ALGEBRA ENVIRONMENTS I. GEOMETRIC AND TOPOLOGICAL STRUCTURES

MIRCEA MARTIN

Communicated by Dan Timotin

Algebra environments capture properties of non-commutative conditional expectations in a general algebraic setting. Their study relies on algebraic geometry, topology, and differential geometry techniques. The structure algebraic and Banach manifolds of algebra environments and their Zariski and smooth tangent vector bundles are particular objects of interest. A description of derivations on algebra environments compatible with geometric structures is an additional issue. Grassmann and flag manifolds of unital involutive algebras and spaces of projective compact group representations in C^*-algebras are analyzed as structure manifolds of associated algebra environments.

AMS 2020 Subject Classification: 08A05, 22C05, 22D20, 32K05.

Key words: algebraic and Banach manifolds, Zariski and smooth tangent spaces, principal fiber bundles, Ehresmann connections, projective compact group representations.

INTRODUCTION

The concept of algebra environments deconstructs non-commutative conditional expectations. We start their study based on an algebraic geometry approach and eventually refine the investigation by relying on topology and differential geometry techniques. The general results turn out to be useful in analyzing spaces of projective compact group representations in C^*-algebras from a geometric perspective. Specific examples include n-fold product, Clifford algebra, and cyclic group algebra environments.

Algebra environments provide frameworks with relevance in Clifford analysis, differential geometry, and operator theory. Though subsequently we will not elaborate on such issues, the last section of the article includes comments and references that would make the points.

The term algebra without other specifications refers to associative, distributive complex algebras. Definitions and results for real algebras are derived by making appropriate adjustments. We may assume that an algebra $\mathfrak{U}$ is unital with unit $1_{\mathfrak{U}}$, or involutive with an involution operation $*$. The norms of

REV. ROUMAINE MATH. PURES APPL. **69** (2024), *1*, 17–60
doi: 10.59277/RRMPA.2024.17.60

Banach algebras will be denoted by $\|\cdot\|$, with a few exceptions imposed by existing standards. We will be using Lie algebras too, for which the attributes associative, distributive, or involutive have their particular interpretations.

To motivate the concept of algebra environments, we recall the definition of non-commutative conditional expectations in different algebraic settings.

Definition (Non-commutative conditional expectations). Suppose $\mathfrak{E}$ is an algebra and $\mathfrak{A} \subseteq \mathfrak{E}$ a subalgebra. A linear operator $\Pi : \mathfrak{E} \to \mathfrak{E}$ with range $\mathrm{Ran}(\Pi) = \mathfrak{A}$ is called a non-commutative conditional expectation from $\mathfrak{E}$ onto $\mathfrak{A}$ provided

(i) $\Pi(a\alpha) = a\Pi(\alpha)$, $\Pi(\alpha a) = \Pi(\alpha)a$, $a \in \mathfrak{A}$, $\alpha \in \mathfrak{E}$,

(ii) $\Pi(1_{\mathfrak{E}}) = 1_{\mathfrak{A}}$, if $\mathfrak{E}$, $\mathfrak{A}$ are unital,

(iii) $\Pi(\alpha^*) = \Pi(\alpha)^*$, $\alpha \in \mathfrak{E}$, whenever $\mathfrak{E}$, $\mathfrak{A}$ are involutive,

(iv) $||\Pi(\alpha)|| \leq ||\alpha||$ and $\Pi(\alpha^*\alpha) \geq 0$, $\alpha \in \mathfrak{E}$, if $\mathfrak{E}$, $\mathfrak{A}$ are Banach $*$-algebras.

By design, algebra environments are close relatives of non-commutative conditional expectations. Anticipating Definition 1.1 in Section 1, we assume that $\Pi : \mathfrak{E} \to \mathfrak{A}$ is a linear mapping from an algebra $\mathfrak{E}$ into a unital algebra $\mathfrak{A}$, subject to two requirements,

(i) $\mathfrak{E}$ is a unital $\mathfrak{A}$-bimodule,

(ii) $\Pi : \mathfrak{E} \to \mathfrak{A}$ is an $\mathfrak{A}$-bilinear mapping,

and refer to $(\mathfrak{E}, \Pi, \mathfrak{A})$ as an environment with total algebra $\mathfrak{E}$, base algebra $\mathfrak{A}$, and environment projection Π. Additional assumptions yield special classes of environments. When $\mathfrak{E}$ and $\mathfrak{A}$ are involutive and Π preserves the involutions, $(\mathfrak{E}, \Pi, \mathfrak{A})$ is called an involutive environment. If $\mathfrak{E}$ and $\mathfrak{A}$ are Banach algebras and Π is continuous, $(\mathfrak{E}, \Pi, \mathfrak{A})$ is referred to as a Banach algebra environment.

For any environment $(\mathfrak{E}, \Pi, \mathfrak{A})$, we introduce an algebraic set $\mathcal{S}(\mathfrak{E}) \subseteq \mathfrak{E}$ consisting of idempotents of $\mathfrak{E}$ that satisfy a specific system of equations. The elements of $\mathcal{S}(\mathfrak{E})$ are called geometric $\mathfrak{E}$-structures on algebra $\mathfrak{A}$. If $(\mathfrak{E}, \Pi, \mathfrak{A})$ is involutive, the set $\mathcal{S}_*(\mathfrak{E}) \subseteq \mathcal{S}(\mathfrak{E})$ of self-adjoint structures is an algebraic set, too. Section 1 analyzes the structure manifolds $\mathcal{S}(\mathfrak{E})$ and $\mathcal{S}_*(\mathfrak{E})$ by relying on algebraic geometry concepts. The goal is to describe the Zariski tangent spaces $\mathrm{T}^{\mathrm{alg}}_{\alpha}\,\mathcal{S}(\mathfrak{E})$, $\alpha \in \mathcal{S}(\mathfrak{E})$, and $\mathrm{T}^{\mathrm{alg}}_{\alpha}\,\mathcal{S}_*(\mathfrak{E})$, $\alpha \in \mathcal{S}_*(\mathfrak{E})$. This is accomplished by assigning to each structure $\alpha \in \mathcal{S}(\mathfrak{E})$ a linear mapping $\Sigma_\alpha : \mathfrak{E} \to \mathfrak{A}$ called the symbol operator, that will serve as an important tool for many other purposes. The main result is stated as Theorem A in Subsection 1.3. An application concerning complex structures on Grassmann and flag manifolds of involutive unital algebras is outlined in Subsection 1.4. In addition, associated with each $\alpha \in \mathcal{S}(\mathfrak{E})$, we construct an involution $\Gamma_\alpha : \mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A}) \to \mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A})$ on the Lie algebra $\mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A})$ of derivations on $(\mathfrak{E}, \Pi, \mathfrak{A})$, and show that its eigenspace corresponding to $\lambda = 1$ is the Lie subalgebra of derivations compatible with α.

Section 2 develops a differential geometry approach. Based on Section 1, we prove that the structure manifolds $\mathcal{S}(\mathfrak{E})$ and $\mathcal{S}_*(\mathfrak{E})$ of an involutive Banach algebra environment $(\mathfrak{E}, \Pi, \mathfrak{A})$ are Banach manifolds, and their connected components are base spaces of principal fiber bundles defined in terms of base algebra $\mathfrak{A}$ and geometric $\mathfrak{E}$-structures on $\mathfrak{A}$. The symbol operators $\Sigma_\alpha : \mathfrak{E} \to \mathfrak{A}$ prove critical in introducing Ehresmann connections on such principal fiber bundles. Smooth curves $\gamma : \mathbb{R} \to \mathcal{S}(\mathfrak{E})$ have local horizontal lifts that will be used to define and characterize geodesics. We prove that when $\mathfrak{A}$ is a C^*-algebra, a geodesic path $\gamma_0 : [0, \tau] \to \mathcal{S}_*(\mathfrak{E})$ with length $\mathrm{L}(\gamma_0) < \pi$ is minimal compared with smooth curves $\gamma : [0, \tau] \to \mathcal{S}_*(\mathfrak{E})$ with the same endpoints.

Section 3 analyzes the spaces $\mathcal{R}(G, \varepsilon, \mathfrak{A})$ and $\mathcal{R}_*(G, \varepsilon, \mathfrak{A})$ of continuos projective representations, or unitary representations, of a compact group G with a two-cocycle ε into a unital C^*-algebra $\mathfrak{A}$. We define an involutive Banach algebra environment $\mathfrak{E}[\,G, \varepsilon, \mathfrak{A}\,]$ and prove that the two spaces of representations are the structure manifolds of $\mathfrak{E}[\,G, \varepsilon, \mathfrak{A}\,]$. This result enables us to access techniques developed in Sections 1 and 2 and derive several consequences. For instance, we show that each continuous representation is similar to a unitary representation and $\mathcal{R}_*(G, \varepsilon, \mathfrak{A})$ is a deformation retract of $\mathcal{R}(G, \varepsilon, \mathfrak{A})$.

Section 4 includes comments and appropriate references underlining work done by many authors who developed lines of research with noteworthy consequences. Part of our results and proofs were motivated by–and benefitted from–their insightful contributions. Several applications of algebra environments and a list of related specific issues will be referred to, as well.

1. **ALGEBRA ENVIRONMENTS–GEOMETRIC APPROACH**

Algebra environments provide a suitable framework for implementing an algebraic geometry approach. As specific outcomes, we define two algebraic sets called structure manifolds of algebra environments, describe their Zariski tangent spaces, set up several appropriate tools, and determine the derivations on algebra environments compatible with prescribed geometric structures.

1.1. **Algebra environments–Definitions**

We start by introducing the category of algebra environments, its objects and homomorphisms, and two subcategories with additional properties.

Definition 1.1. An algebra environment $(\mathfrak{E}, \Pi, \mathfrak{A})$ consists of two algebras $\mathfrak{E}$ and $\mathfrak{A}$, called the total and base algebras, and a linear mapping $\Pi : \mathfrak{E} \to \mathfrak{A}$ called the environment projection, subject to the requirements

(i) base algebra $\mathfrak{A}$ is unital with unit $1_{\mathfrak{A}}$,

(ii) total algebra $\mathfrak{E}$ is a unital $\mathfrak{A}$-bimodule,

(iii) $\Pi : \mathfrak{E} \to \mathfrak{A}$ is an $\mathfrak{A}$-bilinear mapping.

Whenever convenient, we refer to $(\mathfrak{E}, \Pi, \mathfrak{A})$ as environment $\mathfrak{E}$ and regard $\mathfrak{A}$ and Π as implicitly related objects. The elements of $\mathfrak{A}$ will be denoted by $a, b, \ldots, x, y, \ldots$, and the elements of $\mathfrak{E}$ by $\alpha, \beta, \ldots, \varphi, \psi, \ldots$. The products of $x, y \in \mathfrak{A}$, or $\varphi, \psi \in \mathfrak{E}$, are denoted by xy and $\varphi \times \psi$, respectively. The left and right products of $a \in \mathfrak{A}$ and $\alpha \in \mathfrak{E}$ are expressed as $a \cdot \alpha$ and $\alpha \cdot a$. For $a \in \mathfrak{A}$ and α, $\beta \in \mathfrak{E}$, requirement (ii) includes the properties $1_{\mathfrak{A}} \cdot \alpha = \alpha \cdot 1_{\mathfrak{A}} = \alpha$, and

$$a \cdot (\alpha \times \beta) = (a \cdot \alpha) \times \beta, \;\; (\alpha \times \beta) \cdot a = \alpha \times (\beta \cdot a), \;\; (\alpha \cdot a) \times \beta = \alpha \times (a \cdot \beta).$$

Definition 1.2. Let $(\mathfrak{E}, \Pi_{\mathfrak{E},\mathfrak{A}}, \mathfrak{A})$ and $(\mathfrak{F}, \Pi_{\mathfrak{F},\mathfrak{B}}, \mathfrak{B})$ be two environments. An environment homomorphism from $(\mathfrak{E}, \Pi_{\mathfrak{E},\mathfrak{A}}, \mathfrak{A})$ to $(\mathfrak{F}, \Pi_{\mathfrak{F},\mathfrak{B}}, \mathfrak{B})$ is a pair (Θ, Θ_0), with $\Theta : \mathfrak{E} \to \mathfrak{F}$ and $\Theta_0 : \mathfrak{A} \to \mathfrak{B}$ algebra homomorphisms such that

(i) $\Theta_0(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$,

(ii) $\Theta(a \cdot \alpha) = \Theta_0(a) \cdot \Theta(\alpha)$, $\Theta(\alpha \cdot a) = \Theta(\alpha) \cdot \Theta_0(a)$, $a \in \mathfrak{A}$, $\alpha \in \mathfrak{E}$,

(iii) $\Theta_0 \circ \Pi_{\mathfrak{E},\mathfrak{A}} = \Pi_{\mathfrak{F},\mathfrak{B}} \circ \Theta$.

Definition 1.3. An environment $(\mathfrak{E}, \Pi, \mathfrak{A})$ is involutive provided the total and base algebras $\mathfrak{E}$ and $\mathfrak{A}$ are involutive,

(i) $(a \cdot \alpha)^* = \alpha^* \cdot a^*$, $(\alpha \cdot a)^* = a^* \cdot \alpha^*$, $a \in \mathfrak{A}$, $\alpha \in \mathfrak{E}$,

(ii) $\Pi(\alpha^*) = \Pi(\alpha)^*$, $\alpha \in \mathfrak{E}$.

Definition 1.4. An environment $(\mathfrak{E}, \Pi, \mathfrak{A})$ is a Banach algebra environment if both $\mathfrak{E}$ and $\mathfrak{A}$ are Banach algebras,

(i) $\|a \cdot \alpha\| \leq \|a\| \, \|\alpha\|$, $\|\alpha \cdot a\| \leq \|\alpha\| \, \|a\|$, $a \in \mathfrak{A}$, $\alpha \in \mathfrak{E}$,

(ii) $\Pi : \mathfrak{E} \to \mathfrak{A}$ is continuous.

For each subcategory, the environment homomorphisms are consistent with the additional structures of the total and base algebras.

1.2. **Stucture manifolds**

Definition 1.5. The structure manifold of an environment $(\mathfrak{E}, \Pi, \mathfrak{A})$ is the set $\mathcal{S}(\mathfrak{E}) = S(\mathfrak{E}, \Pi, \mathfrak{A})$ of elements $\alpha \in \mathfrak{E}$ that satisfy the algebraic equations

(i) $\Pi(\alpha) = 1_{\mathfrak{A}}$,

(ii) $\alpha \times \alpha = \alpha$,

(iii) $\Pi(\varphi \times \alpha) \cdot \alpha = \varphi \times \alpha$, $\varphi \in \mathfrak{E}$,

(iv) $\alpha \cdot \Pi(\alpha \times \psi) = \alpha \times \psi$, $\psi \in \mathfrak{E}$,

(v) $\Pi(\varphi \times \alpha)\, \Pi(\alpha \times \psi) = \Pi(\varphi \times \alpha \times \psi)$, φ, $\psi \in \mathfrak{E}$.

If $(\mathfrak{E}, \Pi, \mathfrak{A})$ is involutive, $\mathcal{S}_*(\mathfrak{E}) = \mathcal{S}_*(\mathfrak{E}, \Pi, \mathfrak{A})$ is defined by also assuming that

(vi) $\alpha^* = \alpha$.

The elements $\alpha \in \mathcal{S}(\mathfrak{E})$ are called geometric $\mathfrak{E}$-structures on base algebra $\mathfrak{A}$.

We note that the list of requirements in Definition 1.5 is not minimal. For instance, instead of (i) and (ii), we may require $\Pi(\alpha \times \alpha) = 1_{\mathfrak{A}}$ and from this derive (i) and (ii) based on (iii) or (iv). Moreover, property (v) is a direct consequence of either (iii) or (iv). For involutive environments, if $\alpha \in \mathcal{S}(\mathfrak{E})$ then $\alpha^* \in \mathcal{S}(\mathfrak{E})$, and assumptions (iii) and (iv) equivalent. Actually, for some classes of algebra environments the algebraic sets $\mathcal{S}(\mathfrak{E})$ and $\mathcal{S}_*(\mathfrak{E})$ are completely defined by requirements (i), (v), or (i), (v), and (vi), respectively.

To each $\alpha \in \mathcal{S}(\mathfrak{E})$, we associate a subalgebra $\mathfrak{A}_\alpha$ of base algebra $\mathfrak{A}$ by

(1.1) $$\mathfrak{A}_\alpha = \{\, x \in \mathfrak{A} : x \cdot \alpha = \alpha \cdot x \,\},$$

and the linear mapping $\pi_\alpha : \mathfrak{A} \to \mathfrak{A}$ defined as

(1.2) $$\pi_\alpha(x) = \Pi(\alpha \cdot x \times \alpha) = \Pi(\alpha \times x \cdot \alpha), \quad x \in \mathfrak{A}.$$

If $(\mathfrak{E}, \Pi, \mathfrak{A})$ is an involutive environment and $\alpha \in \mathcal{S}_*(\mathfrak{E})$, subalgebra $\mathfrak{A}_\alpha$ is involutive and $\pi_\alpha(x^*) = \pi_\alpha(x)^*$, $x \in \mathfrak{A}$.

PROPOSITION 1.6. *Operator $\pi_\alpha : \mathfrak{A} \to \mathfrak{A}$ is a projection with range $\mathfrak{A}_\alpha$,*

(1.3) $$\pi_\alpha^2 = \pi_\alpha, \quad \mathrm{Ran}(\pi_\alpha) = \mathfrak{A}_\alpha,$$

such that

(1.4) $$\pi_\alpha(1_{\mathfrak{A}}) = 1_{\mathfrak{A}},$$

(1.5) $$\pi_\alpha(x\, a) = x\, \pi_\alpha(a), \quad \pi_\alpha(a\, x) = \pi_\alpha(a)\, x, \quad x \in \mathfrak{A}_\alpha,\ a \in \mathfrak{A}.$$

Essentially, Proposition 1.6 points out that each $\pi_\alpha : \mathfrak{A} \to \mathfrak{A},\ \alpha \in \mathcal{S}(\mathfrak{E})$, is a non-commutative conditional expectation from $\mathfrak{A}$ onto $\mathfrak{A}_\alpha$.

Proof. Requirements (iii) and (iv) in Definition 1.5 imply

$$\pi_\alpha(x) \cdot \alpha = \Pi(\alpha \cdot x \times \alpha) \cdot \alpha = \alpha \cdot x \times \alpha = \alpha \times x \cdot \alpha = \alpha \cdot \Pi(\alpha \times x \cdot \alpha) = \alpha \cdot \pi_\alpha(x),$$

for any $x \in \mathfrak{A}$, and consequently,

$$\pi_\alpha(x) \in \mathfrak{A}_\alpha,\ \ x \in \mathfrak{A},\ \text{ i.e., } \operatorname{Ran}(\pi_\alpha) \subseteq \mathfrak{A}_\alpha. \tag{1.6}$$

Moreover, if $x \in \mathfrak{A}_\alpha$, then

$$\pi_\alpha(x) = \Pi(\alpha \cdot x \times \alpha) = \Pi(x \cdot \alpha \times \alpha) = \Pi(x \cdot \alpha) = x\,\Pi(\alpha) = x. \tag{1.7}$$

Statement (1.3) follows from (1.6) and (1.7). The proof of (1.4) reduces to

$$\pi_\alpha(1_\mathfrak{A}) = \Pi(\alpha \cdot 1_\mathfrak{A} \times \alpha) = \Pi(\alpha \times \alpha) = \Pi(\alpha) = 1_\mathfrak{A}.$$

In its turn, (1.5) follows from

$$\pi_\alpha(x\,a) = \Pi(\alpha \cdot x\,a \times \alpha) = \Pi(x \cdot \alpha \cdot a \times \alpha) = x\,\Pi(\alpha \cdot a \times \alpha) = x\,\pi_\alpha(a),$$

and

$$\pi_\alpha(a\,x) = \Pi(\alpha \times a\,x \cdot \alpha) = \Pi(\alpha \times a \cdot \alpha \cdot x) = \Pi(\alpha \times a \cdot \alpha)\,x = \pi_\alpha(a)\,x.$$

The proof is complete. □

We note that each subalgebra $\mathfrak{A}_\alpha,\ \alpha \in \mathcal{S}(\mathfrak{E})$, has a direct complement, the subspace $\mathfrak{A}_\alpha^\perp \subseteq \mathfrak{A}$ given by

$$\mathfrak{A}_\alpha^\perp = \{x \in \mathfrak{A} : \pi_\alpha(x) = 0\}, \tag{1.8}$$

range of the complement projection $\pi_\alpha^\perp : \mathfrak{A} \to \mathfrak{A},\ \pi_\alpha^\perp = \mathrm{Id}_\mathfrak{A} - \pi_\alpha$.

1.3. **Zariski tangent spaces**

We are going to define the Zariski tangent spaces $\mathrm{T}_\alpha^{\mathrm{alg}}\,\mathcal{S}(\mathfrak{E})$ and $\mathrm{T}_\alpha^{\mathrm{alg}}\,\mathcal{S}_*(\mathfrak{E})$ to the algebraic manifolds $\mathcal{S}(\mathfrak{E})$ and $\mathcal{S}_*(\mathfrak{E})$ at $\alpha \in \mathcal{S}(\mathfrak{E})$, or $\alpha \in \mathcal{S}_*(\mathfrak{E})$. Algebraic geometry textbooks, as for instance Munford [53], provide explicit descriptions of Zariski tangent spaces to algebraic sets. We will just do what Pierre de Fermat did, and rely on an approach that uses dual numbers. The algebra of complex dual numbers is $\Lambda^\#(\mathbb{C}) = \Lambda^0(\mathbb{C}) \oplus \Lambda^1(\mathbb{C})$, the exterior algebra of $\mathbb{C}$ with the usual structures, including an involution and a distinguished element $\delta \in \Lambda^1(\mathbb{C})$, the dual unit, with the properties:

$$\delta^* = \delta, \quad \delta^2 = 0. \tag{1.9}$$

The set $\{1,\delta\}$, $1 \in \Lambda^0(\mathbb{C}) \equiv \mathbb{C}$, $\delta \in \Lambda^1(\mathbb{C}) \equiv \mathbb{C}\delta$, provides a linear basis for $\Lambda^{\#}(\mathbb{C}) \equiv \mathbb{C}[\delta] = \mathbb{C} + \mathbb{C}\delta = \{\zeta + \eta\,\delta : \zeta,\eta \in \mathbb{C}\}$.

We next associate an augmentation $\mathfrak{U}[\delta]$ to any algebra $\mathfrak{U}$ defined as

$$\mathfrak{U}[\delta] = \mathfrak{U} \otimes \mathbb{C}[\delta] = \mathfrak{U} + \mathfrak{U}\,\delta = \{\, u + v\,\delta : u,\, v \in \mathfrak{U}\,\},$$

with natural addition, and multiplication consistent with (1.9). Consequently, an algebra environment $(\mathfrak{E},\Pi,\mathfrak{A})$ has an extension $(\mathfrak{E}[\delta],\Pi_{[\delta]},\mathfrak{A}[\delta])$, where the elements of $\mathfrak{E}[\delta]$ are of the form $\alpha + \theta\,\delta$, $\alpha,\,\theta \in \mathfrak{E}$, and

$$\Pi_{[\delta]}(\alpha + \theta\,\delta) = \Pi(\alpha) + \Pi(\theta)\,\delta \in \mathfrak{A}[\delta]. \tag{1.10}$$

The unit of base algebra $\mathfrak{A}[\delta]$ is $1_{\mathfrak{A}}$. If $(\mathfrak{E},\Pi,\mathfrak{A})$ is an involutive environment, the extension $(\mathfrak{E}[\delta],\Pi_{[\delta]},\mathfrak{A}[\delta])$ is involutive, too.

