COMPACTNESS OF INTEGRAL OPERATORS AND UNIFORM INTEGRABILITY ON MEASURE SPACES

WOLFHARD HANSEN

Communicated by Lucian Beznea

Let $(E, \mathcal{E}, \mu)$ be a measure space and let $\mathcal{E}^+$, $\mathcal{E}_b$ denote the set of all measurable numerical functions on E which are positive, bounded respectively. Moreover, let $G\colon E \times E \to [0, \infty]$ be measurable. We show that the set of all $q \in \mathcal{E}^+$ for which $\{G(x, \cdot)q\colon x \in E\}$ is uniformly integrable coincides with the set of all $q \in \mathcal{E}^+$ for which the mapping $f \mapsto G(fq) := \int G(\cdot, y)f(y)q(y)\,\mathrm{d}\mu(y)$ is a compact operator on the space $\mathcal{E}_b$ (equipped with the sup-norm) provided each of these two sets contains strictly positive functions.

AMS 2020 Subject Classification: 31D05, 47B34, 47G10.

Key words: uniform integrability, compact operator, potential.

1. INTRODUCTION, NOTATION AND FIRST PROPERTIES

In the paper [2] on semilinear perturbation of fractional Laplacians $-(-\Delta)^{\alpha/2}$ on $\mathbb{R}^d$, d $\in \mathbb{N}$, $0 < \alpha < 2 \wedge d$, a Kato class $\mathcal{J}^\alpha(D)$ of measurable functions q on an open set D in $\mathbb{R}^d$ is defined by uniform integrability of the functions $G_D(x, \cdot)q$, $x \in D$, with respect to Lebesgue measure on D, where G_D denotes the corresponding Green function on D (see [2, Definition 1.23]). Let $\mathcal{C}_0(D)$, $\mathcal{B}_b(D)$, respectively, denote the space of all real functions on D which are continuous and vanish at infinity with respect to D, are Borel measurable and bounded, respectively.

Suppose that D is regular and let $q\colon D \to [0, \infty]$ be Borel measurable. Then, using the continuity of G_D and Vitali's theorem, it is established that, provided $q \in \mathcal{J}^\alpha(D)$, the mapping

$$K\colon f \mapsto G_D(fq) := \int G_D(\cdot, y)f(y)q(y)\,\mathrm{d}\mu \in \mathcal{C}_0(D)$$

is a compact operator on $\mathcal{B}_b(D)$ such that $K(\mathcal{B}_b(D)) \subset \mathcal{C}_0(D)$ (see the proof of Theorem 2.4 in [2]). In [2, Proposition 1.31] it is stated that, conversely, $q \in \mathcal{J}^\alpha(D)$ if (only) $K1 = G_D q \in \mathcal{C}_0(D)$.

On the other hand, it is easily seen that, as in the classical case $\alpha = 2$, compactness of K on $\mathcal{B}_b(D)$ implies that $G_D q$ is continuous, and hence

doi: 10.59277/RRMPA.2024.11.16

$G_D q \in \mathcal{C}_0(D)$ due to the regularity of D and the domination principle for K (see [7, Corollary 4.5] for bounded D and [1, Corollary 5.2,(b)] for the general case). So $q \in \mathcal{J}^\alpha(D)$ if and only if K is a compact operator on $\mathcal{B}_b(D)$. For some partial statements in the classical case see [4, Corollary to Proposition 3.1, Theorem 3.2].

In this paper, we fix a measure space $(E, \mathcal{E}, \mu)$ and a numerical function $G \geq 0$ on $E \times E$ which is $\mathcal{E} \otimes \mathcal{E}$-measurable. The purpose of this note is to establish that, even in this most general setting, for every measurable $q \geq 0$ on E, uniform integrability of $\{G(x, \cdot)q \colon x \in E\}$ is equivalent to compactness of the mapping $f \mapsto \int G(\cdot, y) f(y) q(y)\, \mathrm{d}\mu$ on the space of bounded measurable functions on E provided that there are strictly positive functions having these properties.

To be more specific, let $\mathcal{E}^+$, $\mathcal{E}_b$, respectively, denote the set of all $\mathcal{E}$-measurable numerical functions f on E such that $f \geq 0$, f is bounded, respectively. We define

$$(1) \qquad Gf(x) := \int G(x, y) f(y)\, \mathrm{d}\mu(y), \qquad f \in \mathcal{E}^+, \ x \in E.$$

