NON-PERTURBATIVE TOPOS AND ITS APPLICATION

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Communicated by Lucian Beznea

Thanks to the Connes–Kreimer renormalization Hopf algebra in Quantum Field Theory and the topology of graphons in infinite combinatorics, we show the existence of a new class of dimensionally computable Heyting algebras which can encode the quantum logic of mixed Hodge–Tate structures derived from solutions of combinatorial Dyson–Schwinger equations.

AMS 2020 Subject Classification: 03G30, 05C63, 06D20, 16T05, 81T16.

Key words: combinatorial Dyson–Schwinger equations, topology of graphons, Heyting algebras, mixed Hodge–Tate structures.

1. INTRODUCTION

On the one hand, Heyting algebras are useful tools for the study of logical foundations of (semi-)classical and quantum physical systems [6, 7, 26]. On the other hand, mixed Hodge–Tate structures and limiting Hodge theory are fundamental tools in Number Theory which have been applied recently for the study of Feynman integrals and related topics in Quantum Field Theory [1, 2, 5, 9, 14, 15]. In this research work, we are going to show a new connection between these two separate topics where we apply combinatorial Dyson–Schwinger equations ([13, 25]) and their graphon representation models ([17, 18, 19, 20]) to determine a new class of dimensionally computable Heyting algebras which encode the quantum logical background of mixed Hodge–Tate structures associated to polylogarithms.

1.1. Quantum Field Theory

Green's functions are the building blocks of non-perturbative gauge field theories. It is possible to formulate Green's functions in terms of infinite formal expansions of iterated Feynman integrals (or Feynman diagrams) together with increasing powers of (running) coupling constants where the strength of

The author's work on this material was supported by Max Planck Institute for Mathematics, Bonn, Germany.

REV. ROUMAINE MATH. PURES APPL. **68** (2023), *3-4*, 329–345
doi: 10.59277/RRMPA.2023.329.345

the couplings has a direct role in the behavior of these expansions. Green's functions are self-similar such that their fixed point equations generate a fundamental class of recursive equations known as Dyson–Schwinger equations (or quantum motions) in Quantum Field Theory. The Connes–Kreimer renormalization Hopf algebra $H_{\mathrm{FG}}(\Phi)$ ([3]), which has provided an alternative formulation for the BPHZ perturbative renormalization ([5]), enables us to reformulate Dyson–Schwinger equations of a given gauge field theory Φ in terms of the Hochschild Cohomology of commutative graded Hopf algebras [13, 15]. The 1PI Green's function generates a sum over all 1PI Feynman diagrams with respect to the types of external edges. It is given by

$$X^a(\lambda g) = \mathbb{I} + \sum_{\mathrm{res}(\Gamma)=a} (\lambda g)^{|\Gamma|} \frac{\Gamma}{|\mathrm{Aut}(\Gamma)|} \tag{1}$$

as a formal series in the running coupling constants λg which has coefficients in the renormalization Hopf algebra such that $0 < \lambda \leq 1$. Since $H_{\mathrm{FG}}(\Phi)$ is of finite type, at any finite order $|\Gamma|$ only a finite number of graphs contribute to the chosen amplitude [13, 25].

Solutions of combinatorial Dyson–Schwinger equations generate infinite formal expansions of Feynman diagrams as objects in $H_{\mathrm{FG}}(\Phi)[[\lambda g]]$ ([13]). Recently, some new applications of the theory of graphons for sparse graphs to Quantum Field Theory have been found. Thanks to these new combinatorial tools, solutions of combinatorial Dyson–Schwinger equations have been described as convergence limits of sequences of random graphs. It is shown that the theory of graphons for sparse graphs ([4, 12, 16]) enables us to describe the non-perturbative solution of a given Dyson–Schwinger equation in terms of the convergent limit of a sequence of random graphs generated from some partial sums with respect to the cut-distance topology. This perspective enables us to formulate a new analytic generalization of the BPHZ renormalization for the computation of non-perturbative parameters [19, 20].

1.2. **Mixed Hodge–Tate structures**

The category $\mathrm{MHT}_{\mathbb{Q}}$ of mixed Hodge–Tate structures over $\mathbb{Q}$ is the smallest Tannakian subcategory of the category of all mixed Hodge structures over $\mathbb{Q}$. $\mathrm{MHT}_{\mathbb{Q}}$ is in fact a mixed Tate category which contains the Hodge-Tate structures $\mathbb{Q}(0), \mathbb{Q}(1)$ while it is also closed under the extensions [9]. $\mathrm{MHT}_{\mathbb{Q}}$ is equivalent to the category of graded comodules over a graded connected commutative Hopf algebra over $\mathbb{Q}$. The universal Connes–Marcolli category $\mathcal{E}^{\mathrm{CM}}$ of flat equi-singular vector bundles is also a neutral Tannakian category which contains a Tannakian subcategory of mixed Tate motives. This category is rich enough to encode those flat equi-singular connections which contribute to the geometric representation of counterterms derived from perturbative renormalization in physical theories and non-perturbative counterterms derived from renormalization of Dyson–Schwinger equations [5, 21, 22]. The Connes–Kreimer renormalization Hopf algebraic setting has been applied to formulate some strong mathematical tools such as weight, Hodge, monodromy filtrations, etc., to deal with Feynman amplitudes in Quantum Field Theory [1, 2, 15]. The relation between mixed Hodge–Tate structures and flat equi-singular connections has been studied in [5]. Furthermore, solutions of combinatorial Dyson–Schwinger equations determine a class of mixed Hodge–Tate structures associated to polylogarithms [14].