Definition 1.7. Let $\mathfrak{E}$ be an environment with structure manifold $\mathcal{S}(\mathfrak{E})$.

(i) The Zariski tangent space $\mathrm{T}^{\mathrm{alg}}_{\alpha}\mathcal{S}(\mathfrak{E})$ to $\mathcal{S}(\mathfrak{E})$ at $\alpha \in \mathcal{S}(\mathfrak{E})$ consists of all elements $\theta \in \mathfrak{E}$ such that $\alpha + \theta\,\delta \in \mathcal{S}(\mathfrak{E}[\delta])$, the structure manifold associated with the augmented environment $(\mathfrak{E}[\delta],\Pi_{[\delta]},\mathfrak{A}[\delta])$.

(ii) For an involutive algebra environment, the Zariski tangent space $\mathrm{T}^{\mathrm{alg}}_{\alpha}\mathcal{S}_*(\mathfrak{E})$ to $\mathcal{S}_*(\mathfrak{E})$ at $\alpha \in \mathcal{S}_*(\mathfrak{E})$ is defined by assuming that $\alpha + \theta\,\delta \in \mathcal{S}_*(\mathfrak{E}[\delta])$.

We refer to $\theta \in \mathrm{T}^{\mathrm{alg}}_{\alpha}\mathcal{S}(\mathfrak{E})$ as tangent $\mathfrak{E}$-structures on $\mathfrak{A}$ at α.

Lemma 1.8. *If $\alpha \in \mathcal{S}(\mathfrak{E})$, then $\theta \in \mathrm{T}^{\mathrm{alg}}_{\alpha}\mathcal{S}(\mathfrak{E})$ only if*

(i) $\Pi(\theta) = 0$,

(ii) $\theta \times \alpha + \alpha \times \theta = \theta$,

(iii) $\Pi(\varphi \times \theta)\cdot\alpha + \Pi(\varphi \times \alpha)\cdot\theta = \varphi \times \theta,\ \varphi \in \mathfrak{E}$,

(iv) $\theta\cdot\Pi(\alpha \times \psi) + \alpha\cdot\Pi(\theta \times \psi) = \theta \times \psi,\ \psi \in \mathfrak{E}$,

(v) $\Pi(\varphi \times \theta)\,\Pi(\alpha \times \psi) + \Pi(\varphi \times \alpha)\Pi(\theta \times \psi) = \Pi(\varphi \times \theta \times \psi),\ \varphi,\psi \in \mathfrak{E}$.

If $(\mathfrak{E},\Pi,\mathfrak{A})$ involutive and $\alpha \in \mathcal{S}_*(\mathfrak{E})$, then $\theta \in \mathrm{T}^{\mathrm{alg}}_{\alpha}\mathcal{S}_*(\mathfrak{E})$ provided, in addition,

(vi) $\theta^* = \theta$.

Proof. The proof of property (i) follows from requirement (i) in Definition 1.5 and equation (1.10). Assuming that $\Pi_{[\delta]}(\alpha + \theta\,\delta) = 1_{\mathfrak{A}}$, we get

$$1_{\mathfrak{A}} = \Pi(\alpha) + \Pi(\theta)\delta = 1_{\mathfrak{A}} + \Pi(\theta)\delta,$$

i.e., $\Pi(\theta) = 0$. Property (ii) is a consequence of $(\alpha + \theta\delta) \times (\alpha + \theta\delta) = \alpha + \theta\delta$, which is requirement (ii) in Definition 1.5. The other properties are derived from requirements (iii)–(vi) in a similar way. $\square$

The comments made right after Definition 1.5 indicate that the list of properties in Lemma 1.8 is redundant. Eventually, we will give some simpler and more reliable descriptions of $\mathrm{T}_\alpha^{\mathrm{alg}}\mathcal{S}(\mathfrak{E})$ and $\mathrm{T}_\alpha^{\mathrm{alg}}\mathcal{S}_*(\mathfrak{E})$ as main results in this section.

Definition 1.9. A derivation on an algebra $\mathfrak{U}$ is a linear mapping $\mathcal{D} : \mathfrak{U} \to \mathfrak{U}$ with the property $\mathcal{D}(uv) = \mathcal{D}(u)v + u\mathcal{D}(v)$, $u, v \in \mathfrak{U}$. If $\mathfrak{U}$ is involutive, we may assume that $\mathcal{D}$ is involution preserving, i.e., $\mathcal{D}(u^*) = \mathcal{D}(u)^*$, $u \in \mathfrak{U}$.

The space of derivations on $\mathfrak{U}$, denoted by $\mathrm{Der}(\mathfrak{U})$, is a Lie algebra with the Lie product $[\,\cdot\,,\cdot\,]$ given by the commutator of two derivations, i.e.,

$$[\,\mathcal{D}, \mathcal{D}'\,] = \mathcal{D}\,\mathcal{D}' - \mathcal{D}'\,\mathcal{D}, \quad \mathcal{D},\, \mathcal{D}' \in \mathrm{Der}(\mathfrak{U}).$$

The subspace $\mathrm{Der}_*(\mathfrak{U})$ of involution preserving derivations is a Lie subalgebra.

Definition 1.10. A derivation on an environment $(\mathfrak{E}, \Pi, \mathfrak{A})$ is a pair $(\mathcal{D},\mathcal{D}_0)$, where $\mathcal{D} \in \mathrm{Der}(\mathfrak{E})$ and $\mathcal{D}_0 \in \mathrm{Der}(\mathfrak{A})$, such that

$$\mathcal{D}(a \cdot \varphi) = \mathcal{D}_0(a) \cdot \varphi + a \cdot \mathcal{D}(\varphi), \quad \mathcal{D}(\varphi \cdot a) = \mathcal{D}(\varphi) \cdot a + \varphi \cdot \mathcal{D}_0(a),$$

for all $a \in \mathfrak{A}$ and $\varphi \in \mathfrak{E}$, and

$$\Pi(\mathcal{D}(\varphi)) = \mathcal{D}_0(\Pi(\varphi)), \quad \varphi \in \mathfrak{E}.$$

A derivation $(\mathcal{D}, \mathcal{D}_0)$ on an involutive environment is involution preserving provided the derivations $\mathcal{D}$ and $\mathcal{D}_0$ are involution preserving.

In case the environment projection $\Pi : \mathfrak{E} \to \mathfrak{A}$ is surjective, Definition 1.10 implies that $\mathcal{D}_0$ is uniquely determined by $\mathcal{D}$. The space $\mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A})$ of derivations on $(\mathfrak{E}, \Pi, \mathfrak{A})$ is a Lie algebra with the Lie product given by

$$[(\mathcal{D}, \mathcal{D}_0), (\mathcal{D}', \mathcal{D}_0')] = ([\mathcal{D}, \mathcal{D}'], [\mathcal{D}_0, \mathcal{D}_0']), \quad (\mathcal{D}, \mathcal{D}_0), (\mathcal{D}', \mathcal{D}_0') \in \mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A}).$$

The subspace $\mathrm{Der}_*(\mathfrak{E}, \Pi, \mathfrak{A})$ of involution preserving derivations on an involutive environment is a Lie subalgebra.

LEMMA 1.11. *Let $(\mathcal{D}, \mathcal{D}_0)$ be a derivation on $(\mathfrak{E}, \Pi, \mathfrak{A})$:*

(i) *If $\alpha \in \mathcal{S}(\mathfrak{E})$, then $\theta = \mathcal{D}(\alpha) \in \mathrm{T}_\alpha^{\mathrm{alg}}\, \mathcal{S}(\mathfrak{E})$.*

(ii) *If $(\mathfrak{E}, \Pi, \mathfrak{A})$ is involutive, $(\mathcal{D}, \mathcal{D}_0)$ is involution preserving, and $\alpha \in \mathcal{S}_*(\mathfrak{E})$, then $\theta = \mathcal{D}(\alpha) \in \mathrm{T}_\alpha^{\mathrm{alg}}\, \mathcal{S}_*(\mathfrak{E})$.*

Proof. We have to show that $\theta = \mathcal{D}(\alpha)$ satisfies the equations in Lemma 1.8. All we need is to apply $(\mathcal{D}, \mathcal{D}_0)$ to each equation in Definition 1.5. Equation (i) in Lemma 1.8, for instance, follows from

$$\Pi(\theta) = \Pi(\mathcal{D}(\alpha)) = \mathcal{D}_0(\Pi(\alpha)) = \mathcal{D}_0(1_{\mathfrak{A}}) = 0.$$

The proof of equation (ii) is elementary too, and goes as follows,

$$\theta \times \alpha + \alpha \times \theta = \mathcal{D}(\alpha) \times \alpha + \alpha \times \mathcal{D}(\alpha) = \mathcal{D}(\alpha \times \alpha) = \mathcal{D}(\alpha) = \theta.$$

The other equations, (iii)–(vi), have similar straightforward proofs. $\square$

Actually, Lemma 1.11 yields the tangent spaces $\mathrm{T}^{\mathrm{alg}}_{\alpha}\,\mathcal{S}(\mathfrak{E})$ and $\mathrm{T}^{\mathrm{alg}}_{\alpha}\,\mathcal{S}_*(\mathfrak{E})$ in their entirety. To justify this expected claim, we need a few new tools.

Each $x \in \mathfrak{A}$ defines the inner derivation $\mathcal{D}_{0,x} = [\,x\,,\cdot\,] \in \mathrm{Der}(\mathfrak{A})$ given by $\mathcal{D}_{0,x}(a) = [\,x\,,a\,] = x\,a - a\,x$, $a \in \mathfrak{A}$, as well as a derivation $\mathcal{D}_x = [\,x\,,\cdot\,] \in \mathrm{Der}(\mathfrak{E})$ defined by $\mathcal{D}_x(\varphi) = [\,x\,,\varphi\,] = x\,\cdot\varphi - \varphi\,\cdot\,x$, $\varphi \in \mathfrak{E}$. We refer to the so defined pair $(\mathcal{D}_x, \mathcal{D}_{0,x})$, which is a derivation on $(\mathfrak{E}, \Pi, \mathfrak{A})$, as the inner derivation associated with $x \in \mathfrak{A}$. We note that $(\mathcal{D}_x, \mathcal{D}_{0,x}) \in \mathrm{Der}_*(\mathfrak{E}, \Pi, \mathfrak{A})$ if and only if $x^* = -x$, i.e., $x \in \mathfrak{A}_{\mathrm{sh}}$, the subspace of skew-hermitian elements of $\mathfrak{A}$. Related to inner derivations on algebra environments we define the following two linear mappings,

$$\text{(1.11)}\qquad \mathfrak{D}_0 : \mathfrak{A} \to \mathrm{Der}(\mathfrak{A}),\ \mathfrak{D}_0(x) = \mathcal{D}_{0,x}, \quad \mathfrak{D} : \mathfrak{A} \to \mathrm{Der}(\mathfrak{E}),\ \mathfrak{D}(x) = \mathcal{D}_x.$$

The next definition introduces a new concept, quite useful in analyzing structure manifolds of algebra environments and their Zariski tangent spaces.

Definition 1.12. Let $(\mathfrak{E}, \Pi, \mathfrak{A})$ be an algebra environment and $\alpha \in \mathcal{S}(\mathfrak{E})$. The associated symbol operator is defined by

$$\text{(1.12)}\qquad \Sigma_\alpha : \mathfrak{E} \to \mathfrak{A}, \quad \Sigma_\alpha(\varphi) = 2^{-1}\,\Pi(\varphi \times \alpha - \alpha \times \varphi), \quad \varphi \in \mathfrak{E}.$$

For an involutive environment $(\mathfrak{E}, \Pi, \mathfrak{A})$ and $\alpha \in \mathcal{S}_*(\mathfrak{E})$, the symbol operator is

$$\text{(1.13)}\qquad \Sigma_{\alpha,*} : \mathfrak{E}_{\mathrm{h}} \to \mathfrak{A}, \quad \Sigma_{\alpha,*} = \Sigma_\alpha | \mathfrak{E}_{\mathrm{h}}, \quad \mathfrak{E}_{\mathrm{h}} = \{\,\varphi \in \mathfrak{E} : \varphi^* = \varphi\,\}.$$

Before proceeding with a technical result, we refer to equations (1.1), (1.2), (1.8) in Subsection 1.2 for definitions of subalgebra $\mathfrak{A}_\alpha \subseteq \mathfrak{A}$, conditional expectation $\pi_\alpha : \mathfrak{A} \to \mathfrak{A}$ with range $\mathfrak{A}_\alpha$, and the direct complement $\mathfrak{A}^{\perp}_{\alpha} = \mathfrak{A} \ominus \mathfrak{A}_\alpha$.

Proposition 1.13. *Let* $(\mathfrak{E}, \Pi, \mathfrak{A})$ *be an environment and* $\alpha \in \mathcal{S}(\mathfrak{E})$. *The symbol operator* $\Sigma_\alpha : \mathfrak{E} \to \mathfrak{A}$ *has the following properties:*

(i) $\mathrm{Ran}\,\Sigma_\alpha = \mathfrak{A}^{\perp}_{\alpha}$.

(ii) *If* $\theta \in \mathrm{T}^{\mathrm{alg}}_{\alpha}\,\mathcal{S}(\mathfrak{E})$ *and* $x = \Sigma_\alpha(\theta)$, *then* $\theta = \mathfrak{D}(x)(\alpha)$.

If $(\mathfrak{E}, \Pi, \mathfrak{A})$ *is involutive and* $\alpha \in \mathcal{S}_*(\mathfrak{E})$, *then*

(iii) $\mathrm{Ran}\,\Sigma_{\alpha,*} = \mathfrak{A}^{\perp}_{\alpha} \cap \mathfrak{A}_{\mathrm{sh}}$, $\quad \mathfrak{A}_{\mathrm{sh}} = \{\,x \in \mathfrak{A} : x^* = -x\,\}$.

(iv) *If* $\theta \in \mathrm{T}^{\mathrm{alg}}_{\alpha}\,\mathcal{S}_*(\mathfrak{E})$ *and* $x = \Sigma_{\alpha,*}(\theta)$, *then* $\theta = \mathfrak{D}(x)(\alpha)$.

Proof. We rely on several previous definitions and results. Suppose $\varphi \in \mathfrak{E}$ and let $x = \Sigma_\alpha(\varphi) \in \mathfrak{A}$. Using equation (1.12) and requirements (iii) and (iv) in Definition 1.5, we get

$$\begin{aligned} \alpha \times x \cdot \alpha &= 2^{-1} \left[\alpha \times \Pi(\varphi \times \alpha) \cdot \alpha - \alpha \cdot \Pi(\alpha \times \varphi) \times \alpha \right] \\ &= 2^{-1} \left(\alpha \times \varphi \times \alpha - \alpha \times \varphi \times \alpha \right) = 0, \end{aligned}$$

hence $\pi_\alpha(x) = \Pi(\alpha \times x \cdot \alpha) = 0$, i.e., $x \in \mathfrak{A}_\alpha^\perp$. Consequently, $\operatorname{Ran}(\Sigma_\alpha) \subseteq \mathfrak{A}_\alpha^\perp$. Next, choose an arbitrary $x \in \mathfrak{A}_\alpha^\perp$ and let $\varphi = x \cdot \alpha - \alpha \cdot x \in \mathfrak{E}$. Observe that

$$\varphi \times \alpha - \alpha \times \varphi = (x \cdot \alpha - \alpha \cdot x) \times \alpha - \alpha \times (x \cdot \alpha - \alpha \cdot x) = x \cdot \alpha - 2\, \alpha \cdot x \times \alpha + \alpha \cdot x.$$

Therefore, since $\Pi(\alpha) = 1$, we have

$$\Sigma_\alpha(\varphi) = 2^{-1}\, \Pi(\varphi \times \alpha - \alpha \times \varphi) = x - \pi_\alpha(x) = x,$$

hence $\mathfrak{A}_\alpha^\perp \subseteq \operatorname{Ran}(\Sigma_\alpha)$. We just proved statement (i), $\operatorname{Ran}(\Sigma_\alpha) = \mathfrak{A}_\alpha^\perp$.

Assume now that $\theta \in \mathrm{T}_\alpha^{\mathrm{alg}}\, \mathcal{S}(\mathfrak{E})$ and let $x = \Sigma_\alpha(\theta) \in \mathfrak{A}_\alpha^\perp$. By parts (i) and (ii) in Lemma 1.8, we know that

$$\Pi(\theta) = 0, \quad \theta \times \alpha + \alpha \times \theta = \theta.$$

Consequently, $\theta \times \alpha - \alpha \times \theta = 2\, \theta \times \alpha - \theta = \theta - 2\, \alpha \times \theta$, and (1.12) implies

$$x = \Sigma_\alpha(\theta) = \Pi(\theta \times \alpha) = -\, \Pi(\alpha \times \theta). \tag{1.14}$$

Using once more requirements (iii) and (iv) in Definition 1.5, we get

$$\mathfrak{D}(x)(\alpha) = x \cdot \alpha - \alpha \cdot x = \Pi(\theta \times \alpha) \cdot \alpha + \alpha \cdot \Pi(\alpha \times \theta) = \theta \times \alpha + \alpha \times \theta = \theta.$$

The proof of statement (ii) is concluded.

For statement (iii), since $\alpha^* = \alpha$, we observe that

$$\Sigma_\alpha(\varphi)^* = -\, \Sigma_\alpha(\varphi^*), \;\; (x \cdot \alpha - \alpha \cdot x)^* = -\, (x^* \cdot \alpha - \alpha \cdot x^*),$$

for any $\varphi \in \mathfrak{E}$ and $x \in \mathfrak{A}_\alpha^\perp$. Therefore, if $\varphi \in \mathfrak{E}_{\mathrm{h}}$, then $x = \Sigma_\alpha(\varphi) \in \mathfrak{A}_{\mathrm{sh}}$, and whenever $x \in \mathfrak{A}_{\mathrm{sh}}$, then $\varphi = x \cdot \alpha - \alpha \cdot x \in \mathfrak{E}_{\mathrm{h}}$ with $\Sigma_\alpha(\varphi) = x$.

Statement (iv) follows from statement (ii). The proof is complete. □

A quick inspection of the proof of Proposition 1.13 shows that the Zariski tangent spaces to structure manifolds are completely determined by only two, or three, requirements in Lemma 1.8.

Corollary 1.14. *Suppose $\alpha \in \mathcal{S}(\mathfrak{E})$ or $\alpha \in \mathcal{S}_*(\mathfrak{E})$ and $\theta \in \mathfrak{E}$.*

(i) $\theta \in \mathrm{T}_\alpha^{\mathrm{alg}}\, \mathcal{S}(\mathfrak{E})$ *if and only if* $\Pi(\theta) = 0$ *and* $\theta \times \alpha + \alpha \times \theta = \theta$.

(ii) $\theta \in \mathrm{T}_\alpha^{\mathrm{alg}}\, \mathcal{S}_*(\mathfrak{E})$ *if, in addition,* $\theta^* = \theta$.

We are now in a position to state the first main result in our article.

THEOREM A (Zariski tangent spaces to $\mathcal{S}(\mathfrak{E})$ and $\mathcal{S}_*(\mathfrak{E})$). *Let $(\mathfrak{E}, \Pi, \mathfrak{A})$ be an algebra environment and assume that $\alpha \in \mathcal{S}(\mathfrak{E})$.*

(i) *The linear mapping $\mathrm{T}_\alpha : \mathfrak{A}_\alpha^\perp \to \mathrm{T}_\alpha^{\mathrm{alg}}\, \mathcal{S}(\mathfrak{E})$ defined by*

$$\mathrm{T}_\alpha(x) = \mathfrak{D}(x)(\alpha) = x \cdot \alpha - \alpha \cdot x, \quad x \in \mathfrak{A}_\alpha^\perp, \tag{1.15}$$

is a vector space isomorphism.

(ii) *The linear mapping $\mathrm{P}_\alpha : \mathfrak{E} \to \mathfrak{E}$ given by*

$$\mathrm{P}_\alpha(\varphi) = \Sigma_\alpha(\varphi) \cdot \alpha - \alpha \cdot \Sigma_\alpha(\varphi), \quad \varphi \in \mathfrak{E}, \tag{1.16}$$

is a projection of $\mathfrak{E}$ onto the Zariski tangent space $\mathrm{T}_\alpha^{\mathrm{alg}}\, \mathcal{S}(\mathfrak{E})$.

Suppose next that $(\mathfrak{E}, \Pi, \mathfrak{A})$ is an involutive environment and $\alpha \in \mathcal{S}_*(\mathfrak{E})$.

(iii) *The linear mapping $\mathrm{T}_{\alpha,*} : \mathfrak{A}_\alpha^\perp \cap \mathfrak{A}_{\mathrm{sh}} \to \mathrm{T}_\alpha^{\mathrm{alg}}\, \mathcal{S}_*(\mathfrak{E})$ defined as the restriction and corestriction of T_α is a vector space isomorphism.*

(iv) *The linear mapping $\mathrm{P}_{\alpha,*} : \mathfrak{E}_{\mathrm{h}} \to \mathfrak{E}_{\mathrm{h}}$ defined as the restriction and corestriction of P_α is a projection of $\mathfrak{E}_{\mathrm{h}}$ onto the Zariski tangent space $\mathrm{T}_\alpha^{\mathrm{alg}}\, \mathcal{S}_*(\mathfrak{E})$.*

Proof. Let $\operatorname{Ran} \mathrm{T}_\alpha$ and $\operatorname{Ker} \mathrm{T}_\alpha$ denote the range and the kernel of T_α. We get $\operatorname{Ran} \mathrm{T}_\alpha \subseteq \mathrm{T}_\alpha^{\mathrm{alg}}\, \mathcal{S}(\mathfrak{E})$ as a consequence of Lemma 1.11. Proposition 1.13 shows that for each $\theta \in \mathrm{T}_\alpha^{\mathrm{alg}}\, \mathcal{S}(\mathfrak{E})$, we have $x = \Sigma_\alpha(\theta) \in \mathfrak{A}_\alpha^\perp$ and $\mathrm{T}_\alpha(x) = \theta$. Therefore, $\operatorname{Ran} \mathrm{T}_\alpha = \mathrm{T}_\alpha^{\mathrm{alg}}\, \mathcal{S}(\mathfrak{E})$. We conclude the proof of statement (i) by observing that $\operatorname{Ker} \mathrm{T}_\alpha = \mathfrak{A}_\alpha^\perp \cap \{\, x \in \mathfrak{A} : x \cdot \alpha = \alpha \cdot x \,\} = \mathfrak{A}_\alpha^\perp \cap \mathfrak{A}_\alpha = \{\, 0 \,\}$. The just completed proof points out that the inverse of $\mathrm{T}_\alpha : \mathfrak{A}_\alpha^\perp \to \mathrm{T}_\alpha^{\mathrm{alg}}\, \mathcal{S}(\mathfrak{E})$ is the restriction $\Sigma_\alpha | \mathrm{T}_\alpha^{\mathrm{alg}}\, \mathcal{S}(\mathfrak{E}) : \mathrm{T}_\alpha^{\mathrm{alg}}\, \mathcal{S}(\mathfrak{E}) \to \mathfrak{A}_\alpha^\perp$ of the symbol operator.

