RELATIVE BRAUER RELATIONS OF ABELIAN P-GROUPS

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The Brauer relations of a finite group G are virtual differences of non-isomorphic G-sets $X - Y$ which induce isomorphic permutation G-representations $\mathbb{Q}[X] \simeq \mathbb{Q}[Y]$ over the rationals. These relations have been classified by Tornehave-Bouc and Bartel-Dokchitser. Motivated by stable homotopy theory, a relative version of Brauer relations for (G, C_p)-bisets which are C_p-free have been classified by Kahn in case G is an elementary Abelian p-group. In this paper, we extend Kahn's classification to the case when G is a finite Abelian p-group.

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1. INTRODUCTION

The Burnside ring $A(G)$ of a finite group G is the Grothendick ring of the category of finite G-sets and is isomorphic up to completion to the stable cohomotopy group of the classifying space B_G according to Segal's Conjecture [6]. The relative Burnside module $A(G, H)$ of a pair of finite groups (G, H) is the Grothendick module of the category of finite (G, H)-bisets that are H-*free*. Up to completion, $A(G, H)$ describes the stable homotopy classes of maps from the classifying space B_G to the classifying space B_H, by the generalized Segal's Conjecture [10].

The representation ring $R_F(G)$ of a finite group G over a field F is the Grothendick ring of the category of finitely generated FG-modules, where FG denotes the group ring of G over F. Let $F = \mathbb{Q}$ be the field of rational numbers. The functor sending each finite G-set X to the permutation G-module $\mathbb{Q}[X]$ induces a ring homomorphism from the Burnside ring $A(G)$ to the rational representation ring $R_{\mathbb{Q}}(G)$:

$$f_G : A(G) \to R_{\mathbb{Q}}(G), \qquad f_G[X] = \mathbb{Q}[X]. \tag{1}$$

The cokernel of f_G is finite of exponent dividing the group order $|G|$ by Artin's induction theorem [16] and it is trivial if G is a finite p-group by Ritter-Segal [13, 15]. The elements of the kernel $K(G)$ of f_G are called

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the *Brauer relations* of the group G or the *G-relations* and these have been classified by Tornehave-Bouc [5, 17] for finite p-groups and Bartel-Dokchitser [3] for arbitrary finite groups.

We note that the image of the submodule $A(G,H) \subset A(G\times H)$ under the map $f_{G\times H}$ is contained in the Grothendieck submodule $R_{\mathbb{Q}}(G,H) \subset R_{\mathbb{Q}}(G\times H)$ of the category of finitely generated $\mathbb{Q}(G\times H)$-modules which are *free right* $\mathbb{Q}H$-modules. The kernel and cokernel of the well defined restricted homomorphism

$$f_{G,H} = f_{G\times H} : A(G,H) \to R_{\mathbb{Q}}(G,H) \tag{2}$$

are also of interest in view of the generalized Segal's Conjecture. In particular, it makes sense to call the elements of the kernel $K(G,H)$ of $f_{G,H}$ the *relative Brauer relations* of the pair (G,H) or the *(G,H)-relations.* The cokernel of $f_{G,H}$ is trivial for G a finite p-group and $H = C_p$ by Anton [1] and the (G, C_p)-relations have been classified by Kahn [7] for G an elementary Abelian p-group, where C_p *denotes the cyclic group of prime order* p. The main result of this paper is

Theorem 1.1. *The relative Brauer relations of the pair (G, C_p) for G a finite Abelian p-group are linear combinations of relative Brauer relations 'indufted' from sub-quotients of $G \times C_p$ of the form $C_p \times C_p \times C_p$.*

We note that there is a natural ring homomorphism mapping $R_F(G)$ to the stable homotopy classes of maps from the classifying space B_G to the plus construction of the classifying space $B_{GL(F)}$, where $GL(F)$ is the infinite general linear group over the field F. Up to completion, this map is an isomorphism for F the (topological) field of complex numbers [2] or for F a finite field [8]. If $F = \mathbb{Q}$, this homomorphism connects Brauer relations with algebraic K-theory [11]. The relative Brauer relations are connected with maps between algebraic K-theory spectra.

The background terminology will be reviewed in Section §2 and a precise formulation of Theorem 1.1 will be given in Section §3. In Section §4, we prove some key rank lemmas. In Section §5, we prove the main theorem up to torsion. In Section §6, we reduce the proof to a set of special generators and finish the argument in §7.

2. BACKGROUND AND TERMINOLOGY

This section is a survey of basic definitions and facts about Burnside and representation modules, many of them being used in this paper.

2.1. **Relative Burnside modules**

Following [3], the *Burnside ring* $A(\Gamma)$ of a group Γ is the free Abelian group generated by the isomorphism classes $[X]$ of finite Γ-sets X modulo the relations $[X \sqcup Y] = [X] + [Y]$ where $\sqcup$ denotes the disjoint union. The product in $A(\Gamma)$ is given by $[X] * [Y] = [X \times Y]$ where $X \times Y$ is the Γ-set under the diagonal Γ-action.

PROPOSITION 2.1 ([3]). *The transitive* Γ*-sets are* left coset spaces Γ/L *of subgroups* $L \leq \Gamma$ *and their isomorphism classes* $[\Gamma/L]$ *form a basis for* $A(\Gamma)$.

There is a bijection sending the conjugacy class of a subgroup $L \leq \Gamma$ to the basis element $[\Gamma/L]$. Using this identification:

We write the elements of $A(\Gamma)$ *as integral linear combinations* $\sum n_i L_i$ *of subgroups* $L_i \leq \Gamma$ *up to conjugacy.*

The product of basis elements in $A(\Gamma)$ is given by the double coset formula

$$L * M = \sum_{x \in L\backslash\Gamma/M} L \cap {}^{x}M \tag{3}$$

where $L, M \leq \Gamma$ and ${}^{x}M = xMx^{-1}$ for $x \in \Gamma$.

Given a pair (G, H) of finite groups, let $\Gamma = G \times H$. A (G, H)*-biset* is a *finite* set X, endowed with a left G-action and a right H-action that commute with each other, i.e., a left Γ-action with the right H-action defined via

$$xh = h^{-1}x \text{ for } x \in X \text{ and } h \in H. \tag{4}$$

Definition 2.2. The *relative Burnside module* $A(G, H)$ of a pair of finite groups (G, H) is the free Abelian group generated by the isomorphism classes $[X]$ of (G, H)-bisets which are *right* H*-free*, modulo the relations

$$[X \sqcup Y] = [X] + [Y].$$

LEMMA 2.3 (Goursat [9]). *The subgroups of a direct product of two finite groups* $\Gamma \times \Omega$ *are in bijection with the quintuples* (K, N, A, B, θ) *of subgroups* $N \trianglelefteq K \leq \Gamma$ *and* $B \trianglelefteq A \leq \Omega$ *and isomorphisms* $\theta : K/N \approx A/B$.

The correspondence in Goursat Lemma is given by the following map

$$(K, N, A, B, \theta) \mapsto S = \{(kn, \theta(kN)b) | n \in N,\ k \in K,\ b \in B\} \leq \Gamma \times \Omega.$$

This lemma explains how to pass from the basis of $A(G\times H)$ to the basis of $A(G, H)$. More precisely, according to [10], the transitive right H-free (G, H)-bisets are *twisted products* $G \times_\rho H$ between G and a group homomorphism $\rho : K \to H$ from a subgroup $K \leq G$. Such a product is the quotient of Γ modulo the relations

$$[gk, h] = [g, \rho(k)h] \text{ for } g \in G, h \in H, \text{ and } k \in K.$$

Here $[g,h]$ denotes the class of (g,h) in $G\times_\rho H$. The map $\Gamma\to G\times H$ given by $(g,h)\mapsto(g,h^{-1})$ induces an isomorphism of left Γ-sets $\Gamma/(K\times\rho)\approx G\times_\rho H$ where *the graph (subgroup) of* ρ in Γ is denoted by

$$K\times\rho=\{(k,\rho(k)):k\in K\}. \tag{5}$$

This is an isomorphism of (G,H)-bisets via (4).