1.3. **Logical setting**

Heyting algebras are introduced in the theory of logical systems to generalize Boolean algebras in terms of replacing the concept of complement with the concept of pseudo-complement for objects in lattices. A Heyting algebra H is computable, if H and its corresponding logical operations are computable. For a given Heyting algebra with one generator, there exist infinitely nonequivalent intuitionistic formulas of one propositional variable. The free Heyting algebras provide a good starting point for investigating the computable dimension of general Heyting algebras because of their connection to the intuitionistic logic. A free Heyting algebra on finite number of generators is computable (in a categorical setting) because any isomorphism is completely determined by the generators. Quantum topos models, which have been designed for the study of the logical foundations of quantum systems, generate an important class of Heyting algebras [6, 7, 8, 10, 11, 26].

1.4. **Achievements**

Thanks to the addressed background, this research work aims to show a new application of Heyting algebras for the study of Dyson–Schwinger equations in strongly coupled gauge field theories. On the one hand, solutions of combinatorial Dyson–Schwinger equations generate an important class of Hopf subalgebras of the Connes–Kreimer renormalization Hopf algebra of Feynman diagrams of a given (strongly coupled) gauge field theory. We organize these Hopf subalgebras in a topos model which is useful for the logical study of non-perturbative aspects. On the other hand, mixed Hodge–Tate structures associated to polylogarithms can be studied in terms of their related period matrices. This class of matrices can be interpreted as solutions of combinatorial Dyson–Schwinger equations. We show that these investigations lead us to build a new class of dimensionally computable Heyting algebras for the logical description of mixed Hodge–Tate structures in gauge field theories.

2. DYSON–SCHWINGER EQUATIONS VIA FEYNMAN GRAPHONS

In this section, we review the structure of combinatorial Dyson–Schwinger equations and the role of graph functions in dealing with non-perturbative solutions of these physical equations.

2.1. Combinatorial Dyson–Schwinger equations

Feynman diagrams and their formal expansions are useful tools for the formulation of gauge field theories in the context of the path integral formalism. Thanks to Feynman rules of a given physical theory, these combinatorial graphs, which encode interactions between elementary particles, are associated to ill-defined iterated integrals in Green's functions.

Definition 2.1. A Feynman diagram Γ is a finite oriented decorated graph which contains (i) a set $V(\Gamma)$ of labeled vertices as symbols for interactions, (ii) a set $E_{\text{int}}(\Gamma)$ of labeled edges with beginning and ending vertices as symbols for virtual particles with assigned momenta, (iii) a set $E_{\text{ext}}(\Gamma)$ of labeled edges with beginning or ending vertex as symbols for elementary particles with assigned momenta. The conservation of momenta is valid on the whole graph.

In general, nested loops in Feynman diagrams are associated to subdivergencies in the corresponding Feynman integrals. The one fundamental challenge is replacing these ill-defined iterated integrals with some regularized or finite values in terms of the theory of perturbative renormalization. The other fundamental challenge is the appearance of strong bare coupling constants (as dimensionless parameters) in the original Lagrangian of the physical theory or strong running coupling constants in the structure of regularized Green's functions. In this situation, we need to deal with high loop order Feynman diagrams as coefficients in power series with respect to (running) coupling constants. If the running or bare coupling constants are small enough, then asymptotically free techniques or higher order perturbation methods can help us generate some finite approximations from those infinite formal expansions. However, the main problem still remains in dealing with strongly coupled Green's functions where quantum motions have non-perturbative behavior [15, 23, 27].

Perturbative renormalization of Feynman diagrams is formulated by the method of the Bogoliubov–Zimmermann's forest formula and the Bogoliubov map where sub-divergencies are removed under Dimensional Regularization and Minimal Subtraction. The Connes–Kreimer Hopf algebra of Feynman diagrams, which is graded with respect to the loop number or the number of internal edges, encapsulates this process in terms of the renormalization coproduct and the theory of Lie groups. The renormalization coproduct is given by

$$\Delta(\Gamma) = \mathbb{I} \otimes \Gamma + \Gamma \otimes \mathbb{I} + \sum_{\gamma} \gamma \otimes \Gamma/\gamma \tag{2}$$

such that $\mathbb{I}$ is the empty graph and the sum is over all disjoint unions of 1PI Feynman subdiagrams. The phrase "Feynman subdiagram" addresses the existence of superficial divergencies. This Hopf algebra, which is connected graded free commutative non-cocommutative, has a Lie algebraic background. It enables us to rebuild complicated Feynman diagrams in terms of its primitive components and the insertion operator [3, 5, 15, 27].

Definition 2.2. For a given primitive Feynman diagram γ, the grafting operator B^+_γ is the basic linear functional which acts on Feynman diagrams such that for each diagram Γ, $B^+_\gamma(\Gamma)$ is the formal sum of Feynman diagrams generated by all possible choices for the insertion of Γ inside γ.

The operators B^+_γ determine a class of Hochschild one cocycles associated to the Hochschild Cohomology of the renormalization Hopf algebra. They are useful to reformulate fixed point equations of Green's functions under an inductive combinatorial setting.