Statement (ii) is yet another consequence of previous results. Suppose $\varphi \in \mathfrak{E}$ and take $x = \Sigma_\alpha(\varphi) \in \mathfrak{A}$. From equation (1.16) and Lemma 1.11 we get $\mathrm{P}_\alpha(\varphi) = \mathfrak{D}(x)(\alpha) \in \mathrm{T}_\alpha^{\mathrm{alg}}\, \mathcal{S}(\mathfrak{E})$, hence $\operatorname{Ran} \mathrm{P}_\alpha \subseteq \mathrm{T}_\alpha^{\mathrm{alg}}\, \mathcal{S}(\mathfrak{E})$. On the other hand, if $\theta \in \mathrm{T}_\alpha^{\mathrm{alg}}$ and $x = \Sigma_\alpha(\theta)$, then Proposition 1.13 implies $\mathrm{P}_\alpha(\theta) = \theta$. The last two observations clearly show that $\mathrm{P}_\alpha^2 = \mathrm{P}_\alpha$ and $\operatorname{Ran} \mathrm{P}_\alpha = \mathrm{T}_\alpha^{\mathrm{alg}}\, \mathcal{S}(\mathfrak{E})$.

The remaining statements are derived from the previous ones. To get (iii) from (i), we note that since $\alpha^* = \alpha$, if $x \in \mathfrak{A}_\alpha^\perp$, then $\theta = \mathrm{T}_\alpha(x) \in \mathrm{T}_\alpha^{\mathrm{alg}}\, \mathcal{S}_*(\mathfrak{E})$, i.e., $\theta^* = \theta$, if and only if $x \in \mathfrak{A}_{\mathrm{sh}}$. With regard to statement (iv), from (ii) we know that $\operatorname{Ran} \mathrm{P}_{\alpha,*} \subseteq \mathrm{T}_\alpha^{\mathrm{alg}}\, \mathcal{S}(\mathfrak{E})$. By Definition 1.12, we have

$$\Sigma_\alpha(\varphi)^* = 2^{-1}\, \Pi(\varphi \times \alpha - \alpha \times \varphi)^* = 2^{-1}\, \Pi(\alpha^* \times \varphi^* - \varphi^* \times \alpha^*),$$

for any $\varphi \in \mathfrak{E}$. Under the additional assumption $\varphi \in \mathfrak{E}_{\mathrm{h}}$, from $\varphi^* = \varphi$ and $\alpha^* = \alpha$ it follows that

$$\Sigma_\alpha(\varphi)^* = 2^{-1}\, \Pi(\alpha \times \varphi - \varphi \times \alpha) = -\, \Sigma_\alpha(\varphi).$$

Therefore, using equation (1.16) we get,

$$\mathrm{P}_{\alpha,*}(\varphi)^* = \alpha \times \Sigma_\alpha(\varphi)^* - \Sigma_\alpha(\varphi)^* \times \alpha = -\,\alpha \times \Sigma_\alpha(\varphi) + \Sigma_\alpha(\varphi) \times \alpha = \mathrm{P}_{\alpha,*}(\varphi),$$

i.e., $\mathrm{P}_{\alpha,*}(\varphi) \in \mathrm{T}^{\mathrm{alg}}_\alpha\, \mathcal{S}_*(\mathfrak{E})$, and consequently $\operatorname{Ran} \mathrm{P}_{\alpha,*} \subseteq \mathrm{T}^{\mathrm{alg}}_\alpha\, \mathcal{S}_*(\mathfrak{E})$. It remains to observe that for each $\theta \in \mathrm{T}^{\mathrm{alg}}_\alpha\, \mathcal{S}_*(\mathfrak{E})$ by equations (1.16) and (1.14) we have,

$$\begin{aligned}\mathrm{P}_{\alpha,*}(\theta) &= \Sigma_\alpha(\theta)\cdot\alpha - \alpha\cdot\Sigma_\alpha(\theta) = \Pi(\theta\times\alpha)\cdot\alpha + \alpha\cdot\Pi(\alpha\times\theta)\\ &= \theta\times\alpha + \alpha\times\theta = \theta.\end{aligned}$$

The immediate conclusions are $\operatorname{Ran} \mathrm{P}_{\alpha,*} = \mathrm{T}^{\mathrm{alg}}_\alpha\, \mathcal{S}_*(\mathfrak{E})$ and $\mathrm{P}^2_{\alpha,*} = \mathrm{P}_{\alpha,*}$. □

1.4. **Grassmann and flag manifolds**

Theorem A will play an important role subsequently. The next application is related to geometric properties of Grassmann and flag manifolds of involutive unital algebras and provides a description of their complex structures.

Suppose $\mathfrak{A}$ is an involutive unital algebra and let $\mathfrak{E}_n(\mathfrak{A}) = \mathfrak{A}\times\mathfrak{A}\times\cdots\mathfrak{A}$, $n \geq 2$, be the n-fold product of $\mathfrak{A}$ with itself. The operations on $\mathfrak{E}_n(\mathfrak{A})$ and the $\mathfrak{A}$-bimodule structure are defined componentwise. We introduce the involutive n-fold product environment $(\mathfrak{E}_n(\mathfrak{A}), \Pi, \mathfrak{A})$ with projection Π given by

$$\Pi(\varphi) = x_1 + x_2 + \cdots + x_n, \quad \varphi = (x_1, x_2, \ldots, x_n) \in \mathfrak{E}_n(\mathfrak{A}).$$

We denote by $\mathcal{E}(\mathfrak{A})$ and $\mathcal{P}(\mathfrak{A})$ the spaces of idempotents and projections of $\mathfrak{A}$,

$$\mathcal{E}(\mathfrak{A}) = \{e \in \mathfrak{A} : e^2 = e\}, \quad \mathcal{P}(\mathfrak{A}) = \{p \in \mathfrak{A} : p^2 = p = p^*\}.$$

Proposition 1.15. *The structure manifold $\mathcal{S}(\mathfrak{E}_n(\mathfrak{A}))$ consists of n-tuples $\mathfrak{e} = (e_1, e_2, \cdots, e_n)$ of elements of $\mathcal{E}(\mathfrak{A})$ that are n-partitions of $1_\mathfrak{A}$, i.e.,*

$$e_i\, e_j = 0,\ 1 \leq i,\ j \leq n,\ i \neq j, \quad e_1 + e_2 + \cdots + e_n = 1_\mathfrak{A}.$$

For each $\mathfrak{e} \in \mathcal{S}(\mathfrak{E}_n(\mathfrak{A}))$, equations (1.1), (1.2) *imply that subalgebra $\mathfrak{A}_\mathfrak{e} \subseteq \mathfrak{A}$ and projection $\pi_\mathfrak{e} : \mathfrak{A} \to \mathfrak{A}$ onto $\mathfrak{A}_\mathfrak{e}$ are given by*

$$\mathfrak{A}_\mathfrak{e} = \{\, x \in \mathfrak{A} : x\cdot\mathfrak{e} = \mathfrak{e}\cdot x \,\} = \{\, x \in \mathfrak{A} : x\, e_i = e_i\, x,\ 1 \leq i \leq n \,\},$$

$$\pi_\mathfrak{e}(x) = \Pi(\mathfrak{e}\cdot x\times\mathfrak{e}) = e_1\, x\, e_1 + e_2\, x\, e_2 + \cdots + e_n\, x\, e_n, \quad x \in \mathfrak{A}.$$

The direct complement $\mathfrak{A}_\mathfrak{e}^\perp = \mathfrak{A} \ominus \mathfrak{A}_\mathfrak{e}$ of $\mathfrak{A}_\mathfrak{e}$ has the next description,

$$\mathfrak{A}_\mathfrak{e}^\perp = \{x \in \mathfrak{A} : \pi_\mathfrak{e}(x) = 0\} = \mathfrak{A}_{\mathfrak{e},+} \oplus \mathfrak{A}_{\mathfrak{e},-},$$

$$\mathfrak{A}_{\mathfrak{e},+} = \oplus_{1\leq i<j\leq n}\ e_i\, \mathfrak{A}\, e_j, \quad \mathfrak{A}_{\mathfrak{e},-} = \oplus_{1\leq j<i\leq n}\ e_i\, \mathfrak{A}\, e_j.$$

According to equation (1.15), $\mathrm{T}_{\mathfrak{e}} : \mathfrak{A}_{\mathfrak{e}}^{\perp} \to \mathrm{T}_{\mathfrak{e}}^{\mathrm{alg}}\, \mathcal{S}(\mathfrak{E}_n(\mathfrak{A}))$ *is defined as*

$$\mathrm{T}_{\mathfrak{e}}(x) = x \cdot \mathfrak{e} - \mathfrak{e} \cdot x = (x\, e_1 - e_1\, x,\, x\, e_2 - e_2\, x, \ldots,\, x\, e_n - e_n\, x), \quad x \in \mathfrak{A}_{\mathfrak{e}}^{\perp}.$$

The structure manifold $\mathcal{S}_*(\mathfrak{E}_n(\mathfrak{A})) \subseteq \mathcal{S}(\mathfrak{E}_n(\mathfrak{A}))$ *consists of n-partitions of* $1_{\mathfrak{A}}$ *denoted by* $\mathfrak{p} = (p_1, p_2, \ldots, p_n)$ *with components* $p_i \in \mathcal{P}(\mathfrak{A})$, $1 \le i \le n$. *The descriptions of* $\mathfrak{A}_{\mathfrak{p}} \subseteq \mathfrak{A}$, $\pi_{\mathfrak{p}} : \mathfrak{A} \to \mathfrak{A}$, *and* $\mathfrak{A}_{\mathfrak{p}}^{\perp} = \mathfrak{A} \ominus \mathfrak{A}_{\mathfrak{p}} = \mathfrak{A}_{\mathfrak{p},+} \oplus \mathfrak{A}_{\mathfrak{p},-}$ *are similar, and* $\mathrm{T}_{\mathfrak{p},*} : \mathfrak{A}_{\mathfrak{p}}^{\perp} \cap \mathfrak{A}_{\mathrm{sh}} \to \mathrm{T}_{\mathfrak{p}}^{\mathrm{alg}}\, \mathcal{S}_*(\mathfrak{E}_n(\mathfrak{A}))$ *has the explicit definition*

$$\mathrm{T}_{\mathfrak{p},*}(x) = x \cdot \mathfrak{p} - \mathfrak{p} \cdot x = (x\, p_1 - p_1\, x,\, x\, p_2 - p_2\, x, \ldots,\, x\, p_n - p_n\, x), \quad x \in \mathfrak{A}_{\mathfrak{e}}^{\perp} \cap \mathfrak{A}_{\mathrm{sh}}.$$

Proof. Calculations based on previous definitions and equations. □

The tangent spaces $\mathrm{T}_{\mathfrak{e}}^{\mathrm{alg}}\, \mathcal{S}(\mathfrak{E}_n(\mathfrak{A}))$, $\mathfrak{e} \in \mathcal{S}(\mathfrak{E}_n(\mathfrak{A}))$, are complex subspaces of $\mathfrak{E}_n(\mathfrak{A})$, whereas the tangent spaces $\mathrm{T}_{\mathfrak{p}}^{\mathrm{alg}}\, \mathcal{S}_*(\mathfrak{E}_n(\mathfrak{A}))$, $\mathfrak{p} \in \mathcal{S}_*(\mathfrak{E}_n(\mathfrak{A}))$, are real subspaces of $\mathfrak{E}_n(\mathfrak{A})_{\mathrm{h}}$. In spite of this, each $\mathrm{T}_{\mathfrak{p}}^{\mathrm{alg}}\, \mathcal{S}_*(\mathfrak{E}_n(\mathfrak{A}))$, $\mathfrak{p} \in \mathcal{S}_*(\mathfrak{E}_n(\mathfrak{A}))$, has an entire collection of complex structures, that can be constructed by adapting classical results due to Borel, Hirzebruch [8] to our algebraic setting.

Using the isomorphism $\mathrm{T}_{\mathfrak{p},*} : \mathfrak{A}_{\mathfrak{p}}^{\perp} \cap \mathfrak{A}_{\mathrm{sh}} \to \mathrm{T}_{\mathfrak{p}}^{\mathrm{alg}}\, \mathcal{S}_*(\mathfrak{E}_n(\mathfrak{A}))$, it would be enough to set up complex structures on $\mathfrak{A}_{\mathfrak{p}}^{\perp} \cap \mathfrak{A}_{\mathrm{sh}}$. To this end, observe first that any $x \in \mathfrak{A}_{\mathfrak{p}}^{\perp}$ has a unique decomposition $x = x_+ + x_-$, $x_+ \in \mathfrak{A}_{\mathfrak{p},+}$, $x_- \in \mathfrak{A}_{\mathfrak{p},-}$. The requirement $x \in \mathfrak{A}_{\mathrm{sh}}$ reduces to $x^* = x_+^* + x_-^* = -\,x = -\,x_+ - x_-$, therefore $x_- = -\,x_+^*$ and $x = x_+ - x_+^*$. Select a subset $C \subseteq \{\, (i,\, j) : 1 \le i < j \le n \,\}$, denote its complement by C^{c}, and define $\mathrm{J}_{\mathfrak{p},C}^{+} : \mathfrak{A}_{\mathfrak{p},+} \to \mathfrak{A}_{\mathfrak{p},+}$ by

$$\mathrm{J}_{\mathfrak{p},C}^{+}(x_+) = \sqrt{-1} \sum_{(i,\,j) \in C} x_{ij} - \sqrt{-1} \sum_{(i,\,j) \in C^{\mathrm{c}}} x_{ij},$$

for any $x_+ = \displaystyle\sum_{1 \le i < j \le n} x_{ij}$, $x_{ij} \in p_i\, \mathfrak{A}\, p_j$. Next, let $\mathrm{J}_{\mathfrak{p},C} : \mathfrak{A}_{\mathfrak{p}}^{\perp} \cap \mathfrak{A}_{\mathrm{sh}} \to \mathfrak{A}_{\mathfrak{p}}^{\perp} \cap \mathfrak{A}_{\mathrm{sh}}$ be the linear mapping given by

$$\mathrm{J}_{\mathfrak{p},C}(x) = \mathrm{J}_{\mathfrak{p},C}^{+}(x_+) - \mathrm{J}_{\mathfrak{p},C}^{+}(x_+)^*, \; x = x_+ - x_+^* \in \mathfrak{A}_{\mathfrak{p}}^{\perp} \cap \mathfrak{A}_{\mathrm{sh}}, \; x_+ \in \mathfrak{A}_{\mathfrak{p},+}.$$

Since $\mathrm{J}_{\mathfrak{p},C}^{2} = -\mathrm{Id}_{\mathfrak{A}_{\mathfrak{p}}^{\perp} \cap \mathfrak{A}_{\mathrm{sh}}}$, we conclude that $\mathrm{J}_{\mathfrak{p},C}$ is a complex structure. The conjugate of $\mathrm{J}_{\mathfrak{p},C}$ is $\mathrm{J}_{\mathfrak{p},C^{\mathrm{c}}}$. The standard complex structure on $\mathcal{S}_*(\mathfrak{E}_n(\mathfrak{A}))$ corresponds to $C = \{(i,\, j) : 1 \le i < j \le n\}$, and its conjugate to $C^{\mathrm{c}} = \emptyset$.

We still need to define the Grassmann and flag manifolds of $\mathfrak{A}$. Specifically, an n-flag in $\mathfrak{A}$, $n \ge 2$, is an n-tuple $\mathcal{P} = (P_1,\, P_2,\, \ldots\, P_n)$ of projections such that $0 \le P_1 \le P_2 \le \cdots \le P_n = 1_{\mathfrak{A}}$, where $P \le P'$, $P, P' \in \mathcal{P}(\mathfrak{A})$, provided $PP' = P$. The equations

$$P_1 = p_1, P_2 = p_1 + p_2, \ldots, P_n = p_1 + p_2 + \cdots + p_n,$$

assign an n-flag to each $\mathfrak{p} = (p_1, p_2, \ldots, p_n) \in \mathcal{S}_*(\mathfrak{E}_n(\mathfrak{A}))$. The process is reversible, hence the space of all n-flags, denoted by $\mathcal{P}_n(\mathfrak{A})$, and the structure manifold $\mathcal{S}_*(\mathfrak{E}_n(\mathfrak{A})$ are in a one-to-one correspondence. This observation shows that $\mathcal{P}_n(\mathfrak{A})$ is an algebraic subset of $\mathfrak{E}_n(\mathfrak{A})_{\mathrm{h}}$ with $2^{n(n-1)/2}$ complex structures. In particular, $\mathcal{P}_2(\mathfrak{A}) = \{(p, 1_{\mathfrak{A}}) : p \in \mathcal{P}(\mathfrak{A})\} \equiv \mathcal{P}(\mathfrak{A})$ – the Grassmann manifold of $\mathfrak{A}$ – has just the standard complex structure and its conjugate.

1.5. **Geometric structures and conjugation operators**

Definition 1.10 introduced the space $\mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A})$ of derivations on an algebra environment $(\mathfrak{E}, \Pi, \mathfrak{A})$. We use the two linear mappings $\mathfrak{D} : \mathfrak{A} \to \mathrm{Der}(\mathfrak{E})$ and $\mathfrak{D}_0 : \mathfrak{A} \to \mathrm{Der}(\mathfrak{A})$ given by equation (1.11) to define an associate mapping,

$$(1.17) \quad (\mathfrak{D}, \mathfrak{D}_0) : \mathfrak{A} \to \mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A}), \quad (\mathfrak{D}, \mathfrak{D}_0)(x) = (\mathfrak{D}(x), \mathfrak{D}_0(x)), \quad x \in \mathfrak{A}.$$

Suppose $\alpha \in \mathcal{S}(\mathfrak{E})$ and for the rest of this subsection let $\Sigma_\alpha : \mathrm{T}^{\mathrm{alg}}_\alpha \mathcal{S}(\mathfrak{E}) \to \mathfrak{A}$ be the restriction of the general symbol operator to $\mathrm{T}^{\mathrm{alg}}_\alpha \mathcal{S}(\mathfrak{E})$. Prompted by Lemma 1.11, we define the mapping

$$\Theta_\alpha : \mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A}) \to \mathrm{T}^{\mathrm{alg}}_\alpha \mathcal{S}(\mathfrak{E}),\ \Theta_\alpha(D, D_0) = D(\alpha),\ (D, D_0) \in \mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A}).$$

Part (ii) of Proposition 1.13 can be restated as follows.

LEMMA 1.16. *The composite mapping given by*

$$\mathrm{T}^{\mathrm{alg}}_\alpha \mathcal{S}(\mathfrak{E}) \xrightarrow{\Sigma_\alpha} \mathfrak{A} \xrightarrow{(\mathfrak{D}, \mathfrak{D}_0)} \mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A}) \xrightarrow{\Theta_\alpha} \mathrm{T}^{\mathrm{alg}}_\alpha \mathcal{S}(\mathfrak{E})$$

satisfies the property $\Theta_\alpha \circ (\mathfrak{D}, \mathfrak{D}_0) \circ \Sigma_\alpha = \mathrm{Id}_{\mathrm{T}^{\mathrm{alg}}_\alpha \mathcal{S}(\mathfrak{E})}$. □

Definition 1.17. A derivation $(\mathcal{D}, \mathcal{D}_0) \in \mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A})$ is called compatible with an $\mathfrak{E}$-structure $\alpha \in \mathcal{S}(\mathfrak{E})$ on $\mathfrak{A}$ provided $\Theta_\alpha(\mathcal{D}, \mathcal{D}_0) = \mathcal{D}(\alpha) = 0$. The space of derivations compatible with $\alpha \in \mathcal{S}(\mathfrak{E})$ is denoted by $Der_{\alpha,0}(\mathfrak{E}, \Pi, \mathfrak{A})$.

$Der_{\alpha,0}(\mathfrak{E}, \Pi, \mathfrak{A})$ is a Lie subalgebra of $\mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A})$, that turns out to be an eigenspace of an involution on $\mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A})$ defined in terms of $\alpha \in \mathcal{S}(\mathfrak{E})$.

Definition 1.18. The conjugation operator on $\mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A})$ associated with a geometric $\mathfrak{E}$-structure $\alpha \in \mathcal{S}(\mathfrak{E})$ on base algebra $\mathfrak{A}$ is defined as

$$\Gamma_\alpha : \mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A}) \to \mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A}), \quad \Gamma_\alpha = \mathrm{Id}_{\mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A})} - 2\, (\mathfrak{D}, \mathfrak{D}_0) \circ \Sigma_\alpha \circ \Theta_\alpha.$$

The derivation $(\mathcal{D}^\dagger, \mathcal{D}_0^\dagger) = \Gamma_\alpha(\mathcal{D}, \mathcal{D}_0)$ is called the α-conjugate of $(\mathcal{D}, \mathcal{D}_0)$.

THEOREM B (Conjugation operator and compatible derivations). *The operator* $\Gamma_\alpha : \mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A}) \to \mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A})$ *has the following properties:*

(i) Γ_α *is an involution on* $\mathrm{Der}(\mathfrak{E},\Pi,\mathfrak{A})$, *i.e.,* $\Gamma_\alpha^2 = \mathrm{Id}_{\mathrm{Der}(\mathfrak{E},\Pi,\mathfrak{A})}$, *or*

$$(\mathcal{D}^{\dagger\,\dagger},\mathcal{D}_0^{\dagger\,\dagger}) = (\mathcal{D},\mathcal{D}_0),\quad (\mathcal{D},\mathcal{D}_0)\in \mathrm{Der}(\mathfrak{E},\Pi,\mathfrak{A}).$$

(ii) $\Theta_\alpha\circ\Gamma_\alpha = -\,\Theta_\alpha$, *i.e.,*

$$\mathcal{D}^{\dagger}(\alpha) = -\,\mathcal{D}(\alpha),\quad (\mathcal{D},\mathcal{D}_0)\in \mathrm{Der}(\mathfrak{E},\Pi,\mathfrak{A}).$$

(iii) $(\mathcal{D},\mathcal{D}_0)\in Der_{\alpha,0}(\mathfrak{E},\Pi,\mathfrak{A})$ *only if* $\Gamma_\alpha(\mathcal{D},\mathcal{D}_0) = (\mathcal{D},\mathcal{D}_0)$, *i.e.,*

$$(\mathcal{D}^{\dagger},\mathcal{D}_0^{\dagger}) = (\mathcal{D},\mathcal{D}_0).$$

Proof. Properties (i) and (ii) are both direct consequences of Lemma 1.16 and Definition 1.18. Specifically,

$$\Gamma_\alpha^2 = \mathrm{Id}_{\mathrm{Der}(\mathfrak{E},\Pi,\mathfrak{A})} - 4\,(\mathfrak{D},\mathfrak{D}_0)\circ\Sigma_\alpha\circ\Theta_\alpha + 4\,((\mathfrak{D},\mathfrak{D}_0)\circ\Sigma_\alpha\circ\Theta_\alpha)^2$$

$$= \mathrm{Id}_{\mathrm{Der}(\mathfrak{E},\Pi,\mathfrak{A})} - 4\,(\mathfrak{D},\mathfrak{D}_0)\circ\Sigma_\alpha\circ\Theta_\alpha + 4\,(\mathfrak{D},\mathfrak{D}_0)\circ\Sigma_\alpha\circ\mathrm{Id}_{\mathrm{T}_\alpha^{\mathrm{alg}}\,\mathcal{S}(\mathfrak{E})}\circ\Theta_\alpha,$$

hence $\Gamma_\alpha^2 = \mathrm{Id}_{\mathrm{Der}(\mathfrak{E},\Pi,\mathfrak{A})}$, and

$$\Theta_\alpha\circ\Gamma_\alpha = \Theta_\alpha - 2\,\Theta\circ(\mathfrak{D},\mathfrak{D}_0)\circ\Sigma_\alpha\circ\Theta_\alpha = \Theta_\alpha - 2\,\mathrm{Id}_{\mathrm{T}_\alpha^{\mathrm{alg}}\,\mathcal{S}(\mathfrak{E})}\circ\Theta_\alpha = -\,\Theta_\alpha.$$

Property (iii), which identifies $\mathrm{Der}_{\alpha,0}(\mathfrak{E},\Pi,\mathfrak{A})$ with the space of self-conjugate derivations in $\mathrm{Der}(\mathfrak{E},\Pi,\mathfrak{A})$, i.e., the eigenspace of Γ_α corresponding to the eigenvalue $\lambda = 1$, follows from Definition 1.16 and the previous results. If $(\mathcal{D},\mathcal{D}_0)\in\mathrm{Der}_{\alpha,0}(\mathfrak{E},\Pi,\mathfrak{A})$, then $\Theta_\alpha\,(\mathcal{D},\mathcal{D}_0) = 0$ and from Definition 1.18, we get $\Gamma_\alpha(\mathcal{D},\mathcal{D}_0) = (\mathcal{D},\mathcal{D}_0)$. Conversely, if $\Gamma_\alpha(\mathcal{D},\mathcal{D}_0) = (\mathcal{D},\mathcal{D}_0)$, property (ii) shows that $\Theta_\alpha\,(\mathcal{D},\mathcal{D}_0) = -\,\Theta_\alpha\,(\mathcal{D},\mathcal{D}_0)$, hence $\Theta_\alpha\,(\mathcal{D},\mathcal{D}_0) = 0$. □

Corollary 1.19. *The operators* Γ_α^+, $\Gamma_\alpha^- : \mathrm{Der}(\mathfrak{E},\Pi,\mathfrak{A})\to\mathrm{Der}(\mathfrak{E},\Pi,\mathfrak{A})$,

$$\Gamma_\alpha^+ = (\mathrm{Id}_{\mathrm{Der}(\mathfrak{E},\Pi,\mathfrak{A})} + \Gamma_\alpha)/2,\quad \Gamma_\alpha^- = (\mathrm{Id}_{\mathrm{Der}(\mathfrak{E},\Pi,\mathfrak{A})} - \Gamma_\alpha)/2),$$

are complementary projections on the space $\mathrm{Der}(\mathfrak{E},\Pi,\mathfrak{A})$, *i.e.,*

$$(\Gamma_\alpha^+)^2 = \Gamma_\alpha^+,\ (\Gamma_\alpha^-)^2 = \Gamma_\alpha^-,\ \Gamma_\alpha^+\,\Gamma_\alpha^- = \Gamma_\alpha^-\,\Gamma_\alpha^+ = 0,\ \Gamma_\alpha^+ + \Gamma_\alpha^- = \mathrm{Id}_{\mathrm{Der}(\mathfrak{E},\Pi,\mathfrak{A})},$$

such that $\mathrm{Der}_{\alpha,0}(\mathfrak{E},\Pi,\mathfrak{A}) = \mathrm{Ran}\,\Gamma_\alpha^+ = \mathrm{Ker}\,\Gamma_\alpha^-$.

