\alpha_j \beta_j^*) \,.$$

c$_3$) *There is a* $\lambda \in \Lambda(\mathbb{Z}_n, E)$ *such that* $f_\alpha = f_\beta \delta\lambda$.

If these equivalent conditions are fulfilled then the map

$$\mathcal{S}(f_\alpha) \longrightarrow \mathcal{S}(f_\beta), \quad X \longmapsto U_\lambda^* X U_\lambda$$

is an $\mathcal{S}$*-isomorphism and*

$$\lambda(1)^n = \prod_{j=1}^{n-1} (\alpha_j \beta_j^*) = \gamma^n \,, \quad p \in \mathbb{Z}_n \Longrightarrow \lambda(p) = \lambda(1)^p \prod_{j=1}^{p-1} (\alpha_j^* \beta_j) \,.$$

d) *Let* $\alpha \in A^{n-1}$ *and put*

$$\beta : \mathbb{N}_{n-1} \longrightarrow A \,, \quad j \longmapsto \begin{cases} 1 & \text{if } \ j < n-1 \\ \left(\prod\limits_{k=1}^{n-1} \alpha_k^* \right)^{n-1} & \text{if } \ j = n-1 \end{cases} .$$

Then α *and* β *fulfill the equivalent conditions of c).*

e) *There is a natural bijection*

$$\{\, \mathcal{S}(f) \mid f \in \mathcal{F}(\mathbb{Z}_n, E) \,\} / \approx_{\mathcal{S}} \longrightarrow A / \{\, x^n \mid x \in A \,\} \,.$$

If $E := \mathcal{C}(\mathbf{T}^m, \mathbb{C})$ *for some* $m \in \mathbb{N}$ *then*

$$\mathrm{Card}\,(\{\, \mathcal{S}(\mathrm{f}) \mid \mathrm{f} \in \mathcal{F}(\mathbb{Z}_\mathrm{n}, \mathrm{E}) \,\} / \approx_{\mathcal{S}}) = \mathrm{mn} \,.$$

f) *Let* $\alpha \in A^{n-1}$, $\beta \in A$ *such that* $\beta^n = \prod\limits_{j=1}^{n-1} \alpha_j$,

$$F := \begin{cases} E & \text{if } \ \mathbb{K} = \mathbb{C} \\ \overset{\circ}{E} & \text{if } \ \mathbb{K} = \mathbb{R} \end{cases} ,$$

where $\overset{\circ}{E}$ *denotes the complexification of* E*, and*

$$w_k : \mathcal{S}(f_\alpha) \longrightarrow F \,, \quad X \longmapsto \sum_{j=1}^{n} \beta^j \left(\prod_{l=1}^{j-1} \bar{\alpha}_l \right) e^{\frac{2\pi i j k}{n}} X_j$$

for every $k \in \mathbb{N}_n (= \mathbb{Z}_n)$.

f$_1$) *If* $\mathbb{K} = \mathbb{C}$ *then the map*

$$\mathcal{S}(f_\alpha) \longrightarrow E^n, \quad X \longmapsto (w_k X)_{k \in \mathbb{Z}_n}$$

is an E*-*C^**-isomorphism.*

f$_2$) *If* $\mathbb{K}=\mathbb{R}$ *and* n *is odd then we may take* $\beta\in\mathbb{R}$ *and the map*

$$\mathcal{S}(f_\alpha)\longrightarrow E\times(\overset{\circ}{E})^{\frac{n-1}{2}},\quad X\longmapsto(w_nX,(w_kX)_{k\in\mathbb{N}_{\frac{n-1}{2}}})$$

is an E*-*C^**-isomorphism.*

f$_3$) *If* $\mathbb{K}=\mathbb{R}$, n *is even, and* $\prod_{j=1}^{n-1}\alpha_j=-1$ *then the map*

$$\mathcal{S}(f_\alpha)\longrightarrow(\overset{\circ}{E})^{\frac{n}{2}},\quad X\longmapsto(w_{k-1}X)_{k\in\mathbb{N}_{\frac{n}{2}}}$$

is an E*-*C^**-isomorphism.*

f$_4$) *If* $\mathbb{K}=\mathbb{R}$, n *is even, and* $\prod_{j=1}^{n-1}\alpha_j=1$, *and* $\beta=1$ *then the map*

$$\mathcal{S}(f_\alpha)\longrightarrow E\times E\times(\overset{\circ}{E})^{\frac{n}{2}-1},\quad X\longmapsto(w_nX,w_{\frac{n}{2}}X,(w_kX)_{k\in\mathbb{N}_{\frac{n}{2}-1}})$$

is an E*-*C^**-isomorphism.*

f$_5$) *If* n *is even then there is a* $\gamma\in A$ *such that* $f_\alpha(\frac{n}{2},\frac{n}{2})=\gamma^2$.

Example 3.4.2. Let $E:=\mathcal{C}(\mathbf{T},\mathbb{C})$, $r\in\mathbb{Z}^{n-1}$, $z:\mathbf{T}\to\mathbb{C}$ the canonical inclusion, and

$$f:\mathbb{Z}_n\times\mathbb{Z}_n\longrightarrow Un\,E^c,\quad (p,q)\longmapsto z^{\left(\sum_{j=p}^{p+q-1}r_j-\sum_{j=1}^{q-1}r_j\right)},$$

where $\mathbb{Z}_n$ and $\mathbb{N}_n$ are canonically identified. Then $f\in\mathcal{F}(\mathbb{Z}_n,E)$. Let further S be the subgroup of $\mathbb{Z}_n$ generated by $\rho(\sum_{j=1}^{n-1}r_j)$, where $\rho:\mathbb{Z}\to\mathbb{Z}_n$ is the quotient map,

$$m:=\mathrm{Card\,S}\,,\qquad \mathrm{h}:=\frac{\mathrm{n}}{\mathrm{m}}\,,\qquad \omega:=\mathrm{e}^{\frac{2\pi\mathrm{i}}{\mathrm{n}}}\,,$$

$$\sigma:\mathbb{N}_n\longrightarrow\mathbb{Z}\,,\quad p\longmapsto\frac{p}{h}\sum_{j=1}^{n-1}r_j-m\sum_{j=1}^{p-1}r_j\,,$$

and

$$\varphi_k:\mathcal{S}(f)\longrightarrow E\,,\quad X\longmapsto\sum_{p=1}^{n}(X_p\circ z^m)z^{\sigma(p)}\omega^{pk}$$

for every $k\in\mathbb{N}_h$. Then the map

$$\varphi:\mathcal{S}(f)\longrightarrow E^h\,,\quad X\longmapsto(\varphi_kX)_{k\in\mathbb{N}_h}$$

is an E-C*-isomorphism.

The next example shows that the set $\{\,\mathcal{S}(f)\mid f\in\mathcal{F}(\mathbb{Z}_n,\mathcal{C}(\mathbf{T},\mathbb{C}))\,\}$ is not reduced by restricting the Schur functions to have the form indicated in Example 3.4.2.

Example 3.4.3. Let $E := \mathcal{C}(\mathbf{T}, \mathbb{C})$ and $g \in \mathcal{F}(\mathbb{Z}_n, E)$. Put

$$\varphi : [0, 2\pi[\longrightarrow \mathbb{R}\,, \quad \theta \longmapsto \log \prod_{j=1}^{n-1} (g(j,1))(e^{i\theta})\,,$$

where we take a fixed (but arbitrary) branch of log. If we define

$$r : \mathbb{N}_{n-1} \longrightarrow \mathbb{Z}\,, \quad j \longmapsto \begin{cases} \lim\limits_{\theta \to 2\pi} \varphi(\theta) - \varphi(0) & \text{if} \quad j = 1 \\ 0 & \text{if} \quad j \neq 1 \end{cases}$$

then there is a $\lambda \in \Lambda(\mathbb{Z}_n, E)$ such that $g = f\delta\lambda$, where f is the Schur function defined in *Example* 3.4.2. In particular, $\mathcal{S}(f) \approx_{\mathcal{S}} \mathcal{S}(g)$.

3.5. $T := \mathbb{Z}$

Example 3.5.1. Let $f \in \mathcal{F}(\mathbb{Z}, E)$.

a) $\mathcal{S}_{\|\cdot\|}(f) \approx \mathcal{C}(\mathbf{T}, E)$.

b) If E is a W*-algebra then

$$\mathcal{S}_W(f) \approx E \bar{\otimes} L^\infty(\mu) \approx L^\infty(\mu, E)\,,$$

where μ denotes the Lebesgue measure on $\mathbf{T}$.

Proof. By Corollary 1.1.6 c) and Proposition 2.2.2 $a_1 \Rightarrow a_2$, we may assume f constant. By Proposition 2.2.10 c),e), we may assume $E := \mathbb{C}$. Let $\alpha : \mathbf{T} \to \mathbb{C}$ be the inclusion map. Then

$$l^2(\mathbb{Z}) \longrightarrow L^2(\mu), \quad \xi \longmapsto \sum_{n \in \mathbb{Z}} \xi_n \alpha^n$$

is an isomorphism of Hilbert spaces. If we identify these Hilbert spaces using this isomorphism then V_1 becomes the multiplicator operator

$$L^2(\mu) \longrightarrow L^2(\mu), \quad \eta \longmapsto \alpha\eta$$

so

$$\mathcal{R}(f) \longrightarrow L^\infty(\mu), \quad X \longmapsto \sum_{n \in \mathbb{Z}} X_n \alpha^n$$

is an injective, involutive algebra homomorphism. The assertion follows. □

4. CLIFFORD ALGEBRAS

4.1. The general case

Throughout this subsection, I is a totally ordered set, $(T_i)_{i\in I}$ is a family of groups, and $(f_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}(T_i, E)$. We put

$$\bar{t} := \{\, i \in I \mid t_i \neq 1_i \,\}$$

for every $t \in \prod\limits_{i\in I} T_i$ (where 1_i denotes the neutral element of T_i) and

$$T := \left\{ t \in \prod_{i\in I} T_i \;\middle|\; \bar{t} \text{ is finite} \right\}, \quad T' := \{\, t \in T \mid t^2 = 1 \,\}.$$

An associated $f \in \mathcal{F}(T, E)$ will be defined in Proposition 4.1.1 b).

T is a subgroup of $\prod\limits_{i\in I} T_i$. We canonically associate to every element $t \in T$ in a bijective way the "word" $t_{i_1} t_{i_2} \cdots t_{i_n}$, where

$$\{i_1, i_2, \cdots, i_n\} = \bar{t} \qquad \text{and} \qquad i_1 < i_2 < \cdots < i_n$$

and use sometimes this representation instead of t (to $1 \in T$ we associate the "empty word").

