WOLD DECOMPOSITIONS AND BROWNIAN TYPE OPERATORS

LAURIAN SUCIU

Communicated by Dan Timotin

We present a class of m-isometries on a Hilbert space which admit Wold-type decompositions in Shimorin's sense. Among these operators, we recover some sub-Brownian m-isometries and theirs m-Brownian unitary extensions. Our context refers to an integer $m \geq 3$, the cases $m = 1$ and $m = 2$ being well-known and studied.

AMS 2020 Subject Classification: 47A05, 47A15, 47A20, 47A63.

Key words: Wold decomposition, Brownian unitary operator, sub-Brownian m-isometry.

1. INTRODUCTION

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ the C^*-algebra of all bounded linear operators on $\mathcal{H}$, where $I(= I_{\mathcal{H}})$ is the identity operator. For $T \in \mathcal{B}(\mathcal{H})$, T^* stands for the adjoint operator of T, while by $\mathcal{R}(T)$, $\mathcal{N}(T)$ we denote the range, respectively the kernel of T. For a closed subspace $\mathcal{M} \subset \mathcal{H}$, $P_{\mathcal{M}} \in \mathcal{B}(\mathcal{H})$ is the orthogonal projection onto $\mathcal{M}$. Also, $\mathcal{M}$ is invariant (reducing) for T when $T\mathcal{M} \subset \mathcal{M}$ (resp. $T\mathcal{M} \subset \mathcal{M}$ and $T^*\mathcal{M} \subset \mathcal{M}$).

If $\mathcal{K}$ is a Hilbert space which contains $\mathcal{H}$ as a closed subspace (in notation $\mathcal{K} \supset \mathcal{H}$), then an operator $S \in \mathcal{B}(\mathcal{K})$ is an extension of T if $S\mathcal{H} \subset \mathcal{H}$ and $S|_{\mathcal{H}} = T$. More generally, S is a power dilation of T (or T is a compression of S on $\mathcal{H}$) if $T^n = P_{\mathcal{H}} S^n|_{\mathcal{H}}$ for every integer $n \geq 0$.

An operator T on $\mathcal{H}$ is said to be a *m-isometry* for an integer $m \geq 1$ if it verifies the identity

$$\Delta_T^{(m)} := \sum_{j=0}^{m} (-1)^{m-j} T^{*j} T^j = 0. \tag{1}$$

In the case $m = 1$, we shortly denote $\Delta_T = \Delta_T^{(1)} = T^*T - I$. So (1) in this case means that T is an isometry, and T is unitary when $\Delta_T = 0$ and $\Delta_{T^*} = 0$. More generally, T is expansive (resp. a contraction) if $\Delta_T \geq 0$ (resp. $\Delta_T \leq 0$).

REV. ROUMAINE MATH. PURES APPL. **68** (2023), *3-4*, 369–381
doi: 10.59277/RRMPA.2023.369.381

If T verifies (1) for $m = 2$ then T is expansive, but this condition is not assured when $m \geq 3$ in (1). These operators are studied in [1, 2, 3] and recently in [4, 5, 6, 7, 8, 10, 12].

In this paper, we refer to expansive m-isometries with $m \geq 3$. Since $T^*T \geq I$, such operator T is injective with $\mathcal{R}(T)$ closed, T^*T being invertible in $\mathcal{B}(\mathcal{H})$. In this case, the operator $T' = T(T^*T)^{-1}$ is called the *Cauchy dual operator* of T. It is clear that $(T^*T)^{-1} = T'^*T'$, $T'^*T = T^*T' = I$ and $\mathcal{N}(T^*) = \mathcal{N}(T'^*)$. Therefore T and T' are left invertible in $\mathcal{B}(\mathcal{H})$, and the maximum invariant subspaces for T (resp. T') on which T (resp. T') is invertible are $\mathcal{H}_\infty$ (resp. $\mathcal{H}'_\infty$) where

$$\mathcal{H}_\infty = \bigcap_{n\geq 1} T^n\mathcal{H}, \quad \mathcal{H}'_\infty = \bigcap_{n\geq 1} T'^n\mathcal{H}.$$

It is known (see [11, Proposition 2.7]) that $\mathcal{H} \ominus \mathcal{H}_\infty = \bigvee_{n\geq 0} T'^n\mathcal{N}(T^*)$ and $\mathcal{H} \ominus \mathcal{H}'_\infty = \bigvee_{n\geq 0} T^n\mathcal{N}(T^*)$. When $\mathcal{H}_\infty = \{0\}$, T is said to be *analytic.*

According to [11] an m-isometry T on $\mathcal{H}$ admits *Wold-type decomposition* if the subspace $\mathcal{H}_\infty$ is reducing for T, $T|_{\mathcal{H}_\infty}$ is unitary and $\mathcal{H}_\infty = \mathcal{H}'_\infty$, that is, it holds the decomposition

$$\mathcal{H} = \mathcal{H}_\infty \oplus \bigvee_{n\geq 0} T^n\mathcal{N}(T^*).$$

This decomposition in the case $m = 1$ is precisely the classical Wold decomposition of an isometry. On the other hand, it follows from [11, Theorem 3.6] that every 2-isometry admits Wold-type decomposition. But it is not known if an expansive m-isometry with $m \geq 3$ admits such a decomposition, in general. In this paper, we present a sufficient condition for such an operator to possess Wold-type decomposition. We apply our result to some Brownian-type m-isometries which are recently studied in [12, 7].

Thus, in the Section 2, we analyze the triangulation of an expansive m-isometry T on $\mathcal{H}$ obtained by means of the isometric invariant part $\mathcal{H}_0$ of T in $\mathcal{N}(\Delta_T)$. We prove that if the spectral radius of the compression of T' (the Cauchy dual of T) on $\mathcal{H} \ominus \mathcal{H}_0$ is strictly less than 1, then T admits Wold-type decomposition. We mention some cases when this condition occurs. Also, we study an asymptotic limit A induced by T and $P_{\mathcal{H}_0}$, for which T^* is an A-isometry, that is $TAT^* = A$. We show that $\mathcal{R}(A) = \mathcal{H}_\infty$, so $\mathcal{H} = \mathcal{R}(A) \oplus \mathcal{N}(A)$ is precisely the Wold decomposition for T when it exists.