Corollary 1.20. *Suppose* $\theta\in\mathrm{T}_\alpha^{\mathrm{alg}}\,\mathcal{S}(\mathfrak{E})$ *and let* $\mathrm{Der}_{\alpha,\theta}(\mathfrak{E},\Pi,\mathfrak{A})$ *be defined as*

$$\mathrm{Der}_{\alpha,\theta}\,(\mathfrak{A}) = \{(\mathcal{D},\mathcal{D}_0)\in\mathrm{Der}(\mathfrak{E},\Pi,\mathfrak{A}) : \Theta_\alpha\,(\mathcal{D},\mathcal{D}_0) = \theta\}.$$

Then $\mathrm{Der}_{\alpha,\theta}(\mathfrak{E},\Pi,\mathfrak{A}) = (\mathfrak{D},\mathfrak{D}_0)\circ\Sigma_\alpha(\theta) + \mathrm{Der}_{\alpha,0}\,(\mathfrak{E},\Pi,\mathfrak{A})$.

1.6. **Functorial properties**

Let $(\Theta, \Theta_0) : (\mathfrak{E}, \Pi_{\mathfrak{E},\mathfrak{A}}, \mathfrak{A}) \to (\mathfrak{F}, \Pi_{\mathfrak{F},\mathfrak{B}}, \mathfrak{B})$ be an environment homomorphism, and let $\mathcal{S}(\mathfrak{E}) \subseteq \mathfrak{E}$ and $\mathcal{S}(\mathfrak{F}) \subseteq \mathfrak{F}$ be the structure manifolds associated with the two environments. Consider $\Theta \,|\, \mathcal{S}(\mathfrak{E}) : \mathcal{S}(\mathfrak{E}) \to \mathfrak{F}$ and denote its range by $\operatorname{Ran} \Theta \,|\, \mathcal{S}(\mathfrak{E})$. By applying (Θ, Θ_0) to each equation in Definition 1.5 and based on Definition 1.2, we derive the next result.

LEMMA 1.21. *Suppose that $\alpha \in \mathcal{S}(\mathfrak{E})$. The element $\beta = \Theta(\alpha) \in \mathfrak{F}$ satisfies the algebraic equations:*

(i) $\Pi_{\mathfrak{F},\mathfrak{B}}(\beta) = 1_{\mathfrak{B}}$,

(ii) $\beta \times \beta = \beta$,

(iii) $\Pi_{\mathfrak{F},\mathfrak{B}}(\Theta(\varphi) \times \beta) \cdot \alpha = \Theta(\varphi) \times \beta$, $\varphi \in \mathfrak{E}$,

(iv) $\beta \cdot \Pi_{\mathfrak{F},\mathfrak{B}}(\beta \times \Theta(\psi)) = \beta \times \Theta(\psi)$, $\psi \in \mathfrak{E}$,

(v) $\Pi_{\mathfrak{F},\mathfrak{B}}(\Theta(\varphi) \times \beta)\Pi_{\mathfrak{F},\mathfrak{B}}(\beta \times \theta(\psi)) = \Pi_{\mathfrak{F},\mathfrak{B}}(\Theta(\varphi) \times \beta \times \Theta(\psi))$, $\varphi, \psi \in \mathfrak{E}$.

If the environments are involutive and $\alpha \in \mathcal{S}_*(\mathfrak{E})$, i.e., $\alpha^* = \alpha$, then

(vi) $\beta^* = \beta$.

From Definition 1.5, we get the following consequence.

COROLLARY 1.22. *If (Θ, Θ_0) is onto, then $\operatorname{Ran} \Theta \,|\, \mathcal{S}(\mathfrak{E}) \subseteq \mathcal{S}(\mathfrak{F})$. If the environments are involutive, then $\operatorname{Ran} \Theta \,|\, \mathcal{S}_*(\mathfrak{E}) \subseteq \mathcal{S}_*(\mathfrak{F})$.*

In the same setting, with regard to Zariski tangent spaces there is also a natural functorial property. This time we rely on either Lemma 1.8, or Corollary 1.14, that characterize tangent vectors to structure manifolds as solutions to some equations. Applying (Θ, Θ_0) to each equation, we have the next result.

COROLLARY 1.23. *Suppose (Θ, Θ_0) is onto, $\alpha \in \mathcal{S}(\mathfrak{E})$, $\beta = \Theta(\alpha) \in \mathcal{S}(\mathfrak{F})$, and let $\mathrm{T}^{\mathrm{alg}}_{\alpha} \mathcal{S}(\mathfrak{E}) \subseteq \mathfrak{E}$ and $\mathrm{T}^{\mathrm{alg}}_{\beta} \mathcal{S}(\mathfrak{F}) \subseteq \mathfrak{F}$ be the associated Zariski tangent spaces. Then $\operatorname{Ran} \Theta \,|\, \mathrm{T}^{\mathrm{alg}}_{\alpha} \mathcal{S}(\mathfrak{E}) \subseteq \mathrm{T}^{\mathrm{alg}}_{\beta} \mathcal{S}(\mathfrak{F})$. If the environments are involutive and $\alpha \in \mathcal{S}_*(\mathfrak{E})$, then $\operatorname{Ran} \Theta \,|\, \mathrm{T}^{\mathrm{alg}}_{\alpha} \mathcal{S}_*(\mathfrak{E}) \subseteq \mathrm{T}^{\mathrm{alg}}_{\beta} \mathcal{S}_*(\mathfrak{F})$.*

2. **ALGEBRA ENVIRONMENTS–TOPOLOGICAL APPROACH**

The goal of this section is to prove that the structure manifolds $\mathcal{S}(\mathfrak{E}) = \mathcal{S}(\mathfrak{E}, \Pi, \mathfrak{A})$ and $\mathcal{S}_*(\mathfrak{E}) = \mathcal{S}_*(\mathfrak{E}, \Pi, \mathfrak{A})$ of an involutive Banach algebra environment $(\mathfrak{E}, \Pi, \mathfrak{A})$ are more than just algebraic subsets of total algebra $\mathfrak{E}$. With topological structures inherited from $\mathfrak{E}$, both turn out to be smooth Banach manifolds. Definition 1.5 makes us to expect that $\mathcal{S}(\mathfrak{E})$ is complex analytic and $\mathcal{S}_*(\mathfrak{E})$ is real analytic. For general results concerning Banach manifolds, principal fiber bundles, and symmetric spaces, we refer to Bourbaki [15], Helgason [25], Kobayashi, Nomizu [28], Lang [29], Upmeier [64].

2.1. **Smooth tangent spaces**

Let $(\mathfrak{E}, \Pi, \mathfrak{A})$ be an involutive Banach algebra environment. Though the smooth structures on $\mathcal{S}(\mathfrak{E})$ and $\mathcal{S}_*(\mathfrak{E})$ are not yet defined, the Zariski tangent spaces $\mathrm{T}_\alpha^{\mathrm{alg}}\mathcal{S}(\mathfrak{E})$, $\alpha \in \mathcal{S}(\mathfrak{E})$, and $\mathrm{T}_\alpha^{\mathrm{alg}}\mathcal{S}_*(\mathfrak{E})$, $\alpha \in \mathcal{S}_*(\mathfrak{E})$, could be described using a standard differential geometry device.

Definition 2.1. An element $\theta \in \mathfrak{E}$ is called a smooth tangent vector to $\mathcal{S}(\mathfrak{E})$ at $\alpha \in \mathcal{S}(\mathfrak{E})$ provided there exists a smooth function $\gamma : \mathbb{R} \to \mathfrak{E}$ such that

$$\gamma(t) \in \mathcal{S}(\mathfrak{E}),\ t \in \mathbb{R}, \quad \gamma(0) = \alpha, \quad \gamma'(0) = \theta, \tag{2.1}$$

where $\gamma' = d\gamma/dt$. The set of all such $\theta \in \mathfrak{E}$ is denoted by $\mathrm{T}_\alpha^\infty \mathcal{S}(\mathfrak{E})$. Similarly, if $\alpha \in \mathcal{S}_*(\mathfrak{E})$, an element $\theta \in \mathfrak{E}$ is called a smooth tangent vector to $\mathcal{S}_*(\mathfrak{E})$ at α provided there exists a smooth function $\gamma : \mathbb{R} \to \mathfrak{E}$ such that

$$\gamma(t) \in \mathcal{S}_*(\mathfrak{E}),\ t \in \mathbb{R}, \quad \gamma(0) = \alpha, \quad \gamma'(0) = \theta. \tag{2.2}$$

The set of all such $\theta \in \mathfrak{E}$ is denoted by $\mathrm{T}_\alpha^\infty \mathcal{S}_*(\mathfrak{E})$.

Lemma 2.2. *The smooth and Zariski tangent spaces coincide, i.e.,*

(i) $\mathrm{T}_\alpha^\infty\mathcal{S}(\mathfrak{E}) = \mathrm{T}_\alpha^{\mathrm{alg}}\mathcal{S}(\mathfrak{E})$, $\alpha \in \mathcal{S}(\mathfrak{E})$,

(ii) $\mathrm{T}_\alpha^\infty\mathcal{S}_*(\mathfrak{E}) = \mathrm{T}_\alpha^{\mathrm{alg}}\mathcal{S}_*(\mathfrak{E})$, $\alpha \in \mathcal{S}_*(\mathfrak{E})$.

Proof. Suppose $\theta \in \mathrm{T}_\alpha^\infty\mathcal{S}(\mathfrak{E})$. From (2.1), we have

$$\Pi(\gamma(t)) = 1_{\mathfrak{A}},\ \gamma(t) \times \gamma(t) = \gamma(t), \quad t \in \mathbb{R}. \tag{2.3}$$

Taking derivatives of each equation in (2.3) at $t = 0$ we get

$$\Pi(\theta) = 0, \quad \theta \times \alpha + \alpha \times \theta = \theta, \tag{2.4}$$

which by Corollary 1.14 implies $\theta \in \mathrm{T}_\alpha^{\mathrm{alg}}\mathcal{S}(\mathfrak{E})$. Next, assume that $\theta \in \mathrm{T}_\alpha^{\mathrm{alg}}\mathcal{S}(\mathfrak{E})$. Using part (i) in Theorem A there exists a unique $x \in \mathfrak{A}_\alpha^\perp$ such that

$$\theta = \mathcal{D}_x(\alpha) = x \cdot \alpha - \alpha \cdot x. \tag{2.5}$$

Direct calculations show that the smooth function

$$\gamma : \mathbb{R} \to \mathfrak{E},\ \gamma(t) = \exp(tx) \cdot \alpha \cdot \exp(tx)^{-1}, \quad t \in \mathbb{R}, \tag{2.6}$$

satisfies requirements (2.1), hence $\theta \in \mathrm{T}_\alpha^\infty \mathcal{S}(\mathfrak{E})$.

The proof of statement (ii) is similar. Suppose first that $\theta \in \mathrm{T}_\alpha^\infty \mathcal{S}_*(\mathfrak{E})$. From (2.2), in addition to (2.3), we have $\gamma(t)^* = \gamma(t)$, $t \in \mathbb{R}$. Therefore, in addition to (2.4), we get $\theta^* = \theta$, which together imply $\theta \in \mathrm{T}_\alpha^{\mathrm{alg}} \mathcal{S}_*(\mathfrak{E})$. Assuming that $\theta \in \mathrm{T}_\alpha^{\mathrm{alg}} \mathcal{S}_*(\mathfrak{E})$, by part (iii) in Theorem A there exists $x \in \mathfrak{A}_\alpha^\perp \cap \mathfrak{A}_{\mathrm{sh}}$ that yields (2.5). Since each $\exp(tx)$, $t \in \mathbb{R}$, is a unitary element of $\mathfrak{A}$, the smooth function defined as in (2.6) satisfies (2.2), hence $\theta \in \mathrm{T}_\alpha^\infty \mathcal{S}_*(\mathfrak{E})$. □

Lemma 2.2 justifies the use of notation $\mathrm{T}_\alpha \mathcal{S}(\mathfrak{E})$ and $\mathrm{T}_\alpha \mathcal{S}_*(\mathfrak{E})$ for either Zariski or smooth tangent spaces to structure manifolds. Perhaps we should note that by only using Definition 2.1, without Lemma 2.2 it would be tedious to even get that the sets of smooth tangent vectors are vector spaces.

2.2. Topological properties of $\mathcal{S}(\mathfrak{E})$ and $\mathcal{S}_*(\mathfrak{E})$

We continue assuming that $(\mathfrak{E}, \Pi, \mathfrak{A})$ is an involutive Banach environment. Both $\mathcal{S}(\mathfrak{E})$ and $\mathcal{S}_*(\mathfrak{E})$ are closed subsets of $\mathfrak{E}$, and the norm of $\mathfrak{E}$ makes each a metric space. We denote by $\mathrm{G}(\mathfrak{A})$ the Banach Lie group of invertible elements of $\mathfrak{A}$, and by $\mathrm{U}(\mathfrak{A}) = \{u \in \mathrm{G}(\mathfrak{A}) : u^{-1} = u^*\}$ the subgroup of unitary elements.

Definition 2.3. Two structures $\alpha, \beta \in \mathcal{S}(\mathfrak{E})$ are $\mathrm{G}(\mathfrak{A})$-equivalent provided there exists $a \in \mathrm{G}(\mathfrak{A})$ such that $\beta = a \cdot \alpha \cdot a^{-1}$. Two structures $\alpha, \beta \in \mathcal{S}_*(\mathfrak{E})$ are called $\mathrm{U}(\mathfrak{A})$-equivalent if there exists $u \in \mathrm{U}(\mathfrak{A})$ such that $\beta = u \cdot \alpha \cdot u^*$. The equivalence classes of a structure α are denoted by $\mathrm{G}(\mathfrak{A})[\alpha]$, or $\mathrm{U}(\mathfrak{A})[\alpha]$.

PROPOSITION 2.4. *Suppose $\alpha, \beta \in \mathcal{S}(\mathfrak{E})$, or $\alpha, \beta \in \mathcal{S}_*(\mathfrak{E})$, satisfy*

$$||\alpha - \beta|| < ||\Pi||^{-1} \, ||\alpha||^{-1}. \tag{2.7}$$

Then α and β are $\mathrm{G}(\mathfrak{A})$-equivalent, or $\mathrm{U}(\mathfrak{A})$-equivalent, respectively.

Proof. For $\alpha, \beta \in \mathcal{S}(\mathfrak{E})$, let $a = \Pi(\beta \times \alpha) \in \mathfrak{A}$. From requirements (iii) and (iv) in Definition 1.5, we get

$$a \cdot \alpha = \beta \cdot a = \beta \times \alpha. \tag{2.8}$$

Since $\Pi(\alpha \times \alpha) = 1_{\mathfrak{A}}$, we have $1_{\mathfrak{A}} - a = \Pi((\alpha - \beta) \times \alpha)$, and (2.7) implies

$$||1_{\mathfrak{A}} - a|| \leq ||\Pi|| \, ||\alpha - \beta|| \, ||\alpha|| < 1. \tag{2.9}$$

Therefore, $a \in \mathrm{G}(\mathfrak{A})$, and according to (2.8), $\beta = a \cdot \alpha \cdot a^{-1}$.

Next, suppose $\alpha, \beta \in \mathcal{S}_*(\mathfrak{E})$ and, as before, let $a = \Pi(\beta \times \alpha)$. Since $\alpha^* = \alpha$ and $\beta^* = \beta$, from (2.8), we get $\alpha \cdot a^* = a^* \cdot \beta$, so

$$\alpha \cdot (a^* a) = a^* \cdot \beta \cdot a = (a^* a) \cdot \alpha. \tag{2.10}$$

Since $a^*a \in \mathrm{G}(\mathfrak{A})$ is positive, using holomorphic functional calculus one defines the absolute value of a as

$$|a| = (a^*a)^{1/2} = \exp(1/2 \log(a^*a)),$$

where exp is the exponential function and log the principal branch of the logarithmic function. We note that $|a|$ is invertible and, from (2.10), we get

$$\alpha \cdot |a| = |a| \cdot \alpha, \quad \alpha \cdot |a|^{-1} = |a|^{-1} \cdot \alpha.$$

To conclude the proof, let $u = a\,|a|^{-1} \in \mathrm{U}(\mathfrak{A})$ and observe that

$$\beta \cdot u = \beta \cdot a\,|a|^{-1} = a \cdot \alpha \cdot |a|^{-1} = a\,|a|^{-1} \cdot \alpha = u \cdot \alpha,$$

i.e., α and β are $\mathrm{U}(\mathfrak{A})$-equivalent. $\square$

COROLLARY 2.5. *The equivalence classes* $\mathrm{G}(\mathfrak{A})[\alpha]$, $\alpha \in \mathcal{S}(\mathfrak{E})$, $\mathrm{U}(\mathfrak{A})[\alpha]$, $\alpha \in \mathcal{S}_*(\mathfrak{E})$, *are open and closed subsets of* $\mathcal{S}(\mathfrak{E})$ *and* $\mathcal{S}_*(\mathfrak{E})$, *respectively.*

Recall that for each structure $\alpha \in \mathcal{S}(\mathfrak{E})$, we introduced the subalgebra $\mathfrak{A}_\alpha \subseteq \mathfrak{A}$ defined by $\mathfrak{A}_\alpha = \{x \in \mathfrak{A} : x \cdot \alpha = \alpha \cdot x\}$. Since $\mathfrak{A}_\alpha$ is a Banach subalgebra of $\mathfrak{A}$, the associated groups $\mathrm{G}(\mathfrak{A}_\alpha)$ and $\mathrm{U}(\mathfrak{A}_\alpha)$ are Lie subgroups of $\mathrm{G}(\mathfrak{A})$ and $\mathrm{U}(\mathfrak{A})$.

THEOREM C (Principal fiber bundle property). *Let* $\rho_{\mathrm{G},\alpha}$, $\alpha \in \mathcal{S}(\mathfrak{E})$, *and* $\rho_{\mathrm{U},\alpha}$, $\alpha \in \mathcal{S}_*(\mathfrak{E})$, *be the mappings defined by:*

$$\rho_{\mathrm{G},\alpha} : \mathrm{G}(\mathfrak{A}) \to \mathrm{G}(\mathfrak{A})[\alpha],\ \rho_{\mathrm{G},\alpha}(a) = a \cdot \alpha \cdot a^{-1}, \quad a \in \mathrm{G}(\mathfrak{A}), \tag{2.11}$$

$$\rho_{\mathrm{U},\alpha} : \mathrm{U}(\mathfrak{A}) \to \mathrm{U}(\mathfrak{A})[\alpha],\ \rho_{\mathrm{U},\alpha}(u) = u \cdot \alpha \cdot u^*, \quad u \in \mathrm{U}(\mathfrak{A}). \tag{2.12}$$

(i) *If* $\alpha \in \mathcal{S}(\mathfrak{E})$, *then* $\rho_{\mathrm{G},\alpha} : \mathrm{G}(\mathfrak{A}) \to \mathrm{G}(\mathfrak{A})[\alpha]$ *is a principal fiber bundle with structure group* $\mathrm{G}(\mathfrak{A}_\alpha)$ *and the equivalence class* $\mathrm{G}(\mathfrak{A})[\alpha]$ *is homeomorphic to the coset space* $\mathrm{G}(\mathfrak{A})/\mathrm{G}(\mathfrak{A}_\alpha)$.

(ii) *If* $\alpha \in \mathcal{S}_*(\mathfrak{E})$, *then* $\rho_{\mathrm{U},\alpha} : \mathrm{U}(\mathfrak{A}) \to \mathrm{U}(\mathfrak{A})[\alpha]$ *is a principal fiber bundle with structure group* $\mathrm{U}(\mathfrak{A}_\alpha)$ *and the equivalence class* $\mathrm{U}(\mathfrak{A})[\alpha]$ *is homeomorphic to the coset space* $\mathrm{U}(\mathfrak{A})/\mathrm{U}(\mathfrak{A}_\alpha)$.

