HIGHER POWER MOMENTS OF COEFFICIENTS ATTACHED TO THE DEDEKIND ZETA FUNCTION

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Communicated by Alexandru Zaharescu

Let K_3 be a non-normal cubic extension over $\mathbb{Q}$. In this paper, we investigate the higher moments of coefficients $a_{K_3}(n)$ of the Dedekind zeta function of the following type

$$\sum_{n\leq x} a_{K_3}^l(n),$$

where $l \geq 4$ is any fixed positive integer.

AMS 2020 Subject Classification: 11F30, 11R42, 11F66.

Key words: non-normal cubic field, Dedekind zeta function, automorphic L-functions.

1. INTRODUCTION

It is an interesting and important topic to study the coefficients of the Dedekind zeta function in modern number theory. Let $K/\mathbb{Q}$ be a number field of degree d. The Dedekind zeta function is defined by

$$\zeta_K(s) = \sum_{\mathfrak{a}\neq 0} (N\mathfrak{a})^{-s} = \prod_{\mathfrak{p}} \left(1-(N\mathfrak{p})^{-s}\right)^{-1}, \quad \Re(s) > 1, \tag{1}$$

where $d = [K:\mathbb{Q}]$ and $N\mathfrak{a}$ is the norm of the integral ideals $\mathfrak{a}$. We can rewrite the Dedekind zeta function as a Dirichlet series

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{a_K(n)}{n^s}, \quad \Re(s) > 1,$$

where $a_K(n)$ denotes the number of integral ideals in K with norm n, which is called the coefficients of the Dedekind zeta function. It is obvious that

This work was financially supported in part by The National Key Research and Development Program of China (Grant No. 2021YFA1000700), Natural Science Basic Research Program of Shaanxi (Program Nos. 2023-JC-QN-0024, 2023-JC-YB-077), Foundation of Shaanxi Educational Committee (2023-JC-YB-013) and Shaanxi Fundamental Science Research Project for Mathematics and Physics (Grant No. 22JSQ010).

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doi: 10.59277/RRMPA.2024.1.10

$a_K(n) \geq 0$ for all $n \geq 1$. And it is well-known that $a_K(n)$ is a real multiplicative function and for any $\varepsilon > 0$,

$$a_K(n) \leq \tau(n)^d \ll n^{\varepsilon},$$

here $\tau(n)$ is the divisor function. In fact, we can expand the expression (1) as an Euler product

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{a_K(n)}{n^s} = \prod_p \left(1 + \frac{a_K(p)}{p^s} + \cdots + \frac{a_K(p^k)}{p^{ks}} + \cdots\right), \quad \Re(s) > 1.$$

The investigation of coefficients of the Dedekind zeta function has a long scientific record. Landau [17] proved the asymptotic formula

$$\sum_{n \leq x} a_K(n) = cx + O\big(x^{1-\frac{2}{d+1}+\varepsilon}\big)$$

for arbitrary algebraic number fields of degree $d \geq 2$, where $c > 0$ is some suitable positive constant depending on K. Chandrasekharan and Narasimhan [1] considered the second higher moment of $a_K(n)$ and they proved that

$$\sum_{n \leq x} a_K^2(n) \ll x \log^{d-1} x.$$

Later, for K being normal extension of $\mathbb{Q}$, Chandrasekharan and Good [2] investigated the higher moments of $a_K(n)$ and established the asymptotic formulas

$$\sum_{n \leq x} a_K^l(n) = xP_K(\log x) + O\big(x^{1-\frac{2}{d^l}+\varepsilon}\big),$$

where $l \geq 2$ is a positive integer and $P_K(t)$ is a polynomial of t with degree $d^{l-1} - 1$. In 2010, Lü and Wang [18] improved the results of Chandrasekharan and Good.

Let $K_3/\mathbb{Q}$ be a non-normal cubic extension, which is given by an irreducible polynomial $h(x) = x^3 + Ax^2 + Bx + C$ of discriminant D. In 2008, Fomenko [8] considered the second and third moments of $a_{K_3}(n)$ under the condition $D < 0$ and he proved that

$$\sum_{n \leq x} a_{K_3}^2(n) = c_1 x \log x + c_2 x + O\big(x^{\frac{9}{11}+\varepsilon}\big), \tag{2}$$

and

$$\sum_{n \leq x} a_{K_3}^3(n) = xP(\log x) + O\big(x^{\frac{73}{79}+\varepsilon}\big), \tag{3}$$

where c_1 and c_2 are some suitable constants, and $P(t)$ is a polynomial of t with degree 4. In 2013, Lü [20] refined the exponents in the error terms of

(2) and (3) to 23/31 and 235/259, respectively. Very recently, Liu [21] further sharpened the exponent of (3) to 1361/1501, and he also considered the general divisor problem related to coefficients of the Dedekind zeta function.