Definition 2.3. For a given family $\{B^+_{\gamma_n}\}_{n\geq 1}$ of Hochschild one cocycles, we define a class of combinatorial recursive equations with the general form

$$\mathrm{DSE}(\lambda g): \; X = \mathbb{I} + \sum_{n\geq 1} (\lambda g)^n w_n B^+_{\gamma_n}(X^{n+1}). \tag{3}$$

It is called a combinatorial Dyson–Schwinger equation under the running coupling constant λg.

This means that for a given amplitude a which requires renormalization, its corresponding combinatorial Dyson–Schwinger equation can be obtained in terms of formal expansions

$$X^a(\lambda g) = \mathbb{I} \pm \sum_{n\geq 1} (\lambda g)^n B^+_{a;n}(X^a(\lambda g) Q^n(\lambda g)). \tag{4}$$

In this formula, $Q(\lambda g) = \frac{X^{\nu}}{\prod_{e \in \mathrm{Ext}(a)} \sqrt{X^{e}(\lambda g)}}$, where X^{ν} is the only independent scattering amplitude of the physical theory which requires renormalization. $B^{+}_{a;n}$ are Hochschild one cocycles generated by primitive (1PI) Feynman diagrams. The strength of running couplings determines perturbative or non-perturbative behavior of Dyson–Schwinger equations. The beta functions of physical theories govern the behavior of running coupling constants. Linear Dyson–Schwinger equations exist in physical theories with the zero beta function while non-zero beta functions generate non-linear Dyson–Schwinger equations. In physical theories with negative beta functions, we can approximate Dyson–Schwinger equations under a linear setting while in physical theories with positive beta functions, we need to deal with non-linear version of these equations [13, 27].

LEMMA 2.4. *Each combinatorial Dyson–Schwinger equation (3) has a unique solution* $X = \sum_{n \geq 0} (\lambda g)^n X_n$ *such that for each* n,

$$X_n = \sum_{m=1}^{n} B^{+}_{\gamma_m} \Big(\sum_{k_1+\ldots+k_{m+1}=n-m,\ k_i \geq 0} X_{k_1} \ldots X_{k_{m+1}} \Big), \quad X_0 = \mathbb{I}. \tag{5}$$

Graphs X_n *are the generators of a free commutative graded Hopf subalgebra of the renormalization Hopf algebra [13].*

Linear Dyson–Schwinger equations generate cocommutative Hopf subalgebras but non-linear version of these equations generate non-cocommutative Hopf subalgebras [13, 25].

2.2. **Graphon models**

Infinite combinatorics aims to study the behavior of sequences of weighted finite weighted sparse or dense graphs with increasing vertex number. The theory of graph functions or graphons is developed to consider these graph sequences in the context of cut-distance topology and some measure theoretic tools [4, 12, 16].

Definition 2.5. For a fixed probability measure space (Ω, μ_{Ω}), a labeled stretched graphon W^{ρ} is a bounded measurable symmetric function on $\Omega \times \Omega$ defined by

$$(x, y) \mapsto W(\rho(x), \rho(y)) \in [a, b] \subset \mathbb{R} \tag{6}$$

such that ρ is an invertible measure-preserving transformation on Ω. The equivalence class $[W]$, which contains all possible labeled graphons W^{ρ}, is called an unlabeled graphon class.

For the closed interval $\Omega = [0, 1]$ equipped with the Lebesgue measure, W^ρ is called a labeled graphon. In the rest of this work, we use the phrase "graphon" in general. The space $\mathcal{W}_{[a,b]}^{(\Omega,\mu_\Omega)}$ of labeled graphons can be equipped with the cut-distance topology determined by the cut norm

$$||W^\rho||_{\text{cut}} := \sup_{A,B}\left| \int_{A\times B} W(\rho(x), \rho(y)) \mathrm{d}\mu_\Omega(x) \mathrm{d}\mu_\Omega(y)\right| \tag{7}$$

such that A, B are μ_Ω-measurable subsets of Ω. The corresponding quotient space $[\mathcal{W}]_{[a,b]}^{(\Omega,\mu_\Omega)}$ of unlabeled graphons can be equipped with a pseudo-metric. Graphons W_1, W_2 are called weakly isomorphic iff there exists measure preserving transformations ρ_1, ρ_2 such that $W_1^{\rho_1} = W_2^{\rho_2}$ almost everywhere.

The quotient space $[\mathcal{W}]_{[a,b]}^{(\Omega,\mu_\Omega)}$ up to the weakly isomorphic relation determines the boundary region for the cut-distance topological space of finite weighted graphs. In other words, graph limits can be interpreted in terms of graphons as analytic graphs [4, 12, 16].

We can study the space of Feynman diagrams of a given gauge field theory in the context of graphon representations to achieve a new concept of convergence for sequences of Feynman diagrams. For this purpose, we apply rooted tree representations of Feynman diagrams. This approach has already been discussed in [17, 18] and here we address the main steps in this direction.