In Section 3, we refer to m-isometries T having Δ_T a scalar multiple of an orthogonal projection. We show that such operator with $\mathcal{N}(\Delta_T)$ invariant for

T admits Wold-type decomposition. Among these operators, we mention sub-Brownian m-isometries T and their m-Brownian unitary extensions B. We analyze in detail the case when $B^*B\mathcal{H} \subset \mathcal{H}$. Also, in this case we describe $\mathcal{N}(T^*)$ in the terms of $\mathcal{N}(B^*)$ and we show that B' is an extension for T'.

Finally, we give an example of expansive 3-isometry T which admits Wold-type decomposition, such that Δ_T is not a scalar multiple of an orthogonal projection and with T' having its spectral radius 1.

2. WOLD-TYPE DECOMPOSITIONS

Recall (see [10, §2]) that for an operator $T \in \mathcal{B}(\mathcal{H})$ and a closed subspace $\mathcal{M} \subset \mathcal{H}$, the following assertions are equivalent:

(a) $T\mathcal{M} \subset \mathcal{M} \subset \mathcal{N}(\Delta_T)$,

(b) $T\mathcal{M} \subset \mathcal{M}$, $T^*T\mathcal{M} \subset \mathcal{M}$ and $T|_{\mathcal{M}}$ is isometric.

We refer to the maximum invariant subspace $\mathcal{H}_0$ for T contained in $\mathcal{N}(\Delta_T)$ as being the *isometric invariant part* of T in $\mathcal{N}(\Delta_T)$. By [10, Lemma 2.1] this subspace is precisely the isometric invariant part in $\mathcal{H}$ of the contraction $C = P_{\mathcal{N}(\Delta_T)}T|_{\mathcal{N}(\Delta_T)}$. This means that

$$\mathcal{H}_0 = \mathcal{N}(I - S_C) = \bigcap_{n\geq 1} \mathcal{N}(\Delta_{C^n}), \tag{2}$$

where $S_C := s - \lim_{n\to\infty} C^{*n}C^n$ is the (strongly) asymptotic limit of C (see [9]).

THEOREM 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be an expansive m-isometry for an integer $m \geq 3$, $\mathcal{H}_0 \subset \mathcal{N}(\Delta_T)$ be the isometric invariant part of T, such that $r(P_{\mathcal{H}_1}T'|_{\mathcal{H}_1}) < 1$ where T' is the Cauchy dual operator of T, $\mathcal{H}_1 = \mathcal{H} \ominus \mathcal{H}_0$ and r is the spectral radius. Then T admits Wold-type decomposition.*

Proof. Firstly, suppose that $\mathcal{H}_0 \neq \{0\}$ into $\mathcal{N}(\Delta_T)$. So $\mathcal{H}_0$ is invariant for T and T^*T, while $V := T|_{\mathcal{H}_0}$ is an isometry. We prove that $\mathcal{H}_0$ is also invariant for $T' = T(T^*T)^{-1}$. Indeed, having in view the last form of $\mathcal{H}_0$ in (2), we have for every $h \in \mathcal{H}_0$ and any integer $n \geq 1$,

$$C^{*n}C^nT'h = C^{*n}C^nT(T^*T)^{-1}h = C^{*n}C^nTh = Th = T(T^*T)^{-1}h = T'h,$$

taking into account that $Th \in \mathcal{H}_0$. So $T'\mathcal{H}_0 \subset \mathcal{H}_0$ and $T'|_{\mathcal{H}_0} = T|_{\mathcal{H}_0} = V$. Hence T and T' have under the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ the block matrices

$$T = \begin{pmatrix} V & T_0 \\ 0 & T_1 \end{pmatrix}, \quad T' = \begin{pmatrix} V & T_0\Delta^{-1} \\ 0 & T_1\Delta^{-1} \end{pmatrix}, \quad \Delta = T_0^*T_0 + T_1^*T_1 = T^*T|_{\mathcal{H}_1}. \tag{3}$$

Since T is an expansive m-isometry, the subspace $\mathcal{H}_\infty = \bigcap_{n\geq 1} T^n\mathcal{H}$ is reducing for T and $T|_{\mathcal{H}_\infty}$ is unitary. In addition, we have

$$\mathcal{H}_\infty \subset \mathcal{H}'_\infty = \bigcap_{n\geq 1} T'^n\mathcal{H}.$$

Now, if $\mathcal{H}'_\infty = \{0\}$ then $\mathcal{H}_\infty = \{0\}$ which means that the operators T and T' are analytic, hence T admits Wold-type decomposition.

Next, we assume that $\mathcal{H}'_\infty \neq \{0\}$. Clearly, $\mathcal{H}'_\infty$ is invariant for T' and for T^*, because $T^*T' = I$. We prove that $\mathcal{H}'_\infty$ is also invariant for T'^*. Indeed, let $h \in \mathcal{H}'_\infty$, so for every integer $n \geq 1$ there exists $h_n \in \mathcal{H}$ such that $h = T'^n h_n$. We write $h = h_0 \oplus h'$ and $h_n = h_n^0 \oplus h'_n$ with $h_0, h_n^0 \in \mathcal{H}_0$, $h', h'_n \in \mathcal{H}_1$. Using the matrix of T' in (3) we obtain

$$h = (V^n h_n^0 + X_n h'_n) \oplus (T_1\Delta^{-1})^n h'_n,$$

where $X_n = P_{\mathcal{H}_0} T'^n|_{\mathcal{H}_1}$. Thus $h' = (T_1\Delta^{-1})^n h'_n$.