In this paper, we consider the average behaviour of higher moments of the arithmetic function $a_{K_3}(n)$ in the following shape

$$\sum_{n\leq x} a_{K_3}^l(n),$$

where $l \geq 4$ is any fixed positive integer.

THEOREM 1.1. *Let $K_3/\mathbb{Q}$ be a non-normal cubic extension which is given by an irreducible polynomial $h(x) = x^3 + Ax^2 + Bx + C$ of discriminant D. If $D < 0$, then for any $\varepsilon > 0$, we have*
(i) Let $l \geq 4$ be an even integer, then

$$\sum_{n\leq x} a_{K_3}^l(n) = xP_{K_3,l}(\log x) + O(x^{\alpha_l+\varepsilon}),$$

where $P_{K_3,l}(t)$ is a polynomial of t with $\deg P_{K_3,l} = \kappa_{l,1} + \kappa_{l,2} - 1$, and $\alpha_l = 1 - \frac{2}{3^l}$, here $\kappa_{l,1}$ and $\kappa_{l,2}$ are defined by

$$\kappa_{l,1} = 1 + \sum_{i=1}^{\frac{l}{2}} \binom{l}{2i} A_i, \qquad \kappa_{l,2} = \sum_{i=1}^{\frac{l}{2}-1} \binom{l}{2i+1} D_i.$$

(ii) Let $l \geq 5$ be an odd integer, then

$$\sum_{n\leq x} a_{K_3}^l(n) = xP^*_{K_3,l}(\log x) + O(x^{\alpha_l+\varepsilon}),$$

where $P^*_{K_3,l}(t)$ is a polynomial of t with $\deg P^*_{K_3,l} = \nu_{l,1} + \nu_{l,2} - 1$, here $\nu_{l,1}$ and $\nu_{l,2}$ are defined by

$$\nu_{l,1} = 1 + \sum_{j=1}^{\frac{l-1}{2}} \binom{l}{2j} A_j, \quad \nu_{l,2} = \sum_{j=1}^{\frac{l-1}{2}} \binom{l}{2j+1} D_j.$$

The constants $A_i, D_i, i \geq 1$ are defined as in (9) and (10), respectively.

Throughout the paper, let $\varepsilon > 0$ denote the arbitrarily small number which may vary in different occurrence. And p always denotes a prime number.

2. PRELIMINARIES

In this section, we review some analytic properties of automorphic L-functions and introduce some useful lemmas which play an important role in the proof of the main results in this paper.

Let f be a holomorphic cusp form of integral weight k for the full modular group $\Gamma = SL(2,\mathbb{Z})$, then f has the Fourier series expansion at the cusp ∞:

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z}, \quad \Im(z) > 0,$$

here, the coefficients $\lambda_f(n)$ are Hecke eigenvalues of Hecke operators. It is well-known that for each p, there exist two complex numbers $\alpha_f(p), \beta_f(p)$ such that

$$\alpha_f(p) + \beta_f(p) = \lambda_f(p), \qquad \alpha_f(p)\beta_f(p) = 1. \tag{4}$$

By Deligne's bound [6], we also have

$$|\lambda_f(n)| \ll n^{\varepsilon}$$

for any $\varepsilon > 0$.

Define the Hecke L-function associated with f as

$$L(f,s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1}$$

for $\Re(s) > 1$. We can also define the jth symmetric power L-function attached to f as follows

$$L(\mathrm{sym}^j f, s) = \prod_p \prod_{m=0}^{j} \left(1 - \alpha_f(p)^{j-m} \beta_f(p)^m p^{-s}\right)^{-1}$$

for $\Re(s) > 1$, where $\alpha_f(p), \beta_f(p)$ are the local parameters given by (4). We may expand it into a Dirichlet series

$$\begin{aligned} L(\mathrm{sym}^j f, s) &= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s} \\ &= \prod_p \left(1 + \frac{\lambda_{\mathrm{sym}^j f}(p)}{p^s} + \cdots + \frac{\lambda_{\mathrm{sym}^j f}(p^k)}{p^{ks}} + \cdots\right), \quad \Re(s) \gg 1. \end{aligned}$$

Apparently, $\lambda_{\mathrm{sym}^j f}(n)$ is a real multiplicative function.