Lemma 2.6. *Each Feynman diagram determines a unique unlabeled graphon class up to the weakly isomorphic relation.*

Proof. For a given gauge field theory Φ, the corresponding renormalization Hopf algebra $H_{\text{FG}}(\Phi)$ can be reformulated in terms of a particular combinatorial Hopf algebra $H_{\text{CK}}(\Phi)$ of non-planar rooted trees decorated by primitive (1PI) Feynman diagrams in Φ. There exists an injective Hopf algebra homomorphism from $H_{\text{FG}}(\Phi)$ to $H_{\text{CK}}(\Phi)$ which enables us to associate a rooted tree representation t_Γ to each Feynman diagram Γ [3, 15].

t_Γ is a finite simple weighted graph. Its corresponding adjacency matrix determines a pixel picture P_{t_Γ} as a labeled graphon. Set $[P_{t_\Gamma}]_\approx$ as the unlabeled Feynman graphon associated to Γ. □

Proposition 2.7. *The space of Feynman diagrams of a given gauge field theory Φ is metrizable.*

Proof. Thanks to Lemma 2.6, consider the space of labeled Feynman graphons corresponding to Feynman diagrams in Φ. Feynman diagrams Γ_1, Γ_2 are called weakly isomorphic iff there exist measure preserving transformations ρ_1, ρ_2 of the ground probability measure space Ω such that $W_{\Gamma_1}^{\rho_1} = W_{\Gamma_2}^{\rho_2}$ almost everywhere.

For a given Feynman diagram Γ, set $[W_\Gamma]_\approx$ as the equivalence class of weakly isomorphic labeled Feynman graphons corresponding to Γ. It contains graph functions W_Γ^ρ for any measure preserving transformations ρ. We call $[W_\Gamma]_\approx$ the unlabeled Feynman graphon class up to the weakly isomorphic relation.

The cut-distance metric between unlabeled Feynman graphons is defined by

$$d_{\text{cut}}([W_{\Gamma_1}]_\approx, [W_{\Gamma_2}]_\approx) =$$

$$\inf_{\varphi,\psi}\sup_{A,B}\left| \int_{A\times B} \Big(W_{\Gamma_1}(\varphi(x),\varphi(y)) - W_{\Gamma_2}(\psi(x),\psi(y)) \Big) \mathrm{d}\mu_\Omega(x)\mathrm{d}\mu_\Omega(y) \right|. \tag{8}$$

The infimum is taken over all measure preserving transformations on Ω and the supremum is taken over all μ_Ω-measurable subsets A, B of Ω. The cut-distance between weakly isomorphic Feynman graphons is zero.

For given Feynman diagrams Γ_1, Γ_2 with the corresponding unlabaled Feynman graphon classes $[W_{\Gamma_1}]_\approx, [W_{\Gamma_2}]_\approx$, define

$$d(\Gamma_1, \Gamma_2) := d_{\text{cut}}([W_{\Gamma_1}]_\approx, [W_{\Gamma_2}]_\approx). \tag{9}$$

□

Corollary 2.8. *The renormalization Hopf algebra can be completed with respect to the cut-distance topology.*

Proof. Thanks to the compactness of the topology of graphons [12, 16] and Proposition 2.7, we can define Feynman graph limits.

Rooted tree representations of Feynman diagrams are sparse decorated weighted graphs. Therefore when n goes to infinity the density of trees tends to zero which means that the corresponding Feynman graphons W_{Γ_n} converges to zero graph function. We can remove this problem by working on the rescaled versions of the canonical labeled Feynman graphons. For example, we can associate the scaled graph function $\frac{W_{\Gamma_n}}{||W_{\Gamma_n}||_{\text{cut}}}$ to each Γ_n to achieve non-zero graphons as Feynman graph limits. There are other techniques in dealing with sequences of sparse graphs in terms of measure theoretic tools [4].

As the consequence, a sequence $\{\Gamma_n\}_{n\geq 1}$ of Feynman diagrams with increasing loop numbers is convergent whenever the corresponding sequence $\{[W_{\Gamma_n}]_\approx\}_{n\geq 1}$ of unlabeled Feynman graphons is cut-distance convergent to a non-zero Feynman graphon up to the weakly isomorphic. In other words, the space of unlabeled Feynman graphon classes, up to the weakly isomorphic relation, completes the renormalization Hopf algebra. □

We set $H^{\rm cut}_{\rm FG}(\Phi)$ as the renormalization Hopf algebra topologically completed with respect to the cut-distance topology. This new enriched Hopf algebra is useful to formulate a new analytic generalization for non-perturbative solutions of combinatorial Dyson–Schwinger equations. In other words, a non-perturbative solution can be interpreted as the cut-distance convergence of the sequence of some partial sums of finite Feynman diagrams.

PROPOSITION 2.9. *The space of Feynman graphons encodes solutions of combinatorial Dyson–Schwinger equations in a gauge field theory.*

Proof. For a given equation DSE(λg) given by Definition 2.3 with the solution $X_{\rm DSE(\lambda g)} = \sum_{n\geq 0}(\lambda g)^n X_n$, set

$$Y_m := \mathbb{I} + (\lambda g)X_1 + ... + (\lambda g)^m X_m, \quad \forall m \geq 1 \tag{10}$$

as the partial sum of order m. In [20], it is shown that the sequence $\{Y_m\}_{m\geq 1}$ of partial sums is cut-distance convergent to $X_{\rm DSE(\lambda g)}$.

For each m, let t_{Y_m} be the forest representation of Y_m with n_m number of vertices. Apply an embedding τ_m to embed these vertices into the closed interval $[0,1]$ to identify nodes $x_1, ..., x_{n_m}$. We build a new random graph R_m on the set $S_m := \{x_1, ..., x_{n_m}\}$ in such a way that there exists an edge between x_i, x_j with the probability $W_{Y_m}(x_i, x_j)$.