Since $T^*T' = I$ one has $T^{*n}T'^n = I$ for $n \geq 1$, which later gives $h_n = T^{*n}T'^n h_n = T^{*n}h$. So $h'_n = P_{\mathcal{H}_1}T^{*n}h$ and it follows that

$$\sup_{n\geq 1} \frac{1}{n^{\frac{m-1}{2}}} \|h'_n\| \leq \sup_{n\geq 1} \frac{1}{n^{\frac{m-1}{2}}} \|T^{*n}h\| = c < \infty,$$

because T is an m-isometry. This and the above expression of h' lead to the inequality

$$\|h'\| = \|(T_1\Delta^{-1})^n h'_n\| \leq cn^{\frac{m-1}{2}} \|(T_1\Delta^{-1})^n\|.$$

Now we use the assumption that $r(T_1\Delta^{-1}) < 1$. This means that there exist two constants α, β with $0 < \alpha < 1$ and $\beta > 0$ such that $\|(T_1\Delta^{-1})^n\| \leq \beta\alpha^n$. Thus we obtain that

$$\|h'\| \leq c\beta n^{\frac{m-1}{2}} \alpha^n \to 0 \quad \text{as} \quad n \to \infty,$$

that is $h' = 0$. Hence $h = h_0 \in \mathcal{H}_0$ and we get that $\mathcal{H}'_\infty \subset \mathcal{H}_0 \subset \mathcal{N}(\Delta_{T'})$, so $T'|_{\mathcal{H}'_\infty}$ is an isometry.

Next, as $T^*\mathcal{H}'_\infty \subset \mathcal{H}'_\infty$ we have $h_n = T^{*n}h \in \mathcal{H}'_\infty$ and finally this gives

$$T'^*h = T'^*T'(T'^{(n-1)}h_n) = T'^{(n-1)}h_n \in \mathcal{H}'_\infty.$$

Hence $\mathcal{H}'_\infty$ is reducing for T' and it is also reducing for $T = T'(T'^*T')^{-1}$. As T' is invertible on $\mathcal{H}'_\infty$ we obtain that $T'|_{\mathcal{H}_\infty}$ is also invertible, which implies that $\mathcal{H}'_\infty \subset \mathcal{H}_\infty$. Since the reverse inclusion holds we conclude that $\mathcal{H}'_\infty = \mathcal{H}_\infty$, therefore $\mathcal{H} = \mathcal{H}_\infty \oplus \bigvee_{n\geq 0} T^n\mathcal{N}(T^*)$. Hence T admits Wold-type decomposition, in the case $\mathcal{H}_0 \neq \{0\}$.

In the case $\mathcal{H}_0 = \{0\}$, we have also $\mathcal{H}_\infty = \{0\}$, because $T|_{\mathcal{H}_\infty}$ is unitary, therefore $\mathcal{H}_\infty \subset \mathcal{H}_0$. Then $\mathcal{H} = \mathcal{H}_1$ and $r(T') < 1$ (by hypothesis, in this case), therefore as before it follows that $\mathcal{H}'_\infty = \mathcal{H}_\infty = \{0\}$. We conclude that T admits also Wold type decomposition, in this case. $\square$

Remark 2.2. If T is an m-isometry on $\mathcal{H}$ then the above subspace $\mathcal{H}_0$ is also the isometric invariant part in $\mathcal{N}(\Delta_{T'}) = \mathcal{N}(\Delta_T)$ of the Cauchy dual operator T' of T, because $\mathcal{H}_0$ is invariant for T' (as we have seen in the previous proof) and also for $T'^*T' = (T^*T)^{-1}$. Thus $\mathcal{H}_0$ has the same property relative to T and T', justifying its usage in the theorem.

Remark 2.3. The condition $r(P_{\mathcal{H}_1}T\Delta^{-1}) < 1$ in the theorem is particularly ensured when $\|\Delta^{-1}\| < 1$, where $\Delta = T^*T|_{\mathcal{H}_1}$. But this condition implies that $\mathcal{R}(\Delta_T)$ is closed. Indeed, if $\|\Delta^{-1}\| < 1$ then $I - \Delta^{-1} = \Delta^{-1}(\Delta - I)$ is invertible, so $\mathcal{R}(\Delta - I) = \mathcal{R}(\Delta_T|_{\mathcal{H}_1})$ is closed. Since $\mathcal{H}_1 = (\mathcal{N}(\Delta_T) \ominus \mathcal{H}_0) \oplus \overline{\mathcal{R}(\Delta_T)}$ it follows that $\Delta_T\mathcal{H} = \Delta_T\mathcal{H}_1 = \Delta_T\overline{\mathcal{R}(\Delta_T)}$, while this, together with the previous conclusion, imply

$$\overline{\mathcal{R}(\Delta_T)} = \overline{\Delta_T\mathcal{H}_1} = \Delta_T\mathcal{H}_1 = \Delta_T\mathcal{H},$$

hence $\mathcal{R}(\Delta_T)$ is closed. Conversely, if $\mathcal{H}_0 = \mathcal{N}(\Delta_T)$ and Δ_T has closed range then $\Delta_T|_{\mathcal{R}(\Delta_T)}$ is invertible, therefore one has

$$(\Delta_T h, h) = ((\Delta - I)h, h) \geq \rho\|h\|^2, \quad h \in \mathcal{R}(\Delta_T) = \mathcal{H}_1,$$

for some constant $\rho > 0$. Hence $\Delta \geq (\rho + 1)I$ i.e. $\Delta^{-1} \leq (\rho + 1)^{-1}I$ and $\|\Delta^{-1}\| < 1$. We derive from these facts the following

Corollary 2.4. *If $T \in \mathcal{B}(\mathcal{H})$ is an expansive m-isometry for an integer $m \geq 3$, such that $\mathcal{N}(\Delta_T)$ is invariant for T and $\mathcal{R}(\Delta_T)$ is closed, then T admits Wold-type decomposition.*

In the following section, we refer to a special class of operators that satisfy the conditions from this corollary. We describe now the subspaces $\mathcal{H}_\infty$ and $\mathcal{H} \ominus \mathcal{H}_\infty$ for some m-isometries, in the terms of an asymptotic limit associated to the adjoint operators and of the subspace $\mathcal{H}_0$ from Theorem 2.1.