Proof. We start with statement (i). Essentially, we need to prove the existence of continuous local cross-sections of $\rho_{\mathrm{G},\alpha}$. Select $\alpha_0 \in \mathrm{G}(\mathfrak{A})[\alpha]$ and define the open ball

$$\mathcal{B}(\alpha_0) = \{\, \beta \in \mathcal{S}(\mathfrak{E}) : ||\alpha_0 - \beta|| < ||\Pi||^{-1}\, ||\alpha_0||^{-1} \,\}.$$

By Proposition 2.4, $\mathcal{B}(\alpha_0)$ is an open subset of $\mathrm{G}(\mathfrak{A})[\alpha_0] = \mathrm{G}(\mathfrak{A})[\alpha]$. Next choose $a \in \mathrm{G}(\mathfrak{A})$ such that $\alpha_0 = a \cdot \alpha \cdot a^{-1}$ and let $\sigma : \mathcal{B}(\alpha_0) \to \mathrm{G}(\mathfrak{A})$ be the continuous mapping given by

$$\sigma(\beta) = \Pi(\beta \times \alpha_0)\, a, \quad \beta \in \mathcal{B}(\alpha_0). \tag{2.13}$$

From the proof of Proposition 2.4, we know that $\Pi(\beta \times \alpha_0) \in \mathrm{G}(\mathfrak{A})$, and

$$\beta = \Pi(\beta \times \alpha_0) \cdot \alpha_0 \cdot \Pi(\beta \times \alpha_0)^{-1}, \tag{2.14}$$

for any $\beta \in \mathcal{B}(\alpha_0)$. Therefore, the mapping σ is well defined, i.e., its values are in $\mathrm{G}(\mathfrak{A})$, and from (2.13) and (2.14), we get

$$\begin{aligned}\rho_{\mathrm{G},\alpha} \circ \sigma(\beta) = \sigma(\beta) \cdot \alpha \cdot \sigma(\beta)^{-1} &= \Pi(\beta \times \alpha_0)\, a \cdot \alpha \cdot a^{-1} \Pi(\beta \times \alpha_0)^{-1} \\ &= \Pi(\beta \times \alpha_0) \cdot \alpha_0 \cdot \Pi(\beta \times \alpha_0)^{-1} = \beta, \quad \beta \in \mathcal{B}(\alpha_0),\end{aligned}$$

i.e., $\rho_{\mathrm{G},\alpha} \circ \sigma = \mathrm{Id}_{\mathcal{B}(\alpha_0)}$, and that concludes the proof.

Statement (ii) has a quite similar proof. Suppose $\alpha \in \mathcal{S}_*(\mathfrak{E})$, and pick an element $\alpha_0 \in \mathrm{U}(\mathfrak{A})[\alpha]$. Define the open ball

$$\mathcal{B}_*(\alpha_0) = \{\, \beta \in \mathcal{S}_*(\mathfrak{E}) : ||\alpha_0 - \beta|| < ||\Pi||^{-1}\, ||\alpha_0||^{-1} \,\},$$

and observe that by Proposition 2.4 $\mathcal{B}_*(\alpha_0)$ is a subset of $\mathrm{U}(\mathfrak{A})[\alpha_0] = \mathrm{U}(\mathfrak{A})[\alpha]$. Choose $u \in \mathrm{U}(\mathfrak{A})$ such that $\alpha_0 = u \cdot \alpha \cdot u^*$ and let $\sigma_* : \mathcal{B}_*(\alpha_0) \to \mathrm{U}(\mathfrak{A})$ be the continuous mapping given by

$$\sigma_*(\beta) = \Pi(\beta \times \alpha_0)\, u, \quad \beta \in \mathcal{B}(\alpha_0).$$

Since $\Pi(\beta \times \alpha_0) \in \mathrm{U}(\mathfrak{A})$, and

$$\beta = \Pi(\beta \times \alpha_0) \cdot \alpha_0 \cdot \Pi(\beta \times \alpha_0)^*,$$

for any $\beta \in \mathcal{B}_*(\alpha_0)$, the mapping σ_* is well defined and simple calculations show that $\rho_{\mathrm{U},\alpha} \circ \sigma_* = \mathrm{Id}_{\mathcal{B}_*(\alpha_0)}$. The proof of Theorem C is complete. □

2.3. Banach manifold structures on $\mathcal{S}(\mathfrak{E})$ and $\mathcal{S}_*(\mathfrak{E})$

In this subsection, we prove that the spaces $\mathcal{S}(\mathfrak{E})$ and $\mathcal{S}_*(\mathfrak{E})$ with their natural topologies are Banach manifolds. We start with a simple observation. The entire space $\mathcal{S}(\mathfrak{E})$ is a union of mutually disjoint $\mathrm{G}(\mathfrak{A})$-equivalence classes. Therefore, in order to get a Banach manifold structure on $\mathrm{G}(\mathfrak{A})$, it is enough to prove that each class $\mathrm{G}(\mathfrak{A})[\alpha]$, $\alpha \in \mathcal{S}(\mathfrak{E})$, is a Banach manifold. Similarly, for $\mathcal{S}_*(\mathfrak{E})$, we have to prove that each equivalence class $\mathrm{U}(\mathfrak{A})[\alpha]$, $\alpha \in \mathcal{S}_*(\mathfrak{E})$, has a Banach manifold structure. Based on Theorem C in the previous subsection, we will reach our goal by proving the next sequel to that theorem.

Theorem D (Banach manifold property of $\mathcal{S}(\mathfrak{E})$ and $\mathcal{S}_*(\mathfrak{E})$).

(i) *If $\alpha \in \mathcal{S}(\mathfrak{E})$, the coset space $\mathrm{G}(\mathfrak{A}) \,/\, \mathrm{G}(\mathfrak{A}_\alpha)$ is a complex analytic manifold. Moreover, the $\mathrm{G}(\mathfrak{A})$-equivalence class $\mathrm{G}(\mathfrak{A})[\alpha]$ with manifold structure inherited from $\mathrm{G}(\mathfrak{A})/\mathrm{G}(\mathfrak{A}_\alpha)$ is a complex analytic submanifold of the Banach space $\mathfrak{E}$.*

(ii) *If* $\alpha \in \mathcal{S}_*(\mathfrak{E})$*, the coset space* $\mathrm{U}(\mathfrak{A})/\mathrm{U}(\mathfrak{A}_\alpha)$ *is a real analytic manifold. Moreover, the* $\mathrm{U}(\mathfrak{A})$*-equivalence class* $\mathrm{U}(\mathfrak{A})[\alpha]$ *with manifold structure inherited from* $\mathrm{U}(\mathfrak{A})/\mathrm{U}(\mathfrak{A}_\alpha)$ *is a real analytic submanifold of the real Banach space* $\mathfrak{E}_\mathrm{h}$*.*

Proof. We rely on a general result labeled 4.12.5 in Bourbaki [15] that we will recall as part of the proof. To begin with, we consider the complex analytic action $\rho_\mathrm{G} : \mathrm{G}(\mathfrak{A}) \times \mathfrak{E} \to \mathfrak{E}$ of group $\mathrm{G}(\mathfrak{A})$ on $\mathfrak{E}$ defined by

$$\rho_\mathrm{G}(a, \varphi) = a \cdot \varphi \cdot a^{-1}, \quad a \in \mathrm{G}(\mathfrak{A}),\ \varphi \in \mathfrak{E}. \tag{2.15}$$

If $\alpha \in \mathcal{S}(\mathfrak{E})$, the equivalence class $\mathrm{G}(\mathfrak{A})[\alpha]$ is the orbit of α under this action. In other words, $\mathrm{G}(\mathfrak{A})[\alpha]$ is the range of the mapping $\rho_{\mathrm{G},\alpha} : \mathrm{G}(\mathfrak{A}) \to \mathfrak{E}$ given by

$$\rho_{\mathrm{G},\alpha}(a) = a \cdot \alpha \cdot a^{-1}, \quad a \in \mathrm{G}(\mathfrak{A}). \tag{2.16}$$

According to 4.12.5 in Bourbaki [15], in order to prove statement (i), we have to show that $\rho_{\mathrm{G},\alpha} : \mathrm{G}(\mathfrak{A}) \to \mathfrak{E}$ is a subimersion. Explicitly, we need to check that the associated tangent mapping at each point $a \in \mathrm{G}(\mathfrak{A})$, i.e., the differential $\mathrm{d}\rho_{\mathrm{G},\alpha}(a) : \mathrm{T}_a\mathrm{G}(\mathfrak{A}) \to \mathrm{T}_{\rho_{\mathrm{G},\alpha}(a)}\mathfrak{E} \equiv \mathfrak{E}$ of $\rho_{\mathrm{G},\alpha}$, has the following two properties:

(i) $\mathrm{Ker}\,\mathrm{d}\rho_{\mathrm{G},\alpha}(a)$ has a closed complement in $\mathrm{T}_a\mathrm{G}(\mathfrak{A})$,

(ii) $\mathrm{Ran}\,\mathrm{d}\rho_{\mathrm{G},\alpha}(a)$ is closed, with a closed complement in $\mathfrak{E}$.

We proceed by assuming that $a \in \mathrm{G}(\mathfrak{A})$ is fixed and note that

$$\mathrm{T}_a\mathrm{G}(\mathfrak{A}) = a\,\mathfrak{A} = \{\, a\,x : x \in \mathfrak{A} \,\}.$$

From (2.16) we get

$$\mathrm{d}\rho_{\mathrm{G},\alpha}(a)(a\,x) = a \cdot (x \cdot \alpha - \alpha \cdot x) \cdot a^{-1}, \quad x \in \mathfrak{A}. \tag{2.17}$$

Therefore, $\mathrm{Ker}\,\mathrm{d}\rho_{\mathrm{G},\alpha}(a) = a\,\mathfrak{A}_\alpha$, where $\mathfrak{A}_\alpha = \{\, x \in \mathfrak{A} : x \cdot \alpha = \alpha \cdot x \,\}$. Since, as we proved in Subsection 1.2, $\mathfrak{A}_\alpha$ has the closed direct complement $\mathfrak{A}_\alpha^\perp$ in $\mathfrak{A}$ defined by equation (1.8), property (i) is completely proved. Property (ii) is also a consequence of results from Section 1. Equation (2.17) is equivalent to

$$\mathrm{d}\rho_{\mathrm{G},\alpha}(a)(a\,x) = a \cdot \mathrm{d}\rho_{\mathrm{G},\alpha}(1_\mathfrak{A})(x) \cdot a^{-1}, \quad x \in \mathfrak{A},$$

hence $\mathrm{Ran}\,\mathrm{d}\rho_{\mathrm{G},\alpha}(a) = a\,\mathrm{Ran}\,\mathrm{d}\rho_{\mathrm{G},\alpha}(1_\mathfrak{A})\,a^{-1}$. By part (i) in Theorem A, we have

$$\mathrm{Ran}\,\mathrm{d}\rho_{\mathrm{G},\alpha}(1_\mathfrak{A}) = \{\, x \cdot \alpha - \alpha \cdot x : x \in \mathfrak{A} \,\} = \mathrm{T}_\alpha\,\mathcal{S}(\mathfrak{E}),$$

which by Lemma 1.8 is a closed subspace of $\mathfrak{E}$. In addition, part (ii) of the same theorem indicates that $\mathrm{T}_\alpha\,\mathcal{S}(\mathfrak{E})$ is the range of the continuous projection $\mathrm{P}_\alpha : \mathfrak{E} \to \mathfrak{E}$, and for that reason $\mathrm{T}_\alpha\,\mathcal{S}(\mathfrak{E})$ has a closed direct complement $\mathrm{T}_\alpha\,\mathcal{S}(\mathfrak{E})^\perp$ in $\mathfrak{E}$. The proof of statement (i) is complete.

Statement (ii) follows from similar arguments and a few appropriate adjustments. This time, we consider the real analytic action $\rho_{\mathrm{U}} : \mathrm{U}(\mathfrak{A}) \times \mathfrak{E}_{\mathrm{h}} \to \mathfrak{E}_{\mathrm{h}}$,

$$\rho_{\mathrm{U}}(u, \varphi) = u \cdot \varphi \cdot u^*, \quad u \in \mathrm{U}(\mathfrak{A}),\ \varphi \in \mathfrak{E}_{\mathrm{h}}.$$

If $\alpha \in \mathcal{S}_*(\mathfrak{E})$, the $\mathrm{U}(\mathfrak{A})$-equivalence class $\mathrm{U}(\mathfrak{A})[\alpha]$ is the range of the mapping $\rho_{\mathrm{U},\alpha} : \mathrm{U}(\mathfrak{A}) \to \mathfrak{E}_{\mathrm{h}}$ given by

$$\rho_{\mathrm{U},\alpha}(u) = u \cdot \alpha \cdot u^*, \quad u \in \mathrm{U}(\mathfrak{A}).$$

We have to show that $\rho_{\mathrm{U},\alpha} : \mathrm{U}(\mathfrak{A}) \to \mathfrak{E}_{\mathrm{h}}$ is a subimersion at each $u \in \mathrm{U}(\mathfrak{A})$, i.e., to check that the differential $\mathrm{d}\rho_{\mathrm{U},\alpha}(u) : \mathrm{T}_u\mathrm{U}(\mathfrak{A}) \to \mathrm{T}_{\rho_{\mathrm{U},\alpha}(u)}\mathfrak{E}_{\mathrm{h}} \equiv \mathfrak{E}_{\mathrm{h}}$ satisfies the following counterparts of properties (i) and (ii),

(iii) $\operatorname{Ker} \mathrm{d}\rho_{\mathrm{U},\alpha}(u)$ has a closed complement in $\mathrm{T}_u\mathrm{U}(\mathfrak{A})$,

(iv) $\operatorname{Ran} \mathrm{d}\rho_{\mathrm{U},\alpha}(u)$ is closed, with a closed complement in $\mathfrak{E}_{\mathrm{h}}$.

It all reduces to justify two claims. First, $\mathfrak{A}_\alpha \cap \mathfrak{A}_{\mathrm{sh}}$ has a closed complement in $\mathfrak{A}_{\mathrm{sh}}$, which implies (iii), and second, $\operatorname{Ran} \mathrm{d}\rho_{\mathrm{U},\alpha}(1_{\mathfrak{A}}) = \mathrm{T}_\alpha \mathcal{S}_*(\mathfrak{E}) = \mathrm{T}_\alpha \mathcal{S}(\mathfrak{E}) \cap \mathfrak{E}_{\mathrm{h}}$ is a closed subspace of $\mathfrak{E}_{\mathrm{h}}$ with a closed complement, which yields (iv). Both claims are consequences of results from Section 1 related to involutive algebra environments and their structure manifolds. The proof is complete. □

Lemma 2.6. *Suppose $\sigma \in \mathcal{S}(\mathfrak{E})$ and let $\alpha = a \cdot \sigma \cdot a^{-1} \in \mathrm{G}(\mathfrak{A})[\sigma]$, where $a \in \mathrm{G}(\mathfrak{A})$. Let $\mathfrak{A}_\sigma, \mathfrak{A}_\alpha \subseteq \mathfrak{A}$ be the associated subalgebras with corresponding projections $\pi_\sigma, \pi_\alpha : \mathfrak{A} \to \mathfrak{A}$. Consider the algebra automomorphism*

$$\tau_{\sigma,\alpha} : \mathfrak{A} \to \mathfrak{A},\ \tau_{\sigma,\alpha}(z) = a\, z\, a^{-1}, \quad z \in \mathfrak{A}.$$

Then $\tau_{\sigma,\alpha} \circ \pi_\sigma = \pi_\alpha \circ \tau_{\sigma,\alpha}$ and $z \in \mathfrak{A}_\sigma^\perp$ only if $\tau_{\sigma,\alpha}(z) \in \mathfrak{A}_\alpha^\perp$.
In case $\sigma \in \mathcal{S}_*(\mathfrak{E})$ and $a = u \in \mathrm{U}(\mathfrak{A})$, then $\alpha \in \mathrm{U}(\mathfrak{A})[\sigma]$, $\tau_{\sigma,\alpha} : \mathfrak{A} \to \mathfrak{A}$ is an involutive automomorphism, and $z \in \mathfrak{A}_\sigma^\perp \cap \mathfrak{A}_{\mathrm{sh}}$ only if $\tau_{\sigma,\alpha}(z) \in \mathfrak{A}_\alpha^\perp \cap \mathfrak{A}_{\mathrm{sh}}$.

Proof. The only statement that requires a proof is $\tau_{\sigma,\alpha} \circ \pi_\sigma = \pi_\alpha \circ \tau_{\sigma,\alpha}$. Using equation (1.2) for the two projections, from $\sigma = a^{-1} \cdot \alpha \cdot a$ we have

$$\begin{aligned}\tau_{\sigma,\alpha} \circ \pi_\sigma(z) &= \tau_{\sigma,\alpha}(\Pi(\sigma \cdot z \times \sigma)) = a\Pi(a^{-1} \cdot \alpha \cdot a\, z \times a^{-1} \cdot \alpha \cdot a)a^{-1} \\ &= \Pi(\alpha \cdot a\, z \times a^{-1} \cdot \alpha) = \Pi(\alpha \cdot a\, z\, a^{-1} \times \alpha) = \pi_\alpha \circ \tau_{\sigma,\alpha}(z), \quad z \in \mathfrak{A}.\end{aligned}$$

Property $z \in \mathfrak{A}_\sigma^\perp$ only if $\tau_{\sigma,\alpha}(z) \in \mathfrak{A}_\alpha^\perp$ is a direct consequence, and the proof of the second part is similar. □

The next result will provide the Banach manifolds $\mathcal{S}(\mathfrak{E})$ and $\mathcal{S}_*(\mathfrak{E})$ with appropriate charts and atlases.

Theorem E (Analytic atlases on $\mathcal{S}(\mathfrak{E})$ and $\mathcal{S}_*(\mathfrak{E})$). *Suppose* $\sigma \in \mathcal{S}(\mathfrak{E})$, $\alpha \in \mathrm{G}(\mathfrak{A})[\sigma] \subseteq \mathcal{S}(\mathfrak{E})$, *and let* $\exp_\alpha : \mathfrak{A}_\alpha^\perp \to \mathrm{G}(\mathfrak{A})[\sigma]$ *be the complex analytic mapping defined by*

$$\exp_\alpha(x) = \exp(x) \cdot \alpha \cdot \exp(x)^{-1}, \quad x \in \mathfrak{A}_\alpha^\perp. \tag{2.18}$$

For $\sigma \in \mathcal{S}_*(\mathfrak{E})$ *and* $\alpha \in \mathrm{U}(\mathfrak{A})[\sigma] \subseteq \mathcal{S}_*(\mathfrak{E})$ *let* $\exp_{\alpha,*} : \mathfrak{A}_\alpha^\perp \cap \mathfrak{A}_{\mathrm{sh}} \to \mathrm{U}(\mathfrak{A})[\sigma]$ *be the real analytic mapping given by*

$$\exp_{\alpha,*}(x) = \exp(x) \cdot \alpha \cdot \exp(x)^*, \quad x \in \mathfrak{A}_\alpha^\perp \cap \mathfrak{A}_{\mathrm{sh}}. \tag{2.19}$$

The two mappings are well defined and have the following properties:

(i) *There exist an open neighborhood* $\mathcal{W}_\alpha$ *of* 0 *in* $\mathfrak{A}_\alpha^\perp$ *and an open neighborhood* $\mathcal{V}_\alpha$ *of* α *in* $\mathrm{G}(\mathfrak{A})[\sigma]$ *such that the mapping*

$$\varepsilon_\alpha : \mathcal{W}_\alpha \to \mathcal{V}_\alpha, \ \varepsilon_\alpha(x) = \exp_\alpha(x), \quad x \in \mathcal{W}_\alpha,$$

is a diffeomorphism. Next, let $\tau_{\sigma,\alpha} : \mathfrak{A} \to \mathfrak{A}$ *be the automorphism in Lemma* 2.6 *and introduce the open neighborhood* $\mathcal{U}_\alpha = \tau_{\sigma,\alpha}^{-1}(\mathcal{W}_\alpha)$ *of* 0 *in* $\mathfrak{A}_\sigma^\perp$. *The complex analytic function*

$$\chi_{\sigma,\alpha} : \mathcal{U}_\alpha \to \mathcal{V}_\alpha, \ \chi_{\sigma,\alpha}(z) = \exp_\alpha \circ \tau_{\sigma,\alpha}(z), \quad z \in \mathcal{V}_\alpha, \tag{2.20}$$

is a chart on $\mathrm{G}(\mathfrak{A})[\sigma]$. *The collection of charts* $\{\chi_{\sigma,\alpha} : \alpha \in \mathrm{G}(\mathfrak{A})[\sigma]\}$ *is an atlas on* $\mathrm{G}(\mathfrak{A})[\sigma]$ *with model Banach space* $\mathfrak{A}_\sigma^\perp$.

(ii) *There exist an open neighborhood* $\mathcal{W}_{\alpha,*}$ *of* 0 *in* $\mathfrak{A}_\alpha^\perp \cap \mathfrak{A}_{\mathrm{sh}}$ *and an open neighborhood* $\mathcal{V}_{\alpha,*}$ *of* α *in* $\mathrm{U}(\mathfrak{A})[\sigma]$ *such that the mapping*

$$\varepsilon_{\alpha,*} : \mathcal{W}_{\alpha,*} \to \mathcal{V}_{\alpha,*}, \ \varepsilon_{\alpha,*}(x) = \exp_{\alpha,*}(x), \quad x \in \mathcal{W}_{\alpha,*}$$

is a diffeomorphism. Introduce the open neighborhood $\mathcal{U}_{\alpha,*} = \tau_{\sigma,\alpha}^{-1}(\mathcal{W}_{\alpha,*})$ *of* 0 *in* $\mathfrak{A}_\sigma^\perp \cap \mathfrak{A}_{\mathrm{sh}}$ *and the real analytic chart on* $\mathrm{U}(\mathfrak{A})[\sigma] \subseteq \mathcal{S}_*(\mathfrak{E})$ *given by*

$$\chi_{\sigma,\alpha,*} : \mathcal{U}_{\alpha,*} \to \mathcal{V}_{\alpha,*}, \ \chi_{\sigma,\alpha,*}(z) = \exp_{\alpha,*} \circ \tau_{\sigma,\alpha}(z), \quad z \in \mathcal{V}_{\alpha,*}. \tag{2.21}$$

The collection of charts $\{\chi_{\sigma,\alpha,*} : \alpha \in \mathrm{U}(\mathfrak{A})[\sigma]\}$ *is an atlas on* $\mathrm{U}(\mathfrak{A})[\sigma]$ *with model Banach space* $\mathfrak{A}_\sigma^\perp \cap \mathfrak{A}_{\mathrm{sh}}$.

Proof. We start with statement (i). We first observe that $\exp_\alpha(0) = \alpha$. Next, we use equation (2.18) and compute the differential of $\exp_\alpha$ at $0 \in \mathfrak{A}_\alpha^\perp$, which is the linear mapping $\mathrm{d}\exp_\alpha(0) : \mathrm{T}_0\, \mathfrak{A}_\alpha^\perp \equiv \mathfrak{A}_\alpha^\perp \to \mathrm{T}_\alpha\, \mathcal{S}(\mathfrak{E})$ given by

$$\mathrm{d}\exp_\alpha(0)(x) = x \cdot \alpha - \alpha \cdot x, \quad x \in \mathfrak{A}_\alpha^\perp.$$

Comparing the last equation with equation (1.15) in part (i) of Theorem A that defines the linear mapping $\mathrm{T}_\alpha : \mathfrak{A}_\alpha^\perp \to \mathrm{T}_\alpha\, \mathcal{S}(\mathfrak{E})$, we get

$$\mathrm{d}\exp_\alpha(0)(x) = \mathrm{T}_\alpha(x), \quad x \in \mathfrak{A}_\alpha^\perp, \text{ i.e., } \mathrm{d}\exp_\alpha(0) = \mathrm{T}_\alpha.$$

Consequently, $\mathrm{d}\exp_\alpha(0) : \mathrm{T}_0\,\mathfrak{A}_\alpha^\perp \to \mathrm{T}_\alpha\,\mathcal{S}(\mathfrak{E})$ is a Banach space isomorphism, and $\exp_\alpha$ yields a diffeomorphism ε_α from an open neighborhood $\mathcal{W}_\alpha$ of 0 in $\mathfrak{A}_\alpha^\perp$ to an open neighborhood $\mathcal{V}_\alpha$ of $\alpha \in \mathrm{G}(\mathfrak{A})[\sigma]$. Using the open set $\mathcal{U}_\alpha$ and equation (2.20), we get the chart $\chi_{\sigma,\alpha} : \mathcal{U}_\alpha \to \mathcal{V}_\alpha$ on $\mathrm{G}(\mathfrak{A})[\sigma]$. Since by Theorem D we already know that $\mathcal{S}(\mathfrak{E})$ is a Banach manifold, different charts $\chi_{\sigma,\alpha}$, $\chi_{\sigma,\beta}$, $\alpha, \beta \in \mathrm{G}(\mathfrak{A})[\sigma]$, with overlapping domains are analytically correlated and for this reason the collection of charts is an analytic atlas on $\mathrm{G}(\mathfrak{A})[\sigma]$.