We can show that the sequence $\{R_m\}_{m\geq 1}$ of finite random graphs is cut-distance convergent to an infinite random graph $R_{\rm DSE(\lambda g)}$. The random graph $R_{\rm DSE(\lambda g)}$ determines the unlabeled Feynman graphon class corresponding to $X_{\rm DSE(\lambda g)}$. This means that the sequence $\{[W_{Y_m}]_{\approx}\}_{m\geq 1}$ is convergent to $W_{X_{\rm DSE(\lambda g)}}$. □

Thanks to Corollary 2.8, it is possible to topologically complete Hopf subalgebras associated to combinatorial Dyson–Schwinger equations. We set $H^{\rm cut}_{\rm DSE(\lambda g)}$ as the Hopf algebra generated by the equation DSE(λg) and completed with respect to the cut-distance topology.

3. A TOPOS MODEL ON DYSON–SCHWINGER EQUATIONS

In this section, we explain the fundamental structure of a new topos model built on a particular small category of topological Hopf algebras. Our study leads us to address a new class of Heyting algebras.

THEOREM 3.1. *For a given (strongly coupled) gauge field theory Φ with the bare coupling constant g, there exists a topos model which logically encodes solutions of all combinatorial Dyson–Schwinger equations under different running coupling constants.*

Proof. Topological Hopf subalgebras $H^{\text{cut}}_{\text{DSE}(\lambda g)}$ associated to arbitrary combinatorial Dyson–Schwinger equations $\text{DSE}(\lambda g)$ and their closed Hopf subsubalgebras can be organized in a poset structure. For all pairs (H_1, H_2) of these objects, we define arrows pointing from H_1 to H_2 iff there exists a homomorphism $H_1 \to H_2$ of Hopf algebras which is continuous with respect to the cut-distance topology. We interpret this poset in terms of a category $\mathcal{C}^{\text{non},g}_{\Phi}$. It is possible to encode the renormalization Hopf algebra $H_{\text{FG}}(\Phi)$ in terms of an infinite system of combinatorial Dyson–Schwinger equations generated by fixed point equations of vertex-type and edge-type Green's functions of the physical theory.

Therefore $\mathcal{C}^{\text{non},g}_{\Phi}$ is a small category which contains sub-objects of each object. These topological subspaces allow us to determine arbitrary small cut-distance topological neighborhoods around solutions of combinatorial Dyson–Schwinger equations.

Now consider the topos of presheaves on the base category $\mathcal{C}^{\text{non},g}_{\Phi}$. The objects of this topos are contravariant functors from the category $\mathcal{C}^{\text{non},g}_{\Phi}$ to the standard category **Set** of sets and functions. The morphisms of this topos are natural transformations between those functors.

The terminal object $\mathbf{1}$ of this topos is defined by $\mathbf{1}(H) := \{*\}$ at all stages H in $\mathcal{C}^{\text{non},g}_{\Phi}$ while if $f : H_1 \to H_2$ is a morphism in $\mathcal{C}^{\text{non},g}_{\Phi}$, then $\mathbf{1}(f) : \{*\} \to \{*\}$.

The spectral presheaf $\underline{\sum}$ of this topos sends each topological Hopf subalgebra $H^{\text{cut}}_{\text{DSE}}$ to its corresponding complex Lie group

$$\underline{\sum}(H^{\text{cut}}_{\text{DSE}}) := \mathbb{G}_{\text{DSE}}(\mathbb{C}) = \text{Hom}(H_{\text{DSE}}, \mathbb{C}) \tag{11}$$

of characters. It also sends each morphism $i_{H_1H_2} : H_1 \subseteq H_2$ in the base category $\mathcal{C}^{\text{non},g}_{\Phi}$ to the map $\underline{\sum}(i_{H_1H_2}) : \underline{\sum}(H_2) \longrightarrow \underline{\sum}(H_1)$. It sends each character on H_2 to a character on H_1 by restriction. The subobjects of the spectral presheaf can be determined in a standard way.

The outer presheaf $\underline{O}$ of this topos sends each topological Hopf subalgebra $H^{\text{cut}}_{\text{DSE}}$ to the set In(DSE) of all infinitesimal characters corresponding to Feynman diagrams in $H^{\text{cut}}_{\text{DSE}}$. It also sends each morphism $i_{H_1H_2} : H_1 \subseteq H_2$ in the base category $\mathcal{C}^{\text{non},g}_{\Phi}$ to the map $\underline{O}(i_{H_1H_2}) : \underline{O}(H_2) \longrightarrow \underline{O}(H_1)$. It sends each infinitesimal character Z_Γ in $\underline{O}(H_2)$ to the infinitesimal character $\delta(Z_\Gamma)$ in $\underline{O}(H_1)$. $\delta(Z_\Gamma)$ is determined as the smallest infinitesimal character

$$\delta(Z_\Gamma) := \bigwedge\{Z_\gamma \in \text{In}(\text{DSE}_2) : \quad \text{Z}_\Gamma \preceq \text{Z}_\gamma\} \tag{12}$$

such that $\preceq$ is the partial order on infinitesimal characters defined with respect to the number of independent loops in their corresponding Feynman diagrams.