It is standard to find that

$$\lambda_f(p^j) = \lambda_{\mathrm{sym}^j f}(p) = \frac{\alpha_f(p)^{j+1} - \beta_f(p)^{j+1}}{\alpha_f(p) - \beta_f(p)} = \sum_{m=0}^{j} \alpha_f(p)^{j-m} \beta_f(p)^m,$$

which can be written as

$$\lambda_f(p^j) = \lambda_{\mathrm{sym}^j f}(p) = U_j(\lambda_f(p)/2), \tag{5}$$

where $U_j(x)$ is the jth Chebyshev polynomial of the second kind.

Let K_3 be a non-normal cubic extension over $\mathbb{Q}$, which is given by an irreducible polynomial $h(x) = x^3 + Ax^2 + Bx + C$ of discriminant D. If $D < 0$, from the paper of Fomenko [8, (1)] we know that

$$\zeta_{K_3}(s) = \zeta(s)L(f, s), \tag{6}$$

where f is a holomorphic cusp form of weight 1 with respect to the congruence group $\Gamma_0(|D|)$.

From (6), we have the convolution

$$a_{K_3}(n) = \sum_{d|n} \lambda_f(d).$$

In particular, we have

$$a_{K_3}(p) = 1 + \lambda_f(p). \tag{7}$$

We also define the L-function

$$L_{K_3,l}(s) = \sum_{n=1}^{\infty} \frac{a^l_{K_3}(n)}{n^s}, \qquad \Re(s) > 1. \tag{8}$$

We firstly state some basic definitions and analytic properties of general L-functions. Let $L(\phi, s)$ be a Dirichlet series (associated with the object ϕ) that admits an Euler product of degree $m \geq 1$, namely

$$L(\phi, s) = \sum_{n=1}^{\infty} \frac{\lambda_\phi(n)}{n^s} = \prod_p \prod_{j=1}^{m} \left(1 - \frac{\alpha_\phi(p, j)}{p^s}\right)^{-1},$$

where $\alpha_\phi(p, j), j = 1, 2, \cdots, m$ are the local parameters of $L(\phi, s)$ at a finite prime p. Suppose that this series and its Euler product are absolutely convergent for $\Re(s) > 1$. We denote the gamma factor by

$$L_\infty(\phi, s) = \prod_{j=1}^{m} \pi^{-\frac{s+\mu_\phi(j)}{2}} \Gamma\left(\frac{s + \mu_\phi(j)}{2}\right)$$

with local parameters $\mu_\phi(j), j = 1, 2, \cdots, m$ of $L(\phi, s)$ at ∞. The complete L-function $\Lambda(\phi, s)$ is defined by

$$\Lambda(\phi, s) = q(\phi)^{\frac{s}{2}} L_\infty(\phi, s) L(\phi, s),$$

where $q(\phi)$ is the conductor of $L(\phi, s)$ depends at most on ϕ and the parity of the L-function. We assume that $\Lambda(\phi, s)$ admits an analytic continuation to the

the whole complex plane $\mathbb{C}$ and is holomorphic everywhere except for possible poles of finite order at $s = 0, 1$. Furthermore, it satisfies a functional equation of the Riemann-type

$$\Lambda(\phi, s) = \epsilon_\phi \Lambda(\tilde{\phi}, 1 - s)$$

where ϵ_ϕ is the root number with $|\epsilon_\phi| = 1$ and $\tilde{\phi}$ is dual of ϕ such that $\lambda_{\tilde{\phi}}(n) = \overline{\lambda_\phi(n)}, L_\infty(\tilde{\phi}, s) = L_\infty(\phi, s)$ and $q(\tilde{\phi}) = q(\phi)$. We call ϕ is an L-function of degree m, and $\phi \in S_e^{\#}$ if it satisfies the above conditions. We call the L-function $L(\phi, s)$ satisfies the Ramanujan conjecture if $\lambda_\phi(n) \ll n^\varepsilon$ for any ε.

Here, we state a very general theorem due to Lau and Lü [19].

Lemma 2.1 ([19] Lemma 2.4). *Let $L(f, s)$ be a product of two L-functions $L_1, L_2 \in S_e^{\#}$ with both* $\deg L_i \geq 2, i = 1, 2$ *and $L(f, s)$ satisfies the Ramanujan conjecture. Then for any $\varepsilon > 0$, we have*

$$\sum_{n \leq x} \lambda_f(n) = M(x) + O\big(x^{1 - \frac{2}{m} + \varepsilon}\big),$$

where $M(x) = Res_{s=1}\{L(f, s)x^s/s\}$ and $m = \deg L$.