PROPOSITION 4.1.1. a) *Let $t_{i_1} t_{i_2} \cdots t_{i_n}$ be a finite sequence of letters with $t_{i_j} \in T_{i_j} \setminus \{1_{i_j}\}$ for every $j \in \mathbb{N}_n$ and use transpositions of successive letters* with distinct indices *in order to bring these indices in an increasing order. If τ denotes the number of used transpositions then $(-1)^\tau$ does not depend on the manner in which this operation was done.*

b) *Let $s, t \in T$ and let*

$$s_{i_1} s_{i_2} \cdots s_{i_m}\,, \qquad t_{i'_1} t_{i'_2} \cdots t_{i'_n}$$

be the canonically associated words of s and t, respectively. We put for every $k \in I$, $\tilde{s}_k := s_{i_j}$ if there is a $j \in \mathbb{N}_m$ with $k = i_j$ and $\tilde{s}_k := 1_k$ if the above condition is not fulfilled and define $\tilde{t}$ in a similar way. Moreover, we put (Proposition 1.1.2 a))

$$f(s,t) := (-1)^\tau \prod_{k\in I} f_k(\tilde{s}_k, \tilde{t}_k)\,,$$

where τ denotes the number of transpositions of successive letters with distinct indices *in the finite sequence of letters*

$$s_{i_1} s_{i_2} \cdots s_{i_m} t_{i'_1} t_{i'_2} \cdots t_{i'_n}$$

in order to bring the indices in an increasing order. Then $f \in \mathcal{F}(T, E)$.

c) *Let I_0 be a subset of I, T_0 the subgroup $\{\, t \in T \mid \bar{t} \subset I_0 \,\}$ of T, and f_0 the element of $\mathcal{F}(T_0, E)$ defined in a similar way as f was defined in b). Then $f_0 = f|(T_0 \times T_0)$ and the map*

$$\mathcal{S}_{\|\cdot\|}(f_0) \longrightarrow \mathcal{S}_{\|\cdot\|}(f), \quad \sum_{t\in T_0}^{\|\cdot\|} (X_t \widetilde{\otimes} 1_K) V_t^{f_0} \longmapsto \sum_{t\in T_0}^{\|\cdot\|} (X_t \widetilde{\otimes} 1_K) V_t^{f}$$

is an injective E-C^{**}-homomorphism with image

$$\{\, X \in \mathcal{S}(f) \mid (t \in T \,\&\, X_t \neq 0) \Rightarrow t \in T_0 \,\}\ .$$

Proof. a) We define a new total order relation on the indices of the given word by putting for all $j,k \in \mathbb{N}_n$

$$i_j \prec i_k :\Longleftrightarrow ((i_j < i_k) \text{ or } (i_j = i_k \text{ and } j < k))\ .$$

Let P be a sequence of transpositions of successive letters in order to bring the indices in an increasing form with respect to the new order and let τ' be the number of used transpositions. Then $\tau - \tau'$ is even and so $(-1)^\tau = (-1)^{\tau'}$. By the theory of permutations $(-1)^{\tau'}$ does not depend on P, which proves the assertion.

b) By a), f is well-defined. Let $r,s,t \in T$ and let

$$r_{i_1} r_{i_2} \cdots r_{i_m}\,, \qquad s_{i'_1} s_{i'_2} \cdots s_{i'_n}\,, \qquad t_{i''_1} t_{i''_2} \cdots t_{i''_p}$$

be the words canonically associated to r, s, and t, respectively. There are $\alpha, \beta \in \{-1,+1\}$ such that

$$f(r,s)f(rs,t) = \alpha \prod_{i\in I} f(\tilde{r}_i, \tilde{s}_i) f(\widetilde{r_i s_i}, \tilde{t}_i)\,,$$

$$f(r,st)f(s,t) = \beta \prod_{i\in I} f_i(\tilde{r}_i, \widetilde{s_i t_i}) f(\tilde{s}_i, \tilde{t}_i)\ .$$

Write the finite sequence of letters

$$r_{i_1} r_{i_2} \cdots r_{i_m} s_{i'_1} s_{i'_2} \cdots s_{i'_n} t_{i''_1} t_{i''_2} \cdots t_{i''_p}$$

and use transpositions of successive letters with distinct indices in order to bring the indices in an increasing order. We can do this acting first on the letters of r and s only and then in a second step also on the letters of t. Then $\alpha = (-1)^\mu$, where μ denotes the number of all performed transpositions. For β we may start first with the letters of s and t and then in a second step also with the letters of r. Then $\beta = (-1)^\nu$, where ν is the number of all effectuated transpositions. By a), $\alpha = (-1)^\mu = (-1)^\nu = \beta$. The rest of the proof is obvious.

c) follows from Corollary 2.1.17 d). □

COROLLARY 4.1.2. *If $I := \mathbb{N}_2$ then for all $s,t \in T$,*

$$f(s,t) = \begin{cases} f_1(s_1,t_1) & \text{if} \quad s_2 = 1_2 \\ f_2(s_2,t_2) & \text{if} \quad t_1 = 1_1 \\ -f_1(s_1,t_1) f_2(s_2,t_2) & \text{if} \quad s_2 \neq 1_2\,, t_1 \neq 1_1 \end{cases}\ .$$

PROPOSITION 4.1.3. *Let $s,t \in T$.*

a) $f(s,t) = (-1)^{\mathrm{Card}\,(\bar{\mathrm{s}}\times\bar{\mathrm{t}}) - \mathrm{Card}\,(\bar{\mathrm{s}}\cap\bar{\mathrm{t}})} f(t,s)$.

b) *$st = ts$ iff* $V_s V_t = (-1)^{\mathrm{Card}\,(\bar{\mathrm{s}}\times\bar{\mathrm{t}}) - \mathrm{Card}\,(\bar{\mathrm{s}}\cap\bar{\mathrm{t}})} V_t V_s$.

c) *Assume* $\bar{s} \subset \bar{t}$. *If* $\mathrm{Card}\,\bar{\mathrm{s}}$ *is even or if* $\mathrm{Card}\,\bar{\mathrm{t}}$ *is odd then* $f(s,t) = f(t,s)$. *If in addition* $st = ts$ *then* $V_s V_t = V_t V_s$.

d) *If* $\mathrm{Card}\,\mathrm{I}$ *is an odd natural number and* T *is commutative then* $V_t \in \mathcal{S}(f)^c$ *for every* $t \in T$ *with* $\bar{t} = I$.

e) *Assume* $t \in T'$. *If* $n := \mathrm{Card}\,\bar{\mathrm{t}}$ *and* $\alpha := \prod_{i\in\bar{t}} f_i(t_i, t_i)$ *then*

$$f(t,t) = (-1)^{\frac{n(n-1)}{2}} \alpha\,, \qquad \tilde{f}(t) = (-1)^{\frac{n(n-1)}{2}} \alpha^*\,,$$
$$(V_t)^2 = (-1)^{\frac{n(n-1)}{2}} (\alpha \widetilde{\otimes} 1_K) V_1\,, \qquad V_t^* = (-1)^{\frac{n(n-1)}{2}} (\alpha^* \widetilde{\otimes} 1_K) V_t\,.$$

Proof. a) For $i \in \bar{s}$,

$$f(s_i, t) = \begin{cases} (-1)^{\mathrm{Card}\,\bar{\mathrm{t}}} f(t, s_i) & \text{if} \quad i \notin \bar{t} \\ (-1)^{\mathrm{Card}\,\bar{\mathrm{t}}-1} f(t, s_i) & \text{if} \quad i \in \bar{t} \end{cases}$$

so

$$f(s,t) = (-1)^{\mathrm{Card}\,(\bar{\mathrm{s}}\times\bar{\mathrm{t}}) - \mathrm{Card}\,(\bar{\mathrm{s}}\cap\bar{\mathrm{t}})} f(t,s).$$

b) By Proposition 2.1.2 b),

$$V_s V_t = (f(s,t) \widetilde{\otimes} 1_K) V_{st}\,, \qquad V_t V_s = (f(t,s) \widetilde{\otimes} 1_K) V_{ts}\,.$$

Thus if $st = ts$ then by a),

$$V_s V_t = ((f(s,t) f(t,s)^*) \widetilde{\otimes} 1_K) V_t V_s = (-1)^{\mathrm{Card}\,(\bar{\mathrm{s}}\times\bar{\mathrm{t}}) - \mathrm{Card}\,(\bar{\mathrm{s}}\cap\bar{\mathrm{t}})} V_t V_s\,.$$

Conversely, if this relation holds then by a),

$$\begin{aligned} V_{st} &= (f(s,t)^* \widetilde{\otimes} 1_K) V_s V_t = (-1)^{\mathrm{Card}\,(\bar{\mathrm{s}}\times\bar{\mathrm{t}}) - \mathrm{Card}\,(\bar{\mathrm{s}}\cap\bar{\mathrm{t}})} (f(t,s)^* \widetilde{\otimes} 1_K) V_s V_t \\ &= (f(t,s)^* \widetilde{\otimes} 1_K) V_t V_s = V_{ts} \end{aligned}$$

and we get $st = ts$ by Theorem 2.1.9 a).

c) follows from a) and b).

d) follows from c) (and Proposition 2.1.2 d)).

e) We have

$$f(t,t) = (-1)^{(n-1)+\cdots+2+1} \alpha = (-1)^{\frac{n(n-1)}{2}} \alpha\,.$$

By Proposition 2.1.2 b),e),

$$(V_t)^2 = (f(t,t) \widetilde{\otimes} 1_K) V_1 = (-1)^{\frac{n(n-1)}{2}} (\alpha \widetilde{\otimes} 1_K) V_1\,,$$
$$V_t^* = \tilde{f}(t) V_{t^{-1}} = f(t,t)^* V_t = (-1)^{\frac{n(n-1)}{2}} (\alpha^* \widetilde{\otimes} 1_K) V_t.$$

□

Proposition 4.1.4. *Let S be a finite subset of $T' \setminus \{1\}$ such that $st = ts$ and* Card $(\bar{\mathrm{s}} \times \bar{\mathrm{t}})$ − Card $(\bar{\mathrm{s}} \cap \bar{\mathrm{t}})$ *is odd for all distinct $s, t \in S$ and for every $t \in S$ let $\alpha_t, \varepsilon_t \in Un\, E^c$ and $X_t \in E$ be such that*

$$\varepsilon_t^2 = 1_E\,, \quad (V_t)^2 = (\alpha_t \widetilde{\otimes} 1_K)V_1\,, \quad X_t^* = \alpha_t X_t\,,$$

$$\sum_{t\in S} |X_t|^2 = \frac{1}{4} 1_E.$$

a)

$$P := \frac{1}{2}V_1 + \sum_{t\in S}((\varepsilon_t X_t)\tilde{\otimes} 1_K)V_t \in Pr\,\mathcal{S}(f)\,,$$
$$V_1 - P = \frac{1}{2}V_1 + \sum_{t\in S}((-\varepsilon_t X_t)\tilde{\otimes} 1_K)V_t \in Pr\,\mathcal{S}(f)\ .$$

b) *If $s \in S$ and $\beta \in E^c$ such that $X_s = 0$ and $\beta^2 = \alpha_s$ then P is homotopic in $Pr\,\mathcal{S}(f)$ to*

$$\frac{1}{2}(V_1 + ((\beta^* \varepsilon_s)\widetilde{\otimes} 1_K)V_s)\ .$$

Proof. a) By Proposition 4.1.3 b),e),

$$P^* = \frac{1}{2}V_1 + \sum_{t\in S}((\varepsilon_t X_t^* \alpha_t^*)\widetilde{\otimes} 1_K)V_t = \frac{1}{2}V_1 + \sum_{1\in S}((\varepsilon_t X_t)\widetilde{\otimes} 1_K)V_t = P,$$

$$\begin{aligned} P^2 &= \frac{1}{4}V_1 + \sum_{t\in S}(X_t^2 \widetilde{\otimes} 1_K)(V_t)^2 + \sum_{t\in S}((\varepsilon_t X_t)\widetilde{\otimes} 1_K)V_t \\ &\quad + \sum_{\substack{s,t\in S\\ s\neq t}}((\varepsilon_s \varepsilon_t X_s X_t)\widetilde{\otimes} 1_K)(V_s V_t + V_t V_s) \\ &= \frac{1}{4}V_1 + \sum_{t\in S}((X_t^2 \alpha_t)\widetilde{\otimes} 1_K)V_1 + \sum_{t\in S}((\varepsilon_t X_t)\widetilde{\otimes} 1_K)V_t \\ &= \frac{1}{4}V_1 + \sum_{t\in S}(|X_t|^2 \widetilde{\otimes} 1_K)V_1 + \sum_{t\in S}((\varepsilon_t X_t)\widetilde{\otimes} 1_K)V_t = P. \end{aligned}$$

b) Remark first that $\beta \in Un\, E^c$ and put

$$Y : [0,1] \longrightarrow E_+^c\,, \quad u \longmapsto (\frac{1}{4}1_E - u^2 \sum_{t\in S}|X_t|^2)^{\frac{1}{2}},$$

$$Z : [0,1] \longrightarrow E^c\,, \quad u \longmapsto \beta^* \varepsilon_s Y(u),$$

$$Q : [0,1] \longrightarrow \mathcal{S}(f)\,, \quad u \longmapsto \frac{1}{2}V_1 + (Z(u)\widetilde{\otimes} 1_K)V_s + \sum_{t\in S\setminus\{s\}}((u\varepsilon_t X_t)\widetilde{\otimes} 1_K)V_t.$$