Theorem 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be an expansive m-isometry for an integer $m \geq 3$, such that $\mathcal{H}_0 = \bigcap_{n\geq 1}\mathcal{N}(\Delta_{C^n}) \neq \{0\}$ where $C = P_{\mathcal{N}(\Delta_T)}T|_{\mathcal{N}(\Delta_T)}$. Then T^* is an A-isometry that is $TAT^* = A$, where*

$$Ah = \lim_{n\to\infty} T^n P_{\mathcal{H}_0} T^{*n} h \quad h \in \mathcal{H} \tag{4}$$

and A is an orthogonal projection. Moreover, T is unitary on $\mathcal{R}(A)$ such that

$$\mathcal{R}(A) = \mathcal{N}(I - A) = \mathcal{N}(I - S_{V^*}) = \bigcap_{n\geq 1} V^n\mathcal{H}_0 = \bigcap_{n\geq 1} T^n\mathcal{H}, \quad V = T|_{\mathcal{H}_0}. \tag{5}$$

Furthermore, if T admits Wold-type decomposition then

$$\mathcal{N}(A) = \ell^2_+(\mathcal{N}(V^*)) \oplus (\mathcal{H} \ominus \mathcal{H}_0) = \bigvee_{n\geq 0} T^n \mathcal{N}(T^*). \tag{6}$$

Proof. For T as above, we have (by (3))

$$TP_{\mathcal{H}_0}T^* = \begin{pmatrix} V & T_0 \\ 0 & T_1 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V^* & 0 \\ T_0^* & T_1^* \end{pmatrix} = \begin{pmatrix} VV^* & 0 \\ 0 & 0 \end{pmatrix} \leq P_{\mathcal{H}_0}.$$

Therefore the sequence $\{T^n P_{\mathcal{H}_0} T^{*n}\}_{n\geq 0}$ is decreasing and bounded, hence it strongly converges in $\mathcal{B}(\mathcal{H})$ to a positive contraction $A \in \mathcal{B}(\mathcal{H})$ with $A \leq P_{\mathcal{H}_0}$, as in (4). Clearly, $\mathcal{R}(A) \subset \mathcal{H}_0$, $\mathcal{H}_1 = \mathcal{H} \ominus \mathcal{H}_0 \subset \mathcal{N}(A)$ and $TAT^* = A$, which ensures that $T\mathcal{R}(A) \subset \mathcal{R}(A)$ and $T|_{\mathcal{R}(A)} = V|_{\mathcal{R}(A)}$ is an isometry, because $V = T|_{\mathcal{H}_0}$ is such.

Next for every $h \in \mathcal{R}(A)$ we have (using the matrix of T in (3)),

$$\begin{aligned} Ah &= \lim_{n\to\infty} T^n P_{\mathcal{H}_0} T^{*n} h = \lim_{n\to\infty} T^n P_{\mathcal{H}_0}(V^{*n}h \oplus h_n) \\ &= \lim_{n\to\infty} T^n V^{*n} h = \lim_{n\to\infty} V^n V^{*n} h = S_{V^*} h. \end{aligned}$$

Here $h_n = P_{\mathcal{H}_1} T^{*n} h$, while S_{V^*} is the asymptotic limit of the coisometry V^*. Since S_{V^*} is an orthogonal projection, $\mathcal{R}(S_{V^*})$ is the unitary part of V in $\mathcal{H}_0$, so $T|_{\mathcal{R}(A)} = V|_{\mathcal{R}(S_{V^*})}$ is unitary (see [9]). Also, for $h \in \mathcal{R}(A)$ it follows that

$$A^2 h = S_{V^*}(Ah) = S^2_{V^*} h = S_{V^*} h = Ah,$$

and we conclude that A is an orthogonal projection in $\mathcal{B}(\mathcal{H})$. Thus we obtain that

$$\mathcal{R}(A) = \mathcal{N}(I - A) \subset \mathcal{R}(S_{V^*}) = \mathcal{N}(I - S_{V^*}) = \bigcap_{n\geq 1} V^n \mathcal{H}_0 = \bigcap_{n\geq 1} T^n \mathcal{H}.$$

In addition, if $h \in \mathcal{N}(I - S_{V^*}) \ominus \mathcal{R}(A)$ then (as above)

$$h = S_{V^*} h = \lim_{n\to\infty} V^n V^{*n} h = \lim_{n\to\infty} T^n P_{\mathcal{H}_0} T^{*n} h = Ah = 0,$$

hence $\mathcal{R}(A) = \mathcal{N}(I - S_{V^*})$ which yields the equalities in (5). Notice that the last equality in (5) follows immediately from the proof of Theorem 2.1, but it was also mentioned in [10].

From (5), we obtain

$$\mathcal{N}(A) = (\mathcal{H}_0 \ominus \mathcal{R}(A)) \oplus \mathcal{H}_1 = \ell^2_+(\mathcal{N}(V^*)) \oplus \mathcal{H}_1 = \bigvee_{n\geq 0} T'^n \mathcal{N}(T^*),$$

having in view that $\mathcal{H}_0 \ominus \mathcal{R}(A)$ is the shift part of V in $\mathcal{H}_0$, and that $\mathcal{N}(A)$ is the analytic part of T in $\mathcal{H}$ (by (5)).

Finally, if T admits Wold-type decomposition (as in Theorem 2.1, for example) then $\mathcal{N}(A) = \bigvee_{n\geq 0} T^n \mathcal{N}(T^*)$, which completes the equality (6). □

Corollary 2.6. *Let $T, A \in \mathcal{B}(\mathcal{H})$ be as in Theorem 2.5. Then T is analytic if and only if $A = 0$.*

3. BROWNIAN-TYPE m-ISOMETRIES

In the sequel, we refer to a special class of m-isometries T which admit Wold-type decompositions, namely to those with Δ_T a scalar multiple of an orthogonal projection. First, we give the following

Proposition 3.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be an m-isometry for an integer $m \geq 3$ such that $\Delta_T = \delta^2 P$ with P an orthogonal projection and a scalar $\delta > 0$. Then $\mathcal{N}(\Delta_T)$ is invariant for T if and only if T is Δ_T-bounded, that is there exists a constant $c > 0$ such that*

$$T^* \Delta_T T \leq c\Delta_T.$$

If this is the case then T admits Wold-type decomposition.