PROPOSITION 2.4 ([10]). *A basis for the submodule $A(G,H)\subset A(\Gamma)$ consists of the isomorphism classes of twisted products $[G\times_\rho H]$.*

There is a bijection between these basis elements and the conjugacy classes of group homomorphisms $\rho:K\to H$ with $K\leq G$ or equivalently of subgroups $K\times\rho\leq\Gamma$. Using this identification:

We write the elements of $A(G,H)$ as integral linear combinations $\sum n_iK_i\times\rho_i$ of graphs up to conjugacy of homomorphisms $\rho_i:K_i\to H$ from $K_i\leq G$.

If Z is a G-set and X is a (G,H)-biset, then the product $[Z][X]=[Z\times X]$ defines a left $A(G)$-module structure on $A(G,H)$, where G acts on $Z\times X$ diagonally from the left and H acts only on X from the right. The $A(G)$-module structure on $A(G,H)$ is made explicit for $M,K\leq G$ and $\rho:K\to H$ by the product:

$$M*(K\times\rho)=\sum_{x\in K\backslash G/M}(K\cap{}^xM)\times\rho. \tag{6}$$

2.2. **Functorial operations on Burnside modules**

These operations are $\mathbb{Z}$-linear maps on $A(\Gamma)$. The *induction* $\operatorname{Ind}{}_L^\Gamma: A(L)\to A(\Gamma)$ from a subgroup $L\leq\Gamma$ is defined on L-sets Y by

$$\operatorname{Ind}{}_L^\Gamma Y=\Gamma\times_L Y \text{ where } \Gamma \text{ acts by left multiplication on } \Gamma.$$

Here $\Gamma\times_L Y$ denotes the quotient of $\Gamma\times Y$ modulo the relations $[\gamma l,y]=[\gamma,ly]$ for $\gamma\in\Gamma, l\in L$ and $y\in Y$.

The *restriction* $\operatorname{Res}{}_L^\Gamma: A(\Gamma)\to A(L)$ is defined on Γ-sets X by

$$\operatorname{Res}{}_L^\Gamma X=X \text{ where } L \text{ acts on } X \text{ as a subgroup of } \Gamma.$$

On basis elements, for $K\leq L$ and $M\leq\Gamma$ we have

$$\operatorname{Ind}{}_L^\Gamma K=K,\qquad \operatorname{Res}{}_L^\Gamma M=\sum_{x\in L\backslash\Gamma/M}L\cap{}^xM. \tag{7}$$

The *inflation* $\operatorname{Inf}{}_\Pi^\Gamma: A(\Pi)\to A(\Gamma)$ from a quotient $\Pi=\Gamma/N$ by a normal subgroup $N\trianglelefteq\Gamma$ is defined on Π-sets Z by

$$\operatorname{Inf}{}_\Pi^\Gamma Z=Z \text{ where } \Gamma \text{ acts on } Z \text{ via its projection in } \Pi.$$

The *deflation* $\operatorname{Def}^{\Gamma}_{\Pi} : A(\Gamma) \to A(\Pi)$ on Γ-sets X is the *orbit space* of N

$$\operatorname{Def}^{\Gamma}_{\Pi} X = N\backslash X \text{ where } [\gamma][x] = [\gamma x] \text{ for } \gamma \in \Gamma \text{ and } x \in X.$$

Here $[\gamma]$ denotes the coset γN in Π and $[x]$ denotes the *orbit* Nx of x in $N\backslash X$. On basis elements, for $N \trianglelefteq K \leq \Gamma$ and $M \leq \Gamma$, we have

$$\operatorname{Inf}^{\Gamma}_{\Pi}(K/N) = K, \qquad \operatorname{Def}^{\Gamma}_{\Pi} M = NM/N. \tag{8}$$

2.3. **Relative representation modules**

Following [16], the *rational representation ring* $R_{\mathbb{Q}}(\Gamma)$ of a finite group Γ is the free Abelian group generated by the isomorphism classes $[V]$ of finitely generated left $\mathbb{Q}\Gamma$-modules V modulo the relations $[V \oplus W] = [V] + [W]$ where $\oplus$ denotes the direct sum. The product is given by $[V] * [W] = [V \otimes W]$ where $\otimes$ denotes the tensor product over $\mathbb{Q}$ and $V \otimes W$ is the $\mathbb{Q}\Gamma$-module under the diagonal Γ-action. The irreducible $\mathbb{Q}\Gamma$-modules are direct summands V_i of the group ring $\mathbb{Q}\Gamma$ and their isomorphism classes $[V_i]$ form a basis for $R_{\mathbb{Q}}(\Gamma)$.

Given a pair of finite groups (G, H), let $\Gamma = G \times H$. A (G, H)*-bimodule over the rationals* is simply a finitely generated left $\mathbb{Q}\Gamma$-module V with $\mathbb{Q}G$ acting on the left via the canonical inclusion in $\mathbb{Q}\Gamma$ and $\mathbb{Q}H$ acting on the right via the rule

$$vh = h^{-1}v, \text{ for } v \in V \text{ and } h \in H. \tag{9}$$

Definition 2.5. The *relative rational representation module* $R(G, H)$ of a pair of finite groups (G, H) is the submodule of $R_{\mathbb{Q}}(\Gamma)$ generated by the isomorphism classes of (G, H)-bimodules over the rationals, which are *right* $\mathbb{Q}H$*-free* modules.

We call a right $\mathbb{Q}H$-free (G, H)-bimodule V over the rationals *irreducible* if V cannot be decomposed as a direct sum of right $\mathbb{Q}H$-free (G, H)-bimodules over the rationals. Hence, the isomorphism classes $[W_i]$ of irreducible right $\mathbb{Q}H$-free (G, H)-bimodules W_i over the rationals form a basis for $R(G, H)$. Notice that $\mathbb{Q}H$ with $\mathbb{Q}H$ acting on the right by the multiplication in H and $\mathbb{Q}G$ acting on the left by the identity $1 \in G$ is an example of an irreducible right $\mathbb{Q}H$-free (G, H)-bimodule over the rationals which is not an irreducible left $\mathbb{Q}\Gamma$-module (unless $H = 1$).

If U is a left $\mathbb{Q}G$-module and V a right $\mathbb{Q}H$-free (G, H)-bimodule over the rationals, then the product $[U] * [V] = [U \otimes V]$ defines a left $R_{\mathbb{Q}}(G)$-module structure on $R(G, H)$, where $\mathbb{Q}G$ acts on $U \otimes V$ diagonally from the left and $\mathbb{Q}H$ acts only on V from the right. Indeed, with these actions, $U \otimes V$ is a right $\mathbb{Q}H$-free (G, H)-bimodule.

From the relative Burnside module $A(G,H)$ to the relative rational representation module $R(G,H)$, we have the natural linear map $f_{G,H}$ by (2).

Definition 2.6. The *relative Brauer relations* of a pair of finite groups (G,H) are the elements of the kernel $K(G,H)$ of the linear map $f_{G,H}$: $A(G,H) \to R(G,H)$.

Recall that the Brauer relations of the finite group $\Gamma = G \times H$ are the elements of the kernel $K(\Gamma)$ of the linear map $f_\Gamma : A(\Gamma) \to R_{\mathbb{Q}}(\Gamma)$ assigning to a Γ-set X the rational permutation representation $\mathbb{Q}[X]$. The map $f_{G,H}$ is the restriction of f_Γ to the submodule $A(G,H) \subseteq A(\Gamma)$ and its kernel is a submodule $K(G,H) \subseteq K(\Gamma)$.