The subobject classifier of this topos is the presheaf $\Omega^{\text{non},g}_{\Phi} : \mathcal{C}^{\text{non},g}_{\Phi} \to \textbf{Set}$ such that for any object H in the base category, $\Omega^{\text{non},g}_{\Phi}(H)$ is identified by the set of all sieves on H. If $f : H_1 \to H$ is a morphism in the base category, then $\Omega_\Phi^{\mathrm{non},g}(f) : \Omega_\Phi^{\mathrm{non},g}(H) \to \Omega_\Phi^{\mathrm{non},g}(H_1)$ is given by

$$\Omega_\Phi^{\mathrm{non},g}(f)(S) := \{h : H_2 \to H_1,\ f \circ h \in S\} \tag{13}$$

for all $S \in \Omega_\Phi^{\mathrm{non},g}(H)$. It is actually the pull-back to H_1 of the sieve S on H by the morphism f. □

Corollary 3.2. *The presheaf $\Omega_\Phi^{\mathrm{non},g}$ enables us to interpret subobjects of any presheaf X in our topos model in terms of natural transformations $\chi : X \to \Omega_\Phi^{\mathrm{non},g}$.*

Proof. Thanks to Theorem 3.1, on the one hand, for any subobject K of X, its associated characteristic morphism χ^K is defined in terms of its components $\chi^K_{H^{\mathrm{cut}}_{\mathrm{DSE}}} : X(H^{\mathrm{cut}}_{\mathrm{DSE}}) \to \Omega_\Phi^{\mathrm{non},g}(H^{\mathrm{cut}}_{\mathrm{DSE}})$ with respect to all Dyson–Schwinger equations in the base category $\mathcal{C}_\Phi^{\mathrm{non},g}$. On the other hand, each natural transformation $\chi : X \to \Omega_\Phi^{\mathrm{non},g}$ defines a subobject K^χ of X which is given by

$$K^\chi(H^{\mathrm{cut}}_{\mathrm{DSE}}) := \chi^{-1}_{H^{\mathrm{cut}}_{\mathrm{DSE}}}\{1_{\Omega_\Phi^{\mathrm{non},g}(H^{\mathrm{cut}}_{\mathrm{DSE}})}\} \tag{14}$$

at each stage of the logical truth with respect to the Dyson–Schwinger equation DSE. □

4. COMPUTABLE HEYTING ALGEBRAS FOR MIXED HODGE–TATE STRUCTURES

In this section, we plan to apply the topos of Dyson–Schwinger equations to explain the construction of a new class of dimensionally computable Heyting algebras which describe the logical background of mixed Hodge–Tate structures derived from solutions of combinatorial Dyson–Schwinger equations in Quantum Field Theory.

Definition 4.1. A lattice $(L, \leq)$ is a non-empty set equipped with an order relation such that for all $a, b \in L$, the least upper bound $a \vee b := \sup(\{a, b\})$ and the greatest lower bound $a \wedge b := \inf(\{a, b\})$ exist. It is called a bounded lattice, if there exist a top element $\top$ and a bottom element $\bot$ such that for all $a \in L$, $\bot \leq a \leq \top$. It is called a complete lattice if for any subset $S \subseteq L$, $\mathrm{Inf}(S)$ and $\mathrm{Sup}(S)$ exist. It is called a distributive lattice, if L obeys the distributive law which means that for all $a, b, c \in L$, we have

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),\ a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c). \tag{15}$$

It is possible to extend these conditions to an infinite level. [8]

For a given bounded lattice L and any $a \in L$, another object $b \in L$ is a complement of a if $a \wedge b = \bot$ and $a \vee b = \top$. In general, the complement is not unique. A bounded distributive lattice L is called Boolean algebra if each $a \in L$ has a unique complement $a' \in L$. Boolean algebras are useful for the description of the logical foundations of Classical Mechanics while we need a weaker version of these algebras (i.e. Heyting algebras) for the construction of quantum logics [6, 8, 10, 26].

Definition 4.2. A bounded distributive lattice H equipped with the implication (as a binary operation) is called a Heyting algebra if for all $a, b, c \in H$, we have

$$c \leq (a \Rightarrow b) \Leftrightarrow (a \wedge c) \leq b. \tag{16}$$

A subset A of natural numbers is called computable if there exists an algorithm to decide whether a natural number belongs to A or not. In other words, A is computable if its corresponding characteristic function is computable.

Definition 4.3. An algebraic structure is called computable if its domain can be identified with a computable set of natural numbers where the (finitely many) operations and relations on the structure are computable. If the structure is infinite, we identify the cardinal of its domain with the symbol ω.

The computable dimension of a computable structure is the number of classically isomorphic computable copies of the structure up to the computable isomorphism [6, 8, 10].

For a given gauge field theory Φ, the subobject classifier $\Omega_{\Phi}^{\mathrm{non},g}$ in the topos of Dyson–Schwinger equations has a natural Heyting algebraic structure.

Lemma 4.4. *For a given combinatorial Dyson–Schwinger equation DSE, the space $\Omega_{\Phi}^{\mathrm{non},g}(H_{\mathrm{DSE}}^{\mathrm{cut}})$ is a Heyting algebra.*