Proof. Let T be an m-isometry with $\Delta_T = \delta^2 P$ where $P = P_{\mathcal{R}(\Delta_T)}$ and $\delta^2 = \|\Delta_T\| > 0$. So T is expansive. Assume that $T\mathcal{N}(\Delta_T) \subset \mathcal{N}(\Delta_T)$. Then for $h \in \mathcal{H}$, $h = h_0 \oplus h_1$ with $h_0 \in \mathcal{N}(\Delta_T)$, $h_1 \in \mathcal{R}(\Delta_T)$ we have

$$\begin{aligned}(T^* \Delta_T T h, h) &= \delta^2 (T^* P T h_1, h_1) \leq \delta^2 (T^* T h_1, h_1) \\ &\leq \|T\|^2 (\delta^2 P h, h) = \|T\|^2 (\Delta_T h, h).\end{aligned}$$

Hence $T^* \Delta_T T \leq c\Delta_T$, that is T is Δ_T-bounded with $c = \|T\|^2 \geq 1$.

Obviously, when T is Δ_T-bounded, $\mathcal{N}(\Delta_T)$ is invariant for T, taking into account that $\Delta_T \geq 0$ (so $T^* \Delta_T T \geq 0$). We conclude by Corollary 2.4 that if T is Δ_T-bounded then T admits Wold-type decomposition. □

Recall from [7] that a 3-isometry T which is Δ_T-bounded is called a sub-Brownian 3-isometry. Obviously, such an operator T is convex (i.e. $\Delta_T^{(2)} \geq 0$), expansive with $T\mathcal{N}(\Delta_T) \subset \mathcal{N}(\Delta_T)$, but $\mathcal{R}(\Delta_T)$ is not necessarily closed. More generally, the *sub-Brownian m-isometries* for $m \geq 3$ were studied in [12]. Such an m-isometry T is $\Delta_T^{(j)}$-bounded with the boundedness constant $c_j \geq 1$ for $j = 1, 2, \cdots, m-2$ (see [12, Theorem 2.5]). Equivalently, by [12, Theorem 2.2] this means that T has an *m-Brownian unitary extension* on a Hilbert space $\mathcal{K} \supset \mathcal{H}$. This extension is an operator B which, under a decomposition $\mathcal{K} = \bigoplus_{j=1}^{m} \mathcal{K}_j$, has a representation of the form

$$B = \begin{pmatrix} V_1 & \sigma E_1 & 0 & \dots & 0 & 0 \\ 0 & V_2 & \sigma E_2 & \dots & 0 & 0 \\ .. & .. & .. & \dots & .. & .. \\ 0 & 0 & 0 & \dots & V_{m-1} & \sigma E_{m-1} \\ 0 & 0 & 0 & \dots & 0 & U \end{pmatrix}, \tag{7}$$

where V_j, E_j are isometries with $\mathcal{N}(V_j^*) = \mathcal{R}(E_j)$ for $j = 1, 2, \cdots, m-1$, U is unitary and $\sigma > 0$ is a scalar.

It is clear that $\Delta_B = \sigma^2 P_{\mathcal{K}_2 \oplus \cdots \oplus \mathcal{K}_m}$, B is a sub-Brownian m-isometry, while $\sigma = \|\Delta_B\|^{1/2}$ is called the *covariance* of B.

For such operators we derive from Proposition 3.1 the following

COROLLARY 3.2. *Every sub-Brownian m-isometry T with $m \geq 3$ and $\Delta_T = \delta^2 P$, where P is an orthogonal projection and $\delta > 0$ admits Wold-type decomposition.*

A more special class of sub-Brownian m-isometries is now described

THEOREM 3.3. *Let $T \in \mathcal{B}(\mathcal{H})$ be a sub-Brownian m-isometry for an integer $m \geq 3$, and let $B \in \mathcal{B}(\mathcal{K})$ be an m-Brownian unitary extension for T of covariance $\sigma = \|\Delta_B\|^{1/2} > 0$. The following statements are equivalent:*

(i) $\Delta_T = \sigma^2 P_{\mathcal{R}(\Delta_T)}$;

(ii) $B^* B \mathcal{H} \subset \mathcal{H}$;

(iii) $\mathcal{R}(\Delta_T) \subset \mathcal{R}(\Delta_B)$.

Moreover, when these conditions hold true $T_1 = P_{\mathcal{R}(\Delta_T)} T|_{\mathcal{R}(\Delta_T)}$ has as a power dilation the $(m-1)$-Brownian unitary $B_1 = P_{\mathcal{R}(\Delta_B)} B|_{\mathcal{R}(\Delta_B)}$.

Proof. Assume that $\Delta_T = \sigma^2 P_{\mathcal{R}(\Delta_T)}$. Since the m-Brownian unitary B is an extension for T, B as well as Δ_B have the representations

$$B = \begin{pmatrix} T & X \\ 0 & Y \end{pmatrix}, \quad \Delta_B = \begin{pmatrix} \Delta_T & T^* X \\ X^* T & X^* X + \Delta_Y \end{pmatrix} = \sigma^2 P_{\mathcal{R}(\Delta_B)}, \tag{8}$$

under the decomposition $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}'$. As $\sigma^{-2}\Delta_B = (\sigma^{-2}\Delta_B)^2$, by using the matrix of Δ_B one obtains that $T^* X = 0$. Therefore

$$B^* B = T^* T \oplus (X^* X + Y^* Y) \quad \text{on} \quad \mathcal{K} = \mathcal{H} \oplus \mathcal{H}',$$

which gives that $B^* B \mathcal{H} \subset \mathcal{H}$. Hence (i) implies (ii).