If $q \in \mathcal{E}^+$ and Gq is bounded, then $K_q \colon f \mapsto G(fq)$ obviously is a bounded operator on $\mathcal{E}_b$, equipped with the sup-norm $\|\cdot\|_\infty$, such that $\|K_q\| = \|Gq\|_\infty$. So we are interested in the sets

$$\begin{aligned} \mathcal{F}_{co}(G) &:= \{q \in \mathcal{E}^+ \colon Gq \text{ is bounded and } K_q \text{ is a compact operator on } \mathcal{E}_b\}, \\ \mathcal{F}_{ui}(G) &:= \{q \in \mathcal{E}^+ \colon \{G(x, \cdot)q, \ x \in E\} \text{ is uniformly integrable}\}. \end{aligned}$$

Our main results are that $\mathcal{F}_{ui}(G) \subset \mathcal{F}_{co}(G)$ if there exists a strictly positive function in $\mathcal{F}_{co}(G)$ (Theorem 2.1), and $\mathcal{F}_{co}(G) \subset \mathcal{F}_{ui}(G)$ if there exists a strictly positive function in $\mathcal{F}_{ui}(G)$ (Theorem 2.2). In Section 3, we provide general examples (covering the situation in [2]), where the assumptions are satisfied.

Before studying the relation between $\mathcal{F}_{ui}(G)$ and $\mathcal{F}_{co}(G)$, let us recall that a subset $\mathcal{F}$ of $\mathcal{E}^+$ is called *uniformly integrable* if, for every $\varepsilon > 0$, there exists an integrable $g \in \mathcal{E}^+$ such that $\int_{\{f \geq g\}} f\, d\mu \leq \varepsilon$ for every $f \in \mathcal{F}$. Further, we note some simple facts, which are trivial for $\mathcal{F}_{ui}(G)$.

LEMMA 1.1. *Let $\mathcal{F} = \mathcal{F}_{ui}(G)$ or $\mathcal{F} = \mathcal{F}_{co}(G)$. The following hold:*

(1) *$\mathcal{F}$ is a convex cone and $G(1_{\{q=\infty\}} q) = 0$ for all $q \in \mathcal{F}$.*

(2) *If $q \in \mathcal{F}$ and $q' \in \mathcal{E}^+$ with $q' \leq q$, then $q' \in \mathcal{F}$.*

(3) *If $q \in \mathcal{E}^+$ and, for every $\varepsilon > 0$, there are $q' \in \mathcal{F}$ and $q'' \in \mathcal{E}^+$ with $q = q' + q''$ and $Gq'' \leq \varepsilon$, then $q \in \mathcal{F}$.*

Clearly, (1) holds for $\mathcal{F}_{co}(G)$ as well, and to get (2) it suffices to observe that taking $h := 1_{\{q>0\}}q'/q$, we have $G(fq') = G(fhq)$ for every $f \in \mathcal{E}_b$. For (3) it suffices to note that $\|K_{q''}\| = \|Gq''\|_\infty$ and every limit of compact operators on $\mathcal{E}_b$ is compact.

2. RELATION BETWEEN $\mathcal{F}_{co}(G)$ AND $\mathcal{F}_{ui}(G)$

THEOREM 2.1. *If there is a strictly positive function in $\mathcal{F}_{co}(G)$, then $\mathcal{F}_{co}(G)$ contains $\mathcal{F}_{ui}(G)$.*

Proof. Let $q_0 \in \mathcal{F}_{co}(G)$, $q_0 > 0$, and $q \in \mathcal{F}_{ui}(G)$. To prove $q \in \mathcal{F}_{co}(G)$ we may, by Lemma 1.1(1), assume that $q < \infty$. Let $\varepsilon > 0$ and let $g \in \mathcal{E}^+$ be integrable such that $A_x := \{G(x,\cdot)q > g\}$ satisfies

(2) $$\int_{A_x} G(x,\cdot)q\,d\mu < \varepsilon, \qquad x \in E.$$

Since $\{Mq_0 < q\} \downarrow \emptyset$ as $M \uparrow \infty$, there is $M \in \mathbb{N}$ such that, defining $A := \{Mq_0 < q\}$,

(3) $$\int_A g\,\mathrm{d}\mu < \varepsilon.$$

Let $q' := 1_{E\setminus A}q$ and $q'' := 1_A q$. Then $q = q' + q''$, $q' \le Mq_0$. By Lemma 1.1(2), $q' \in \mathcal{F}_{co}(G)$. Moreover, $G(x,\cdot)q \le g$ on $E \setminus A_x$. So, by (2) and (3),

$$Gq''(x) = \int_A G(x,\cdot)q\,\mathrm{d}\mu \le \int_{A_x} G(x,\cdot)q\,\mathrm{d}\mu + \int_{A\setminus A_x} g\,\mathrm{d}\mu < 2\varepsilon, \qquad x \in E.$$

By Lemma 1.1(3), the proof is finished. $\square$

THEOREM 2.2. *If there is a strictly positive function in $\mathcal{F}_{ui}(G)$, then $\mathcal{F}_{ui}(G)$ contains $\mathcal{F}_{co}(G)$.*