2.4. **Functorial operations on representations**

On the representation ring $R_{\mathbb{Q}}(\Gamma)$, we define functorial operations, which are $\mathbb{Z}$-linear maps. The *induction* $\mathrm{Ind}\,_L^\Gamma : R_{\mathbb{Q}}(L) \to R_{\mathbb{Q}}(\Gamma)$ from a subgroup $L \leq \Gamma$ is defined on $\mathbb{Q}L$-modules W by

$$\mathrm{Ind}\,_L^\Gamma W = \mathbb{Q}\Gamma \otimes_{\mathbb{Q}L} W \text{ where } \mathbb{Q}\Gamma \text{ acts by left multiplication on } \mathbb{Q}\Gamma.$$

The *restriction* $\mathrm{Res}\,_L^\Gamma : R_{\mathbb{Q}}(\Gamma) \to R_{\mathbb{Q}}(L)$ is defined on $\mathbb{Q}\Gamma$-modules V by

$$\mathrm{Res}\,_L^\Gamma V = V \text{ where } \mathbb{Q}L \text{ acts on } V \text{ as a subring of } \mathbb{Q}\Gamma.$$

The *inflation* $\mathrm{Inf}\,_\Pi^\Gamma : R_{\mathbb{Q}}(\Pi) \to R_{\mathbb{Q}}(\Gamma)$ from a quotient $\Pi = \Gamma/N$ by a normal subgroup $N \trianglelefteq \Gamma$ is defined on $\mathbb{Q}\Pi$-modules U by

$$\mathrm{Inf}\,_\Pi^\Gamma U = U \text{ where } \mathbb{Q}\Gamma \text{ acts on } U \text{ via its projection in } \mathbb{Q}\Pi.$$

The *deflation* $\mathrm{Def}\,_\Pi^\Gamma : R_{\mathbb{Q}}(\Gamma) \to R_{\mathbb{Q}}(\Pi)$ is given on $\mathbb{Q}\Gamma$-modules V by

$$\mathrm{Def}\,_\Pi^\Gamma V = \mathbb{Q}\Pi \otimes_{\mathbb{Q}\Gamma} V \text{ where } \mathbb{Q}\Gamma \text{ acts on } \mathbb{Q}\Pi \text{ via its projection.}$$

Notation 2.7. For any group C we denote by 1_C the trivial rational representation $[\mathbb{Q}]$ where C acts on $\mathbb{Q}$ by the identity $1 \in C$. Also $\mathrm{Ind}\,^\Gamma = \mathrm{Ind}\,_C^\Gamma$ and $R(\Gamma) = R_{\mathbb{Q}}(\Gamma)$. Similarly, $\mathrm{Inf}\,^\Gamma = \mathrm{Inf}\,_\Pi^\Gamma$, $\mathrm{Res}\,^\Gamma = \mathrm{Res}\,_C^\Gamma$, $\mathrm{Def}\,^\Gamma = \mathrm{Def}\,_\Pi^\Gamma$ where Π is a quotient of Γ.

Proposition 2.8 ([3]). *The linear map* $f_\Gamma : A(\Gamma) \to R(\Gamma)$ *commutes with the operations* $\mathrm{Ind}\,^\Gamma$, $\mathrm{Inf}\,^\Gamma$, $\mathrm{Res}\,^\Gamma$, $\mathrm{Def}\,^\Gamma$.

For example, if $L \leq \Gamma$ is a subgroup and K is a basis element in $A(\Gamma)$ given by the conjugacy class of a subgroup in Γ, we get Mackey's formula [16]

$$\mathrm{Res}\,_L^\Gamma(f_\Gamma K) = \mathrm{Res}\,_L^\Gamma \mathrm{Ind}\,^\Gamma(1_K) = \sum_{x \in L\backslash\Gamma/K} \mathrm{Ind}\,^L(1_{L\cap {}^xK}) = f_L(\mathrm{Res}\,_L^\Gamma(K)).$$

However, for $\Gamma = G \times H$ these operations restricted to the submodules $A(G,H)$ and $R(G,H)$ do not land in the same submodules except in special cases. For example, if $K \leq G$ is a subgroup, and $N \trianglelefteq G$ is a normal subgroup with $\Pi = G/N$, we have a commutative diagram of linear maps

$$\begin{array}{ccccc} A(K,H) & \xrightarrow{\operatorname{Ind}{}_K^G} & A(G,H) & \xleftarrow{\operatorname{Inf}{}_\Pi^G} & A(\Pi,H) \\ \downarrow{\scriptstyle f_{K,H}} & & \downarrow{\scriptstyle f_{G,H}} & & \downarrow{\scriptstyle f_{\Pi,H}} \\ R(K,H) & \xrightarrow{\operatorname{Ind}{}_K^G} & R(G,H) & \xleftarrow{\operatorname{Inf}{}_\Pi^G} & R(\Pi,H) \end{array} \tag{10}$$

as $\operatorname{Ind}{}_K^G = \operatorname{Ind}{}_{K\times H}^{\Gamma}$ and $\operatorname{Inf}{}_\Pi^G = \operatorname{Inf}{}_{\Pi\times H}^{\Gamma}$ do not change the H-structure.

THEOREM 2.9 (Artin's Induction [16]). *The vector space $\mathbb{Q} \otimes R(\Gamma)$ has a basis given by* $\operatorname{Ind}{}^{\Gamma}(1_C)$ *where C runs over the conjugacy classes of cyclic subgroups of Γ.*

It is a known fact [16] that $\operatorname{Ind}{}^{\Gamma}(1_C)$ do not generate $R(\Gamma)$ in general.

COROLLARY 2.10. *For $\Gamma = G \times H$, the rank of the $\mathbb{Z}$-module $R(\Gamma)$ equals the number of conjugacy classes of cyclic subgroups of Γ and the rank of $K(\Gamma)$ equals the number of conjugacy classes of non-cyclic subgroups of Γ.*

Proof. The conjugacy classes of subgroups $L_i \leq \Gamma$ form a basis for $A(\Gamma)$ and their images under $f_\Gamma : A(\Gamma) \to R(\Gamma)$ are given by $f_\Gamma(L_i) = \operatorname{Ind}{}^{\Gamma}(1_{L_i})$. By Artin's Induction, it follows that in the exact sequence below, Coker (f_Γ) is torsion:

$$0 \to K(\Gamma) \to A(\Gamma) \xrightarrow{f_\Gamma} R(\Gamma) \to \operatorname{Coker}(f_\Gamma) \to 0.$$

This concludes the proof as the alternating sum of the ranks is zero. □

3. THE FORMULATION OF THE MAIN THEOREM

We denote by C_p the multiplicative cyclic group of prime order p. Recall that an element $\Theta = \sum n_i H_i$ is a G-relation if and only if

$$\sum n_i \operatorname{Ind}^G(1_{H_i}) = 0, \qquad 1_{H_i} = \text{ trivial } H_i\text{-module } \mathbb{Q}.$$

According to Proposition 2.8, Brauer relations can be induced by $\operatorname{Ind}{}^{G'}$ from subgroups $H \leq G'$ (or restricted by $\operatorname{Res}{}_H$) and can be lifted by $\operatorname{Inf}{}^{\tilde{G}}$ from quotients $G = \tilde{G}/N$ (or projected by $\operatorname{Def}{}_{\tilde{G}/N}$). A Brauer relation of G is called *primitive* if it cannot be induced from a proper subgroup or lifted from a proper sub-quotient of G.

Theorem 3.1 (Bouc-Tornehave [5, 17]). *All Brauer relations of a p-group G are linear combinations of the form*

$$\Theta = \sum n_P \mathrm{Ind}_K^G \,\mathrm{Inf}_P^K(\Theta_P), \quad P = K/N \text{ sub-quotient of } G,\ n_P \in \mathbb{Z}$$

of relations $\mathrm{Ind}_K^G \,\mathrm{Inf}_P^K(\Theta_P)$ 'indufted' *from primitive P-relations Θ_P where*

1. $P \approx C_p \times C_p$ *and* $\Theta_P = 1 - \sum_{C=\text{ cyclic}} C + pP$ *or*

2. $P \approx$ *the Heisenberg group of order* p^3 *or* $P \approx$ *the dihedral group of order* 2^n *with* $n \geq 4$ *and* $\Theta_P = I - IZ - J + JZ$.