For $u\in[0,1]$,

$$\alpha_s Z(u)=\beta^2\beta^*\varepsilon_s Y(u)=\beta\varepsilon_s Y(u)=Z(u)^*\,,$$
$$|Z(u)|^2+\sum_{t\in S\setminus\{s\}}|uX_t|^2=\frac{1}{4}1_E$$

so by a), $Q(u)\in Pr\,\mathcal{S}(f)$. Moreover

$$Q(0)=\frac{1}{2}(V_1+((\beta^*\varepsilon_s)\widetilde{\otimes}1_K)V_s)\,,\qquad Q(1)=P.$$

□

COROLLARY 4.1.5. *Let $s,t\in T'\setminus\{1\}$, $s\neq t$, $st=ts$, $\alpha_s,\alpha_t,\varepsilon_s,\varepsilon_t\in Un\;E^c$ such that*

$$\varepsilon_s^2=\varepsilon_t^2=1_E,\quad (V_s)^2=(\alpha_s^2\widetilde{\otimes}1_K)V_1,\quad (V_t)^2=(\alpha_t^2\widetilde{\otimes}1_K)V_1\,,$$

and put

$$P_s:=\frac{1}{2}(V_1+((\varepsilon_s\alpha_s^*)\widetilde{\otimes}1_K)V_s),\quad P_t:=\frac{1}{2}(V_1+((\varepsilon_t\alpha_t^*)\widetilde{\otimes}1_K)V_t)\,.$$

a) *$P_s,P_t\in Pr\,\mathcal{S}(f)$; we denote by $P_s\wedge P_t$ the infimum of P_s and P_t in $\mathcal{S}(f)_+$* (by b) and c) it exists).

b) *If $V_sV_t\neq V_tV_s$ then $P_s\wedge P_t=0$.*

c) *If $V_sV_t=V_tV_s$ then $P_s\wedge P_t=P_sP_t\in Pr\,\mathcal{S}(f)$.*

Proof. a) follows from Proposition 2.1.20 $b\Rightarrow a$.

b) By Proposition 4.1.3 b), $V_sV_t=-V_tV_s$. Let $X\in\mathcal{S}(f)_+$ with $X\leq P_s$ and $X\leq P_t$. By [1, Proposition 4.2.7.1 $d\Rightarrow c$],

$$X=P_sX=\frac{1}{2}X+\frac{1}{2}((\varepsilon_s\alpha_s^*)\widetilde{\otimes}1_K)V_sX,$$

$$\begin{aligned}X=((\varepsilon_s\alpha_s^*)\widetilde{\otimes}1_K)V_sX&=((\varepsilon_s\varepsilon_t\alpha_s^*\alpha_t^*)\widetilde{\otimes}1_K)V_sV_tX\\&=-((\varepsilon_s\varepsilon_t\alpha_s^*\alpha_t^*)\widetilde{\otimes}1_K)V_tV_sX=-X\end{aligned}$$

so $X=0$ and $P_s\wedge P_t=0$.

c) We have $P_sP_t=P_tP_s$ so $P_sP_t\in Pr\,\mathcal{S}(f)$ and $P_sP_t=P_s\wedge P_t$ by [1, Corollary 4.2.7.4 $a\Rightarrow b\&d$]. □

COROLLARY 4.1.6. *Let $m,n\in\mathbb{N}$, $\mathbb{N}_{m+n}\subset I$, $(\alpha_i)_{i\in\mathbb{N}_m}\in(Un\;E^c)^m$, and for every $i\in\mathbb{N}_m$ let $t_i\in T'$ with $\bar{t}_i:=\mathbb{N}_n\cup\{n+i\}$ and $t_it_j=t_jt_i$ for all $i,j\in\mathbb{N}_m$. If for every $i\in\mathbb{N}_m$,*

$$(V_{t_i})^2=(\alpha_i^2\otimes 1_K)V_1$$

then

$$\frac{1}{2}\left(V_1+\frac{1}{\sqrt{m}}\sum_{i\in\mathbb{N}_m}(\alpha_i^*\otimes 1_K)V_{t_i}\right)\in Pr\,\mathcal{S}(f)\,.$$

Proof. For distinct $i, j \in \mathbb{N}_m$,

$$\mathrm{Card}\,(\bar{\mathrm{t}}_\mathrm{i} \times \bar{\mathrm{t}}_\mathrm{j}) - \mathrm{Card}\,(\bar{\mathrm{t}}_\mathrm{i} \cap \bar{\mathrm{t}}_\mathrm{j}) = (\mathrm{n}+1)^2 - \mathrm{n} = \mathrm{n}(\mathrm{n}+1)+1$$

is odd. For every $i \in \mathbb{N}_m$ put $X_i := \frac{1}{2\sqrt{m}}\alpha_i^*$. Then

$$\alpha_i^2 X_i = \frac{1}{2\sqrt{m}}\alpha_i = X_i^*\,, \qquad |X_i|^2 = \frac{1}{4m}1_E\,, \qquad \sum_{i\in\mathbb{N}_m} |X_i|^2 = \frac{1}{4}1_E$$

and the assertion follows from Proposition 4.1.4 a). □

THEOREM 4.1.7. *Let $n \in \mathbb{N}$ such that $\mathbb{N}_{2n}$ is an ordered subset of I, $S := \{\, t \in T \mid \bar{t} \subset \mathbb{N}_{2n-2}\,\}$, $g := f|(S\times S)$, $a, b \in T$ such that $a^2 = b^2 = 1$,*

$$\bar{a} = \mathbb{N}_{2n-1}, \quad \bar{b} = \mathbb{N}_{2n-2} \cup \{2n\}, \quad i \in \mathbb{N}_{2n-2} \Longrightarrow a_i = b_i\,,$$

$\omega : \mathbb{Z}_2 \times \mathbb{Z}_2 \to T$ the (injective) group homomorphism defined by $\omega(1,0) := a$, $\omega(0,1) := b$, $\alpha_1 := f(a,a)$, $\alpha_2 := f(b,b)$, $\beta_1, \beta_2 \in Un\ E^c$ such that $\alpha_1\beta_1^2 + \alpha_2\beta_2^2 = 0$,

$$\gamma := \frac{1}{2}(\alpha_1^*\beta_1^*\beta_2 - \alpha_2^*\beta_1\beta_2^*) = \alpha_1^*\beta_1^*\beta_2 = -\alpha_2^*\beta_1\beta_2^*\,,$$

$$X := \frac{1}{2}((\beta_1\widetilde{\otimes}1_K)V_a + (\beta_2\widetilde{\otimes}1_K)V_b), \quad P_+ := X^*X\,, \quad P_- := XX^*.$$

We consider $\mathcal{S}(g)$ as an E-C^{**}-subalgebra of $\mathcal{S}(f)$ (Corollary 2.1.17 e)).

a) *$ab = ba$, $\gamma^2 = -\alpha_1^*\alpha_2^*$. We put $c := ab = \omega(1,1)$.*

b) *X, V_c, $P_\pm \in \mathcal{S}(g)^c$.*

c) *We have*

$$P_\pm = \frac{1}{2}(V_1 \pm (\gamma\widetilde{\otimes}1_K)V_c) \in Pr\,\mathcal{S}(f), \quad P_+ + P_- = V_1, \quad P_+P_- = 0,$$

$$X^2 = 0,\ XP_+ = X,\ P_-X = X,\ P_+X = XP_- = 0,\ X + X^* \in Un\,\mathcal{S}(f)\,.$$

d) *The map*

$$E \longrightarrow P_\pm\mathcal{S}(f)P_\pm, \quad x \longmapsto P_\pm(x\widetilde{\otimes}1_K)P_\pm$$

is an injective unital C^{**}-homomorphism. We identify E with its image with respect to this map and consider $P_\pm\mathcal{S}(f)P_\pm$ as an E-C^{**}-algebra.

e) *The map*

$$\varphi_\pm : \mathcal{S}(g) \longrightarrow P_\pm\mathcal{S}(f)P_\pm\,, \quad Y \longmapsto P_\pm Y P_\pm = P_\pm Y = Y P_\pm$$

is an injective unital C^{**}-homomorphism. If $Y_1, Y_2 \in Un\ \mathcal{S}(g)$ then $\varphi_+Y_1 + \varphi_-Y_2 \in Un\,\mathcal{S}(f)$.

f) *The map*

$$\psi : \mathcal{S}(f) \longrightarrow \mathcal{S}(f)\,, \quad Z \longmapsto (X+X^*)Z(X+X^*)$$

*is an E-C**-isomorphism such that*

$$\psi^{-1} = \psi,\ \psi(P_+\mathcal{S}(f)P_+) = P_-\mathcal{S}(f)P_-,\ \psi\circ\varphi_+ = \varphi_-,\ \psi\circ\varphi_- = \varphi_+\,.$$

If $Y_1, Y_2 \in \mathcal{S}(g)$ *then*

$$\varphi_+Y_1 + \varphi_-Y_2 = (\varphi_+Y_1 + \varphi_-V_1)\psi(\varphi_+Y_2 + \varphi_-V_1).$$

g) *If* $p \in Pr\,\mathcal{S}(g)$ *then*

$$(X(\varphi_+p)^*(X(\varphi_+p)) = \varphi_+p\,, \qquad (X(\varphi_+p))(X(\varphi_+p))^* = \varphi_-p\,.$$

h) *Let* R *be the subgroup* $\{1,a,b,c\}$ *of* T, $h := f|(R\times R)$, $d\in T$ *such that* $\bar{d} = \mathbb{N}_{2n-2}$ *and* $d_i = a_i$ *for every* $i \in \mathbb{N}_{2n-2}$, *and*

$$\alpha := f(d,d)\,, \qquad \alpha' := f_{2n-1}(2n-1, 2n-1)\,, \qquad \alpha'' := f_{2n}(2n,2n).$$