Proof. Let $q_0 \in \mathcal{F}_{ui}(G)$, $q_0 > 0$, $q \in \mathcal{F}_{co}(G)$ and $\varepsilon > 0$. By Lemma 1.1,(1), $G(1_{\{q>Mq_0\}}q) \downarrow 0$ pointwise as $M \uparrow \infty$, hence uniformly on E, by compactness of K_q. So there exists $M \in \mathbb{N}$ such that

(4) $$A := \{q > Mq_0\} \ \text{ satisfies } \ G(1_A q) \le \varepsilon.$$

By assumption, there is an integrable $g \in \mathcal{E}^+$ such that, for every $x \in E$,

(5) $$A_x := \{G(x,\cdot)q_0 \ge g\} \ \text{ satisfies } \ \int_{A_x} G(x,\cdot)q_0\,\mathrm{d}\mu \le \varepsilon/M.$$

Let us fix $x \in E$ and define $B_x := \{G(x,\cdot)q \ge Mg\}$. Then

$$B_x \setminus A \subset \{Mg \le G(x,\cdot)q \le G(x,\cdot)Mq_0\} \subset A_x,$$

and hence, by (5),

$$\int_{B_x\setminus A} G(x,\cdot)q\,\mathrm{d}\mu \le M\int_{A_x} G(x,\cdot)q_0\,\mathrm{d}\mu \le \varepsilon.$$

Thus $\int_{B_x} G(x,\cdot)q\,\mathrm{d}\mu \le 2\varepsilon$, by (4). Since Mg is integrable, the proof is finished. □

Corollary 2.3. *If both $\mathcal{F}_{ui}(G)$ and $\mathcal{F}_{co}(G)$ contain strictly positive functions, then $\mathcal{F}_{co}(G) = \mathcal{F}_{ui}(G)$.*

Let $G'\colon E\times E \to [0,\infty]$ be $\mathcal{E}\otimes\mathcal{E}$-measurable and suppose that $G' \le G$. Then, of course, $\mathcal{F}_{ui}(G) \subset \mathcal{F}_{ui}(G')$. So Theorems 2.2 and 2.1 also imply the following.

Corollary 2.4. *If both $\mathcal{F}_{ui}(G)$ and $\mathcal{F}_{co}(G')$ contain strictly positive functions, then $\mathcal{F}_{co}(G) \subset \mathcal{F}_{co}(G')$.*

Remark 2.5. Of course, the preceding results immediately yield corresponding statements for arbitrary $\mathcal{E}$-measurable numerical functions q on E using $q = q^+ - q^-$.

3. EXAMPLES IN POTENTIAL THEORY

3.1. First example

Let E be a Borel set in $\mathbb{R}^d$, $\mathrm{d} \ge 1$, let $\mathcal{E}$ be the σ-algebra of all Borel sets in E and μ be the restriction of Lebesgue measure on $(E,\mathcal{E})$. Further, let $G\colon E\times E\to[0,\infty]$ and $\varphi\colon [0,\infty)\to[0,\infty]$ be measurable such that

$$G(x,y) \le \varphi(|x-y|), \qquad x,y\in E,$$

and, for some $a, r, \in (0,\infty)$, $\varphi \le a$ on (r,∞) and $\int_0^r \varphi(t)t^{\mathrm{d}-1}\mathrm{d}t < \infty$.

Proposition 3.1.

$$\mathcal{F} := \{q\in\mathcal{E}^+ \colon q \text{ integrable},\ q\le 1,\ \lim_{|x|\to\infty} q(x) = 0\}$$

is contained in $\mathcal{F}_{ui}(G)$, and $\mathcal{F}_{co}(G)\subset\mathcal{F}_{ui}(G)$.

Proof. By Theorem 2.2, it clearly suffices to prove $\mathcal{F}\subset\mathcal{F}_{ui}(G)$. To that end, we may assume without loss of generality that

$$E = \mathbb{R}^d \quad\text{and}\quad G(x,y) = \varphi(|x-y|), \qquad x,y\in\mathbb{R}^d$$

(first extend G to $\mathbb{R}^d\times\mathbb{R}^d$ by $G(x,y) := 0$, if x or y are in the complement of E).

Let $q \in \mathcal{F}$. For $x \in E$ and $r > 0$, we define $B(x,r) := \{y \in E \colon |y-x| < r\}$. Defining $f_x := 1_{B(x,r)} G(x,\cdot)$, we then have

$$G(x,\cdot)q \le aq + f_x q, \qquad x \in E.$$

So it suffices to show that the functions $f_x q$, $x \in E$, are uniformly integrable.