In the second case, Z is the center of P and I, J are non-conjugate subgroups of P of order p (or order 2) intersecting Z trivially.

This result of Bouc [5] was proven independently and with methods that are used in this paper. The work of Tornehave [17] was done with a different approach.

Corollary 3.2 ([5]). *Cyclic groups have no Brauer relations except zero. Products $P = C_p \times C_p$ of cyclic groups of order p have one independent Brauer relation given below where C runs over all non-trivial cyclic subgroups of P*

$$\Theta_P = 1 - \sum_{C=\text{ cyclic}} C + pP.$$

This corollary can be easily obtained and does not need all the strength of Bouc's result. For example, since the rank of the $\mathbb{Z}$-module $K(G)$ of Brauer relations equals the number of non-cyclic subgroups of G by Corollary 2.10, the first part of the corollary is immediate.

Let G be a finite p-group and $K(G, C_p)$ be the module of relative (G, C_p)-Brauer relations as in Definition 2.6. In this case, we have the following commutative diagram of short exact sequences of $\mathbb{Z}$-modules and linear maps

$$(11)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & K(G\times C_p) & \xrightarrow{incl.} & A(G\times C_p) & \xrightarrow{f_{G\times C_p}} & R(G\times C_p) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle incl.} & & \uparrow{\scriptstyle incl.} & & \uparrow{\scriptstyle incl.} & & \\ 0 & \longrightarrow & K(G, C_p) & \xrightarrow{incl.} & A(G, C_p) & \xrightarrow{f_{G,C_p}} & R(G, C_p) & \longrightarrow & 0 \end{array}$$

The surjectivity of the maps $f_{G\times C_p}$ and f_{G,C_p} has been established in [15] and [1]. By estimating the ranks of the $\mathbb{Z}$-modules involved in the diagram and searching for Brauer P-relations Θ'_P for sub-quotients $P = K/N$ of the group $G \times C_p$ such that $\mathrm{Ind}\,_K^{G\times C_p}\mathrm{Inf}\,_P^K(\Theta'_P)$ are relative Brauer (G, C_p)-relations, we formulate the following

Conjecture 3.3. *Let p be a prime and G a finite p-group. All relative Brauer (G, C_p)-relations for a p-group G are linear combinations*

$$\Theta = \sum n_P \mathrm{Ind}_K^{G\times C_p} \mathrm{Inf}_P^K(\Theta'_P), \quad P = K/N \text{ sub-quotient of } G \times C_p,\ n_P \in \mathbb{Z}$$

of (G, C_p)-relations $\mathrm{Ind}_K^{G\times C_p} \mathrm{Inf}_P^K(\Theta'_P)$ 'indufted' *from P-relations Θ'_P where*

1. *$P \approx C_p \times C_p \times C_p$ or*
2. *$P \approx$ (the Heisenberg group of order p^3) $\times C_p$ or*
3. *$P \approx$ (the dihedral group of order 2^n with $n \geq 4$) $\times C_2$.*

In [7], this conjecture was proved for G an elementary Abelian p-group by giving an explicit description of $K(G, C_p)$. The simplest example shows an intricate network of subgroups behind the relative Brauer relations.

Proposition 3.4 (Kahn [7]). *$K(C_2 \times C_2, C_2)$ has a basis with four elements $e_1 - e_3 - e_5 - e_7 + 2e_{12}$, $e_3 - e_{12} - e_{13} - e_4 + e_{14} + e_{15}$, $e_5 - e_{12} - e_{14} - e_6 + e_{13} + e_{15}$, $e_7 - e_{12} - e_{15} - e_8 + e_{13} + e_{14}$ where e's label distinct subgroups of $C_2 \times C_2 \times C_2$.*

In this paper, we prove Conjecture 3.3 for G any finite Abelian p-group by giving the following description of $K(G, C_p)$:

Theorem 3.5. *Let G be a finite Abelian p-group. The $\mathbb{Z}$-module $K(G, C_p)$ of relative Brauer (G, C_p)-relations is generated by elements of the form*

$$\mathrm{Ind}_K^{G\times C_p} \mathrm{Inf}_P^K(\Theta'_P)$$

where $P = K/N \approx C_p \times C_p \times C_p$ are sub-quotients of $G \times C_p$ and Θ'_P are elements of $K(P)$, the module of Brauer P-relations.

4. RANK LEMMAS AND THEIR PROOFS

In this section, we fix G to be a finite Abelian p-group and endow the cyclotomic field $\mathbb{Q}(\zeta)$ with ζ a fixed primitive p-root of unity adjoined by the (G, C_p)-bimodule structure where C_p-action is given by multiplication by ζ and the G-action is *trivial.* Notice that a (G, C_p)-bimodule W over the rationals is a right $\mathbb{Q}[C_p]$-*free* module if and only if the module

$$ {}_0W = \mathrm{Res}_{1\times C_p} W \tag{12}$$

with the G-action forgotten is a left $\mathbb{Q}[C_p]$-free module.

Lemma 4.1. *$1_{G\times C_p} \oplus \mathbb{Q}(\zeta)$ represents an element in $R(G, C_p)$.*

Proof. We have the following $\mathbb{Q}[C_p]$-isomorphism

$$_0\left(1_{G\times C_p}\oplus\mathbb{Q}(\zeta)\right)=1_{C_p}\oplus\mathbb{Q}(\zeta)\approx\mathbb{Q}\times\mathbb{Q}(\zeta)=\mathbb{Q}[C_p].$$

This concludes the proof. □

LEMMA 4.2. $R(G\times C_p)=R(G,C_p)\oplus\mathbb{Z}\cdot 1_{G\times C_p}$.

Proof. Since G is a finite Abelian p-group, the group ring $\mathbb{Q}[G\times C_p]$ is a product of cyclotomic field extensions of $\mathbb{Q}$ obtained by adjoining primitive p^ν-roots of unity ξ_ν where $\nu\geq 0$ are integers. In particular, any irreducible $\mathbb{Q}[G\times C_p]$-module W is a cyclotomic field, say $W=\mathbb{Q}(\xi_\nu)$, whose degree over $\mathbb{Q}$ is $d_\nu=p^{\nu-1}(p-1)$ and whose degree over $\mathbb{Q}(\zeta)$ is $p^{\nu-1}$.

More precisely, there is a group homomorphism $\chi: G\times C_p\to\mathbb{Q}(\xi_\nu)^\times$ into the multiplicative group of the field $W=\mathbb{Q}(\xi_\nu)$ such that the elements $y\in G\times C_p$ act on W by multiplication by $\chi(y)$. If we fix a generator y_0 of C_p, then C_p acts on W by multiplication by $\chi(y_0)$. Since $y_0^p=1$, we know that $\chi(y_0)$ is a p-root of unity. In particular, we distinguish two cases.

If $\chi(y_0)=1$, we have the $\mathbb{Q}[C_p]$-isomorphisms

$$_0\left(W\oplus d_\nu\mathbb{Q}(\zeta)\right)=({}_0W)\oplus d_\nu\mathbb{Q}(\zeta)\approx d_\nu 1_{C_p}\oplus d_\nu\mathbb{Q}(\zeta)\approx d_\nu\mathbb{Q}[C_p].$$

If $\chi(y_0)=\zeta$ is a primitive p-root of unity, we have the $\mathbb{Q}[C_p]$-isomorphisms

$$_0\left(W\oplus p^{\nu-1}1_{G\times C_p}\right)\approx p^{\nu-1}\mathbb{Q}(\zeta)\oplus p^{\nu-1}1_{C_p}\approx p^{\nu-1}\mathbb{Q}[C_p].$$

By Lemma 4.1, we deduce that for any irreducible $\mathbb{Q}[G\times C_p]$-module W, either $W-d_\nu 1_{G\times C_p}$ or $W+p^{\nu-1}1_{G\times C_p}$ represents an element in $R(G,C_p)$. □

Let $\mathcal{S}_G$ be the graph with a vertex K for each subgroup $K\leq G$ and an edge (K,N) for each pair of subgroups $N\leq K$ having index $[K:N]=p$.