Then $\alpha_1 = \alpha\alpha'$, $\alpha_2 = \alpha\alpha''$, $-\alpha'\alpha'' = (\alpha^*\gamma^*)^2$,

h	a	b	c
a	$\alpha\alpha'$	α	α'
b	$-\alpha$	$\alpha\alpha''$	$-\alpha''$
c	$-\alpha'$	α''	$-\alpha'\alpha''$

is the table of h, $P_\pm \in Pr\,\mathcal{S}(h)$, *and the map*

$$\varphi : \mathcal{S}(h) \longrightarrow E_{2,2}\,, \quad Z \longmapsto \begin{bmatrix} Z_0 + \gamma^*Z_c & \alpha\alpha' Z_a - \alpha\gamma^* Z_b \\ Z_a + \alpha'^*\gamma^* Z_b & Z_0 - \gamma^* Z_c \end{bmatrix}$$

*is an E-C**-isomorphism. In particular*

$$\varphi P_+ = \begin{bmatrix} 1_E & 0 \\ 0 & 0 \end{bmatrix}, \qquad \varphi P_- = \begin{bmatrix} 0 & 0 \\ 0 & 1_E \end{bmatrix}$$

and $E_{2,2}$ *is E-C**-isomorphic to an E-C**-subalgebra of* $\mathcal{S}(f)$.

i) *Assume* $I = \mathbb{N}_{2n}$ *and* $T_{2n-1} = T_{2n} = \mathbb{Z}_2$. *Then* $T \approx S\times\mathbb{Z}_2\times\mathbb{Z}_2$, $\varphi_\pm$ *is an E-C*-isomorphism with inverse*

$$P_\pm\mathcal{S}(f)P_\pm \longrightarrow \mathcal{S}(f_0), \quad Z \longmapsto 2\sum_{u\in T_0}(Z_u\otimes 1_K)V_u\,,$$

and $\mathcal{S}(f) \approx_E \mathcal{S}(g)_{2,2}$

Proof. a) is easy to see.
b) follows from Proposition 4.1.3 b).

c) follows from a) and Theorem 2.2.18 b),h).

d) follows from Theorem 2.2.18 c).

e) By b) and c), the map is well-defined. The assertion follows now from Theorem 2.2.18 d),h).

f) follows from b), c), and Theorem 2.2.18 h).

g) follows from b) and Proposition 2.2.17 d).

h) follows from c), d), Proposition 3.2.1 a), Corollary 3.2.2 d), and Proposition 3.2.3 c).

i) follows from Theorem 2.2.18 f). □

PROPOSITION 4.1.8. *We use the notation and the hypotheses of* Theorem 4.1.7 *and assume* $I := \mathbb{N}_2$, $T_1 := \mathbb{Z}_2$, *and* $T_2 := \mathbb{Z}_{2m}$ *with* $m \in \mathbb{N}$.

a) $a = (1,0)$, $b = (0,m)$, $c = (1,m)$, $\alpha = 1_E$, $\alpha' = \alpha_1 = f_1(1,1)$, $\alpha'' = \alpha_2 = f_2(m,m)$, *and*

$$P_\pm \mathcal{S}(f) P_\pm = \left\{ (x \widetilde{\otimes} 1_K) P_\pm \;\middle|\; x \in E \right\} .$$

b) *If* $m = 1$ *then there are* $\alpha, \beta, \gamma, \delta \in Un\, E^c$ *such that* f *is given by the following table:*

f	$(0,1)$	$(0,2)$	$(0,3)$	$(1,0)$	$(1,1)$	$(1,2)$	$(1,3)$
$(0,1)$	α	β	γ	-1_E	$-\alpha$	$-\beta$	$-\gamma$
$(0,2)$	β	$\alpha^*\beta\gamma$	$\alpha^*\gamma$	-1_E	$-\beta$	$-\alpha^*\beta\gamma$	$-\alpha^*\gamma$
$(0,3)$	γ	$\alpha^*\gamma$	$\beta^*\gamma$	-1_E	$-\gamma$	$-\alpha^*\gamma$	$-\beta^*\gamma$
$(1,0)$	1_E	1_E	1_E	δ	δ	δ	δ
$(1,1)$	α	β	γ	$-\delta$	$-\alpha\delta$	$-\beta\delta$	$-\gamma\delta$
$(1,2)$	β	$\alpha^*\beta\gamma$	$\alpha^*\gamma$	$-\delta$	$-\beta\delta$	$-\alpha^*\beta\gamma\delta$	$-\alpha^*\gamma\delta$
$(1,3)$	γ	$\alpha^*\gamma$	$\beta^*\gamma$	$-\delta$	$-\gamma\delta$	$-\alpha^*\gamma\delta$	$-\beta^*\gamma\delta$

.

c) *We assume* $\mathbb{K} := \mathbb{C}$ *and* $m := 1$ *and put for all* $j, k \in \{0,1\}$

$$\varphi_{j,k} : \mathcal{S}(f) \longrightarrow E\,, \quad Z \longmapsto Z_0 + (-1)^j Z_b + i^j Z_{(k,1)} - i^j Z_{(k,3)}\,,$$

$$\phi : \mathcal{S}(f) \longrightarrow E^4\,, \quad Z \longmapsto (\varphi_{0,0}Z,\, \varphi_{0,1}Z,\, \varphi_{1,0}Z,\, \varphi_{1,1}Z)\,.$$

If we take $\alpha := \beta := \gamma := -\delta := \beta_1 := \beta_2 := 1_E$ *in b) then the map*

$$\mathcal{S}(f) \longrightarrow E_{2,2} \times E^4, \quad Z \longmapsto \left(\begin{bmatrix} Z_0 + Z_{(1,2)} & Z_{(1,0)} - Z_b \\ Z_{(1,0)} + Z_b & Z_0 - Z_{(1,2)} \end{bmatrix}, \phi Z \right)$$

is an E-C^{**}*-isomorphism.*

Proof. a) Use Corollary 4.1.2 and Proposition 2.1.2 b).

b) Use Proposition 3.4.1 a) and Proposition 4.1.1.

c) follows from b) and Proposition 3.4.1 f_1. □

4.2. **A special case**

Throughout this subsection, we denote by S a totally ordered set, put $T := (\mathbb{Z}_2)^{(S)}$, and fix a map $\rho : S \to Un\ E^c$. We define for every $s \in S$, $f_s \in \mathcal{F}(\mathbb{Z}_2, E)$ by putting $f_s(1,1) = \rho(s)$ (Proposition 3.1.1 a)). Moreover, we denote by f_ρ the Schur function f defined in Proposition 4.1.1 b) (with I replaced by S) and put $\mathcal{C}l(\rho) := \mathcal{S}(f_\rho)$.

Remark. If $S := \mathbb{N}_2$ then $T = \mathbb{Z}_2 \times \mathbb{Z}_2$ so $\mathcal{C}l(\rho)$ is a special case of the example treated in Subsection 3.2. With the notation used in the left table of Proposition 3.2.1 this case appears for $a := (1,0)$ and $b := (0.1)$ exactly when $\varepsilon = -1_E$, $\alpha = -\rho(b)$, $\beta = \rho(a)$, and $\gamma = 1_E$.

LEMMA 4.2.1. *$\mathfrak{P}_f(S)$ endowed with the composition law*

$$\mathfrak{P}_f(S) \times \mathfrak{P}_f(S) \longrightarrow \mathfrak{P}_f(S), \quad (A,B) \longmapsto A \triangle B := (A \setminus B) \cup (B \setminus A)$$

*is a locally finite commutative group (*Definition 2.1.18*) with $\emptyset$ as neutral element and the map*

$$\mathfrak{P}_f(S) \longrightarrow T, \quad A \longmapsto e_A$$

is a group isomorphism with inverse

$$T \longrightarrow \mathfrak{P}_f(S), \quad x \longmapsto \{\, s \in S \mid x(s) = 1 \,\} \ .$$

We identify T with $\mathfrak{P}_f(S)$ by using this isomorphism and write s instead of $\{s\}$ for every $s \in S$. For $A, B \in T$,

$$f_\rho(A,B) = (-1)^\tau \prod_{s \in A \cap B} \rho(s)\,,$$

where τ is defined in Proposition 4.1.1 b).

PROPOSITION 4.2.2. *Assume S finite and let F be an E-C^*-algebra. Let further $(x_s)_{s \in S}$ be a family in F such that for all distinct $s, t \in S$ and for every $y \in E$,*

$$x_s x_t = -x_t x_s, \quad x_s^2 = \rho(s) 1_F, \quad x_s^* = \rho(s)^* x_s, \quad x_s y = y x_s.$$

Then there is a unique E-C^*-homomorphism $\varphi : \mathcal{C}l(\rho) \to F$ such that $\varphi V_s = x_s$ for all $s \in S$. If the family $\left(\prod_{s \in A} x_s\right)_{A \subset S}$ is E-linearly independent (resp. generates F as an E-C^*-algebra) then φ is injective (resp. surjective).

Proof. Put $\varphi V_A := x_{s_1} x_{s_2} \cdots x_{s_m}$ for every $A := \{s_1, s_2, \cdots, s_m\}$, where $s_1 < s_2 < \cdots < s_m$, and

$$\varphi : \mathcal{C}l(\rho) \longrightarrow F\,, \quad X \longmapsto \sum_{A \subset S} X_A \, \varphi V_A \ .$$

It is easy to see that $(\varphi V_s)(\varphi V_t) = \varphi(V_s V_t)$ and $y\,\varphi V_s = (\varphi V_s)y$ for all $s, t \in S$ and $y \in E$ (Proposition 2.1.2 b)). Let $A := \{s_1, s_2, \cdots, s_m\} \subset S$, $B := \{t_1, t_2, \cdots, t_n\} \subset S$, $\{r_1, r_2, \cdots, r_p\} := A \triangle B$, where the letters are written in strictly increasing order. Then

$$(\varphi V_A)(\varphi V_B) = x_{s_1} x_{s_2} \cdots x_{s_m} x_{t_1} x_{t_2} \cdots x_{t_n} = f_\rho(A, B) x_{r_1} x_{r_2} \cdots x_{r_p}$$
$$= f_\rho(A, B)\varphi V_{A\triangle B} = \varphi((f_\rho(A, B)\widetilde{\otimes} 1_K)V_{A\triangle B}) = \varphi(V_A V_B),$$

$$(\varphi V_A)^* = x^*_{s_m} \cdots x^*_{s_2} x^*_{s_1} = (-1)^{\frac{m(m-1)}{2}} x^*_{s_1} x^*_{s_2} \cdots x^*_{s_m}$$
$$= (-1)^{\frac{m(m-1)}{2}} \prod_{i\in\mathbb{N}_m} \rho(s_i)^* x_{s_1} x_{s_2} \cdots x_{s_m} = (-1)^{\frac{m(m-1)}{2}} \prod_{i\in\mathbb{N}_m} \rho(s_i)^* \varphi V_A$$
$$= \varphi((-1)^{\frac{m(m-1)}{2}} ((\prod_{i\in\mathbb{N}_m} \rho(s_i)^*)\widetilde{\otimes} 1_K)V_A) = \varphi(V_A^*)$$

by Proposition 4.1.3 e).