With regard to statement (ii), similar calculations show that

$$\mathrm{d}\exp_{\alpha,*}(0)(x) = \mathrm{T}_{\alpha,*}(x), \quad x \in \mathfrak{A}_\alpha^\perp \cap \mathfrak{A}_{\mathrm{sh}}, \text{ i.e., } \mathrm{d}\exp_{\alpha,*}(0) = \mathrm{T}_{\alpha,*},$$

where $\mathrm{T}_{\alpha,*} : \mathfrak{A}_\alpha^\perp \cap \mathfrak{A}_{\mathrm{sh}} \to \mathrm{T}_\alpha\mathcal{S}_*(\mathfrak{E})$ is the Banach space isomorphism in part (iii) in Theorem A. The rest of the proof is obvious. $\square$

2.4. **Standard Ehresmann connections and geodesics**

Theorem C and part of the proof of Theorem D imply the existence of natural Ehresmann connections on the Banach principal fiber bundles associated with structure manifolds of algebra environments. The article by Ehresmann [22] is a recommended reference in this regard. The related concepts of horizontal lifts of smooth curves and smooth geodesics would just upgrade our tool kit.

Supose $(\mathfrak{E}, \Pi, \mathfrak{A})$ is a Banach algebra environment, $\alpha \in \mathcal{S}(\mathfrak{E})$, and let

$$\rho_{\mathrm{G},\alpha} : \mathrm{G}(\mathfrak{A}) \to \mathrm{G}(\mathfrak{A})[\alpha], \; \rho_{\mathrm{G},\alpha}(a) = a \cdot \alpha \cdot a^{-1}, \quad a \in \mathrm{G}(\mathfrak{A}),$$

be the principal fiber bundle introduced in Theorem C. As part of the proof of Theorem D, we noted that $\mathrm{d}\rho_{\mathrm{G},\alpha}(a) : \mathrm{T}_a\mathrm{G}(\mathfrak{A}) \to \mathrm{T}_{\rho_{\mathrm{G},\alpha(a)}}\mathrm{G}(\mathfrak{A})[\alpha]$, where $\mathrm{T}_a\mathrm{G}(\mathfrak{A}) = a\mathfrak{A} = \{a\,x : x \in \mathfrak{A}\}$, $a \in \mathrm{G}(\mathfrak{A})$, is given by

$$\mathrm{d}\rho_{\mathrm{G},\alpha}(a)(a\,x) = a \cdot (x \cdot \alpha - \alpha \cdot x) \cdot a^{-1}, \quad x \in \mathfrak{A}.$$

According to custom, the vertical subspace of $\mathrm{T}_a\mathrm{G}(\mathfrak{A})$ is defined as

$$\mathrm{V}_a\mathrm{G}(\mathfrak{A}) = \operatorname{Ker}\mathrm{d}\rho_{\mathrm{G},\alpha}(a) = a\,\mathfrak{A}_\alpha, \quad \mathfrak{A}_\alpha = \{x \in \mathfrak{A} : x \cdot \alpha = \alpha \cdot x\}.$$

Definition 2.7. The standard Ehresmann connection on $\mathrm{G}(\mathfrak{A})$ assigns to each $a \in \mathrm{G}(\mathfrak{A})$ the closed direct complement of $\mathrm{V}_a\mathrm{G}(\mathfrak{A})$ in $\mathrm{T}_a\mathrm{G}(\mathfrak{A})$ given by

$$\mathrm{H}_a\mathrm{G}(\mathfrak{A}) = a\,\mathfrak{A}_\alpha^\perp, \quad \mathfrak{A}_\alpha^\perp = \mathfrak{A} \ominus \mathfrak{A}_\alpha = \{\, x \in \mathfrak{A} : \pi_\alpha(x) = \Pi(\alpha \cdot x \times \alpha) = 0\},$$

called the horizontal subspace at a.

Definition 2.8. Let $\gamma : \mathbb{R} \to \mathcal{S}(\mathfrak{E})$ be a smooth curve with $\gamma(0) = \alpha$. A local horizontal lift of γ centered at α is a smooth mapping $\Gamma : (-\,\varepsilon, \varepsilon) \to \mathrm{G}(\mathfrak{A})$, $\varepsilon > 0$, satisfying the following requirements,

(i) $\gamma(t) = \Gamma(t) \cdot \alpha \cdot \Gamma(t)^{-1}$, $t \in (-\,\varepsilon, \varepsilon)$, $\Gamma(0) = 1_{\mathfrak{A}}$,

(ii) $\Gamma'(t) \in \mathrm{H}_{\Gamma(t)}\mathrm{G}(\mathfrak{A}) = \Gamma(t)\,\mathfrak{A}_\alpha^\perp$, $t \in (-\varepsilon, \varepsilon)$,

where $\Gamma'(t) = d\,\Gamma(t)/d\,t$ is computed by regarding Γ as an $\mathfrak{A}$-valued function.

Proposition 2.9. *Requirements in Definition* 2.8 *are equivalent to*

$$\Gamma'(t) = \Sigma_{\gamma(t)}(\gamma'(t))\Gamma(t),\ \ \gamma'(t) = d\,\gamma(t)/d\,t,\ \ t \in (-\varepsilon, \varepsilon),\ \ \Gamma(0) = 1_{\mathfrak{A}}, \tag{2.22}$$

where $\gamma'(t) = d\,\gamma(t)/d\,t$ is computed by regarding γ as an $\mathfrak{E}$-valued function.

Proof. Requirement (ii) states that $\Gamma(t)^{-1}\,\Gamma'(t) \in \mathfrak{A}_\alpha^\perp$, i.e.,

$$\pi_\alpha(\,\Gamma(t)^{-1}\,\Gamma'(t)\,) = \Pi(\,\alpha \cdot \Gamma(t)^{-1}\,\Gamma'(t) \times \alpha\,) = 0, \quad t \in (-\,\varepsilon, \varepsilon). \tag{2.23}$$

Since Π is $\mathfrak{A}$-bilinear, left multiplication by $\Gamma(t) \in \mathrm{G}(\mathfrak{A})$ and requirement (i) yield the equivalent equation

$$\Pi(\gamma(t) \cdot \Gamma'(t) \times \alpha) = 0. \tag{2.24}$$

From (i) we have $\gamma(t) \cdot \Gamma(t) = \Gamma(t) \cdot \alpha$, hence

$$\gamma'(t) \cdot \Gamma(t) + \gamma(t) \cdot \Gamma'(t) = \Gamma'(t) \cdot \alpha,\ \text{ i.e., }\ \gamma(t) \cdot \Gamma'(t) = \Gamma'(t) \cdot \alpha - \gamma'(t) \cdot \Gamma(t).$$

Substituting the last equation into (2.24), we get

$$\Pi(\Gamma'(t) \cdot \alpha \times \alpha - \gamma'(t) \times \Gamma(t) \cdot \alpha) = 0. \tag{2.25}$$

Since $\alpha \times \alpha = \alpha$, $\Gamma(t) \cdot \alpha = \gamma(t) \cdot \Gamma(t)$, and $\Pi(\alpha) = 1_{\mathfrak{A}}$, equation (2.25) becomes

$$\Gamma'(t) = \Pi(\,\gamma'(t) \times \gamma(t)\,) \cdot \Gamma(t). \tag{2.26}$$

Since $\gamma'(t) \in \mathrm{T}_{\gamma(t)}\mathcal{S}(\mathfrak{E})$, according to equation (1.14) in Subsection 1.3, we have

$$\Sigma_{\gamma(t)}(\gamma'(t))\Gamma(t) = \Pi(\gamma'(t) \times \gamma(t)) = -\Pi(\gamma(t) \times \gamma'(t)),$$

hence (2.26) reduces to equation (2.22).

We next prove that any local solution to the initial value problem (2.22) is a local horizontal lift of γ centered at $\alpha = \gamma(0)$. Since $\Gamma(0) = 1_{\mathfrak{A}}$, we assume that $\varepsilon > 0$ is small enough such that $\Gamma(t) \in \mathrm{G}(\mathfrak{A})$, $t \in (-\varepsilon, \varepsilon)$, and define

$$\delta : (-\,\varepsilon, \varepsilon) \to \mathcal{S}(\mathfrak{E}),\ \delta(t) = \Gamma(t)^{-1} \cdot \gamma(t) \cdot \Gamma(t), \quad t \in (-\varepsilon, \varepsilon).$$

Using equation (2.22), simple calculations show that

$$\begin{aligned}
\delta'(t) &= -\Gamma(t)^{-1}\Gamma'(t)\Gamma(t)^{-1} \cdot \gamma(t) \cdot \Gamma(t) + \Gamma(t)^{-1} \cdot \gamma'(t) \cdot \Gamma(t) \\
&\quad + \Gamma(t)^{-1} \cdot \gamma(t) \cdot \Gamma'(t) \\
&= -\Gamma(t)^{-1}\,\Sigma_{\gamma(t)}(\gamma'(t)) \cdot \gamma(t) \cdot \Gamma(t) + \Gamma(t)^{-1} \cdot \gamma'(t) \cdot \Gamma(t) \\
&\quad + \Gamma(t)^{-1} \cdot \gamma(t) \cdot \Sigma_{\gamma(t)}(\gamma'(t))\,\Gamma(t).
\end{aligned}$$

Left and right multiplications by $\Gamma(t)$ and $\Gamma(t)^{-1}$, both in $\mathrm{G}(\mathfrak{A})$, yield

$$\Gamma(t)\,\delta'(t)\Gamma(t)^{-1} = -\Sigma_{\gamma(t)}(\gamma'(t))\cdot\gamma(t) + \gamma'(t) + \gamma(t)\cdot\Sigma_{\gamma(t)}(\gamma'(t)).$$

We now adapt statement (ii) in Proposition 1.13 to this situation and get

$$\gamma'(t) = \Sigma_{\gamma(t)}(\gamma'(t))\cdot\gamma(t) - \gamma(t)\cdot\Sigma_{\gamma(t)}(\gamma'(t)),$$

an equation that implies $\Gamma(t)\delta'(t)\Gamma(t)^{-1} = 0$, hence $\delta'(t) = 0$. Consequently, $\delta : (-\,\varepsilon,\varepsilon) \to \mathcal{S}(\mathfrak{E})$ is a constant function and

$$\delta(t) = \Gamma(t)^{-1}\cdot\gamma(t)\cdot\Gamma(t) = \delta(0) = \alpha, \text{ i.e., } \gamma(t) = \Gamma(t)\cdot\alpha\cdot\Gamma(t)^{-1}. \tag{2.27}$$

We just derived requirement (i). From equation (2.22), we next have

$$\pi_\alpha(\Gamma(t)^{-1}\Gamma'(t)) = \Pi[\alpha\cdot\Gamma(t)^{-1}\Gamma'(t)\times\alpha] = \Pi[\alpha\cdot\Gamma(t)^{-1}\Sigma_{\gamma(t)}(\gamma'(t))\times\Gamma(t)\cdot\alpha].$$

Using (2.27), we substitute $\alpha = \Gamma(t)^{-1}\cdot\gamma(t)\cdot\Gamma(t)$ and get

$$\begin{aligned}\pi_\alpha(\Gamma(t)^{-1}\Gamma'(t)) &= \Pi[\Gamma(t)^{-1}\cdot\gamma(t)\cdot\Sigma_{\gamma(t)}(\gamma'(t))\times\gamma(t)\cdot\Gamma(t)]\\ &= \Gamma(t)^{-1}\Pi[\gamma(t)\cdot\Sigma_{\gamma(t)}(\gamma'(t))\times\gamma(t)]\Gamma(t)\\ &= \Gamma(t)^{-1}\pi_{\gamma(t)}(\Sigma_{\gamma(t)}(\gamma'(t)))\Gamma(t).\end{aligned}$$

By Proposition 1.13, $\operatorname{Ran}\Sigma_{\gamma(t)} = \mathfrak{A}^{\perp}_{\gamma(t)}$, i.e., $\pi_{\gamma(t)}(\Sigma_{\gamma(t)}(\gamma'(t)) = 0$, hence $\pi_\alpha(\Gamma(t)^{-1}\Gamma'(t)) = 0$, i.e., $\Gamma(t)^{-1}\Gamma'(t) \in \mathfrak{A}^{\perp}_\alpha$, or $\Gamma'(t) \in \Gamma(t)\mathfrak{A}^{\perp}_\alpha = \mathrm{H}_{\Gamma(t)}\mathrm{G}(\mathfrak{A})$, which is requirement (ii) in Definition 2.8. The proof is complete. □

Existence and uniqueness of local solutions of (2.22) leads to the following.

COROLLARY 2.10. *Any smooth curve* $\gamma : \mathbb{R} \to \mathcal{S}(\mathfrak{E})$ *has a unique local horizontal lift* $\Gamma : (-\varepsilon,\varepsilon) \to \mathrm{G}(\mathfrak{A})$, $\varepsilon > 0$, *centered at* $\alpha = \gamma(0)$.

Assume next that $(\mathfrak{E},\Pi,\mathfrak{A})$ is an involutive Banach algebra environment and $\alpha \in \mathcal{S}_*(\mathfrak{E})$. The principal fiber bundle introduced in Theorem C is

$$\rho_{\mathrm{U},\alpha} : \mathrm{U}(\mathfrak{A}) \to \mathrm{U}(\mathfrak{A})[\alpha],\ \rho_{\mathrm{U},\alpha}(u) = u\cdot\alpha\cdot u^*, \quad u \in \mathrm{U}(\mathfrak{A}).$$

For each $u \in \mathrm{U}(\mathfrak{A})$, $\mathrm{T}_u\mathrm{U}(\mathfrak{A}) = u\,\mathfrak{A}_{\mathrm{sh}} = \{u\,x : x \in \mathfrak{A}_{\mathrm{sh}}\}$. The differential of $\rho_{\mathrm{U},\alpha}$, $\mathrm{d}\rho_{\mathrm{U},\alpha}(u) : \mathrm{T}_u\mathrm{U}(\mathfrak{A}) \to \mathrm{T}_{\rho_{\mathrm{U},\alpha(u)}}\mathrm{U}(\mathfrak{A})[\alpha]$, is given by

$$\mathrm{d}\rho_{\mathrm{U},\alpha}(u)(u\,x) = u\cdot(x\cdot\alpha - \alpha\cdot x)\cdot u^*, \quad x \in \mathfrak{A}_{\mathrm{sh}}.$$

The vertical subspace of $\mathrm{T}_u\mathrm{U}(\mathfrak{A})$ is $\mathrm{V}_u\mathrm{U}(\mathfrak{A}) = \operatorname{Ker}\mathrm{d}\rho_{\mathrm{U},\alpha}(u) = u\mathfrak{A}_\alpha \cap \mathfrak{A}_{\mathrm{sh}}$.

Definition 2.11. The Ehresmann connection on $\mathrm{U}(\mathfrak{A})$ is defined using as complements of $\mathrm{V}_u\mathrm{U}(\mathfrak{A})$ the horizontal subspaces $\mathrm{H}_u\mathrm{U}(\mathfrak{A}) = u\,\mathfrak{A}^{\perp}_\alpha \cap \mathfrak{A}_{\mathrm{sh}}$.

The local horizontal lift $\Gamma : (-\varepsilon,\varepsilon) \to \mathrm{U}(\mathfrak{A})$ of a smooth curve $\gamma : \mathbb{R} \to \mathcal{S}_*(\mathfrak{E})$ centered at $\alpha = \gamma(0)$ is introduced by adapting Definition 2.8, and is the unique local solution of the initial value problem

$$\Gamma'(t) = \Sigma_{\gamma(t),*}(\gamma'(t))\,\Gamma(t), \quad t \in (-\varepsilon,\varepsilon),\ \Gamma(0) = 1_{\mathfrak{A}}.$$

The standard Ehresmann connections on $\mathrm{G}(\mathfrak{A})$ and $\mathrm{U}(\mathfrak{A})$ provide canonical linear connections on the structure manifolds $\mathcal{S}(\mathfrak{E})$ or $\mathcal{S}_*(\mathfrak{E})$ that could be used to define geodesics. We are not going to follow this traditional approach. Since structure manifolds are closed submanifolds of Banach spaces, we would rely on a geometric definition that uses the projections $\mathrm{P}_\alpha : \mathfrak{E} \to \mathfrak{E}$, $\alpha \in \mathcal{S}(\mathfrak{E})$, or $\mathrm{P}_{\alpha,*} : \mathfrak{E}_{\mathrm{h}} \to \mathfrak{E}_{\mathrm{h}}$, $\alpha \in \mathcal{S}_*(\mathfrak{E})$, onto the tangent spaces $\mathrm{T}_\alpha\mathcal{S}(\mathfrak{E})$ or $\mathrm{T}_\alpha\mathcal{S}_*(\mathfrak{E})$ defined in parts (ii) and (iv) of Theorem A. Their complementary subspaces, denoted by $\mathrm{T}^\perp_\alpha\mathcal{S}(\mathfrak{E})$ or $\mathrm{T}^\perp_\alpha\mathcal{S}_*(\mathfrak{E})$, are called normal spaces.

Definition 2.12. Suppose $(\mathfrak{E}, \Pi, \mathfrak{A})$ is a Banach algebra environment. A smooth curve $\gamma : (\tau, \varepsilon) \to \mathcal{S}(\mathfrak{E})$, $-\infty \le \tau < \varepsilon \le \infty$, is a geodesic provided

$$\gamma''(t) \in \mathrm{T}^\perp_{\gamma(t)}\mathcal{S}(\mathfrak{E}), \text{ i.e., } \mathrm{P}_{\gamma(t)}(\gamma''(t)) = 0, \quad t \in (\tau, \varepsilon). \tag{2.28}$$

If $(\mathfrak{E}, \Pi, \mathfrak{A})$ is involutive, the smooth geodesics $\gamma : (\tau, \varepsilon) \to \mathcal{S}_*(\mathfrak{E})$ satisfy

$$\gamma''(t) \in \mathrm{T}^\perp_{\gamma(t)}\mathcal{S}_*(\mathfrak{E}), \text{ i.e., } \mathrm{P}_{\gamma(t),*}(\gamma''(t)) = 0, \quad t \in (\tau, \varepsilon). \tag{2.29}$$

Proposition 2.13. *Suppose $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{S}(\mathfrak{E})$ is smooth with $\gamma(0) = \alpha$ and $\gamma'(0) = \theta \in \mathrm{T}_\alpha\mathcal{S}(\mathfrak{E})$. Let $x = \Sigma_\alpha(\theta) \in \mathfrak{A}^\perp_\alpha$, and let $\Gamma : (-\varepsilon, \varepsilon) \to \mathrm{G}(\mathfrak{A})$ be the horizontal lift of γ centered at α. The following statements are equivalent:*

(i) *γ is a geodesic.*

(ii) *$\Sigma_{\gamma(t)}(\gamma''(t)) = 0$, $t \in (-\varepsilon, \varepsilon)$.*

(iii) *$\Sigma_{\gamma(t)}(\gamma'(t)) = x$, $t \in (-\varepsilon, \varepsilon)$.*

(iv) *$\Gamma(t) = \exp(tx)$, $t \in (-\varepsilon, \varepsilon)$.*

Proof. Equation (1.16) in Subsection 1.3 that defines $\mathrm{P}_{\gamma(t)}$ implies

$$\mathrm{P}_{\gamma(t)}(\gamma''(t)) = \Sigma_{\gamma(t)}(\gamma''(t)) \cdot \gamma(t) - \gamma(t) \cdot \Sigma_{\gamma(t)}(\gamma''(t)), \quad t \in (-\varepsilon, \varepsilon).$$

Consequently, (2.28) in Definition 2.11 is equivalent to $\Sigma_{\gamma(t)}(\gamma''(t)) \in \mathfrak{A}_{\gamma(t)}$. By part (i) in Proposition 1.13, we have $\Sigma_{\gamma(t)}(\gamma''(t)) \in \mathfrak{A}^\perp_{\gamma(t)}$, regardless any assumptions. Therefore, statements (i) and (ii) are equivalent.

Direct calculations lead to

$$\frac{\mathrm{d}}{\mathrm{d}t}\Sigma_{\gamma(t)}(\gamma'(t)) = 2^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\Pi(\gamma'(t) \times \gamma(t) - \gamma(t) \times \gamma'(t)) = \Sigma_{\gamma(t)}(\gamma''(t)).$$

Assuming (ii), we get that $\Sigma_{\gamma(t)}(\gamma'(t))$ is a constant function with value at $t = 0$ equal to $\Sigma_\alpha(\theta) = x$, which is statement (iii). Since obviously (ii) follows from (iii), statements (ii) and (iii) are equivalent. The initial value problem (2.22) in Proposition 2.9 implies that (iii) is equivalent to $\Gamma'(t) = x\,\Gamma(t)$, $\Gamma(0) = 1_{\mathfrak{A}}$, with the unique solution $\Gamma(t) = \exp(tx)$. The proof is complete. □

Corollary 2.14. *For any $\alpha \in \mathcal{S}(\mathfrak{E})$ and $\theta \in \mathrm{T}_\alpha\mathcal{S}(\mathfrak{E})$ there exists a unique geodesic $\gamma : (-\infty, \infty) \to \mathcal{S}(\mathfrak{E})$ with $\gamma(0) = \alpha$ and $\gamma'(0) = \theta$ given by*

$$\gamma(t) = \exp(tx) \cdot \alpha \cdot \exp(tx)^{-1}, \quad x = \Sigma_\alpha(\theta) \in \mathfrak{A}_\alpha^\perp, \quad t \in (-\infty, \infty). \tag{2.30}$$

Whenever $\alpha \in \mathcal{S}_*(\mathfrak{E})$ and $\theta \in \mathrm{T}_\alpha\mathcal{S}_*(\mathfrak{E})$, the previous equation defines the unique geodesic $\gamma : (-\infty, \infty) \to \mathcal{S}_*(\mathfrak{E})$ with initial values $\gamma(0) = \alpha$, $\gamma'(0) = \theta$, where $x = \Sigma_{\alpha,*}(\theta) \in \mathfrak{A}_\alpha^\perp \cap \mathfrak{A}_{\mathrm{sh}}$ and $\exp(tx) \in \mathrm{U}(\mathfrak{A})$, $t \in (-\infty, \infty)$.
In particular, all geodesics on $\mathcal{S}(\mathfrak{E})$ or $\mathcal{S}_*(\mathfrak{E})$ are complete.

We will conclude Section 2 with a result that points out an expected property of short geodesic paths on $\mathcal{S}_*(\mathfrak{E})$ under the assumption that $\mathfrak{A}$ is a C^*-algebra. As prerequisites, we need to define the length of smooth paths on $\mathcal{S}_*(\mathfrak{E})$ and find a convenient way of computing it.