LEMMA 4.3. rank $A(G\times C_p)-$ rank $A(G,C_p)=$ rank $A(G)$.

Proof. By Proposition 2.1, we have

$$\text{rank } A(G)=\#\text{ vertices in }\mathcal{S}_G,$$
$$\text{rank } A(G\times C_p)=\#\text{ vertices in }\mathcal{S}_{G\times C_p}.$$

By Proposition 2.4, a basis for $A(G,C_p)$ is given by graphs of homomorphisms $\rho: K\to C_p$ from subgroups $K\leq G$. Each such ρ is either trivial $\rho=1$ or factors through a canonical map $K\to K/N$ and an automorphism of C_p where $N\leq K$ is a subgroup of index $[K:N]=p$. The number of automorphisms of C_p is $p-1$. Hence, the number of graphs $K\times\rho$ equals the number of subgroups $K\leq G$ ($\rho=1$) plus $(p-1)$ times the number of pairs (K,N) with $N\leq K$ having index p ($\rho\neq 1$):

$$\text{rank } A(G,C_p)=\#\text{ vertices in }\mathcal{S}_G+(p-1)\cdot\#\text{ edges in }\mathcal{S}_G.$$

By Goursat's Lemma 2.3, the vertices of $\mathcal{S}_{G\times C_p}$ are in bijection with quintuples (K, N, A, B, θ) of subgroups $N \le K \le G$ and $B \le A \le C_p$ and isomorphisms $\theta : K/N \approx A/B$. We distinguish two cases:

1) $A = B$, $K = N$, $\theta = 1$ and

2) $A = C_p$, $B = 1$, (K, N) is an edge in $\mathcal{S}_G$ and $\theta : K/N \to C_p$ is an isomorphism.

Since C_p has only two subgroups, we conclude that

$$\#\text{ vertices in } \mathcal{S}_{G\times C_p} = 2 \cdot \#\text{ vertices in } \mathcal{S}_G + (p-1)\cdot \#\text{ edges in } \mathcal{S}_G.$$

The statement now follows by combining the formulas above. □

PROPOSITION 4.4. rank $K(G \times C_p)$ − rank $K(G, C_p)$ = rank $A(G) - 1$.

Proof. From the diagram of exact sequences (11), we deduce the relations

$$\text{rank } A(G\times C_p) = \text{rank } K(G\times C_p) + \text{rank } R(G\times C_p)$$
$$\text{rank } A(G, C_p) = \text{rank } K(G, C_p) + \text{rank } R(G, C_p)$$

By Lemma 4.2, rank $R(G \times C_p)$ − rank $R(G, C_p) = 1$. Hence, the difference between the two equations above gives the result by Lemma 4.3. □

5. GENERATORS UP TO TORSION

In this section, we start with elements in $K(P)$ for specific groups P and apply all the biset operations to generate $K(G, C_p)$ for G a finite Abelian p-group. For the rest of the paper, we use the notation $\epsilon : G \to 1$ for the trivial map and the notation (5) for the graph of a homomorphism. By Goursat Lemma 2.3 the subgroups of $G \times C_p$ have the following structure

$$1\times\epsilon,\ L\times\epsilon,\ 1\times C_p,\ L\times C_p,\ L\times\lambda \tag{13}$$

where $\lambda : L \to C_p$ is a surjective homomorphism and $1 \ne L \le G$.

LEMMA 5.1. *We have the following list of possible pairs of subgroups (K, N) of $G \times C_p$ with $K/N \approx C_p \times C_p$:*

$$\begin{array}{lll} K = G'\times C_p, & N = L\times\rho & \text{with } \rho : L\to C_p,\ \rho(G'^p)=1,\ G'/L\approx C_p \\ K = G'\times C_p, & N = L\times C_p & \text{with } G'/L\approx C_p\times C_p \\ K = G'\times\lambda, & N = L\times\lambda & \text{with } \lambda : G'\to C_p \text{ surjective},\ G'/L\approx C_p\times C_p \\ K = G'\times\epsilon, & N = L\times\epsilon & \text{with } G'/L\approx C_p\times C_p \end{array}$$

where (G', L) are pairs of subgroups of G with $L < G'$.

Proof. The structure of the subgroup K is given by (13). If K is not a graph, the structure of a subgroup of K is again given by (13). If K is a graph, any subgroup of K is a subgraph. The constraint $K/N \approx C_p \times C_p$ translates to $G'/L \approx C_p \times C_p$ except for the case of a homomorphism $\rho : L \to C_p$ with $G'/L \approx C_p$. In this case, we always have $G'^p \leq L$ as $G'/L \approx C_p$. If $\rho(G'^p) = 1$, then an isomorphism $(G' \times C_p)/(L \times \rho) \approx C_p \times C_p$ is given by

$$(yx_0^i, c) \mapsto (c_0^i, c\rho(y)^{-1}), \ \ y \in L, \ c \in C_p, \ i \in \mathbb{Z}$$

where $x_0 \in G'$ is a generator of $G'/L \approx C_p$ and $c_0 \in C_p$ is a generator of C_p. Indeed, each element of G' is of the form yx_0^i and the map is well defined since any other representation $y'x_0^j = yx_0^i$ gives $y'y^{-1} = x_0^{i-j}$ with $i - j = kp$ for some $k \in \mathbb{Z}$. Hence,

$$\rho(y'y^{-1}) = \rho(x_0^{kp}) = \rho(x_0^p)^k = 1.$$

If $\rho(G'^p) \neq 1$ then $(G' \times C_p)/(L \times \rho) \approx C_{p^2}$ is generated by $(x_0, 1)$. Indeed, $(x_0^p, 1) \notin L \times \rho$ as $\rho(x_0^p) \neq 1$, but $(x_0^{p^2}, 1) \in L \times \rho$ as $\rho(x_0^{p^2}) = \rho(x_0^p)^p = 1$ (recall that $x_0^p \in L$). □

Now we make a sublist $\mathcal{L}_{G\times C_p}$ of pairs (K, N') selected from Lemma 5.1 such that each K appears *exactly once* in $\mathcal{L}_{G\times C_p}$. For each non-trivial subgroup $G' \leq G$, we *choose* a subgroup $L' < G'$ such that $G'/L' \approx C_p \times C_p$ and if this is impossible, we *choose* a subgroup $L' < G'$ such that $G'/L' \approx C_p$.

Definition 5.2. With the choices above, the list $\mathcal{L}_{G\times C_p}$ of pairs (K, N') of subgroups of $G \times C_p$ with $K/N' \approx C_p \times C_p$ is defined by

if $G'/L' \approx C_p \times C_p$	$K = G' \times C_p, \ N' = L' \times C_p$
	$K = G' \times \lambda, \ N' = L' \times \lambda$ with $\lambda : G' \to C_p$ surjective
	$K = G' \times \epsilon, \ N' = L' \times \epsilon$
if $G'/L' \approx C_p$	$K = G' \times C_p, \ N' = L' \times \epsilon$

Lemma 5.3. *The number of pairs (K, N') of subgroups of $G \times C_p$ in the list $\mathcal{L}_{G\times C_p}$ with K not a graph is one less than the number of subgroups of G.*

Proof. Each non-trivial subgroup $G' \leq G$ falls into one of the two categories of the Definition 5.2. Namely, G' is non-cyclic if and only if admits a quotient $G'/L' \approx C_p \times C_p$. If G' is cyclic, then it is non-trivial if and only if admits a quotient $G'/L' \approx C_p$. Since each product $K = G' \times C_p$ appears exactly once in the list $\mathcal{L}_{G\times C_p}$, this concludes the proof. □

Lemma 5.4. rank $K(G\times C_p) =$ *number of pairs (K, N') in the list $\mathcal{L}_{G\times C_p}$.*