Let assume now that $B^* B \mathcal{H} \subset \mathcal{H}$. As $T\mathcal{N}(\Delta_T) \subset \mathcal{N}(\Delta_T)$, T being a sub-Brownian m-isometry, we have $B\mathcal{N}(\Delta_T) \subset \mathcal{N}(\Delta_T)$ and $B|_{\mathcal{N}(\Delta_T)} = T|_{\mathcal{N}(\Delta_T)} =:$

V is an isometry. So the above matrices of B and Δ_B can be expressed under the decomposition $\mathcal{K} = \mathcal{N}(\Delta_T) \oplus \overline{\mathcal{R}(\Delta_T)} \oplus \mathcal{H}'$ in the form

$$(9)\qquad B = \begin{pmatrix} V & T_0 & X_0 \\ 0 & T_1 & X_1 \\ 0 & 0 & Y \end{pmatrix}, \quad \Delta_B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta - I & E \\ 0 & E^* & G \end{pmatrix},$$

where $\Delta = T^*T|_{\overline{\mathcal{R}(\Delta_T)}}$, $E = T_0^* X_0 + T_1^* X_1$ and $G = X_0^* X_0 + X_1^* X_1 + \Delta_Y$. Since $\Delta_B \mathcal{H} \subset \mathcal{H}$ we need to have $E = 0$, and this implies $\mathcal{R}(\Delta_T) \subset \mathcal{R}(\Delta_B)$. So (ii) implies (iii).

Finally, we suppose that $\mathcal{R}(\Delta_T) \subset \mathcal{R}(\Delta_B)$. Then $\mathcal{R}(\Delta_T)$ is closed. Indeed, if $h \in \overline{\mathcal{R}(\Delta_T)}$ then $h = \Delta_B h' = \sigma^2 h'$ for some element $h' \in \mathcal{H}$, so $h = P_{\mathcal{H}} \Delta_B h' = \Delta_T h'$, taking into consideration that $B|_{\mathcal{H}} = T$ and $P_{\mathcal{H}} B^*|_{\mathcal{H}} = T^*$. Hence $h \in \mathcal{R}(\Delta_T)$ and it follows that $\mathcal{R}(\Delta_T)$ is closed. Next for $h \in \mathcal{R}(\Delta_T)$ and having in view the assumption (iii) one obtains

$$\Delta_T h = P_{\mathcal{R}(\Delta_T)} \Delta_B h = \sigma^2 P_{\mathcal{R}(\Delta_T)} P_{\mathcal{R}(\Delta_B)} h = \sigma^2 P_{\mathcal{R}(\Delta_T)} h = \sigma^2 h,$$

whence we infer that $\Delta_T = \sigma^2 P_{\mathcal{R}(\Delta_T)}$. Hence (iii) implies (i). The equivalences (i)-(iii) are proved.

Next it is clear from the representation (7) of B that $\mathcal{K}_1 = \mathcal{N}(\Delta_B)$ so $\mathcal{R}(\Delta_B) = \mathcal{K}_2 \oplus \cdots \oplus \mathcal{K}_m$, and that the compression $B_1 = P_{\mathcal{R}(\Delta_B)} B|_{\mathcal{R}(\Delta_B)}$ is an $(m-1)$-Brownian unitary on $\mathcal{R}(\Delta_B)$. Thus, under the assumption (iii) as well as the fact that $T\mathcal{N}(\Delta_T) \subset \mathcal{N}(\Delta_T)$ we have for $T_1 = P_{\mathcal{R}(\Delta_T)} T|_{\mathcal{R}(\Delta_T)}$ and $n \geq 1$,

$$T_1^{*n} = T^{*n}|_{\mathcal{R}(\Delta_T)} = P_{\mathcal{H}} B^{*n}|_{\mathcal{R}(\Delta_T)} = P_{\mathcal{R}(\Delta_T)} B_1^{*n}|_{\mathcal{R}(\Delta_T)},$$

which by duality means $T_1^n = P_{\mathcal{R}(\Delta_T)} B_1^n|_{\mathcal{R}(\Delta_T)}$, that is B_1 is a power dilation of T_1. $\square$

Some relations between $\mathcal{N}(T^*)$ and $\mathcal{N}(B^*)$ can be obtained under the conditions from the previous theorem.

THEOREM 3.4. *Let T on $\mathcal{H}$ be a sub-Brownian m-isometry ($m \geq 3$) and B on $\mathcal{K} \supset \mathcal{H}$ be an m-Brownian unitary extension for T of covariance $\sigma > 0$, such that T and B satisfy (one of) the conditions (i)-(iii) of Theorem 3.3. Then*

$$(10)\qquad \mathcal{N}(T^*) = \overline{P_{\mathcal{H}} \mathcal{N}(B^*)} + \overline{P_{\mathcal{H}} B(\mathcal{K} \ominus \mathcal{H})}$$

and the Cauchy dual operator B' of B is an extension for the Cauchy dual operator T' of T.

Moreover, the two subspaces in (8) are orthogonal if and only if

$$(11)\qquad \mathcal{N}(B^*) = \mathcal{N}(T^*) \cap \mathcal{N}(P_{\mathcal{K}\ominus\mathcal{H}} B^*|_{\mathcal{H}}) \oplus \mathcal{N}(B^*|_{\mathcal{K}\ominus\mathcal{H}}).$$

Proof. Consider T and B as above satisfying the conditions (i)-(iii) of Theorem 3.3. From the representation of B in (8) we infer that $P_{\mathcal{H}}\mathcal{N}(B^*) \subset \mathcal{N}(T^*)$. Also, denoting $X = P_{\mathcal{H}}B|_{\mathcal{K}\ominus\mathcal{H}}$ (as in (8)) we have by (ii) that $T^*X = 0$ that is $\mathcal{R}(X) \subset \mathcal{N}(T^*)$. Thus we obtain on one hand that

$$\overline{P_{\mathcal{H}}\mathcal{N}(B^*)} + \overline{\mathcal{R}(X)} \subset \mathcal{N}(T^*).$$