Let $\varepsilon > 0$ and $b := 1 + \int f_0 \, d\mu$. Then $b < \infty$, by assumption on φ, and we may choose $R > 0$ such that $q \le \varepsilon/b$ on $B(0,R)^c$. If $x \in B(0,R+r)^c$, then $B(x,r) \cap B(0,R) = \emptyset$, and hence

$$\int f_x q \, \mathrm{d}\mu \le \frac{\varepsilon}{b} \int f_x \, \mathrm{d}\mu = \frac{\varepsilon}{b} \int f_0 \, \mathrm{d}\mu \le \varepsilon. \tag{6}$$

Suppose now that $x \in B(0,R+r)$, and hence $B(x,r) \subset B(0,R+2r)$. Let $M > a$ such that $\int_{\{f_0 > M\}} f_0 \, \mathrm{d}\mu < \varepsilon$ and $g := M 1_{B(0,R+2r)}$. Then

$$\int_{\{f_x q \ge g\}} f_x q \, \mathrm{d}\mu \le \int_{\{f_x \ge M\}} f_x \, \mathrm{d}\mu = \int_{\{f_0 \ge M\}} f_0 \, \mathrm{d}\mu < \varepsilon. \tag{7}$$

Thus, the functions $f_x q$, $x \in E$, are uniformly integrable. □

3.2. **Second example**

Let $(X, \mathcal{W})$ be a balayage space such that $\mathcal{W}$ contains a function $0 < w_0 \le 1$ (see [3, 5, 6, 7]), and let $G \colon X \times X \to [0,\infty]$ be Borel measurable such that, for every $y \in X$, $G(\cdot, y)$ is a potential on X which is harmonic on $X \setminus \{y\}$.

Let μ be a positive Radon measure on X and let $\mathcal{B}(X)$, $\mathcal{C}(X)$, respectively, denote the set of all Borel measurable numerical functions, continuous real functions, respectively, on X. We recall that, for every positive $f \in \mathcal{B}(X)$, the function $Gf := \int G(\cdot,y) f(y) \, \mathrm{d}\mu(y)$ is lower semicontinuous, by Fatou's Lemma.

Proposition 3.2. *If there exists $q \in \mathcal{B}(X)$, $q > 0$, such that $Gq \in \mathcal{C}(X)$, then $\mathcal{F}_{ui}(G) \subset \mathcal{F}_{co}(G)$.*

Proof. By Theorem 2.1, it suffices to find a strictly positive function $q_0 \in \mathcal{F}_{co}(G)$.

We choose compact sets L_n, $n \in \mathbb{N}$, covering X. Let $n \in \mathbb{N}$ and

$$q_n := 1_{L_n} q.$$

Since $Gq_n + G(1 - q_n) = Gq \in \mathcal{C}(X)$, we know that q_n is a continuous real potential. It is harmonic on $X \setminus L_n$. Let $a_n := \sup Gq_n(L_n) / \inf w_0(L_n)$.

Obviously, $Gq_n \leq a_n w_0$ on L_n, hence on X. So $Gq_n \leq a_n$ and, by [7, Proposition 4.1], $q_n \in \mathcal{F}_{co}(G)$. Using Lemma 1.1, we finally obtain that

$$q_0 := \sum\nolimits_{n\in\mathbb{N}} (a_n 2^n)^{-1} q_n \in \mathcal{F}_{co}(G).$$

□

REFERENCES

[1] K. Bogdan and W. Hansen, *Solution of semilinear equations by potential-theoretic methods.* In preparation.

[2] K. Bogdan, S. Jahros, and E. Kania, *Semilinear Dirichlet problem for the fractional Laplacian.* Nonlinear Anal. **193** (2020), Article ID 111512.

[3] J. Bliedtner and W. Hansen, *Potential Theory – An Analytic and Probabilistic Approach to Balayage.* Universitext. Springer, Berlin, Heidelberg, New York, Tokyo, 1986.

[4] K.L. Chung and Z. Zhao, *From Brownian motion to Schrödinger's equation.* Grundlehren der mathematischen Wissenschaften **312** (Corrected 2nd Printing 2001). Springer, Berlin-Heidelberg, 1995.

[5] W. Hansen, Balayage spaces – a natural setting for potential theory. Potential Theory - Surveys and Problems, Proceedings, Prague 1987. Lecture Notes **1344** (1987), 98–117.

[6] W. Hansen, *Three views on potential theory.* A course at Charles University (Prague), Spring 2008. http://www.karlin.mff.cuni.cz/ hansen/lecture/ course-07012009.pdf.

[7] W. Hansen, *Equicontinuity of harmonic functions and compactness of potential kernels.* Rev. Roumaine Math. Pures Appl. **66** (2021), *1*, 179–189.

Received 15 September 2022

Universität Bielefeld
Fakultät für Mathematik
33501 Bielefeld, Germany
hansen@math.uni-bielefeld.de