Proof. By Corollary 2.10, the rank of $K(G \times C_p)$ equals the number of non-cyclic subgroups of $G \times C_p$. In the list $\mathcal{L}_{G\times C_p}$ the non-cyclic subgroups $K \leq G \times C_p$ appear exactly once. Indeed, for a graph subgroup $G' \times \rho \leq G \times C_p$ to admit a quotient $G'/L' \approx C_p \times C_p$ is equivalent with being non-cyclic. And a subgroup $G' \times C_p$ that admits a quotient $G'/L' \approx C_p$ is non-cyclic as a direct product. The two cases cover all the possibilities of non-cyclic subgroups without overlap. □

By Proposition 3.2, each pair (K, N) from the Lemma 5.1 produces an element $\mathrm{Induf}(\Theta_{K/N})$ of $K(G \times C_p)$ which is defined as follows

$$\mathrm{Induf} : K(K/N) \to K(G \times C_p), \quad S/N \mapsto S \text{ for } N \leq S \leq K$$
$$\Theta_{K/N} = (N/N) - \sum_{C'} (C'/L) + p(K/N)$$

where $N \leq C' \leq K$ such that $C'/N \approx C_p$. Indeed, by (7) and (8) we have the following calculation

$$\mathrm{Ind}_K^{G\times C_p} \mathrm{Inf}_{K/N}^K(\Theta_{K/N}) = \mathrm{Induf}(\Theta_{K/N}) = N - \sum_{C'} C' + pK. \tag{14}$$

THEOREM 5.5. *Let G be a finite Abelian p-group. Then $K(G, C_p)[\frac{1}{p}]$ is a free $\mathbb{Z}[\frac{1}{p}]$-module whose basis is given by the elements* $\mathrm{Induf}(\Theta_{K/N'})$ *indexed by the pairs $(K, N') = (G' \times \rho, L' \times \rho)$ in the list $\mathcal{L}_{G\times C_p}$ where $\rho : G' \to C_p$ is a homomorphism and $G'/L' \approx C_p \times C_p$ is a sub-quotient of G.*

Proof. By Proposition 4.4, we have

$$\text{rank } K(G, C_p) = \text{ rank } K(G \times C_p) - \text{ rank } A(G) + 1$$

where rank $A(G)$ = the number of subgroups of G. By Lemmas 5.3 and 5.4, we deduce that rank $K(G, C_p)$ = the number of pairs (K, N') with K a graph, which are listed in $\mathcal{L}_{G\times C_p}$. Observe that K is a graph if there is a homomorphism $\rho : G' \to C_p$ such that $G' \leq G$ and $K = G' \times \rho$. In this situation, any subgroup of K must be a subgraph of the form $L \times \rho \leq K$ where $L \leq G'$ and ρ is restricted to L. Hence,

$$\mathrm{Induf}(\Theta_{K/N'}) = L' \times \rho - \sum_{C'} C' \times \rho + p(G' \times \rho) \tag{15}$$

where $L' < C' < G'$ such that $C'/L' \approx C_p$ according to (14). We deduce that each element (15) belongs to $A(G, C_p)$. For $(K, N') = (G' \times \rho, L' \times \rho)$ in the list $\mathcal{L}_{G\times C_p}$ the number of these elements equals the rank of $K(G, C_p)$. Since their dominant terms pK under inclusion form a sub-basis of $A(G \times C_p)[\frac{1}{p}]$, the statement follows. □

COROLLARY 5.6 ([7]). *For G a cyclic p-group, $K(G, C_p) = 0$.*

Proof. Since G is cyclic, G has no sub-quotients of the form $G'/L' \approx C_p \times C_p$. By Theorem 5.5, rank $K(G, C_p) = 0$. Recall that $A(G \times C_p)$ is a free $\mathbb{Z}$-module and thus, $K(G, C_p) \subset A(G \times C_p)$ is a free $\mathbb{Z}$-submodule. Hence, $K(G, C_p) = 0$. $\square$

6. THE REDUCTION TO TYPE 2 GENERATORS

We denote by $K'(G, C_p) \subset K(B, C_p)$ the submodule generated by the elements of $K(G, C_p)$ that are 'indufted' from sub-quotients of $G \times C_p$ isomorphic to $C_p \times C_p \times C_p$. Theorem 3.5 states that $K'(G, C_p) = K(G, C_p)$. Since $K(G, C_p) \subset K(G \times C_p)$, by Theorem 3.1, we know that each element x of $K(G, C_p)$ is a $\mathbb{Z}$-linear combination of elements of the form $\mathrm{Induf}(\Theta_{K/N})$ where $\Theta_{K/N}$ are defined as in (14) for each pair (K, N) given by Lemma 5.1. A careful analysis of the elements $\mathrm{Induf}(\Theta_{K/N})$ reveals the following classification:

Type 1. For each pair of subgroups $L < G' < G$ with $G'/L \approx C_p \times C_p$ and each homomorphism $\alpha : G' \to C_p$ we define

$$A_{G',L,\alpha} = L \times \alpha - \sum_{L<C'<G'} C' \times \alpha + pG' \times \alpha.$$

Here C' runs over the subgroups $L < C' < G'$ with $C'/L \approx C_p$.

Type 2. For each pair of subgroups $C < G' < G$ with $G'/C \approx C_p$ and each homomorphism $\beta : C \to C_p$ with $\beta(G'^p) = 1$ we define

$$B_{G',C,\beta} = C \times \beta - \sum_{\tilde{\beta}|C=\beta} G' \times \tilde{\beta} - C \times C_p + pG' \times C_p.$$

Here $\tilde{\beta}$ runs over the homomorphisms $\tilde{\beta} : G' \to C_p$ with $\tilde{\beta}|C = \beta$.

Type 3. For each pair of subgroups $L < G' < G$ with $G'/L \approx C_p \times C_p$ we define

$$D_{G',L} = L \times C_p - \sum_{L<C'<G'} C' \times C_p + pG' \times C_p.$$

Here C' runs over the subgroups $L < C' < G'$ with $C'/L \approx C_p$.

LEMMA 6.1. *The* Type 1 *elements $A_{G',L,\alpha}$ belong to $K'(G, C_p)$.*

Proof. For each sub-quotient $G'/L \approx C_p \times C_p$ of G and homomorphism $\alpha : G' \to C_p$, we have the following isomorphism

$$P = (G' \times C_p)/(L \times \alpha) \approx C_p \times C_p \times C_p,$$

which comes from $\varphi : G' \to G'/L \approx C_p \times C_p$ by sending $(x, c) \in G' \times C_p$ to the element $(\varphi(x), c\rho(x)^{-1}) \in C_p \times C_p \times C_p$. In this context, the element $A_{G',L,\alpha}$ is of the form $A_{G',L,\alpha} = \mathrm{Induf}(\Theta'_P) \in K(G, C_p)$ for some $\Theta'_P \in K(P)$. □

Corollary 6.2. *For each $x \in K(G, C_p)$, either x or px belongs to $K'(G, C_p)$.*

Proof. By Theorem 5.5 and its proof, we know that for each $x \in K(G, C_p)$ either x or px is a $\mathbb{Z}$-linear combination of Type 1 elements and we apply Lemma 6.1. □

Lemma 6.3. *For each pair $L < G' < G$ with $G'/L \approx C_p \times C_p$, we have*

$$(p+1)D_{G',L} \equiv \sum_{L<C'<G'} B_{G',C',\epsilon} - \sum_{L<C'<G'} B_{C',L,\epsilon} \mod K'(G, C_p) \tag{16}$$

Here C' runs over the subgroups $L < C' < G'$ with $C'/L \approx C_p$.