Associated with a primitive cusp form f, there is an automorphic cuspidal representation π_f of $GL_2(\mathbb{A}_\mathbb{Q})$ and hence, an automorphic L-function $L(\pi_f, s)$ which coincides with $L(f, s)$, namely

$$L(\pi_f, s) = L(f, s).$$

It is predicted by the Langlands functoriality conjecture that π_f gives rise to a symmetric power lift $\mathrm{sym}^j \pi_f$–an automorphic representation whose L-function is the symmetric power L-function attached to f,

$$L(\mathrm{sym}^j \pi_f, s) = L(\mathrm{sym}^j f, s).$$

For the known cases, the lifts are cuspidal, namely, there exists an automorphic cuspidal self-dual representation, denoted by $\mathrm{sym}^j \pi_f$ of $GL_{j+1}(\mathbb{A}_\mathbb{Q})$ whose L-function is the same as $L(\mathrm{sym}^j f, s)$. For $j = 1, 2, 3, 4$, this special Langlands functoriality conjecture that $\mathrm{sym}^j f$ is automorphic is shown by a series of important works. See, for example, Gelbert and Jacquet [9], Kim [16], Kim and Shahidi [14, 15], and Shahidi [24]. Later, Dieulefait [7] and Clozel and Thorne [3, 4, 5] investigated the cases $j \leq 8$. Very recently, Newton and Thorne [22, 23] proved that $\mathrm{sym}^j f$ corresponds with a cuspidal automorphic representation of $GL_{j+1}(\mathbb{A}_\mathbb{Q})$ for all $j \geq 1$. Hence, for $j \geq 1$, the L-function $L(\mathrm{sym}^j f, s)$ is an entire function and satisfies a functional equation of certain Riemann zeta-type with degree $j + 1$. In particular, in the paper [23, Theorem A] the authors established the existence of the symmetric power liftings

$\mathrm{sym}^n\pi_f$ for all $n \geq 1$ under the assumption that π is non-CM type. Regarding the case of automorphy of the symmetric power lifting for cuspidal Hecke eigenforms of weight 1, or with CM, they also proved the result holds in these cases by using Kim and Shahidi's results on tensor product and symmetric power functoriality (e.g., [23, Theorem A.1]).

Now, we need to give the decomposition of the L-function $L_{K_3,l}(s)$, which plays an essential role in the determination of the main results in this paper.

LEMMA 2.2. *Let K_3 be a non-normal cubic extension over $\mathbb{Q}$ as in Theorem 1.1. And let $L_{K_3,l}(s)$ be the L-function defined by (8). Let $l \geq 4$ be an even integer, we have*

$$L_{K_3,l}(s) = L^*_{K_3,l}(s)G_{l,1}(s),$$

where $L^*_{K_3,l}(s)$ is another L-function of degree 3^l that can be represented as the product of automorphic L-functions of the types $\zeta(s)^{\kappa_{l,1}}, L(sym^{i_1} f, s)^{j_1}$, where the exponents $\kappa_{l,1}, i_1, j_1 \geq 1$ are some suitable positive integers. Here, the exponent in $\zeta(s)$ is given by

$$\kappa_{l,1} = 1 + \sum_{i=1}^{\frac{l}{2}} \binom{l}{2i} A_i, \quad A_i = \frac{(2i)!}{i!(i+1)!}. \tag{9}$$

And the exponent in $L(sym^3 f, s)$ is given by

$$\kappa_{l,2} = \sum_{i=1}^{\frac{l}{2}-1} \binom{l}{2i+1} D_i, \quad D_1 = 1, \qquad D_i = \frac{4 \cdot (2i+1)!}{(i-1)!(i+3)!}, \ i \geq 2. \tag{10}$$

Here, the Dirichlet series $G_{l,1}(s)$ converges absolutely and uniformly in the half-plane $\Re(s) \geq \frac{1}{2} + \epsilon$ and $G_{l,1}(s) \neq 0$ when $\Re(s) = 1$.