Conversely, let $h \in \mathcal{N}(T^*)$, $h = h_0 \oplus h_1$ where $h_0 \in \overline{P_{\mathcal{H}}\mathcal{N}(B^*)}$ and with h_1 orthogonal on $P_{\mathcal{H}}\mathcal{N}(B^*)$. As $h_1 = h - h_0 \in \mathcal{H}$ it follows that h_1 is orthogonal on $\mathcal{N}(B^*)$, so $h_1 \in \mathcal{R}(B)$. Thus $h_1 = Bk$ with $k \in \mathcal{K}$. Setting $k = h_2 \oplus h'$ with $h_2 \in \mathcal{H}$, $h' \in \mathcal{K} \ominus \mathcal{H}$ we get (by (8)) that

$$h_1 = B(h_2 \oplus h') = Th_2 + Xh',$$

whence $T^*h_1 = T^*Th_2$, taking into account that $T^*X = 0$. On the other hand, as $h_0 \in \mathcal{N}(T^*)$ by the above inclusion, we have $T^*h_1 = T^*(h_0 \oplus h_1) = T^*h = 0$, hence $T^*Th_2 = 0$ that is $h_2 = 0$ (T being injective). Thus $k = h' \in \mathcal{K} \ominus \mathcal{H}$ which later gives

$$h_1 = Bh' = P_{\mathcal{H}}Bh' = Xh'.$$

Finally, one obtains that $h = h_0 \oplus h_1 \in \overline{P_{\mathcal{H}}\mathcal{N}(B^*)} + \overline{\mathcal{R}(X)}$, and we conclude that the relation (10) is true.

In order to show the next assertion of theorem, we assume that $\mathcal{R}(X)$ and $P_{\mathcal{H}}\mathcal{N}(B^*)$ are orthogonal subspaces in $\mathcal{H}$. Then

$$\overline{P_{\mathcal{H}}\mathcal{N}(B^*)} \subset \mathcal{N}(T^*) \cap \mathcal{N}(X^*),$$

so for $k = h \oplus h' \in \mathcal{N}(B^*)$ with $h \in \mathcal{H}$, $h' \in \mathcal{K} \ominus \mathcal{H}$ we have $X^*h = 0$ and

$$Y^*h' = X^*h + Y^*h' = P_{\mathcal{K}\ominus\mathcal{H}}B^*k = 0, \quad (Y \text{ from (8)}).$$

Therefore $\overline{P_{\mathcal{K}\ominus\mathcal{H}}\mathcal{N}(B^*)} \subset \mathcal{N}(Y^*)$ and finally we get

$$\mathcal{N}(B^*) = \overline{P_{\mathcal{H}}\mathcal{N}(B^*)} \oplus \overline{P_{\mathcal{K}\ominus\mathcal{H}}\mathcal{N}(B^*)} = \mathcal{N}(T^*) \cap \mathcal{N}(X^*) \oplus \mathcal{N}(Y^*),$$

having in view that always $\mathcal{N}(T^*) \cap \mathcal{N}(X^*)$, $\mathcal{N}(Y^*) \subset \mathcal{N}(B^*)$.

Conversely, if the equality (11) holds, then $P_{\mathcal{H}}\mathcal{N}(B^*) \subset \mathcal{N}(X^*)$, hence $P_{\mathcal{H}}\mathcal{N}(B^*)$ and $\mathcal{R}(X)$ are orthogonal in $\mathcal{H}$ (even in $\mathcal{N}(T^*)$).

To end the proof it remains to show that B' is an extension for T' (the Cauchy duals of B, T). Thus, by the assertion (i) of Theorem 3.3 and the matrix of Δ_B in (9) we have $B^*B = I \oplus (\sigma^2+1)I \oplus (G+I)$ on $\mathcal{K} = \mathcal{N}(\Delta_T) \oplus \mathcal{R}(\Delta_T) \oplus \mathcal{H}'$ where $G = \Delta_B|_{\mathcal{H}'}$ and $\sigma^2 = \|\Delta_B\|$. This implies

$$(B^*B)^{-1} = I \oplus (\sigma^2+1)^{-1}I \oplus (G+I)^{-1}.$$

On the other hand, as $T^*T = I \oplus (\sigma^2+1)I$ on $\mathcal{H} = \mathcal{N}(\Delta_T) \oplus \mathcal{R}(\Delta_T)$ we get $(T^*T)^{-1} = I \oplus (\sigma^2+1)^{-1}I$. Therefore $(B^*B)^{-1} = (T^*T)^{-1} \oplus (G+I)^{-1}$ under

$\mathcal{K} = \mathcal{H} \oplus \mathcal{H}'$. This later leads to our conclusion, that is

$$B' = B(B^*B)^{-1} = \begin{pmatrix} T & X \\ 0 & Y \end{pmatrix} \begin{pmatrix} (T^*T)^{-1} & 0 \\ 0 & (G+I)^{-1} \end{pmatrix} = \begin{pmatrix} T' & X(G+I)^{-1} \\ 0 & Y(G+I)^{-1} \end{pmatrix}$$

on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}'$, which proves that B' is an extension for T'. □

By Theorem 3.3, the condition $B^*B\mathcal{H} \subset \mathcal{H}$ ensures that Δ_T is a scalar multiple of an orthogonal projection. But if $\|\Delta_B\| > \|\Delta_T\|$ for an m-Brownian extension B of a sub-Brownian m-isometry T (what is possible by [12, Theorem 2.2]) such that $\Delta_T = \|\Delta_T\| P_{\mathcal{R}(\Delta_T)}$, then $\mathcal{H}$ is not invariant for B^*B. Thus we infer from Theorem 3.3 and Corollary 3.2 the following

COROLLARY 3.5. *If T on $\mathcal{H}$ is a sub-Brownian m-isometry (with $m \geq 3$) which has an m-Brownian unitary extension B on $\mathcal{K} \supset \mathcal{H}$ such that $B^*B\mathcal{H} \subset \mathcal{H}$, then T admits Wold-type decomposition.*

The results above refer to a special class of m-isometries which are sub-Brownian and have Wold-type decomposition, namely those with $\Delta_T = \delta^2 P_{\mathcal{R}(\Delta_T)}$ and $\delta > 0$. But this last condition is not necessary for a sub-Brownian m-isometry with Wold-type decomposition, as can be seen even in the case $m = 3$.