Proof. Recall that $A(G, C_p)$ is generated by graph-subgroups $K \times \rho < G \times C_p$ as in the Proposition 2.4. In this context, for each of the $p+1$ subgroups $L < C' < G'$ (see the next section) with $C'/L \approx C_p$, we have

$$\begin{aligned} B_{G',C',\epsilon} &\equiv -C' \times C_p + pG' \times C_p \mod A(G, C_p) \\ -B_{C',L,\epsilon} &\equiv L \times C_p - pC' \times C_p \mod A(G, C_p) \\ D_{G',L} &\equiv L \times C_p - \sum_{L<C'<G'} C' \times C_p + pG' \times C_p. \end{aligned}$$

Hence, if we apply the operator $\sum_{L<C'<G'}$ to the first two equations, we get

$$\begin{aligned} \sum_{L<C'<G'} B_{G',C',\epsilon} &\equiv - \sum_{L<C'<G'} C' \times C_p + (p+1)pG' \times C_p \mod A(G, C_p) \\ - \sum_{L<C'<G'} B_{C',L,\epsilon} &\equiv (p+1)L \times C_p - p \sum_{L<C'<G'} C' \times C_p \mod A(G, C_p). \end{aligned}$$

By definitions, $K(G.C_p) = A(G, C_p) \cap K(G \times C_p)$ where $K(G \times C_p)$ contains the Type 2 and Type 3 elements. Hence, by adding the last two equations, we get the relation (16) mod $K(G, C_p)$ where all the terms are 'indufted' from the sub-quotient $(G' \times C_p)/(L \times \epsilon) \approx C_p \times C_p \times C_p$. This proves (16) mod $K'(G, C_p)$. □

Now we can reduce the proof of Theorem 3.5 to $\mathbb{Z}$-linear combinations of Type 2 generators. The precise statement is

Proposition 6.4. *Each element $x \in K(G, C_p)$ is a $\mathbb{Z}$-linear combination mod $K'(G, C_p)$ of* Type 2*-elements of the form $B_{G',C,\epsilon}$ with $G'/C \approx C_p$.*

Proof. Each element $x \in K(G, C_p)$ is a $\mathbb{Z}$-linear combination of the form

$$x = \text{ Type 1 combination } + \text{ Type 2 combination } + \text{ Type 3 combination.}$$

By Lemma 6.3, we have the following reduction

$$(p+1)(\text{Type 3 combination}) \equiv \text{ Type 2 combination} \mod K'(G, C_p).$$

By Corollary 6.2, we have $px \equiv 0 \mod K'(G, C_p)$. By putting together the equations above and Lemma 6.1, we get

$$x = (p+1)x - px \equiv \text{ Type 2 combination} \mod K'(G, C_p).$$

If $G' < G$ is cyclic, then $C = G'^p < G'$ is the unique subgroup of index p. In this case, $B_{G',C,\beta} = B_{G',C,\epsilon}$. If $G' < G$ is non-cyclic and $\beta : C \to C_p$ with $C < G'$ is such that $G'/C \approx C_p$ and $\beta \neq \epsilon$, then the difference

$$B_{G',C,\beta} - B_{G',C,\epsilon} = C \times \beta - \sum_{\tilde{\beta}|C=\beta} G' \times \tilde{\beta} - C \times \epsilon + \sum_{\tilde{\epsilon}|C=\epsilon} G' \times \tilde{\epsilon}$$

is 'indufted' from $(G' \times C_p)/(L \times \beta) \approx C_p \times C_p \times C_p$ if we take $L = \ker \beta < C$. This shows that the difference belongs to $K'(G, C_p)$ concluding the proof. □

Now we are ready to prove Theorem 3.5. To that end, let $x \in K(G, C_p)$ be given. By Proposition 6.4, we can represent x by a Type 2 combination mod $K'(G, C_p)$. In what follows, we will show how to eliminate all the Type 2 elements from that combination, concluding that $x \in K'(G, C_p)$. This proves Theorem 3.5.

7. THE ELIMINATION ALGORITHM

The Type 2 elements $B_{G',L,\epsilon}$ generate a $\mathbb{Z}$-submodule $\mathcal{M} \subset K(G \times C_p)$ and each such generator is uniquely determined by a pair of subgroups $L < G'$ with $G'/L \approx C_p$. Hence, we can drop the ϵ from the notation $B_{G'L} = B_{G',L,\epsilon}$. Moreover, its image mod $A(G, C_p)$ is given by the formula

$$B_{G'L} \equiv -L \times C_p + pG' \times C_p \mod A(G, C_p). \tag{17}$$

Definition 7.1. The *signature* homomorphism $\sigma : A(G \times C_p) \to A(G)$ is sending $L \times C_p \mapsto L$ for $L < G$ and any other basis elements to zero.

For example, the Type 2 generator $B_{G'L}$ has the signature $-L + pG$.

LEMMA 7.2. *The signature homomorphism $\sigma : A(G \times C_p) \to A(G)$ is surjective and its kernel is $A(G, C_p)$.*

Proof. Observe that σ has a well defined section $\ell : A(G) \to A(G, C_p)$ sending $L \mapsto L \times C_p$ for $L < G$. Since the identity map $\sigma \circ \ell : A(G) \to A(G)$ is surjective, so is σ. Moreover, by Proposition 2.4 the $A(G \times C_p)$ is the direct sum of its submodules $A(G, C_p)$ and $\ell A(G)$. Hence, $\ker \sigma = A(G, C_p)$. □

Corollary 7.3. *The kernel of $\sigma : \mathcal{M} \to A(G)$ is $\mathcal{M} \cap K(G, C_p)$.*

Proof. By the previous lemma, $\mathcal{M} \cap A(G, C_p)$ is the kernel of the restriction $\sigma|\mathcal{M}$. Since $\mathcal{M} \subset K(G \times C_p)$ and $K(G, C_p) = K(G \times C_p) \cap A(G, C_p)$, we get the statement $\ker \sigma|\mathcal{M} = \mathcal{M} \cap K(G, C_p)$. □

Definition 7.4. We call a *resolution* starting at a subgroup L and ending at a subgroup G' any chain of intermediate subgroups

$$L = G_q < G_{q-1} < ... < G_1 < G_0 = G'$$

such that each subgroup G_i has index p in the next subgroup G_{i+1} of the chain.

Here are a couple of basic facts [12]. Between any two comparable subgroups $L < G'$ of a finite p-group there is at least one resolution starting at L and ending at G'. If L has index p^2 in G', then $G'/L \approx C_{p^2}$ if from L to G' there is only one resolution and $G'/L \approx C_p \times C_p$ if there are at least two resolutions. In the latter case, there will be exactly $p+1$ such resolutions.

Lemma 7.5. *Given any resolution $L = G_e < G_{e-1} < ... < G_1 < G_0 = G'$ starting at a subgroup L and ending at a subgroup G' of the group G, we have*

$$\sigma\left(B_{G_{e-1}G_e} + pB_{G_{e-2}G_{e-1}} + ... + p^{e-1}B_{G_0G_1}\right) = -G_e + p^e G_0.$$

Proof. Notice that $\sigma\left(B_{G_iG_{i+1}}\right) = -G_{i+1} + pG_i$ and apply a telescopic sum. □

Lemma 7.6. *Given any two resolutions starting at a subgroup L and ending at a subgroup G' of G, say*

$$L = G_e < G_{e-1} < ... < G_1 < G_0 = G'$$
$$L = H_e < H_{e-1} < ... < H_1 < H_0 = G'$$

we have the following relation

$$\sum_{j=1}^{e} p^{e-j} B_{G_{j-1}G_j} \equiv \sum_{j=1}^{e} p^{e-j} B_{H_{j-1}H_j} \mod K'(G, C_p).$$

Proof. By [12] there is a sequence of resolutions

$$L = G_e^{(i)} < G_{e-1}^{(i)} < ... < G_1^{(i)} < G_0^{(i)} = G$$

for $i = 0, 1, 2, ..., n$ such that

(1) for each k we have $G_k^{(0)} = G_k$, $G_k^{(n)} = H_k$, and

(2) for each i there is λ with $G_\lambda^{(i+1)} \neq G_\lambda^{(i)}$ and $G_k^{(i+1)} = G_k^{(i)}$ if $k \neq \lambda$.