To this end, let $\gamma : [\,0, \varepsilon\,] \to \mathcal{S}_*(\mathfrak{E})$ be a smooth path with associated horizontal lift $\Gamma : [\,0, \varepsilon\,] \to \mathrm{U}(\mathfrak{A})$, $\Gamma(0) = 1_{\mathfrak{A}}$. The tangent space $\mathrm{T}_\alpha\mathcal{S}_*(\mathfrak{E}) \subseteq \mathfrak{E}_{\mathrm{h}}$ has a norm inherited from $\mathfrak{E}_{\mathrm{h}}$, which is not adequate with regard to our purposes. Instead, we introduce on $\mathrm{T}_\alpha\mathcal{S}_*(\mathfrak{E})$ a norm $||\cdot||_\alpha$ using the norm $||\cdot||$ of $\mathfrak{A}$. From Theorem A, we know that $\Sigma_{\alpha,*}|\mathrm{T}_\alpha\mathcal{S}_*(\mathfrak{E}) : \mathrm{T}_\alpha\mathcal{S}_*(\mathfrak{E}) \to \mathfrak{A}_\alpha^\perp \cap \mathfrak{A}_{\mathrm{sh}}$ is a vector space isomorphism. Consequently, the function

$$||\cdot||_\alpha : \mathrm{T}_\alpha\mathcal{S}_*(\mathfrak{E}) \to [\,0, \infty\,), \;\; ||\,\theta\,||_\alpha = ||\,\Sigma_{\alpha,*}(\theta)\,||, \quad \theta \in \mathrm{T}_\alpha\mathcal{S}_*(\mathfrak{E}),$$

is a bona fide norm. Returning to $\gamma : [\,0, \varepsilon\,] \to \mathcal{S}_*(\mathfrak{E})$, we define its lenght as

$$\mathrm{L}(\gamma) = \int_0^\varepsilon ||\,\gamma'(t)\,||_{\gamma(t)}\,\mathrm{d}t. \tag{2.31}$$

Lemma 2.15. *Under previous assumptions, using $\Gamma' : [0, \varepsilon] \to \mathfrak{A}$, we have*

$$\mathrm{L}(\gamma) = \int_0^\varepsilon ||\,\Gamma'(t)\,||\,\mathrm{d}t. \tag{2.32}$$

Proof. Let $\alpha = \gamma(0)$. The derivative of $\gamma(t) = \Gamma(t) \cdot \alpha \cdot \Gamma(t)^{-1}$ equals

$$\gamma'(t) = \Gamma'(t) \cdot \alpha \cdot \Gamma(t)^{-1} - \Gamma(t) \cdot \alpha \cdot \Gamma(t)^{-1}\,\Gamma'(t)\,\Gamma(t)^{-1}.$$

Substituting $\Gamma'(t) = \Sigma_{\gamma(t),*}(\gamma'(t))\,\Gamma(t)$, we get

$$\begin{aligned}\gamma'(t) &= \Sigma_{\gamma(t),*}(\gamma'(t))\,\Gamma(t) \cdot \alpha \cdot \Gamma(t)^{-1} - \Gamma(t) \cdot \alpha \cdot \Gamma(t)^{-1} \cdot \Sigma_{\gamma(t),*}(\gamma'(t)) \\ &= \Sigma_{\gamma(t),*}(\gamma'(t)) \cdot \gamma(t) - \gamma(t) \cdot \Sigma_{\gamma(t),*}(\gamma'(t)).\end{aligned}$$

Therefore, $||\,\gamma'(t)\,||_{\gamma(t)} = ||\,\Sigma_{\gamma(t),*}(\gamma'(t))\,||$. Since $\mathfrak{A}$ is a C^*-algebra,

$$||\,\Gamma'(t)\,||^2 = ||\,\Gamma'(t)\Gamma'(t)^*\,|| = ||\,\Sigma_{\gamma(t),*}(\gamma'(t))\Gamma(t)\Gamma(t)^*\Sigma_{\gamma(t),*}(\gamma'(t))^*\,||.$$

Since $\Gamma(t) \in \mathrm{U}(\mathfrak{A})$, we get $||\,\Gamma'(t)\,||^2 = ||\,\Sigma_{\gamma(t),*}(\gamma'(t))\,||^2$, and consequently $||\,\gamma'(t)\,||_{\gamma(t)} = ||\,\Sigma_{\gamma(t),*}(\gamma'(t))\,|| = ||\,\Gamma'(t)\,||$. (2.32) is derived from (2.31). $\square$

The anticipated property of geodesics is the minimality of their length.

PROPOSITION 2.16. *Let* $\gamma_0 : [\,0,\varepsilon\,] \to \mathcal{S}_*(\mathfrak{E})$ *be a smooth geodesic with* $\mathrm{L}(\gamma_0) < \pi$. *Suppose* $\gamma : [\,0,\varepsilon\,] \to \mathcal{S}_*(\mathfrak{E})$ *is another smooth path that has the same endpoints,* $\gamma(0) = \gamma_0(0)$, $\gamma(\varepsilon) = \gamma_0(\varepsilon)$. *Then* $\mathrm{L}(\gamma_0) \le \mathrm{L}(\gamma)$.

Proof. Let $\Gamma_0, \Gamma : [0,\varepsilon] \to \mathrm{U}(\mathfrak{A})$, $\Gamma_0(0) = \Gamma(0) = 1_{\mathfrak{A}}$ be the horizontal lifts of γ_0 and γ. Define $\alpha = \gamma_0(0)$, $\theta = \gamma_0'(0)$, and $x = \Sigma_{\alpha,*}(\theta) \in \mathfrak{A}_\alpha^\perp \cap \mathfrak{A}_{\mathrm{sh}}$. Part (iii) in Proposition 2.13 implies $\Gamma_0'(t) = \exp(tx)\,x$, hence $||\,\Gamma_0'(t)\,|| = ||\,x\,||$.

Let ϕ be a state on $\mathfrak{A}$ such that $\phi(x^*\,x) = ||\,x\,||^2$, and consider the associated Gelfand–Naimark–Segal cyclic representation $\rho_\phi : \mathfrak{A} \to \mathfrak{L}(\mathcal{H}_\phi)$, where $\mathcal{H}_\phi$ is a Hilbert space with inner product $\langle\,\cdot\,|\,\cdot\,\rangle$ and $\xi_\phi \in \mathcal{H}_\phi$ is a unit cyclic vector. The link between state ϕ and representation ρ_ϕ is given by

$$\phi(a) = \langle\, \rho_\phi(a)\,\xi_\phi \,|\, \xi_\phi \,\rangle, \quad a \in \mathfrak{A}, \tag{2.33}$$

We pick $u \in \mathrm{U}(\mathfrak{A})$ such that $\Gamma(\varepsilon) = u\,\Gamma_0(\varepsilon)\,u^*$, and introduce the smooth paths

$$c_0,\, c : [\,0,\varepsilon\,] \to \mathcal{H}_\phi, \quad c_0(t) = \rho_\phi(\Gamma_0(t))\,\xi_\phi, \quad c(t) = \rho_\phi(u^*\,\Gamma(t)\,u)\,\xi_\phi. \tag{2.34}$$

Since $\Gamma_0(t), \Gamma(t), u \in \mathrm{U}(\mathfrak{A})$ and ξ_ϕ is a unit vector, the values of c_0 and c are on the unit sphere $\mathbb{S}_\phi$ of $\mathcal{H}_\phi$. The two paths start at $c(0) = c_0(0) = \xi_\phi$ and end at $c(\varepsilon) = c_0(\varepsilon) = \rho_\phi(\Gamma_0(\varepsilon))\,\xi_\phi$. We claim that $\mathrm{L}(\gamma_0) = \mathrm{L}(c_0)$, and $c_0 : [\,0,\varepsilon\,] \to \mathbb{S}_\phi$ is a geodesic on $\mathbb{S}_\phi$. First, we note the following two consequences of (2.34),

$$c_0'(t) = \rho_\phi(\Gamma_0'(t))\,\xi_\phi = \rho_\phi(\exp(tx))\,\rho_\phi(x)\,\xi_\phi,$$
$$||\,c_0'(t)\,||^2 = ||\,\rho_\phi(x)\,\xi_\phi\,||^2 = \langle\,\rho_\phi(x)\,\xi_\phi \,|\, \rho_\phi(x)\,\xi_\phi\,\rangle = \langle\,\rho_\phi(x^*x)\,\xi_\phi \,|\, \xi_\phi\,\rangle.$$

By (2.33) and from the assumption on ϕ we get $||\,c_0'(t)\,||^2 = \phi(x^*x) = ||\,x\,||^2$. Since we just proved that $||\,\Gamma_0'(t)\,|| = ||\,c_0'(t)\,||$, equation (2.32) implies

$$\mathrm{L}(\gamma_0) = \int_0^\varepsilon ||\,\Gamma_0'(t)\,||\,\mathrm{d}t = \int_0^\varepsilon ||\,c_0'(t)\,||\,\mathrm{d}t = \mathrm{L}(c_0). \tag{2.35}$$

To prove the second half of our claim we need to check that $c_0''(t)$ is a normal vector to $\mathbb{S}_\phi$, or, equivalently, that $c_0''(t)$ and $c_0(t)$ are colinear vectors in $\mathcal{H}_\phi$. Using (2.34) again, we get

$$c_0''(t) = \rho_\phi(\Gamma_0''(t))\,\xi_\phi = \rho_\phi(\exp(tx))\,\rho_\phi(x^2)\,\xi_\phi = -\,\rho_\phi(\exp(tx))\,\rho_\phi(x^*x)\,\xi_\phi,$$

an equation with two consequences. Because $||\,c_0(t)\,|| = 1$, we have

$$||\,c_0''(t)\,|| \cdot ||\,c_0(t)\,|| = ||\,c_0''(t)\,|| \le ||\,x^*x\,|| = ||\,x\,||^2,$$

$$\begin{aligned} |\,\langle\,c_0''(t) \,|\, c_0(t)\,\rangle\,| &= |\,\langle\,-\,\rho_\phi(\exp(tx))\,\rho_\phi(x^*x)\,\xi_\phi \,|\, \exp(tx))\,\xi_\phi\,\rangle\,| \\ &= |\,\langle\,\rho_\phi(x^*x)\,\xi_\phi \,|\, \xi_\phi\,\rangle\,| = \phi(x^*x) = ||\,x\,||^2. \end{aligned}$$

Therefore,

$$|| c_0''(t) || \cdot || c_0(t) || \leq | \langle c_0''(t) \,|\, c_0(t) \rangle |. \tag{2.36}$$

Combined with Cauchy–Schwarz inequality, (2.36) turns out to be an equality, hence $c_0''(t)$ and $c_0(t)$ are colinear vectors in $\mathcal{H}_\phi$.

The path $c : [\,0, \varepsilon\,] \to \mathbb{S}_\phi$ defined in (2.34) has the same endpoints as c_0. Under assumption $\mathrm{L}(\gamma_0) < \pi$, which according to (2.35) reduces to $\mathrm{L}(c_0) < \pi$, c_0 is the shortest path on the unit sphere $\mathbb{S}_\phi$ from $c_0(0)$ to $c_0(\varepsilon)$, hence

$$\mathrm{L}(c_0) \leq \mathrm{L}(c). \tag{2.37}$$

It remains to observe that

$$|| c'(t) || = || \rho_\phi(u^* \,\Gamma'(t)\, u)\, \xi_\phi || = || \rho_\phi(u)^* \,\rho_\phi(\Gamma'(t))\, \rho_\phi)u)\, \xi_\phi || \leq || \Gamma'(t) ||,$$

which leads to

$$\mathrm{L}(c) = \int_0^\varepsilon || c'(t) || \, \mathrm{d}t \leq \int_0^\varepsilon || \Gamma'(t) || \, \mathrm{d}t = \mathrm{L}(\gamma). \tag{2.38}$$

The proof is completed by assembling (2.35), (2.37), and (2.38). □

3. PROJECTIVE COMPACT GROUP REPRESENTATIONS

In this section, we analyze the spaces $\mathcal{R}(G, \varepsilon, \mathfrak{A})$ and $\mathcal{R}_*(G, \varepsilon, \mathfrak{A})$ of projective continuos representations, or projective continuous unitary representations of a compact group G with a two-cocycle ε in a unital C^*-algebra $\mathfrak{A}$. To this end, we will introduce an involutive Banach algebra environment $\mathfrak{E}[\,G, \varepsilon, \mathfrak{A}\,]$ associated with G, ε, and $\mathfrak{A}$. The goal is to prove that the two spaces of representations are the structure manifolds of $\mathfrak{E}[\,G, \varepsilon, \mathfrak{A}\,]$. Specifically, $\mathcal{R}(G, \varepsilon, \mathfrak{A}) = \mathcal{S}(\mathfrak{E}[G, \varepsilon, \mathfrak{A}])$ and $\mathcal{R}_*(G, \varepsilon, \mathfrak{A}) = \mathcal{S}_*(\mathfrak{E}[G, \varepsilon, \mathfrak{A}])$. This result will enable us to rely on the algebraic and differential geometry techniques developed in Sections 1 and 2.

3.1. Group with two-cocycle convolution algebras

Suppose G is a separable compact group with unit e and let μ be the Haar measure of G with $\mu(G) = 1$. Let $\mathfrak{A}$ be a unital C^*-algebra and denote by $\mathrm{G}(\mathfrak{A})$ and $\mathrm{U}(\mathfrak{A})$ the groups of invertible and unitary elements of $\mathfrak{A}$. In particular, $\mathrm{U}(\mathbb{C}) = \{\zeta \in \mathbb{C} \setminus \{0\} : \zeta^{-1} = \zeta^-\}$, where ζ^- is the complex conjugate of $\zeta \in \mathbb{C}$.

Definition 3.1. A continuous function $\varepsilon : G \times G \to \mathbb{C} \setminus \{0\}$ is called a unitary two-cocycle of G provided

(i) $\varepsilon(g,e)=\varepsilon(e,h)=1$, $g,h\in G$,

(ii) $\varepsilon(g,h)\,\varepsilon(gh,k)=\varepsilon(g,hk)\,\varepsilon(h,k)$, $g,h,k\in G$,

(iii) $\varepsilon(g,h)^{-1}=\varepsilon(g,h)^{-}$, i.e., $\varepsilon(g,h)\in \mathrm{U}(\mathbb{C})$, $g,h\in G$.

The complex conjugate ε^{-} of a two-cocycle ε is a two-cocycle, too. The constant function $\varepsilon_0: G\times G\to \mathrm{U}(\mathbb{C})$, $\varepsilon_0(g,h)=1$, $g,h\in G$, is referred to as the trivial two-cocycle of G.

Definition 3.2. A continuous projective representation of group G with unitary two-cocycle ε in C^*-algebra $\mathfrak{A}$ is a continuous mapping $\alpha: G\to\mathfrak{A}$ such that

(i) $\alpha(e)=1_{\mathfrak{A}}$,

(ii) $\alpha(g)\alpha(h)=\varepsilon(g,h)\alpha(gh)$, $g,h\in G$,

(iii) $\alpha(g^{-1})=\varepsilon(g,g^{-1})\alpha(g)^{-1}$, $g\in G$.

The space of continuous representations is denoted by $\mathcal{R}(G,\varepsilon,\mathfrak{A})$. If in addition

(iv) $\alpha(g)^{-1}=\alpha(g)^{*}$, $g\in G$,

we refer to α as a unitary projective representation of G with two-cocycle ε, and denote the space of unitary representations by $\mathcal{R}_*(G,\varepsilon,\mathfrak{A})$.

If $\varepsilon=\varepsilon_0$ one gets the usual spaces of continuous representations of G in $\mathfrak{A}$.

Properties (i) and (ii) imply $\alpha(g)\in \mathrm{G}(\mathfrak{A})$, $g\in G$, and property (iii) is an obvious consequence. Under requirement (iv), we have $\alpha(g)\in\mathrm{U}(\mathfrak{A})$, $g\in G$. We note that the defining properties of the unitary two-cocycle ε are consistent with, and could be recovered from, requirements (i), (ii), and (iii) in Definition 3.2. For instance, property (ii) in Definition 3.1 is a consequence of requirement (ii) in Definition 3.2 in conjunction with

$$[\alpha(g)\,\alpha(h)]\alpha(k)=\alpha(g)[\alpha(h)\alpha(k)],\quad g,h,k\in G.$$

We next introduce the involutive Banach algebra $C(G,\varepsilon,\mathfrak{A})$ of continuos functions from G to $\mathfrak{A}$ with multiplication provided by the convolution ε-product,
(3.1)
$$\varphi\times\psi(g)=\int_G \varepsilon(gk^{-1},k)^{-}\,\varphi(gk^{-1})\psi(k)\,\mathrm{d}\mu(k),\varphi,\psi\in C(G,\varepsilon,\mathfrak{A}),\quad g\in G,$$

and involution defined as

$$\varphi^*(g)=\varepsilon(g,g^{-1})\varphi(g^{-1})^*,\quad \varphi\in C(G,\varepsilon,\mathfrak{A}),\ g\in G. \tag{3.2}$$

The natural norm on the convolution algebra $C(G,\varepsilon,\mathfrak{A})$ is given by

$$||\varphi||_1 = \int_G ||\varphi(g)||\, \mathrm{d}\mu(g), \quad \varphi \in C(G,\varepsilon,\mathfrak{A}).$$

Actually, instead of $||\cdot||_1$ we will be using the equivalent norm

$$||\varphi||_\infty = \sup\{||\varphi(g)|| : g \in G\}, \quad \varphi \in C(G,\varepsilon,\mathfrak{A}).$$

Algebra $C(G,\varepsilon,\mathfrak{A})$ has unit only if group G is finite. When that is not the case, we select a collection $\{K_n,\, n \geq 1\}$ of compact neighborhoods of unit $e \in G$ and a collection of continuous functions $\{u_n : G \to \mathbb{R},\, n \geq 1\}$ with the properties

$$K_{n+1} \subseteq K_n,\ \bigcap_{n=1}^{\infty} K_n = \{e\},\ \text{support } u_n \subseteq K_n,\ \int_{K_n} u_n(k)\, \mathrm{d}\mu(k) = 1.$$

The functions $\{\delta_n \in C(G,\varepsilon,\mathfrak{A}) : \delta_n(g) = u_n(g)1_{\mathfrak{A}},\, g \in G,\, n \geq 1\}$ with supports $\{K_n,\, n \geq 1\}$ provide an approximate unit on $C(G,\varepsilon,\mathfrak{A})$, i.e.,

$$\lim_n \delta_n \times \varphi = \lim_n \varphi \times \delta_n = \varphi, \quad \varphi \in C(G,\varepsilon,\mathfrak{A}). \tag{3.3}$$

Associated with any $\delta \in C(G,\varepsilon,\mathfrak{A})$ and $g,h \in G$, we introduce the functions

$$L_g(\delta), R_h(\delta) : G \to \mathfrak{A},\ L_g(\delta)(k) = \delta(gk),\ R_h(\delta)(k) = \delta(kh), \quad k \in G.$$

Proposition 3.3. *The continuous linear operator*

$$\Pi : C(G,\varepsilon,\mathfrak{A}) \to \mathfrak{A},\ \Pi(\varphi) = \varphi(e), \quad \varphi \in C(G,\varepsilon,\mathfrak{A}), \tag{3.4}$$

has the following properties:

(i) $\Pi(a\varphi) = a\Pi(\varphi), \quad \Pi(\varphi\, a) = \Pi(\varphi)a,\ a \in \mathfrak{A},\ \varphi \in \mathfrak{E},$

(ii) $\Pi(\varphi^*) = \Pi(\varphi)^*,\ \varphi \in \mathfrak{E},$

(iii) $\Pi(\varphi^* \times \varphi) \geq 0,\ \varphi \in \mathfrak{E}.$

Moreover, for any $\varphi \in \mathfrak{E},\ g,h \in G,$ *we have*

(iv) $\lim_n \Pi(L_g(\delta_n) \times \varphi) = \varepsilon(g^{-1},g)^- \varphi(g),$

(v) $\lim_m \Pi(\varphi \times R_h(\delta_m)) = \varepsilon(h,h^{-1})^- \varphi(h),$

(vi) $\lim_{n,m} \Pi(L_g(\delta_n) \times \varphi \times R_h(\delta_m)) = \varepsilon(g^{-1},g)^- \varepsilon(g,h)\varepsilon(h,h^{-1})^- \varphi(gh).$

Proof. Properties (i) and (ii) are obvious, and property (iii) follows from (3.1) and (3.2) by observing that

$$\Pi(\varphi^* \times \varphi) = \int_G \varepsilon(g^{-1},g)^-\, \varphi^*(g^{-1})\varphi(g)\, \mathrm{d}\mu(g)$$

$$= \int_G \varepsilon(g^{-1},g)^- \varepsilon(g^{-1},g)\varphi(g)^*\varphi(g)\,\mathrm{d}\mu(g)$$
$$= \int_G \varphi(g)^*\varphi(g)\,\mathrm{d}\mu(g) \geq 0.$$

To get property (iv), we note that

$$\Pi(L_g(\delta_n)\times\varphi) = \int_G \varepsilon(g^{-1}k,k^{-1}g)^-\, L_g(\delta_n)(g^{-1}k)\varphi(k^{-1}g)\,\mathrm{d}\mu(k)$$
$$= \int_{K_n} \varepsilon(g^{-1}k,k^{-1}g)^-\, \delta_n(k)\varphi(k^{-1}g)\,\mathrm{d}\mu(k),$$

and then just take the limit as $n\to\infty$, hence $k\to e$. Property (v) is proved in a similar way. Property (vi) is an extension of (iv) and (v). By (iv) we have

$$\begin{aligned}
&\lim_n \Pi(L_g(\delta_n)\times\varphi\times R_h(\delta_m))\\
&\quad= \varepsilon(g^{-1},g)^-\varphi\times R_h(\delta_m)(g)\\
&\quad= \varepsilon(g^{-1},g)^- \int_G \varepsilon(ghk^{-1},kh^{-1})^-\varphi(ghk^{-1})\,R_h(\delta_m)(kh^{-1})\,\mathrm{d}\mu(k)\\
&\quad= \varepsilon(g^{-1},g)^- \int_{K_m} \varepsilon(ghk^{-1},kh^{-1})^-\varphi(ghk^{-1})\,\delta_m(k)\,\mathrm{d}\mu(k).
\end{aligned}$$

Consequently, if $m\to\infty$, hence $k\to e$, we get

$$\lim_{n,m} \Pi(L_g(\delta_n)\times\varphi\times R_h(\delta_m)) = \varepsilon(g^{-1},g)^-\,\varepsilon(gh,h^{-1})^-\varphi(gh). \tag{3.5}$$

It remains to observe that requirements (ii) and (i) in Definition 3.1 imply

$$\varepsilon(g,h)\,\varepsilon(gh,h^{-1}) = \varepsilon(h,h^{-1}),\ \text{ i.e., }\ \varepsilon(gh,h^{-1})^- = \varepsilon(g,h)\,\varepsilon(h,h^{-1})^-,$$

and conclude that equation (3.5) proves property (vi). □

3.2. **Projective compact group algebra environments**

Definition 3.4. Suppose G is a compact group with a continuous unitary two-cocycle ε and let $\mathfrak{A}$ be a unital C^*-algebra. The associated involutive Banach algebra environment $\mathfrak{C}[G,\varepsilon,\mathfrak{A}] = (\mathfrak{C},\Pi,\mathfrak{A})$ has total algebra $\mathfrak{C} = C(G,\varepsilon,\mathfrak{A})$, base algebra $\mathfrak{A}$, and environment projection $\Pi : C(G,\varepsilon,\mathfrak{A}) \to \mathfrak{A}$ given by (3.4).

If $\varepsilon = \varepsilon_0$, by regarding each $a\in\mathfrak{A}$ as a constant function in $C(G,\varepsilon_0,\mathfrak{A})$, equations (3.1) and (3.2) make $\mathfrak{A}\subseteq C(G,\varepsilon_0,\mathfrak{A})$ an involutive subalgebra, and Lemma 3.3 shows that $\Pi : C(G,\varepsilon_0,\mathfrak{A}) \to \mathfrak{A}$ is a non-commutative conditional expectation from $C(G,\varepsilon_0,\mathfrak{A})$ onto $\mathfrak{A}$.