For $X, Y \in \mathcal{C}l(\rho)$ (by Theorem 2.1.9 c),g)),

$$(\varphi X)(\varphi Y) = \left(\sum_{A\in T} X_A \varphi V_A\right)\left(\sum_{B\in T} Y_B \varphi V_B\right) = \sum_{A,B\in T} X_A Y_B (\varphi V_A)(\varphi V_B)$$
$$= \sum_{A,B\in T} X_A Y_B \varphi(V_A V_B) = \sum_{A,B\in T} X_A Y_B f_\rho(A, B)\varphi V_{A\triangle B}$$
$$= \sum_{A,C\in T} X_A Y_{A\triangle C} f_\rho(A, A\triangle C)\varphi V_C = \sum_{C\in T}\left(\sum_{A\in T} f_\rho(A, A\triangle C) X_A Y_{A\triangle C}\right)\varphi V_C$$
$$= \sum_{C\in T} (XY)_C \varphi V_C = \varphi(XY),$$

$$(\varphi X)^* = \sum_{A\in T} X_A^* (\varphi V_A)^* = \sum_{A\in T} X_A^* \varphi(V_A)^*$$
$$= \sum_{A\in T} \tilde{f}_\rho(A)^* (X^*)_A \tilde{f}_\rho(A)\varphi V_A = \sum_{A\in T} (X^*)_A \varphi V_A = \varphi(X^*)$$

(Proposition 4.1.3 e)) i.e. φ is an E-C*-homomorphism. The uniqueness and the last assertions are obvious (by Theorem 2.1.9 a)). □

PROPOSITION 4.2.3. *Let* $m, n \in \mathbb{N}\cup\{0\}$, $S := \mathbb{N}_{2n}$, $S' := \mathbb{N}_{2n+m}$, $K' := l^2(\mathfrak{P}(S'))$, $(\alpha_i)_{i\in\mathbb{N}_m} \in (Un\ E^c)^m$,

$$\rho' : S' \longrightarrow Un\ E^c, \quad s \longmapsto \begin{cases} \rho(s) & \text{if} \quad s \in S \\ \alpha_i^2 \tilde{f}_\rho(S) & \text{if} \quad s = 2n + i \text{ with } i \in \mathbb{N}_m \end{cases},$$

and $A_i := A \cup \{2n + i\}$ *for every* $A \subset S$ *and* $i \in \mathbb{N}_m$.

a) $i \in \mathbb{N}_m \Longrightarrow \tilde{f}_{\rho'}(S_i) = \alpha_i^{*2}\,, \;\; (V_{S_i}^{\rho'})^2 = (\alpha_i^2 \otimes 1_{K'})V_\emptyset^{\rho'}$.

b) $P := \frac{1}{2}V_\emptyset^{\rho'} + \frac{1}{2\sqrt{m}} \sum\limits_{i\in\mathbb{N}_m} (\alpha_i^* \otimes 1_{K'})V_{S_i}^{\rho'} \in Pr\,\mathcal{C}l(\rho')$.

c) *There is a unique injective E-C*-homomorphism* $\varphi : \mathcal{C}l(\rho) \to P\mathcal{C}l(\rho')P$ *such that* $\varphi V_s^\rho = PV_s^{\rho'}P = PV_s^{\rho'} = V_s^{\rho'}P$ *for every* $s \in S$.

d) *If* $m \in \mathbb{N}_2$ *then* φ *is an E-C*-isomorphism.*

Proof. a) By Proposition 4.1.3 e),

$$\tilde{f}_{\rho'}(S_i) = (-1)^{n(2n+1)} \prod_{s\in S_i} \rho'(s)^* = \left((-1)^{n(2n-1)} \prod_{s\in S} \rho(s)^*\right) \alpha_i^{*2} \tilde{f}_\rho(S)^* = \alpha_i^{*2}\,,$$

$$(V_{S_i}^{\rho'})^2 = (\alpha_i^2 \otimes 1_{K'})V_\emptyset^{\rho'}\,.$$

b) follows from a) and Corollary 4.1.6.

c) By Proposition 4.1.3 c), for $s \in S$, $V_s^{\rho'}V_{S_i}^{\rho'} = V_{S_i}^{\rho'}V_s^{\rho'}$ for every $i \in \mathbb{N}_m$ so $V_s^{\rho'}P = PV_s^{\rho'}$. By b), for distinct $s, t \in S$ (Proposition 4.1.3 b)),

$$(PV_s^{\rho'})(PV_t^{\rho'}) = PV_s^{\rho'}V_t^{\rho'} = -PV_t^{\rho'}V_s^{\rho'} = -(PV_t^{\rho'})(PV_s^{\rho'})\,,$$

$$(PV_s^{\rho'})^2 = P(V_s^{\rho'})^2 = P(\rho'(s) \otimes 1_{K'})V_\emptyset^{\rho'} = (\rho(s) \otimes 1_{K'})P\,,$$

$$(PV_s^{\rho'})^* = P(V_s^{\rho'})^* = P(\rho'(s)^* \otimes 1_{K'})V_s^{\rho'} = (\rho(s) \otimes 1_{K'})^* PV_s^{\rho'}\,.$$

By Proposition 4.2.2 there is a unique E-C*-homomorphism $\varphi : \mathcal{C}l(\rho) \to P\mathcal{C}l(\rho')P$ with the given properties.

Let $X \in \mathcal{C}l(\rho)$ with $\varphi X = 0$. Then

$$\begin{aligned} 0 &= \left(\sum_{A\subset S} (X_A \otimes 1_{K'})V_A^{\rho'}\right) P \\ &= \frac{1}{2}\sum_{A\subset S} (X_A \otimes 1_{K'})V_A^{\rho'} + \frac{1}{2\sqrt{m}} \sum_{i\in\mathbb{N}_m} \sum_{A\subset S} (X_A \otimes 1_{K'}) f_{\rho'}(A, S_i) V_{A\triangle S_i}^{\rho'} \end{aligned}$$

and this implies $X_A = 0$ for all $A \subset S$ (Theorem 2.1.9 a)). Thus φ is injective.

d) The case $m = 1$.

Let $Y \in P\mathcal{C}l(\rho')P$. Then (by Proposition 2.1.2 b))

$$Y = YP = \frac{1}{2}Y + \frac{1}{2}\sum_{A\subset S'} (\alpha_1^* \otimes 1_{K'})V_{S_1}^{\rho'}Y\,,$$

$$\begin{aligned} Y = &\sum_{A\subset S} ((\alpha_1^* f_{\rho'}(S_1, A) Y_A) \otimes 1_{K'})V_{S_1\triangle A}^{\rho'} + \\ &+ \sum_{A\subset S} (((\alpha_1^* f_{\rho'}(S_1, A_1) Y_{A_1})) \otimes 1_{K'})V_{S\triangle A}^{\rho'} \end{aligned}$$

so

$$\begin{cases} Y_A = \alpha_1^* f_{\rho'}(S_1, (S\triangle A)_1) Y_{(S\triangle A)_1} \\ Y_{A_1} = \alpha_1^* f_{\rho'}(S_1, S\triangle A) Y_{S\triangle A} \end{cases}$$

for every $A \subset S$. If we put

$$X := 2 \sum_{A\subset S} (Y_A \otimes 1_K) V_A^\rho \in \mathcal{C}l(\rho)$$

then

$$\begin{aligned} \varphi X &= \frac{1}{2}\varphi X + \sum_{A\subset S} ((\alpha_1^* f_{\rho'}(S_1, A) Y_A) \otimes 1_{K'}) V_{S_1\triangle A}^{\rho'} \\ &= \sum_{A\subset S} (Y_A \otimes 1_{K'}) V_A^{\rho'} + \sum_{A\subset S} ((\alpha_1^* f_{\rho'}(S_1, S\triangle A) Y_{S\triangle A}) \otimes 1_{K'}) V_{A_1}^{\rho'} \\ &= \sum_{A\subset S} (Y_A \otimes 1_{K'}) V_A^{\rho'} + \sum_{A\subset S} (Y_{A_1} \otimes 1_{K'}) V_{A_1}^{\rho'} = Y. \end{aligned}$$

Thus φ is surjective.

The case $m = 2$.
Let $Y \in P\mathcal{C}l(\rho')P$. Then

$$\begin{cases} Y = PY = \frac{1}{2}Y + \frac{1}{2\sqrt{2}}((\alpha_1^* \otimes 1_{K'}) V_{S_1}^{\rho'} + (\alpha_2^* \otimes 1_{K'}) V_{S_2}^{\rho'}) Y \\ Y = YP = \frac{1}{2}Y + \frac{1}{2\sqrt{2}} Y ((\alpha_1^* \otimes 1_{K'}) V_{S_1}^{\rho'} + (\alpha_2^* \otimes 1_{K'}) V_{S_2}^{\rho'}) , \end{cases}$$

$$\sqrt{2}Y = (\alpha_1^* \otimes 1_{K'}) V_{S_1}^{\rho'} Y + (\alpha_2^* \otimes 1_{K'}) V_{S_2}^{\rho'} Y = (\alpha_1^* \otimes 1_{K'}) Y V_{S_1}^{\rho'} + (\alpha_2^* \otimes 1_{K'}) Y V_{S_2}^{\rho'}.$$

For every $B \subset S$ put $B_a := B \cup \{2n+1\}$, $B_b := B \cup \{2n+2\}$, $B_c := B \cup \{2n+1\, ,\ 2n+2\}$. Then

$$\begin{aligned} V_{S_1}^{\rho'} Y = &\sum_{B\subset S} ((Y_B f_{\rho'}(S_1, B)) \otimes 1_{K'}) V_{(S\triangle B)_a}^{\rho'} + \sum_{B\subset S} ((Y_{B_a} f_{\rho'}(S_1, B_a)) \otimes 1_{K'}) V_{S\triangle B}^{\rho'} \\ &+ \sum_{B\subset S} ((Y_{B_b} f_{\rho'}(S_1, B_b)) \otimes 1_{K'}) V_{(S\triangle B)_c}^{\rho'} + \sum_{B\subset S} ((Y_{B_c} f_{\rho'}(S_1, B_c)) \otimes 1_{K'}) V_{(S\triangle B)_b}, \end{aligned}$$

$$\begin{aligned} V_{S_2}^{\rho'} Y = &\sum_{B\subset S} ((Y_B f_{\rho'}(S_2, B)) \otimes 1_{K'}) V_{(S\triangle B)_b}^{\rho'} + \sum_{B\subset S} ((Y_{B_a} f_{\rho'}(S_2, B_a)) \otimes 1_{K'}) V_{(S\triangle B)_c}^{\rho'} \\ &+ \sum_{B\subset S} ((Y_{B_b} f_{\rho'}(S_2, B_b)) \otimes 1_{K'}) V_{S\triangle B}^{\rho'} + \sum_{B\subset S} ((Y_{B_c} f_{\rho'}(S_2, B_c)) \otimes 1_{K'}) V_{(S\triangle B)_a}^{\rho'}, \end{aligned}$$

$$\begin{aligned} Y V_{S_1}^{\rho'} = &\sum_{B\subset S} ((Y_B f_{\rho'}(B, S_1)) \otimes 1_{K'}) V_{(S\triangle B)_a}^{\rho'} + \sum_{B\subset S} ((Y_{B_a} f_{\rho'}(B_a, S_1)) \otimes 1_{K'}) V_{S\triangle B}^{\rho'} \\ &+ \sum_{B\subset S} ((Y_{B_b} f_{\rho'}(B_b, S_1)) \otimes 1_{K'}) V_{(S\triangle B)_c}^{\rho'} + \sum_{B\subset S} ((Y_{B_c} f_{\rho'}(B_c, S_1)) \otimes 1_{K'}) V_{(S\triangle B)_b}^{\rho'}, \end{aligned}$$