Example 3.6. Let $\ell^2_+(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$ where $\mathcal{H}_n = \mathcal{H}$ for $n \geq 0$ and let T be the weighted forward shift on $\ell^2_+(\mathcal{H})$ defined by

$$T(h_0, h_1, \cdots) = (0, \alpha_1 h_0, \alpha_2 h_1, \cdots), \quad \{h_n\} \in \ell^2_+(\mathcal{H}),$$

where the weights α_n are given by

$$\alpha_n = \frac{n+1}{n}, \quad n \geq 1.$$

It is easy to see that T is a 3-isometry which is not a 2-isometry. The adjoint T^* of T is the weighted backward shift defined by

$$T^*(h_0, h_1, h_2, \cdots) = (\alpha_1 h_1, \alpha_2 h_2, \cdots).$$

In this case, we have $\mathcal{H} = \bigvee_{n \geq 0} T^n \mathcal{N}(T^*)$ where $\mathcal{N}(T^*) = \mathbb{C}\{e_0\}$, $e_0 = (1, 0, \cdots)$. Hence T admits Wold-type decomposition with $\mathcal{H}_\infty = \{0\}$.

Since T^*T has the representation

$$T^*T(h_0, h_1, \cdots) = (\alpha_1^2 h_0, \alpha_2^2 h_1, \cdots),$$

while the operator Cauchy dual of T on $\ell^2_+(\mathcal{H})$ is given by

$$T'(h_0, h_1, \cdots) = (0, \alpha_1^{-1} h_0, \alpha_2^{-1} h_1, \cdots),$$

that is T' is the forward shift with the weights $\{\alpha_n^{-1}, n \geq 1\}$. Since $\mathcal{N}(\Delta_T) = \{0\}$ and $r(T') = 1$, T does not satisfy the hypothesis of Theorem 2.1. So the condition on the spectral radius in this theorem is only sufficient for a Wold-type decomposition.

The conclusion of having a Wold-type decomposition for T results also from [8, Theorem 3.9], because T is expansive what by [8, Remark 11] means that

$$\sum_{n=1}^{\infty} T'^n \Delta_T^{(2)} T'^{*n} \leq \Delta_T.$$

On the other hand, for $h = \{h_n\}_{n\geq 0} \in \ell_+^2(\mathcal{H})$ we have

$$\begin{aligned} T^*\Delta_T T h &= T^*(0, (\alpha_2^2 - 1)\alpha_1 h_0, (\alpha_3^2 - 1)\alpha_2 h_1, \cdots) \\ &= (\alpha_1^2(\alpha_2^2 - 1)h_0, \alpha_2^2(\alpha_3^2 - 1)h_1, \cdots). \end{aligned}$$

Since $\alpha_{n+1} < \alpha_n$ for $n \geq 1$, we infer that

$$\begin{aligned} (T^*\Delta_T T h, h) &= \sum_{n=1}^{\infty} \alpha_n^2(\alpha_{n+1}^2 - 1)\|h_{n-1}\|^2 \\ &\leq \alpha_1^2 \sum_{n=1}^{\infty} (\alpha_n^2 - 1)\|h_{n-1}\|^2 = \alpha_1^2(\Delta_T h, h). \end{aligned}$$

Thus $T^*\Delta_T T \leq \alpha_1^2 \Delta_T$ that is T is Δ_T-bounded, hence a sub-Brownian 3-isometry. This inequality also shows that $\|\Delta_T^{1/2}\| = \sqrt{\alpha_1^2 - 1} = \sqrt{3}$, but $\Delta_T \neq 3I$.

Acknowledgments. We would like to thank the referee for a careful reading of the manuscript and for useful suggestions.

REFERENCES

[1] J. Agler and M. Stankus, *m-isometric transformations of Hilbert spaces*. Integral Equations Operator Theory **21** (1995), *4*, 383–429.

[2] J. Agler and M. Stankus, *m-isometric transformations of Hilbert spaces II*. Integral Equations Operator Theory **23** (1995), *1*, 1–48.

[3] J. Agler and M. Stankus, *m-isometric transformations of Hilbert spaces III*. Integral Equations Operator Theory **24** (1996), *4*, 379–421.

[4] C. Badea, V. Müller, and L. Suciu, *High order isometric liftings and dilations*. Studia Math. **258** (2021), *1*, 87–101.

[5] C. Badea and L. Suciu, *Hilbert space operators with two-isometric dilations*. J. Operator Theory **86** (2021), *1*, 93–123.

[6] S. Chavan and S. Trivedi, *Failure of the wandering subspace property for analytic norm-increasing* 3*-isometries*. arXiv:2212.04446.

[7] A. Crăciunescu and L. Suciu, *Brownian extensions in the context of 3-isometries*. J. Math. Anal. Appl. **529** (2024), 1–19, 127591.

[8] S. Ghara, R. Gupta, and R. Reza, *Analytic m-isometries and weighted Dirichlet-type spaces*. J. Operator Theory **88** (2022), *2*, 445–477.

[9] C.S. Kubrusly, *An Introduction to Models and Decompositions in Operator Theory*. Birkhäuser, Boston, 1997.

[10] W. Majdak and L. Suciu, *Brownian type parts of operators in Hilbert spaces*. Results Math. **75** (2020), *5*.

[11] S. Shimorin, *Wold-type decompositions and wandering subspaces for operators close to isometries*. J. Reine Angew. Math. **531** (2001), 147–189.

[12] L. Suciu, *Brownian type extensions for a class of m-isometries*. Results Math. **78** (2023), *144*.

Received March 15, 2023

"Lucian Blaga" University of Sibiu
Department of Mathematics and Informatics
Sibiu, Romania
laurians2002@yahoo.com