Proof. From Theorem 3.1, $\Omega^{\mathrm{non},g}(H_{\mathrm{DSE}}^{\mathrm{cut}})$ contains all sieves on $H_{\mathrm{DSE}}^{\mathrm{cut}}$. For arbitrary collections S_1, S_2 of sieves on $H_{\mathrm{DSE}}^{\mathrm{cut}}$, the partial order relation on $\Omega^{\mathrm{non},g}(H_{\mathrm{DSE}}^{\mathrm{cut}})$ is given by

$$S_1 \leq S_2 \Leftrightarrow S_1 \subseteq S_2. \tag{17}$$

This ordering allows us to define the following elementary logical statements

$$\begin{aligned} &S_1 \wedge S_2 := S_1 \cap S_2, \quad S_1 \vee S_2 := S_1 \cup S_2, \\ &S_1 \Rightarrow S_2 := \{f : H_{\mathrm{DSE},2}^{\mathrm{cut}} \to H_{\mathrm{DSE}}^{\mathrm{cut}} | \ \forall g : H_{\mathrm{DSE},3}^{\mathrm{cut}} \to H_{\mathrm{DSE},2}^{\mathrm{cut}}, \\ &\qquad\qquad \text{if } f \circ g \in S_1, \text{then } f \circ g \in S_2\}. \end{aligned} \tag{18}$$

The negation of an element S is defined by the proposition $\neg S := S \Rightarrow 0$ which means that

$$\neg S := \{f : H_{\mathrm{DSE},2}^{\mathrm{cut}} \to H_{\mathrm{DSE}}^{\mathrm{cut}} \mid \forall g : H_{\mathrm{DSE},3}^{\mathrm{cut}} \to H_{\mathrm{DSE},2}^{\mathrm{cut}}, \quad f \circ g \notin S\}. \tag{19}$$

Therefore for arbitrary pair (S_1, S_2), there exists a proposition $S_1 \Rightarrow S_2$ with the property that for all S in $\Omega^{\mathrm{non},g}(H^{\mathrm{cut}}_{\mathrm{DSE}})$,

$$S \leq (S_1 \Rightarrow S_2) \Leftrightarrow S \wedge S_1 \leq S_2. \tag{20}$$

In addition, the principal sieve on $H^{\mathrm{cut}}_{\mathrm{DSE}}$ is the unit element of this Heyting algebra while the null element is the empty sieve. □

Therefore for any object $H^{\mathrm{cut}}_{\mathrm{DSE}}$ in the base category $\mathcal{C}^{\mathrm{non},g}_{\Phi}$, $\Omega^{\mathrm{non},g}_{\Phi}(H^{\mathrm{cut}}_{\mathrm{DSE}})$ can be equipped by the intuitionistic logic. It enables us to build a new logical platform for the evaluation of propositions associated to cut-distance topological regions of Feynman diagrams which contribute to the solution of the equation DSE.

Definition 4.5. A mixed Hodge–Tate structure over $\mathbb{Q}$ is a $\mathbb{Q}$-vector space V which is equipped with a weight filtration $W_{\bullet}$ and a Hodge filtration $F^{\bullet}$ of its complexification $V_{\mathbb{C}}$ such that for odd m, $\mathrm{gr}^W_m V = 0$ and for even m, $\mathrm{gr}^W_{2n} V$ is a direct sum of the Hodge–Tate structures $\mathbb{Q}(-n)$ in terms of the Hodge filtration $F^{\bullet}$ [5, 9].

We are going to apply the subobject classifier $\Omega^{\mathrm{non},g}_{\Phi}$ of the topos of Dyson–Schwinger equations in a given gauge field theory Φ (i.e. Theorem 3.1) to formulate a new class of Heyting algebras which are capable to encode the quatum logics of mixed Hodge–Tate structures associated to polylogarithms.

THEOREM 4.6. *There exists a class of dimensionally computable Heyting algebras which encode the logical background of mixed Hodge–Tate structures associated to polylogarithms.*

Proof. At the first step of the proof, we plan to show that the logical propositions about the solution of the equation DSE can be evaluated by a computable Heyting algebra. For this purpose, we want to lift the Heyting algebra $\Omega^{\mathrm{non},g}_{\Phi}(H^{\mathrm{cut}}_{\mathrm{DSE}})$ onto an enriched version $\hat{\Omega}^{\mathrm{non},g}_{\Phi}(H^{\mathrm{cut}}_{\mathrm{DSE}})$ which is computable at the level of its dimension.

The structure of an algorithmic model of constructing a computable version of a given infinite Heyting algebra at the level of its dimension has been discussed in [24]. Here, we modify that construction process to build our promised computable Heyting algebra $\hat{\Omega}^{\mathrm{non},g}_{\Phi}$ on the basis of the subobject classifier $\Omega^{\mathrm{non},g}_{\Phi}$.

Consider the propositional intuitionistic logic over the given language

$$(\Omega^{\mathrm{non},g}_{\Phi}(H^{\mathrm{cut}}_{\mathrm{DSE}}), \wedge, \vee, \Rightarrow, \perp, \top). \tag{21}$$

$\Omega^{\mathrm{non},g}_{\Phi}(H^{\mathrm{cut}}_{\mathrm{DSE}})$ can be seen as the collection of propositional formulas in infinitely many variables modulo equivalence under the intuitionistic logic where $\wedge, \vee, \Rightarrow$

are the logical connectives, $\perp$ is false and $\top$ is true. The codes for formulas such as $\phi \wedge \psi, \phi \vee \psi$ or $\phi \Rightarrow \psi$ are always greater than the codes for ϕ and ψ.