$$YV^{\rho'}_{S_2} = \sum_{B\subset S}((Y_B f_{\rho'}(B,S_2))\otimes 1_{K'})V^{\rho'}_{(S\triangle B)_b} + \sum_{B\subset S}((Y_{B_a} f_{\rho'}(B_a,S_2))\otimes 1_{K'})V^{\rho'}_{(S\triangle B)_c}$$
$$+\sum_{B\subset S}((Y_{B_b} f_{\rho'}(B_b,S_2))\otimes 1_{K'})V^{\rho'}_{S\triangle B} + \sum_{B\subset S}((Y_{B_c} f_{\rho'}(B_c,S_2))\otimes 1_{K'})V^{\rho'}_{(S\triangle B)_a},$$

$$\begin{aligned}\sqrt{2}Y = &\sum_{B\subset S}((\alpha_1^* Y_{B_a} f_{\rho'}(S_1,B_a) + \alpha_2^* Y_{B_b} f_{\rho'}(S_2,B_b))\otimes 1_{K'})V^{\rho'}_{S\triangle B}\\ &+\sum_{B\subset S}((\alpha_1^* Y_{B} f_{\rho'}(S_1,B) + \alpha_2^* Y_{B_c} f_{\rho'}(S_2,B_c))\otimes 1_{K'})V^{\rho'}_{(S\triangle B)_a}\\ &+\sum_{B\subset S}((\alpha_1^* Y_{B_c} f_{\rho'}(S_1,B_c) + \alpha_2^* Y_{B} f_{\rho'}(S_2,B))\otimes 1_{K'})V^{\rho'}_{(S\triangle B)_b}\\ &+\sum_{B\subset S}((\alpha_1^* Y_{B_b} f_{\rho'}(S_1,B_b) + \alpha_2^* Y_{B_a} f_{\rho'}(S_2,B_a))\otimes 1_{K'})V^{\rho'}_{(S\triangle B)_c},\end{aligned}$$

$$\begin{aligned}\sqrt{2}Y = &\sum_{B\subset S}((\alpha_1^* Y_{B_a} f_{\rho'}(B_a,S_1) + \alpha_2^* Y_{B_b} f_{\rho'}(B_b,S_2))\otimes 1_{K'})V^{\rho'}_{S\triangle B}\\ &+\sum_{B\subset S}((\alpha_1^* Y_{B} f_{\rho'}(B,S_1) + \alpha_2^* Y_{B_c} f_{\rho'}(B_c,S_2))\otimes 1_{K'})V^{\rho'}_{(S\triangle B)_a}\\ &+\sum_{B\subset S}((\alpha_1^* Y_{B_c} f_{\rho'}(B_c,S_1) + \alpha_2^* Y_{B} f_{\rho'}(B,S_2))\otimes 1_{K'})V^{\rho'}_{(S\triangle B)_b}\\ &+\sum_{B\subset S}((\alpha_1^* Y_{B_b} f_{\rho'}(B_b,S_1) + \alpha_2^* Y_{B_a} f_{\rho'}(B_a,S_2))\otimes 1_{K'})V^{\rho'}_{(S\triangle B)_c}.\end{aligned}$$

It follows for $B\subset S$,

$$\begin{aligned}\sqrt{2}Y_{B_a} &= \alpha_1^* Y_{S\triangle B} f_{\rho'}(S\triangle B,S_1) + \alpha_2^* Y_{(S\triangle B)_c} f_{\rho'}((S\triangle B)_c,S_2)\,,\\ \sqrt{2}Y_{B_b} &= \alpha_1^* Y_{(S\triangle B)_c} f_{\rho'}((S\triangle B)_c,S_1) + \alpha_2^* Y_{S\triangle B} f_{\rho'}(S\triangle B,S_2)\,,\\ \sqrt{2}Y_{B_c} &= \alpha_1^* Y_{(S\triangle B)_b} f_{\rho'}(S_1,(S\triangle B)_b) + \alpha_2^* Y_{(S\triangle B)_a} f_{\rho'}(S_2,(S\triangle B)_a)\\ &= \alpha_1^* Y_{(S\triangle B)_b} f_{\rho'}((S\triangle B)_b,S_1) + \alpha_2^* Y_{(S\triangle B)_a} f_{\rho'}((S\triangle B)_a,S_2),\end{aligned}$$

so by Proposition 4.1.3 a),b), $Y_{B_c}=0$. If we put

$$X := 2\sum_{B\subset S}(Y_B\otimes 1_K)V^\rho_B \in \mathcal{C}l(\rho)$$

then

$$\begin{aligned}\varphi X &= \left(2\sum_{B\subset S}(Y_B\otimes 1_{K'})V^{\rho'}_B\right)P\\ &= \sum_{B\subset S}(Y_B\otimes 1_{K'})V^{\rho'}_B + \frac{1}{\sqrt{2}}\sum_{B\subset S}((\alpha_1^* Y_B f_{\rho'}(B,S_1))\otimes 1_{K'})V^{\rho'}_{S_1\triangle B}\end{aligned}$$

$$+\frac{1}{\sqrt{2}}\sum_{B\subset S}((\alpha_2^* Y_B f_{\rho'}(B,S_2))\otimes 1_{K'})V^{\rho'}_{S_2\triangle B},$$

and so for $B\subset S$,

$$(\varphi X)_B = Y_B, \quad (\varphi X)_{B_a} = \frac{1}{\sqrt{2}}\alpha_1^* Y_{S\triangle B} f_{\rho'}(S\triangle B, S_1) = Y_{B_a}\,,$$

$$(\varphi X)_{B_b} = \frac{1}{\sqrt{2}}\alpha_2^* Y_{S\triangle B} f_{\rho'}(S\triangle B, S_2) = Y_{B_b}, \quad (\varphi X)_{B_c} = 0 = Y_{B_c}.$$

Thus $\varphi X = Y$ and φ is surjective. □

Remark. If $m=3$ then φ may be not surjective.

PROPOSITION 4.2.4. *Let* $\mathbb{K} := \mathbb{R}$, $n\in\mathbb{N}\cup\{0\}$, $S := \mathbb{N}_{2n}$, *and*

$$\rho' : \mathbb{N}_{2n+1} \longrightarrow Un\ E^c\,, \quad s\longmapsto \begin{cases} \rho(s) & \text{if} \quad s\in S \\ -\tilde f_\rho(S) & \text{if} \quad s = 2n+1 \end{cases}.$$

Let $\overbrace{\mathcal{C}l(\rho)}^{\circ}$ *be the complexification of* $\mathcal{C}l(\rho)$, *considered as a real* E-C^*-*algebra* ([1, Theorem 4.1.1.8 a)]) *by using the embedding*

$$E\longrightarrow \overbrace{\mathcal{C}l(\rho)}^{\circ}, \quad x\longmapsto ((x\otimes 1_K)V^\rho_\emptyset, 0).$$

Then there is a unique E-C^*-*isomorphism* $\varphi : \mathcal{C}l(\rho')\to \overbrace{\mathcal{C}l(\rho)}^{\circ}$ *such that* $\varphi V^{\rho'}_s = (V^\rho_s, 0)$ *for every* $s\in S$ *and* $\varphi V^{\rho'}_{2n+1} = (0, -(\tilde f_\rho(S)\otimes 1_K)V^\rho_S)$.

Proof. We put

$$x_s := \begin{cases} (V^\rho_s, 0) & \text{if} \quad s\in S \\ (0, -(\tilde f_\rho(S)\otimes 1_K)V^\rho_S) & \text{if} \quad s = 2n+1 \end{cases}.$$

For $s\in S$, by Proposition 4.1.3 b),

$$\begin{aligned} x_s x_{2n+1} &= (V^\rho_s, 0)(0, -(\tilde f_\rho(S)\otimes 1_K)V^\rho_S) = (0, -(\tilde f_\rho(S)\otimes 1_K)V^\rho_s V^\rho_S) \\ &= (0, (\tilde f_\rho(S)\otimes 1_K)V^\rho_S V^\rho_s) = (0, (\tilde f_\rho(S)\otimes 1_K)V^\rho_s)(V^\rho_s, 0) = -x_{2n+1}x_s. \end{aligned}$$

By Proposition 2.1.2 b),e),

$$\begin{aligned} x^2_{2n+1} &= (-((\tilde f_\rho(S)\otimes 1_K)V^\rho_S)^2, 0) \\ &= (-(\tilde f_\rho(S)^2\otimes 1_K)(f_\rho(S,S)\otimes 1_K)V^\rho_\emptyset, 0) = (\rho'(2n+1)\otimes 1_K)(V^\rho_\emptyset, 0), \end{aligned}$$

$$\begin{aligned} x^*_{2n+1} &= (0, ((\tilde f_\rho(S)\otimes 1_K)V^\rho_S)^*) \\ &= (0, (\tilde f_\rho(S)^*\otimes 1_K)(\tilde f_\rho(S)\otimes 1_K)V^\rho_S) = (\rho'(2n+1)^*\otimes 1_K)x_{2n+1}, \end{aligned}$$

and the assertion follows from Proposition 4.2.2. □

PROPOSITION 4.2.5. *Let* $n \in \mathbb{N} \cup \{0\}$, $S := \mathbb{N}_n$, $S' := \mathbb{N}_{n+2}$, $K' := l^2(\mathfrak{P}(S'))$, $\alpha_1, \alpha_2 \in Un\, E^c$, *and*

$$\rho' : S' \longrightarrow Un\, E^c\,, \quad s \longmapsto \begin{cases} \rho(s) & if \quad s \in S \\ \alpha_1^2 & if \quad s = n+1 \\ -\alpha_2^2 & if \quad s = n+2 \end{cases} .$$

a) *There is a unique E-C*-isomorphism* $\varphi : \mathcal{C}l(\rho') \to \mathcal{C}l(\rho)_{2,2}$ *such that*

$$\varphi V_s^{\rho'} = \begin{bmatrix} V_s^\rho & 0 \\ 0 & -V_s^\rho \end{bmatrix}$$

for every $s \in S$ *and*

$$\varphi V_{n+1}^{\rho'} = (\alpha_1 \otimes 1_K) \begin{bmatrix} 0 & V_\emptyset^\rho \\ V_\emptyset^\rho & 0 \end{bmatrix}, \ \varphi V_{n+2}^{\rho'} = (\alpha_2 \otimes 1_K) \begin{bmatrix} 0 & -V_\emptyset^\rho \\ V_\emptyset^\rho & 0 \end{bmatrix} .$$

b)

$$\varphi \frac{1}{2}(V_\emptyset^{\rho'} + ((\alpha_1^* \alpha_2^*) \otimes 1_{K'}) V_{\{n+1,\, n+2\}}^{\rho'}) = \begin{bmatrix} V_\emptyset^\rho & 0 \\ 0 & 0 \end{bmatrix},$$

$$\varphi \frac{1}{2}(V_\emptyset^{\rho'} - ((\alpha_1^* \alpha_2^*) \otimes 1_{K'}) V_{\{n+1,\, n+2\}}^{\rho'}) = \begin{bmatrix} 0 & 0 \\ 0 & V_\emptyset^\rho \end{bmatrix}.$$

Proof. a) Put

$$x_s := \begin{bmatrix} V_s^\rho & 0 \\ 0 & -V_s^\rho \end{bmatrix}$$

for every $s \in S$ and

$$x_{n+1} := (\alpha_1 \otimes 1_K) \begin{bmatrix} 0 & V_\emptyset^\rho \\ V_\emptyset^\rho & 0 \end{bmatrix}, \quad x_{n+2} := (\alpha_2 \otimes 1_K) \begin{bmatrix} 0 & -V_\emptyset^\rho \\ V_\emptyset^\rho & 0 \end{bmatrix} .$$

For distinct $s, t \in S$ and $i \in \mathbb{N}_2$,

$$x_s x_t = -x_t x_s\,, \quad x_s^2 = (\rho'(s) \otimes 1_K) \begin{bmatrix} V_\emptyset^\rho & 0 \\ 0 & V_\emptyset^\rho \end{bmatrix}, \quad x_s^* = (\rho'(s) \otimes 1_K)^* x_s\,,$$

$$x_s x_{n+i} = -x_{n+i} x_s\,, \quad x_{n+i}^2 = (\rho'(n+i) \otimes 1_K) \begin{bmatrix} V_\emptyset^\rho & 0 \\ 0 & V_\emptyset^\rho \end{bmatrix},$$

$$x_{n+i}^* = (\rho'(n+i) \otimes 1_K)^* x_{n+i}, \quad x_{n+1}\, x_{n+2} = -x_{n+2}\, x_{n+1}.$$

By Proposition 4.2.2 there is a unique E-C*-homomorphism $\varphi : \mathcal{C}l(\rho') \to \mathcal{C}l(\rho)_{2,2}$ satisfying the given conditions.