In this context, notice that the following combinations belong to $K'(G,C_p)$

$$\sum_{j=1}^{e} p^{e-j} B_{G_{j-1}^{(i+1)} G_j^{(i+1)}} - \sum_{j=1}^{e} p^{e-j} B_{G_{j-1}^{(i)} G_j^{(i)}}$$

$$= p^{e-\lambda-1} \left(pB_{G_{\lambda-1}^{(i+1)} G_\lambda^{(i+1)}} - pB_{G_{\lambda-1}^{(i)} G_\lambda^{(i)}} + B_{G_\lambda^{(i+1)} G_{\lambda+1}^{(i+1)}} - B_{G_\lambda^{(i)} G_{\lambda+1}^{(i)}} \right) \tag{18}$$

since the terms on the right hand side of the equation are associated with two resolutions starting at $G_{\lambda+1}^{(i+1)} = G_{\lambda+1}^{(i)}$ and ending at $G_{\lambda-1}^{(i+1)} = G_{\lambda-1}^{(i)}$ and thus, they are 'indufted' from $\left(G_{\lambda-1}^{(i)} \times C_p\right) / \left(G_{\lambda+1}^{(i)} \times \epsilon\right) \approx C_p \times C_p \times C_p$ as noted in basic facts. By adding up all the relations (18) for $i = 0, 1, 2, ..., n$, we get the result. $\square$

Let the order of G be p^n and for each $k = 0, 1, 2, ..., n$ define $\mathcal{G}_k$ to be the set of all subgroups of index p^k in G. According to the formula (17), the Type 2 elements are in bijection with their signatures as listed for each pair (X_i, X_{i+1}) with $X_{i+1} < X_i$ and $X_k \in \mathcal{G}_k$ in the table below

Table 1

$-X_n + pX_{n-1}$	$-X_{n-1} + pX_{n-2}$	.....	$-X_2 + pX_1$	$-X_1 + pX_0$

The Type 2 elements generate a submodule $\mathcal{M} \subset K(G \times C_p)$. Using elementary operations, we build a new system of generators for $\mathcal{M}$. Namely, by basic facts, each pair $X_{i+1} < X_i$ can be extended to a resolution

$$X_{i+1} < X_i < X_{i-1} < ... < X_1 < X_0 = G \tag{19}$$

and using this resolution, we replace $B_{X_i X_{i+1}}$ by

$$B_{X_i X_{i+1}} + pB_{X_{i-1} X_i} + ... + p^i B_{X_0 X_1}. \tag{20}$$

By Lemma 7.5, the signature table of the new system of generators is

Table 2

$-X_n + p^n X_0$	$-X_{n-1} + p^{n-1} X_0$	.....	$-X_2 + p^2 X_0$	$-X_1 + pX_0$

In this new table, the signatures appear with repetitions. More precisely, any two resolutions starting at X_{i+1} and ending at X_0, say resolution (20) and resolution

$$X_{i+1} < Y_i < Y_{i-1} < ... < Y_1 < X_0 = G \tag{21}$$

produce the signature $-X_{i+1}+p^{i+1}X_0$ to generator (20) and generator

$$B_{Y_iX_{i+1}}+pB_{Y_{i-1}Y_i}+\ldots+p^iB_{X_0Y_1}. \tag{22}$$

Using the generator (20) as a pivot and subtracting that generator from the generator (22), we can remove the signature duplicate in Table 2. According to Lemma 7.6, the zero represents an element in $K'(G,C_p)$. By using this procedure, we eliminate all the repetitions in Table 2 mod $K'(G,C_p)$. Let $\mathcal{S}$ be the system thus obtained of generators for $\mathcal{M}$. By Proposition 6.4, given an element $x\in K(G,C_p)$, we can write it as a $\mathbb{Z}$-linear combination $y\in\mathcal{M}$ of elements in $\mathcal{S}$ mod $K'(G,C_p)$ as follows

$$x\equiv y \mod K'(G,C_p),\quad \sigma(y)=\sum_{i=1}^{n}\sum_{X\in\mathcal{G}_i} m_X(-X+p^iX_0)$$

where $m_X\in\mathbb{Z}$ are the coefficients mod $K'(G,C_p)$ of the combination y. By Corollary 7.3, we must have $\sigma(y)=0$. Since the collection of subgroups $\cup_{i=1}^n\mathcal{G}_i$ is a sub-basis for $A(G)$, we deduce that $m_X=0$ for each X. This proves that $y\in K'(G,C_p)$ and thus, $x\in K'(G,C_p)$ proving Theorem 3.5.

8. AN EXAMPLE

By Theorem 5.5, a basis for $K(C_p\times C_p,C_p)[\frac{1}{p}]$ is given by

$$(1\times 1)\times\epsilon-\sum_C(C\times\epsilon)+p(C_p\times C_p)\times\epsilon$$
$$(1\times 1)\times\lambda-\sum_C(C\times\lambda)+p(C_p\times C_p)\times\lambda.$$

Here C runs over the cyclic subgroups of order p of $C_p\times C_p$ and λ over the surjective homomorphisms $\lambda: C_p\times C_p\to C_p$. By direct counting,

$$\operatorname{rank} K(C_p\times C_p,C_p)=1+(p+1)(p-1)=p^2.$$

In particular, for $p=2$ we have rank $K(C_2\times C_2,C_2)=4$. Specifically, the lattice of subgroups for $e_{16}=(C_2\times C_2)\times C_2$ is given by

$$e_9=(1\times C_2)\times C_2,\ e_{10}=(C_2\times 1)\times C_2,\ e_{11}=\Delta\times C_2$$
$$e_{12}=(C_2\times C_2)\times\epsilon,\ e_{13}=(C_2\times C_2)\times p_1,\ e_{14}=(C_2\times C_2)\times p_2$$
$$e_{15}=(C_2\times C_2)\times\sigma$$
$$e_2=(1\times 1)\times C_2,\ e_3=(1\times C_2)\times\epsilon,\ e_4=(1\times C_2)\times p_2$$
$$e_5=(C_2\times 1)\times\epsilon,\ e_6=(C_2\times 1)\times p_1,\ e_7=\Delta\times\epsilon,\ e_8=\Delta\times\delta$$
$$e_1=(1\times 1)\times\epsilon$$

where $\Delta \subset C_2 \times C_2$ is the diagonal subgroup, $\rho : C_2 \times C_2 \to (C_2 \times C_2)/\Delta$ is the canonical projection, $\delta : \Delta \to C_2$ is the unique isomorphism, $p_1, p_2 : C_2 \times C_2 \to C_2$ are the projections on the first and the second component.

The generators of $K(C_2 \times C_2 \times C_2)$ are

$$\begin{aligned}
E_9 &= e_1 - e_2 - e_3 - e_4 + 2e_9 & E_{10} &= e_1 - e_2 - e_5 - e_6 + 2e_{10} \\
E_{11} &= e_1 - e_2 - e_7 - e_8 + 2e_{11} & E_{12} &= e_1 - e_3 - e_5 - e_7 + 2e_{12} \\
E_{13} &= e_1 - e_3 - e_6 - e_8 + 2e_{13} & E_{14} &= e_1 - e_4 - e_5 - e_8 + 2e_{14} \\
E_{15} &= e_1 - e_4 - e_6 - e_7 + 2e_{15} & E_2 &= e_2 - e_9 - e_{10} - e_{11} + 2e_{16} \\
E_3 &= e_3 - e_9 - e_{12} - e_{13} + 2e_{16} & E_4 &= e_4 - e_9 - e_{14} - e_{15} + 2e_{16} \\
E_5 &= e_5 - e_{10} - e_{12} - e_{14} + 2e_{16} & E_6 &= e_6 - e_{10} - e_{13} - e_{15} + 2e_{16} \\
E_7 &= e_7 - e_{11} - e_{12} - e_{15} + 2e_{16} & E_8 &= e_8 - e_{11} - e_{13} - e_{14} + 2e_{16}
\end{aligned}$$

As in Proposition 3.4, a basis for $K(C_2 \times C_2, C_2)$ can be given by

$$E_{15},\ E_4 - E_3,\ E_6 - E_5,\ E_8 - E_7.$$

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