We know that the propositional intuitionistic logic is decidable which provides a computable copy of the free Heyting algebra on ω generators [6, 10]. Therefore we consider elements of $\Omega_{\Phi}^{\mathrm{non},g}(H_{\mathrm{DSE}}^{\mathrm{cut}})$ as the equivalence classes $[\phi]$ under provable equivalence in the intuitionistic logic which leads us to the following computational operations:

$$[\phi] \leq [\psi] \Longleftrightarrow \phi \Rightarrow \psi \text{ is provable under the intuitionistic logic,}$$

$$[\phi] \wedge [\psi] = [\phi \wedge \psi], \ [\phi] \vee [\psi] = [\phi \vee \psi]. \tag{22}$$

The plan is to build $\hat{\Omega}_{\Phi}^{\mathrm{non},g}(H_{\mathrm{DSE}}^{\mathrm{cut}})$ as a computable copy which is not computability isomorphic to $\Omega_{\Phi}^{\mathrm{non},g}(H_{\mathrm{DSE}}^{\mathrm{cut}})$. Let $\alpha_s(n)$ be a label at stage s determined by the domain of the subobject classifier in the construction process. It is indeed a propositional formula in the intuitionistic logic. The resulting sequence $\{\alpha_s(n)\}_{n\geq 0}$ is convergent which means that $\alpha(n) = \lim_s \alpha_s(n)$. For $n \neq m$, the propositional formulas $\alpha(n)$ and $\alpha(m)$ are not intuitionistically equivalent. For each intuitionistic propositional formula ϕ, there exists some n such that $\alpha(n)$ is intuitionistically equivalent to ϕ.

In addition, once we define the join, meet or relative pseudo-complement of elements, these relationships never change in the future stages.

As the consequence, the function α, which indicates an isomorphism between $\hat{\Omega}_{\Phi}^{\mathrm{non},g}(H_{\mathrm{DSE}}^{\mathrm{cut}})$ and $\Omega_{\Phi}^{\mathrm{non},g}(H_{\mathrm{DSE}}^{\mathrm{cut}})$, makes $\hat{\Omega}_{\Phi}^{\mathrm{non},g}(H_{\mathrm{DSE}}^{\mathrm{cut}})$ computable.

At this stage, we should diagonalize against all possible computable isomorphisms and for this purpose we need to concern morphisms such as

$$\phi_e : \hat{\Omega}_{\Phi}^{\mathrm{non},g}(H_{\mathrm{DSE}}^{\mathrm{cut}}) \to \Omega_{\Phi}^{\mathrm{non},g}(H_{\mathrm{DSE}}^{\mathrm{cut}}) \tag{23}$$

which is not an isomorphism. We remove this issue by adapting the inductive machinery applied in [24]. At the end of the day, $\hat{\Omega}_{\Phi}^{\mathrm{non},g}(H_{\mathrm{DSE}}^{\mathrm{cut}})$ is our promised computable Heyting algebra.

At the second step of the proof, we are going to apply the close connection between solutions of combinatorial Dyson–Schwinger equations and polylogarithms and their corresponding Mixed Hodge–Tate structures. This allows us to adapt the built computable Heyting algebra in the previous part for the level of Mixed Hodge–Tate structures.

Mixed Hodge–Tate structures associated to polylogarithms can be described in terms of their related period matrices [1, 5, 14]. It is shown that this class of matrices can be interpreted as solutions of Dyson–Schwinger equations [14, 15]. Therefore the period matrix M_{MHT} of each mixed Hodge–Tate structure MHT determines a topological Hopf subalgebra $H_{\mathrm{DSE}_{\mathrm{MHT}}}^{\mathrm{cut}}$ in the base category $\mathcal{C}_{\Phi}^{\mathrm{non},g}$ of the topos of Dyson–Schwinger equations. It means that the Heyting algebra $\Omega_{\Phi}^{\mathrm{non},g}(H^{\mathrm{cut}}_{\mathrm{DSE}_{\mathrm{MHT}}})$ encodes the quantum logic platform for MHT and its corresponding period matrix. □

Theorem 4.6 constructs a new computable modification for those Heyting algebras derived from the subobject classifier $\Omega_{\Phi}^{\mathrm{non},g}$. This logical setting shows a fundamental connection between a class of mixed Hodge–Tate structures and solutions of combinatorial Dyson–Schwinger equations.

5. CONCLUSION

Topological Hopf subalgebras associated to solutions of combinatorial Dyson–Schwinger equations of a given gauge field theory can be organized in a small category. This small category is applied as the base category for a new topos model which encodes the logical evaluation of the cut-distance topological regions of Feynman diagrams which contribute to solutions of Dyson–Schwinger equations. Representations of this new topos model address the logical foundations of strongly coupled gauge field theories. It allows us to evaluate logical propositions about a physical phenomena which contains interactions among a finite number of particles in a quantum system with infinite degrees of freedom.

Perturbative renormalization deals with ill-defined iterated Feynman integrals in terms of methods such as Dimensional Regularization and Minimal Subtraction scheme. Applying parametric representations enables us to formulate Feynman integrals in the context of Kirchhoff–Symanzik polynomials to extract finite values. Logarithmically divergent and projective integrals address a deep connection between renormalization theory and limiting Hodge theory. This setting leads to analyze the motivic nature of periods associated to renormalized amplitudes [1, 2, 15, 23, 27]. Thanks to the topos model given by Theorem 3.1, which is defined on the basis of the cut-distance topological regions of Feynman diagrams which contribute to solutions of Dyson–Schwinger equations, the internal logic of $\Omega_{\Phi}^{\mathrm{non},g}$ provides the logical background of perturbative renormalization in the context of limiting Hodge theory.

Acknowledgments. The author is grateful to Max Planck Institute for Mathematics, Bonn, Germany for the support and hospitality during the work on this research.

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Received October 25, 2021

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