We put, for every $A \subset S$ and $i \in \mathbb{N}_2$, $|A| :=$ Card A, $A_i := A \cup \{n+i\}$, $A_3 := A \cup \{n+1,\, n+2\}$. For $A \subset S$,

$$\varphi V_{A_1}^{\rho'} = (\alpha_1 \otimes 1_K) \begin{bmatrix} V_A^\rho & 0 \\ 0 & (-1)^{|A|} V_A^\rho \end{bmatrix} \begin{bmatrix} 0 & V_\emptyset^\rho \\ V_\emptyset^\rho & 0 \end{bmatrix}$$

$$= (\alpha_1 \otimes 1_K) \begin{bmatrix} 0 & V_A^\rho \\ (-1)^{|A|} V_A^\rho & 0 \end{bmatrix},$$

$$\varphi V_{A_2}^{\rho'} = (\alpha_2 \otimes 1_K) \begin{bmatrix} V_A^\rho & 0 \\ 0 & (-1)^{|A|} V_A^\rho \end{bmatrix} \begin{bmatrix} 0 & -V_\emptyset^\rho \\ V_\emptyset^\rho & 0 \end{bmatrix}$$

$$= (\alpha_2 \otimes 1_K) \begin{bmatrix} 0 & -V_A^\rho \\ (-1)^{|A|} V_A^\rho & 0 \end{bmatrix},$$

$$\varphi V_{A_3}^{\rho'} = ((\alpha_1\alpha_2) \otimes 1_K) \begin{bmatrix} 0 & V_A^\rho \\ (-1)^{|A|} V_A^\rho & 0 \end{bmatrix} \begin{bmatrix} 0 & -V_\emptyset^\rho \\ V_\emptyset^\rho & 0 \end{bmatrix}$$

$$= ((\alpha_1\alpha_2) \otimes 1_K) \begin{bmatrix} V_A^\rho & 0 \\ 0 & -(-1)^{|A|} V_A^\rho \end{bmatrix}.$$

Then for $Y \in \mathcal{C}l(\rho')$,

$$\begin{cases} (\varphi Y)_{11} = \sum\limits_{A \subset S} ((Y_A + (\alpha_1\alpha_2) Y_{A_3}) \otimes 1_K) V_A^\rho \\ (\varphi Y)_{12} = \sum\limits_{A \subset S} ((\alpha_1 Y_{A_1} - \alpha_2 Y_{A_2}) \otimes 1_K) V_A^\rho \\ (\varphi Y)_{21} = \sum\limits_{A \subset S} (-1)^{|A|}) ((\alpha_1 Y_{A_1} + \alpha_2 Y_{A_2}) \otimes 1_K) V_A^\rho \\ (\varphi Y)_{22} = \sum\limits_{A \subset S} (-1)^{|A|} ((Y_A - \alpha_1\alpha_2 Y_{A_3}) \otimes 1_K) V_A^\rho \,. \end{cases}$$

It follows from the above identities that φ is bijective.

b) By the above,

$$\varphi V_{\{n+1,\, n+2\}}^{\rho'} = \varphi V_{\emptyset_3}^{\rho'} = ((\alpha_1\alpha_2) \otimes 1_K) \begin{bmatrix} V_\emptyset^\rho & 0 \\ 0 & -V_\emptyset^\rho \end{bmatrix}$$

and the assertion follows. □

Corollary 4.2.6. *Let $m, n \in \mathbb{N} \cup \{0\}$, $S := \mathbb{N}_n$, $(\alpha_i)_{i \in \mathbb{N}_{2m}} \in (Un\ E^c)^{2m}$, and*

$$\rho' : \mathbb{N}_{n+2m} \longrightarrow Un\ E^c\,, \quad s \longmapsto \begin{cases} \rho(s) & \text{if} \quad s \in S \\ -(-1)^i \alpha_i^2 & \text{if} \quad s = n+i \end{cases}.$$

Then $\mathcal{C}l(\rho') \approx_E \mathcal{C}l(\rho)_{2^m, 2^m}$.

Proposition 4.2.7. *Let $\mathbb{K} := \mathbb{R}$, $n \in \mathbb{N} \cup \{0\}$, $S := \mathbb{N}_{2n}$, $S' := \mathbb{N}_{2n+2}$, $\alpha_1, \alpha_2 \in Un\ E^c$, and*

$$\rho' : S' \longrightarrow Un\ E^c\,, \quad s \longmapsto \begin{cases} \rho(s) & \text{if} \quad s \in S \\ -\alpha_l^2 \tilde{f}_\rho(S) & \text{if} \quad s = 2n + l \text{ with } l \in \mathbb{N}_2 \end{cases}.$$

Then there is a unique E-C^*-isomorphism $\varphi : \mathcal{C}l(\rho') \to \mathcal{C}l(\rho) \otimes \mathbb{H}$ such that

$$\varphi V_s^{\rho'} = \begin{cases} V_s^\rho \otimes 1_{\mathbb{H}} & \text{if} \quad s \in S \\ (((\alpha_1 \tilde{f}_\rho(S)) \otimes 1_K) V_S^\rho) \otimes i & \text{if} \quad s = 2n+1 \\ (((\alpha_2 \tilde{f}_\rho(S)) \otimes 1_K) V_S^\rho) \otimes j & \text{if} \quad s = 2n+2 \end{cases},$$

where i, j, k are the canonical unitaries of $\mathbb{H}$.

Proof. Put

$$x_s := \begin{cases} V_s^\rho \otimes 1_{\mathbb{H}} & \text{if} \quad s \in S \\ (((\alpha_1 \tilde{f}_\rho(S)) \otimes 1_K)V_S^\rho) \otimes i & \text{if} \quad s = 2n+1 \\ (((\alpha_2 \tilde{f}_\rho(S)) \otimes 1_K)V_S^\rho) \otimes j & \text{if} \quad s = 2n+2 \end{cases}.$$

For distinct $s, t \in S$ and $l \in \mathbb{N}_2$, by Proposition 4.1.3 b),

$$x_s x_t = -x_t x_s\,, \qquad x_s^2 = (\rho'(s) \otimes 1_K)(V_\emptyset^\rho \otimes 1_{\mathbb{H}})\,, \qquad x_s^* = (\rho'(s) \otimes 1_K)^* x_s\,,$$

$$x_s x_{2n+l} = -x_{2n+l} x_s,\ x_{2n+1} x_{2n+2} = (((\alpha_1 \alpha_2 \tilde{f}_\rho(S)) \otimes 1_K)V_\emptyset^\rho) \otimes k = -x_{2n+2} x_{2n+1},$$

$$\begin{aligned} (x_{2n+l})^2 &= (((\alpha_l^2 \tilde{f}_\rho(S)^2) \otimes 1_K)(\tilde{f}_\rho(S)^* \otimes 1_K)V_\emptyset^\rho) \otimes (-1_{\mathbb{H}}) \\ &= (\rho'(2n+l) \otimes 1_K)(V_\emptyset^\rho \otimes 1_{\mathbb{H}}), \\ (x_{2n+l})^* &= (((\alpha_l^* \tilde{f}_\rho(S)^*) \otimes 1_K)(\tilde{f}_\rho(S) \otimes 1_K)V_S^\rho) \otimes -(i \text{ or } j) \\ &= (\rho'(2n+l) \otimes 1_K)^* x_{2n+l}. \end{aligned}$$

By Proposition 4.2.2 there is a unique E-C*-homomorphism $\varphi : \mathcal{C}l(\rho') \to \mathcal{C}l(\rho) \otimes \mathbb{H}$ satisfying the given conditions.

For $X \in \mathcal{C}l(\rho')$,

$$\begin{aligned} \varphi X = {} & \left(\sum_{A \subset S} (X_A \otimes 1_K) V_A^\rho \right) \otimes 1_{\mathbb{H}} \\ & + \left(\sum_{A \subset S} ((X_{A \cup \{2n+1\}} \alpha_1 \tilde{f}_\rho(S) f_\rho(A,S)) \otimes 1_K) V_{S \triangle A} \right) \otimes i \\ & + \left(\sum_{A \subset S} ((X_{A \cup \{2n+2\}} \alpha_2 \tilde{f}_\rho(S) f_\rho(A,S)) \otimes 1_K) V_{S \triangle A}^\rho \right) \otimes j \\ & + \left(\sum_{A \subset S} ((X_{A \cup \{2n+1,\, 2n+2\}} \alpha_1 \alpha_2 \tilde{f}_\rho(S)) \otimes 1_K) V_A^\rho \right) \otimes k \end{aligned}$$

and so φ is bijective. $\square$

Proposition 4.2.8. *Let $n \in \mathbb{N} \cup \{0\}$, $S := \mathbb{N}_{2n}$, $A' := A \cup \{2n+1\}$ for every $A \subset S$,*

$$\rho' : S' \longrightarrow Un\, E^c\,, \quad s \longmapsto \begin{cases} \rho(s) & \text{if} \quad s \in S \\ \tilde{f}(S) & \text{if} \quad s = 2n+1 \end{cases},$$

$P_\pm := \frac{1}{2}(V_\emptyset^{\rho'} \pm V_{S'}^{\rho'})$, and $\theta_\pm : \bigoplus_{A \subset S} \breve{E} \to \bigoplus_{A \subset S'} \breve{E}$ defined by

$$(\theta_\pm \xi)_A := \frac{1}{\sqrt{2}} \xi_A\,, \qquad (\theta_\pm \xi)_{A'} := \pm \frac{1}{\sqrt{2}} f_\rho(S \triangle A, S) \xi_{S \triangle A}$$

for every $\xi \in \bigoplus_{A \subset S} \breve{E}$ *and* $A \subset S$.