Proof. Since $a^l_{K_3}(n), l \geq 4$ are real multiplicative functions and satisfy the trivial bound $O(n^{\varepsilon})$, then for $\Re(s) > 1$, we have the Euler product

$$L_{K_3,l}(s) = \prod_p \left(1 + \frac{a^l_{K_3}(p)}{p^s} + \cdots + \frac{a^l_{K_3}(p^k)}{p^{ks}} + \cdots\right).$$

In the half-plane $\Re(s) > \frac{1}{2}$, the corresponding coefficients of p^{-s} determine analytic properties of $L_{K_3,l}(s)$. From (7), then we have

$$a^l_{K_3}(p) = (1 + \lambda_f(p))^l.$$

By the binomial expansion, we have

$$(1 + \lambda_f(p))^l = \sum_{i=0}^{l} \binom{l}{i} \lambda^i_f(p).$$

Then from Lau and Lü [19, Lemma 7.1], we can determine the corresponding exponents of the automorphic L-functions $\zeta(s), L(\mathrm{sym}^3 f, s)$. $\square$

Lemma 2.3. *Let K_3 be a non-normal cubic extension over $\mathbb{Q}$ as in Theorem* 1.1. *And let $L_{K_3,l}(s)$ be the L-function defined by* (8). *Let $l \geq 5$ be an odd integer, we have*

$$L_{K_3,l}(s) = L^{**}_{K_3,l}(s) H_{l,2}(s),$$

where $L^{**}_{K_3,l}(s)$ is another L-function of degree 3^l that can be represented as the product of automorphic L-functions of the types $\zeta(s)^{\nu_{l,1}}, L(sym^{i_2} f, s)^{j_2}$, where the exponents $\nu_{l,1}, i_2, j_2 \geq 1$ are some suitable positive integers. Here, the exponent in $\zeta(s)$ is given by

$$\nu_{l,1} = 1 + \sum_{j=1}^{\frac{l-1}{2}} \binom{l}{2j} A_j, \tag{11}$$

where A_j is defined by (9). *And the exponent in $L(sym^3 f, s)$ is given by*

$$\nu_{l,2} = \sum_{j=1}^{\frac{l-1}{2}} \binom{l}{2j+1} D_j, \tag{12}$$

where D_j is defined by (10). *Here, the Dirichlet series $H_{l,2}(s)$ converges absolutely and uniformly in the half-plane $\Re(s) \geq \frac{1}{2} + \epsilon$ and $H_{l,2}(s) \neq 0$ when $\Re(s) = 1$.*

Proof. This follows essentially the same argument as that of Lemma 2.2. $\square$

Remark 2.4. In the half-plane $\Re(s) > \frac{1}{2}$, we learn from Fomenko [8] that $L(\mathrm{sym}^3 f, s)$ has an analytic continuation to that half-plane except for a simple pole at $s = 1$. Then, we learn from Lemma 2.2 and Lemma 2.3 that the factorization of $L_{K_3,l}(s)$ in the same half-plane have a pole of finite order at $s = 1$ which comes from the L-functions $\zeta(s)$ and $L(\mathrm{sym}^3, s)$.

3. PROOF OF THEOREM 1.1

In this section, we give the proof of Theorem 1.1 in the case $l \geq 4$ being an even integer, since for $l \geq 5$ an odd integer can be treated in the similar approach.

Let $l \geq 4$ be an even integer, by Lemma 2.2, we know that $L_{K_3,l}(s)$ is a general L-function satisfying the conditions in Lemma 2.1, then we have

$$\sum_{n \leq x} a_{K_3}^l(n) = xP_{K_3,l}(\log x) + O\big(x^{1-\frac{2}{3^l}+\varepsilon}\big),$$

where $xP_{K_3,l}(\log x)$ comes from the residue of the integrand at the pole $s = 1$ which takes the form

$$\mathrm{Res}_{s=1}\left\{\frac{L_{K_3,l}(s)}{s}x^s\right\},$$

and the L-functions $\zeta(s)$, $L(\mathrm{sym}^3 f, s)$ (with exponents) contributes to the pole at $s = 1$ in the decomposition of $L_{K_3,l}(s)$, here $P_{K_3,l}(t)$ is a polynomial of t with degree $\kappa_{l,1} + \kappa_{l,2} - 1$, here $\kappa_{l,1}$ and $\kappa_{l,2}$ are defined by (9) and (10), respectively.

Acknowledgments. The first author would like to express his gratitude to Professors Guangshi Lü, Bingrong Huang, Yujiao Jiang, and Research fellow Zhiwei Wang, for their constant encouragement and valuable suggestions. The authors are extremely grateful to the anonymous referees for their meticulous checking, for thoroughly reporting countless typos and inaccuracies as well as for their valuable comments. These corrections and additions have made the manuscript clearer and more readable.

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Received March 22, 2022

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