a)

$$\tilde{f}_{\rho'}(S') = 1_E, \quad (V_{S'}^{\rho'})^2 = V_{\emptyset}^{\rho'}, \quad P_{\pm} \in Pr\,\mathcal{C}l(\rho')^c,$$
$$P_+ + P_- = V_{\emptyset}^{\rho'}, \quad V_{S'}^{\rho'} \in \mathcal{C}l(\rho')^c, \quad V_{S'}^{\rho'} P_{\pm} = \pm P_{\pm}\,.$$

b) *For* $A \subset S$, $f_\rho(A,S)^* = f_{\rho'}(S',A)^* = f_{\rho'}(S',(S \triangle A)')$.

c) $\theta_{\pm} \in \mathcal{L}_E(\bigoplus_{A \subset S} \breve{E}, \bigoplus_{A \subset S'} \breve{E})$ *and for* $\eta \in \bigoplus_{A \subset S'} \breve{E}$ *and* $A \subset S$,

$$(\theta_{\pm}^* \eta)_A = \frac{1}{\sqrt{2}}(\eta_A \pm f_\rho(A,S)^* \eta_{(S \triangle A)'}) = \sqrt{2}(P_{\pm}\eta)_A\,.$$

d) $\theta_{\pm}^* \theta_{\pm}$ *is the identity map of* $\bigoplus_{A \subset S} \breve{E}$.

e) $\theta_{\pm}\theta_{\pm}^* = P_{\pm}$.

f) *For every* $A \subset S$, $\theta_{\pm} V_A^{\rho} \theta_{\pm}^* = V_A^{\rho'} P_{\pm} = P_{\pm} V_A^{\rho'} = P_{\pm} V_A^{\rho'} P_{\pm}$.

g) *For every closed ideal* F *of* E *the map*

$$\varphi : \mathcal{C}l(\rho,F) \longrightarrow P_{\pm}\mathcal{C}l(\rho',F)P_{\pm}\,, \quad X \longmapsto \theta_{\pm} X \theta_{\pm}^*$$

is an E-C^**-isomorphism with inverse*

$$P_{\pm}\mathcal{C}l(\rho',F)P_{\pm} \longrightarrow \mathcal{C}l(\rho,F), \quad Y \longmapsto \theta_{\pm}^* Y \theta_{\pm}$$

and the map $\psi : \mathcal{C}l(\rho',F) \longrightarrow \mathcal{C}l(\rho,F) \times \mathcal{C}l(\rho,F)$

$$Y \longmapsto (\theta_+^* P_+ Y P_+ \theta_+,\, \theta_-^* P_- Y P_- \theta_-) = (\theta_+ Y \theta_+,\, \theta_-^* Y \theta_-)$$

is an E-C^**-isomorphism.*

Proof. a) By Proposition 4.1.3 d),e), $V_{S'}^{\rho'} \in \mathcal{C}l(\rho')^c$,

$$\tilde{f}_{\rho'}(S') = (-1)^{n(2n+1)} \prod_{s \in S'} \rho'(s)^* = (-1)^{n(2n-1)} \left(\prod_{s \in S} \rho(s)^*\right) \rho'(2n+1)^* = 1_E\,,$$

$$(V_{S'}^{\rho'})^* = \tilde{f}_{\rho'}(S') V_{S'}^{\rho'} = V_{S'}^{\rho'}\,, \qquad (V_{S'}^{\rho'})^2 = \tilde{f}(S')^* V_{\emptyset}^{\rho'} = V_{\emptyset}^{\rho'}\,,$$

so

$$P_{\pm} \in Pr\,\mathcal{C}l(\rho')^c\,, \qquad V_{S'}^{\rho'} P_{\pm} = \pm P_{\pm}\,.$$

b) By a), Proposition 4.1.3 c),d), Proposition 4.1.1 b), and Proposition 1.1.2 b),

$$\begin{aligned} f_\rho(A,S)^* &= f_{\rho'}(A,S)^* = f_{\rho'}(A,S')^* \\ &= f_{\rho'}(S',A)^* = f_{\rho'}(S',(S \triangle A)')\tilde{f}_{\rho'}(S') = f_{\rho'}(S',(S \triangle A)'). \end{aligned}$$

c) For $\xi \in \bigoplus_{A\subset S} \breve{E}$,

$$\langle\, \theta\xi \mid \eta\,\rangle = \sum_{A\subset S} \eta_A^* \frac{1}{\sqrt{2}}\xi_A \pm \sum_{A\subset S} \eta_{A'}^* \frac{1}{\sqrt{2}} f_\rho(S\triangle A, S)\xi_{S\triangle A}$$
$$= \sum_{A\subset S} \eta_A^* \frac{1}{\sqrt{2}}\xi_A \pm \sum_{A\subset S} \eta_{(S\triangle A)'}^* \frac{1}{\sqrt{2}} f_\rho(A, S)\xi_A$$
$$= \sum_{A\subset S} \frac{1}{\sqrt{2}}(\eta_A \pm f_\rho(A,S)^*\eta_{(S\triangle A)'})^*\xi_A$$

so $\theta \in \mathcal{L}_E(\bigoplus_{A\subset S} \breve{E}, \bigoplus_{A\subset S'} \breve{E})$ and

$$(\theta^*\eta)_A = \frac{1}{\sqrt{2}}(\eta_A \pm f_\rho(A,S)^*\eta_{(S\triangle A)'})\ .$$

By a) and b),

$$(P_\pm\eta)_A = \frac{1}{2}\eta_A \pm \frac{1}{2} f_{\rho'}(S', (S\triangle A)')\eta_{(S\triangle A)'}$$
$$= \frac{1}{2}(\eta_A \pm f_\rho(A,S)^*\eta_{(S\triangle A)'}) = \frac{1}{\sqrt{2}}(\theta_\pm^*\eta)_A .$$

d) For $\xi \in \bigoplus_{A\subset S} \breve{E}$ and $A \subset S$, by c),

$$(\theta_\pm^*\theta_\pm\xi)_A = \frac{1}{\sqrt{2}}((\theta\xi)_A \pm f_\rho(A,S)^*(\theta\xi)_{(S\triangle A)'})$$
$$= \frac{1}{2}(\xi_A + f_\rho(A,S)^* f_\rho(A,S)\xi_A) = \xi_A .$$

e) For $\eta \in \bigoplus_{A\subset S'} \breve{E}$ and $A \subset S$, by b) and c),

$$(\theta_\pm\theta_\pm^*\eta)_A = \frac{1}{\sqrt{2}}(\theta_\pm^*\eta)_A = (P_\pm\eta)_A\,,$$

$$(\theta_\pm\theta_\pm^*\eta)_{A'} = \pm\frac{1}{\sqrt{2}} f_\rho(S\triangle A, S)(\theta_\pm^*\eta)_{S\triangle A}$$
$$= \pm\frac{1}{2} f_\rho(S\triangle A,S)(\eta_{S\triangle A} \pm f_\rho(S\triangle A,S)^*\eta_{A'}) = \pm\frac{1}{2} f_\rho(S\triangle A,S)\eta_{S\triangle A} + \frac{1}{2}\eta_{A'}$$
$$= \frac{1}{2}(\eta_{A'} \pm f_{\rho'}(S', S\triangle A)\eta_{S\triangle A}) = \frac{1}{2}((V_\emptyset^{\rho'}\eta)_{A'} \pm (V_{S'}^{\rho'}\eta)_{A'}) = (P_\pm\eta)_{A'},$$

so $\theta_\pm\theta_\pm^* = P_\pm$.

f) For $\eta \in \bigoplus_{B\subset S'} \breve{E}$ and $B \subset S$, by a),b),c),e) and Proposition 4.1.1 b) (and Corollary 2.1.17 e)),

$$(V_A^{\rho'} P_\pm\eta)_B = f_{\rho'}(A, A\triangle B)(P_\pm\eta)_{A\triangle B} = f_\rho(A, A\triangle B)(\theta_\pm\theta_\pm^*\eta)_{A\triangle B}$$

$$= \frac{1}{\sqrt{2}} f_\rho(A, A\triangle B)(\theta_\pm^* \eta)_{A\triangle B} = \frac{1}{\sqrt{2}} (V_A^\rho \theta_\pm^* \eta)_B = (\theta_\pm V_A^\rho \theta_\pm^* \eta)_B ,$$

$$\begin{aligned}(\theta_\pm V_A^\rho \theta_\pm^* \eta)_{B'} &= \pm \frac{1}{\sqrt{2}} f_\rho(S\triangle B, S)(V_A^\rho \theta_\pm^* \eta)_{S\triangle B} \\ &= \pm \frac{1}{\sqrt{2}} f_\rho(S\triangle B, S) f_\rho(A, S\triangle A\triangle B)(\theta_\pm^* \eta)_{S\triangle A\triangle B} \\ &= \pm f_\rho(S\triangle B, S) f_\rho(A, S\triangle A\triangle B)(P_\pm \eta)_{S\triangle A\triangle B} \\ &= \pm f_\rho(S\triangle B, S)(V_A^{\rho'} P_\pm \eta)_{S\triangle B} = \pm f_{\rho'}(S', S'\triangle B')(V_A^{\rho'} P_\pm \eta)_{S'\triangle B'} \\ &= \pm (V_{S'}^{\rho'} V_A^{\rho'} P_\pm \eta)_{B'} = \pm (V_A^{\rho'} V_{S'}^{\rho'} P_\pm \eta)_{B'} = (V_A^{\rho'} P_\pm \eta)_{B'}\end{aligned}$$

so by a),

$$\theta_\pm V_A^\rho \theta_\pm^* = V_A^{\rho'} P_\pm = P_\pm V_A^{\rho'} P_\pm = P_\pm V_A^{\rho'} .$$

g) The assertion concerning φ as well as the identity in the definition of ψ follow from a),d),e), and f). Thus ψ is a surjective E-C*-homomorphism. For $Y \in Ker\,\psi$,

$$\theta_+^* Y \theta_+ = \theta_-^* Y \theta_- = 0 ,$$

so by a) and e),

$$P_+ Y = P_- Y = 0$$

and we get

$$Y = P_+ Y + P_- Y = 0$$

i.e. ψ is injective. □

REFERENCES

[1] C. Constantinescu, *C*-algebras.* North-Holland Mathematical Library, Amsterdam, Elsevier, 2001.

[2] C. Constantinescu, *W*-tensor products and selfdual Hilbert right W*-modules.* Rev. Roumaine Math. Pures Appl. **51** (2006), *5-6*, 583–596.

[3] C. Constantinescu, *Selfdual Hilbert right W*-modules and their W*-tensor products.* Rev. Roumaine Math. Pures Appl. **55** (2010), *3*, 159–196.

[4] R.V. Kadison and J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras.* Academic Press, 1983–1986.

[5] C.E. Lance, *Hilbert C*-modules. A toolkit for operator algebraist.* London Math. Soc. Lecture Note Ser. **210**, Cambridge, Cambridge Univ. Press, 1995.

[6] I. Schur, *Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen.* J. Reine Angew. Math. **127** (1904), 20–50.

[7] M. Takesaki, *Theory of Operator Algebra I.* Encyclopaedia of Mathematical Sciences, Berlin, Springer, 2002.

[8] N.E. Wegge-Olsen, *K-theory and C*-algebras.* Oxford Science Publications, Oxford, Oxford Univ. Press, 1993.

Received August 02, 2021

Bodenacherstr. 53
CH 8121 Benglen
`constant@math.ethz.ch`