Theorem F (Spaces of representations as structure manifolds). *Let $\mathcal{S}(\mathfrak{E}[G,\varepsilon,\mathfrak{A}])$ and $\mathcal{S}_*(\mathfrak{E}[G,\varepsilon,\mathfrak{A}])$ be the structure manifolds of the involutive environment $\mathfrak{E}[G,\varepsilon,\mathfrak{A}]$. Then*

(i) $\mathcal{R}(G,\varepsilon,\mathfrak{A}) = \mathcal{S}(\mathfrak{E}[G,\varepsilon,\mathfrak{A}])$,

(ii) $\mathcal{R}_*(G,\varepsilon,\mathfrak{A}) = \mathcal{S}_*(\mathfrak{E}[G,\varepsilon,\mathfrak{A}])$.

Proof. We first prove that $\alpha \in \mathcal{R}(G,\varepsilon,\mathfrak{A})$ implies $\alpha \in \mathcal{S}(\mathfrak{E}[G,\varepsilon,\mathfrak{A}])$. We just need to check requirements (i)–(v) in Definition 1.5 of $\mathcal{S}(\mathfrak{E}[G,\varepsilon,\mathfrak{A}])$. Requirement (i) is obvious, and requirement (ii) follows from

$$\begin{aligned}\alpha \times \alpha(g) &= \int_G \varepsilon(gk,k^{-1})^- \alpha(gk)\alpha(k^{-1})\,\mathrm{d}\mu(k)\\ &= \int_G \varepsilon(gk,k^{-1})^- \varepsilon(gk,k^{-1})\alpha(g)\,\mathrm{d}\mu(k)\\ &= \int_G \alpha(g)\,\mathrm{d}\mu(k) = \alpha(g), \quad g\in G.\end{aligned}$$

With regard to requirement (iii), we note that (3.1) and Definition 3.2 imply

$$\Pi(\varphi\times\alpha)\alpha(g) = \varphi\times\alpha(e)\,\alpha(g) = \int_G \varepsilon(k^{-1},k)^-\varphi(k^{-1})\alpha(k)\alpha(g)\,\mathrm{d}\mu(k)$$

$$= \int_G \varepsilon(k^{-1},k)^-\varepsilon(k,g)\varphi(k^{-1})\alpha(kg)\,\mathrm{d}\mu(k), \quad \varphi\in C(G,\varepsilon,\mathfrak{A}),\ g\in G.$$

Direct calculations based on Definition 3.1 show that

$$\varepsilon(k^{-1},k)^-\varepsilon(k,g) = \varepsilon(k^{-1},kg)^-, \quad k,g\in G,$$

and the previous equation reduces to $\Pi(\varphi\times\alpha)\alpha(g) = \varphi\times\alpha(g)$ for any $g\in G$. Requirement (iv) has a similar proof, and (v) follows from (iii) and (iv).

Our next task is to show that $\alpha\in\mathcal{S}(\mathfrak{E}[G,\varepsilon,\mathfrak{A}])$ implies $\alpha\in\mathcal{R}(G,\varepsilon,\mathfrak{A})$. Properties $\Pi(\alpha) = 1_{\mathfrak{A}}$ and $\alpha(e) = 1_{\mathfrak{A}}$ are equivalent. Requirement (v) in Definition 1.5 yields

$$\Pi(L_g(\delta_n)\times\alpha)\,\Pi(\alpha\times R_h(\delta_m)) = \Pi(L_g(\delta_n)\times\alpha\times R_h(\delta_m)),\ \ n,m\ge 1.$$

Using the last three statements in Proposition 3.3 and taking limits, we get $\alpha(g)\alpha(h) = \varepsilon(g,h)\,\alpha(gh)$, $g,h\in G$, an equation that proves statement (i). For statement (ii), we only have to check that property (iv) in Definition 3.2 and requirement (vi) in Definition 1.5, i.e., $\alpha^* = \alpha$, are equivalent. □

Corollary 3.5. *Spaces $\mathcal{R}(G,\varepsilon,\mathfrak{A})$ and $\mathcal{R}_*(G,\varepsilon,\mathfrak{A})$ are Banach manifolds. Their tangent spaces have the following descriptions:*

(i) $\mathrm{T}_\alpha \mathcal{R}(G,\varepsilon,\mathfrak{A})$, $\alpha \in \mathcal{R}(G,\varepsilon,\mathfrak{A})$, *consists of all* $\theta \in C(G,\varepsilon,\mathfrak{A})$ *such that*

$$\theta(e) = 0, \;\; \theta(g)\alpha(h) + \alpha(g)\theta(h) = \varepsilon(g,h)\,\theta(gh), \quad g,h \in G.$$

(ii) *If* $\alpha \in \mathcal{R}_*(G,\varepsilon,\mathfrak{A})$, $\mathrm{T}_\alpha \mathcal{R}_*(G,\varepsilon,\mathfrak{A}) = \mathrm{T}_\alpha \mathcal{R}(G,\varepsilon,\mathfrak{A}) \cap C(G,\varepsilon,\mathfrak{A})_{\mathrm{h}}$.

Proof. Direct consequences of Corollary 1.14. □

As a brief comment related to Theorem G, we note that Definition 2.3 of $\mathrm{G}(\mathfrak{A})$ or $\mathrm{U}(\mathfrak{A})$ equivalent geometric structures on $\mathfrak{A}$ reduces to the concepts of similar, or unitarily equivalent group representations. Proposition 2.4 implies that any $\alpha,\beta \in \mathcal{R}(G,\varepsilon,\mathfrak{A})$, or $\alpha,\beta \in \mathcal{R}_*(G,\varepsilon,\mathfrak{A})$, with $||\alpha-\beta||_\infty < ||\Pi||^{-1}||\alpha||_\infty^{-1}$ are similar or unitarily equivalent, respectively.

On the other hand, we recall that there is a concept of co-homologous two-cocycles on a group, and co-homologous two-cocycles yield isomorphic group algebras. Perhaps the reader would be interested to figure out when the environments associated with different two-cocycles on a compact group G and arbitrary C^*-algebras $\mathfrak{A}$ are isomorphic.

3.3. **Representations and conjugation operators**

We introduce a class of derivations on $\mathfrak{E}[\,G,\varepsilon,\mathfrak{A}\,]$ related to derivations on $\mathfrak{A}$.

Proposition 3.6. *Suppose* $\mathcal{D}_0 = \partial \in \mathrm{Der}(\mathfrak{A})$ *and define the linear mapping* $\mathcal{D}$ *with domain* $C(G,\varepsilon,\mathfrak{A})$ *by*

$$\mathcal{D}(\varphi) = \partial \circ \varphi, \quad \varphi \in C(G,\varepsilon,\mathfrak{A}). \tag{3.6}$$

Then $(\mathcal{D},\mathcal{D}_0)$ *is a derivation on algebra environment* $\mathfrak{E}[G,\varepsilon,\mathfrak{A}]$.

Proof. Consistent with Definition 1.10, there are several things we have to check. First, we claim that $\mathcal{D}$ is an operator on $C(G,\varepsilon,\mathfrak{A})$. The critical part is to show that $\mathcal{D}(\varphi) \in C(G,\varepsilon,\mathfrak{A})$ for any $\varphi \in C(G,\varepsilon,\mathfrak{A})$. Since $\mathfrak{A}$ is a C^*-algebra, we rely on a result proved by Sakai [60], which states that each $\partial \in \mathrm{Der}(\mathfrak{A})$ is continuos. The other properties,

$$\mathcal{D}(\varphi \times \psi) = \mathcal{D}(\varphi) \times \psi + \varphi \times \mathcal{D}(\psi), \quad \varphi,\psi \in C(G,\varepsilon,\mathfrak{A}),$$
$$\mathcal{D}(a\varphi) = \partial(a)\varphi + a\mathcal{D}(\varphi), \quad a \in \mathfrak{A},\; \varphi \in C(G,\varepsilon,\mathfrak{A}),$$

are consequences of (3.6) and (3.1). Perhaps we should note that the continuity of $\partial \in \mathrm{Der}(\mathfrak{A})$ makes it possible to apply ∂ to both sides of equation (3.1) and to move it inside the integral. □

Lemma 1.11 in Section 1 provides alternate justifications of statements (i) and (ii) in Corollary 3.5.

Corollary 3.7. *Suppose* $\partial \in \mathrm{Der}(\mathfrak{A})$ *and let* $\alpha \in \mathcal{R}(G,\varepsilon,\mathfrak{A})$. *Then* $\mathcal{D}(\alpha) \in \mathrm{T}_\alpha \mathcal{R}(G,\varepsilon,\mathfrak{A})$. *In addition, if* $\partial \in \mathrm{Der}_*(\mathfrak{A})$ *and* $\alpha \in \mathcal{R}_*(G,\varepsilon,\mathfrak{A})$, *then* $\mathcal{D}(\alpha) \in \mathrm{T}_\alpha \mathcal{R}_*(G,\varepsilon,\mathfrak{A})$.

To derive Corollary 3.5, we just apply ∂ to each equation in Definition 3.2.

Suppose $\alpha \in \mathcal{R}(G,\varepsilon,\mathfrak{A})$ and let $\mathrm{Der}_{\alpha,0}(\mathfrak{A}) = \{\partial \in \mathrm{Der}(\mathfrak{A}) : \partial \circ \alpha = 0\}$ be the Lie subalgebra of derivations on $\mathfrak{A}$ compatible with α. We next introduce natural counterparts of the operators Σ_α, Θ_α, and Γ_α defined in Section 1. For convenience, we denote the new operators, adapted to the current setting, using the same symbols.

Definition 3.8. Suppose that $\alpha \in \mathcal{R}(G,\varepsilon,\mathfrak{A})$.

(i) The operator $\Sigma_\alpha : \mathrm{T}_\alpha \mathcal{R}(G,\varepsilon,\mathfrak{A}) \to \mathfrak{A}_\alpha^\perp$ is defined by

$$\Sigma_\alpha(\theta) = 2^{-1}\,\Pi(\theta\times\alpha-\alpha\times\theta) = 2^{-1}\,(\varphi\times\alpha-\alpha\times\varphi)(e), \quad \theta \in \mathrm{T}_\alpha \mathcal{R}(G,\varepsilon,\mathfrak{A}).$$

(ii) The operator $\Theta_\alpha : \mathrm{Der}(\mathfrak{A}) \to \mathrm{T}_\alpha \mathcal{R}(G,\varepsilon,\mathfrak{A})$ is given by

$$\Theta_\alpha(\partial) = \partial \circ \alpha, \quad \partial \in \mathrm{Der}(\mathfrak{A}).$$

(iii) The conjugation operator $\Gamma_\alpha : \mathrm{Der}(\mathfrak{A}) \to \mathrm{Der}(\mathfrak{A})$ is defined as

$$\Gamma_\alpha = \mathrm{Id}_{\mathrm{Der}(\mathfrak{A})} - 2\,\mathfrak{D} \circ \Sigma_\alpha \circ \Theta_\alpha,$$

where $\mathfrak{D} : \mathfrak{A} \to \mathrm{Der}(\mathfrak{A})$ assigns inner derivations to elements of $\mathfrak{A}$, i.e.,

$$\mathfrak{D}(x)(a) = xa - ax, \quad x,\, a \in \mathfrak{A}.$$

The next proposition collects consequences of results from Subsection 1.4.

Proposition 3.9. *Assume that* $\alpha \in \mathcal{R}(G,\varepsilon,\mathfrak{A})$ *and* $\partial \in \mathrm{Der}(\mathfrak{A})$:

(i) $\Theta_\alpha \circ \mathfrak{D} \circ \Sigma_\alpha = \mathrm{Id}_{\mathrm{T}_\alpha \mathcal{R}(G,\varepsilon,\mathfrak{A})}$.

(ii) $\Gamma_\alpha^2 = \mathrm{Id}_{\mathrm{Der}(\mathfrak{A})}$, $\Theta_\alpha \circ \Gamma_\alpha = -\,\Theta_\alpha$, *and* $\Theta_\alpha(\partial) = 0$ *only if* $\Gamma_\alpha(\partial) = \partial$.

(iii) *The operators* $\Gamma_\alpha^+, \Gamma_\alpha^- : \mathrm{Der}(\mathfrak{A}) \to \mathrm{Der}(\mathfrak{A})$,

$$\Gamma_\alpha^+ = (\mathrm{Id}_{\mathrm{Der}(\mathfrak{A}))} + \Gamma_\alpha)/2, \quad \Gamma_\alpha^- = (\mathrm{Id}_{\mathrm{Der}(\mathfrak{A})} - \Gamma_\alpha)/2,$$

are complementary projections on the space $\mathrm{Der}(\mathfrak{A})$, *i.e.*,

$$(\Gamma_\alpha^+)^2 = \Gamma_\alpha^+,\ (\Gamma_\alpha^-)^2 = \Gamma_\alpha^-,\ \Gamma_\alpha^+\,\Gamma_\alpha^- = \Gamma_\alpha^-\,\Gamma_\alpha^+ = 0,\ \Gamma_\alpha^+ + \Gamma_\alpha^- = \mathrm{Id}_{\mathrm{Der}(\mathfrak{A})},$$

and $\mathrm{Der}_{\alpha,0}(\mathfrak{A}) = \mathrm{Ran}(\Gamma_\alpha^+) = \mathrm{Ker}(\Gamma_\alpha^-)$ *is a Lie subalgebra of* $\mathrm{Der}(\mathfrak{A})$.

3.4. **Case study–Clifford algebras**

This subsection includes a brief description of Clifford algebras as an example of group with two-cocycle convolution algebras that deserve special attention due to their algebraic and geometric properties. For additional details, we refer to Atiyah, Bott, Shapiro [7], Chevalley [17], and Louenesto [31].

The real Clifford algebra $\mathfrak{C}_{n,m}(\mathbb{R})$, $n, m \geq 0$, of signature (n, m) is defined by assuming that $\mathbb{R} \oplus \mathbb{R}^{n+m} \subseteq \mathfrak{C}_{n,m}(\mathbb{R})$, $1_{\mathfrak{C}_{n,m}(\mathbb{R})} = 1 \in \mathbb{R}$, and the orthonormal basis $\{e_1, e_2, \ldots, e_{n+m}\}$ for $\mathbb{R}^{n+m}$ is a complete set of generators with relations

$$e_i\, e_j + e_j\, e_i = 0, \quad 1 \leq i, j \leq n+m,\ i \neq j,$$
$$e_k^2 = -1,\ 1 \leq k \leq n, \quad e_k^2 = 1,\ n+1 \leq k \leq n+m.$$

In particular, $\mathfrak{C}_{0,0}(\mathbb{R}) = \mathbb{R}$. Assuming that $(n, m) \neq (0, 0)$ and $0 \leq p \leq n+m$, we let $\mathfrak{I}^p_{n+m}$ denote the collection of all p-element subsets $I \subseteq \{1, 2, \ldots, n+m\}$. If $p = 0$, $\mathfrak{I}^0_{n+m} = \{\emptyset\}$. Each $I \in \mathfrak{I}^p_{n+m}$, $p \geq 1$, is expressed as a p-tuple

$$I = (i_1, i_2, \ldots, i_p), \quad 1 \leq i_1 < i_2 < \cdots < i_p \leq n+m.$$

We form the union $\mathfrak{I}_{n+m} = \bigcup_{p=0}^{n+m} \mathfrak{I}^p_{n+m}$, and next assign to every $I \in \mathfrak{I}_{n+m}$ the element $e_I \in \mathfrak{C}_{n,m}(\mathbb{R})$ given by

$$e_\emptyset = 1,\ e_I = e_{i_1} e_{i_2} \cdots e_{i_p}, I = (i_1, i_2, \ldots, i_p), \quad 1 \leq p \leq n+m.$$

Since these are all possible reduced products of generators of $\mathfrak{C}_{n,m}(\mathbb{R})$, the set $\{e_I : I \in \mathfrak{I}_{n+m}\}$ is a linear basis for $\mathfrak{C}_{n,m}(\mathbb{R})$. Operation Δ of symmetric difference introduces a group structure on $\mathfrak{I}_{n+m}$, and the Clifford relations yield a function

$$\varepsilon = \varepsilon_{n,m} : \mathfrak{I}_{n+m} \times \mathfrak{I}_{n+m} \to \{1, -1\},$$

uniquely determined by the requirements

$$e_I\, e_J = \varepsilon(I, J)\, e_{I \Delta J}, \quad I, J \in \mathfrak{I}_{n+m}.$$

Function $\varepsilon_{n,m}$ is a two-cocycle on $\mathfrak{I}_{n+m}$. The defining properties

$$\varepsilon(I, \emptyset) = \varepsilon(\emptyset, I) = 1, \quad I \in \mathfrak{I}_{n+m},$$
$$\varepsilon(I, J)\varepsilon(I \Delta J, K) = \varepsilon(I, J \Delta K)\varepsilon(J, K), \quad I, J, K \in \mathfrak{I}_{n+m},$$

are derived from

$$e_I e_\emptyset = e_\emptyset e_I = e_I, \quad I \in \mathfrak{I}_{n+m},$$
$$(e_I e_J) e_K = e_I (e_J e_K), \quad I, J, K \in \mathfrak{I}_{n+m}.$$

Consequently, $\mathfrak{C}_{n,m}(\mathbb{R})$ is the group algebra of $\mathfrak{I}_{n+m}$ with two-cocycle $\varepsilon_{n,m}$. The natural involution and norm provided by the general definitions for compact groups with two-cocycles make $\mathfrak{C}_{n,m}(\mathbb{R})$ a $\mathbb{Z}_2$-graded unital C^*-algebra. The linear basis $\{e_I : I \in \mathfrak{I}_{n+m}\}$ consists of unitary elements of $\mathfrak{C}_{n,m}(\mathbb{R})$. Moreover, $\mathfrak{C}_{n,m}(\mathbb{R})$ has an inner product $\langle \cdot | \cdot \rangle$ such that $\{e_I : I \in \mathfrak{I}_{n+m}\}$ is an orthonormal basis, and the function $\tau : \mathfrak{C}_{n,m}(\mathbb{R}) \to \mathbb{R}$, $\tau(\,\cdot\,) = \langle\, e_\emptyset \,|\, \cdot \,\rangle$ is a faithful trace. The associated environments $\mathfrak{E}[\mathfrak{I}_{n+m}, \varepsilon_{n,m}, \mathfrak{A}]$ with an arbitrary base algebra $\mathfrak{A}$ and total algebra $\mathfrak{C}_{n,m}(\mathbb{R}) \otimes_{\mathbb{R}} \mathfrak{A}$ are called Clifford algebra environments.

Clifford algebras are usually called geometric algebras. The algebras $\mathfrak{C}_{n,0}(\mathbb{R})$ in particular form the class of Euclidean Clifford algebras. The first three non-trivial such algebras are $\mathfrak{C}_{1,0}(\mathbb{R}) = \mathbb{C}$, the complex numbers, $\mathfrak{C}_{2,0}(\mathbb{R}) = \mathbb{H}$, the Hamilton quaternions, and $\mathfrak{C}_{3,0}(\mathbb{R}) = \mathbb{H} \oplus \mathbb{H}$, the split biquaternions. Complex Clifford algebras, $\mathfrak{C}_{n,m}(\mathbb{C}) = \mathfrak{C}_{n,m}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$, are complexifications of the real ones, and $\mathfrak{C}_{n,0}(\mathbb{C})$, $n \geq 1$, are called hermitian Clifford algebras.

3.5. **Additional properties of spaces of representations**

This subsection points out properties of $\mathcal{R}(G, \varepsilon, \mathfrak{A})$ and $\mathcal{R}_*(G, \varepsilon, \mathfrak{A})$ that are not derived from the structure manifolds $\mathcal{S}(\mathfrak{E}[G, \varepsilon, \mathfrak{A}])$ and $\mathcal{S}_*(\mathfrak{E}[G, \varepsilon, \mathfrak{A}])$.

Suppose $\alpha \in \mathcal{R}(G, \varepsilon, \mathfrak{A})$ and note that $\alpha^* \in \mathcal{R}(G, \varepsilon, \mathfrak{A})$. Define $a \in \mathfrak{A}$ as

$$a = \Pi(\alpha^* \times \alpha) = \alpha^* \times \alpha(e) = \int_G \alpha(g)^* \alpha(g)\, \mathrm{d}\mu(g).$$

Requirements (iii) and (iv) in Definition 1.5 imply

$$a \cdot \alpha = \Pi(\alpha^* \times \alpha) \cdot \alpha = \alpha^* \times \alpha = \alpha^* \cdot \Pi(\alpha^* \times \alpha) = \alpha^* \cdot a.$$

We observe that $a \in \mathrm{G}(\mathfrak{A})$, hence α^* and α are similar representations. In addition, a is positive and by using functional calculus we define powers of a with real exponents by

$$a^t = \exp(t \log(a)) \in \mathrm{G}(\mathfrak{A}), \quad t \in \mathbb{R},$$

and introduce the smooth path on $\mathcal{R}(G, \varepsilon, \mathfrak{A})$ defined by

$$\gamma : [0,1] \to \mathcal{R}(G, \varepsilon, \mathfrak{A}),\ \gamma(t) = a^t \cdot \alpha \cdot a^{-t}, \quad 0 \leq t \leq 1. \tag{3.7}$$

Obviously, by (3.7) each $\gamma(t)$ is a representation similar to α. We claim that the midpoint of γ, $\gamma(1/2) = a^{1/2} \cdot \alpha \cdot a^{-1/2}$, is in $\mathcal{R}_*(G, \varepsilon, \mathfrak{A})$, i.e., $\gamma(1/2)$ is a unitary representation. One expects this because $\gamma(0) = \alpha$ and $\gamma(1) = \alpha^*$. For the proof, we just note that $\gamma(1/2)^* = a^{-1/2} \cdot \alpha^* \cdot a^{1/2} = a^{-1/2}\, a \cdot \alpha \cdot a^{-1}\, a^{1/2} = \gamma(1/2)$. The following result summarizes our previous observations.

Proposition 3.10. *$\mathcal{R}(G, \varepsilon, \mathfrak{A})$ and $\mathcal{R}_*(G, \varepsilon, \mathfrak{A})$ have the next properties:*

(i) *Any $\alpha \in \mathcal{R}(G, \varepsilon, \mathfrak{A})$ and its conjugate $\alpha^* \in \mathcal{R}(G, \varepsilon, \mathfrak{A})$ are similar.*

(ii) *Each $\alpha \in \mathcal{R}(G, \varepsilon, \mathfrak{A})$ is similar to a unitary representation.*

(iii) $\mathcal{R}_*(G,\varepsilon,\mathfrak{A})$ *is a deformation retract of* $\mathcal{R}(G,\varepsilon,\mathfrak{A})$.

We end this subsection with some comments and an example. Suppose that G is abelian and $\varepsilon = \varepsilon_0$. Under these assumptions, we can use the Pontryagin dual group $\hat{G}$ of G, Pontryagin duality theorem, and Fourier transforms to switch from representations of G to representations of $\hat{G}$. The geometric properties of spaces of representations of G have natural counterparts in terms of $\hat{G}$. Moriss [52] is an excellent reference in this regard.

To make a point, let $\mathbb{U}_n = \{\zeta \in \mathbb{C} : \zeta^n = 1\}$, $n \geq 2$, be the cyclic group of n-th roots of unity, with generator $\omega = \exp(2\pi\sqrt{-1}/n)$. Each unitary representation $\alpha \in \mathcal{R}_*(\mathbb{U}_n,\mathfrak{A})$ of group $\mathbb{U}_n$ into $\mathfrak{A}$ is determined by the element $\alpha(\omega) = u \in \mathrm{U}(\mathfrak{A})$ with $u^n = 1_{\mathfrak{A}}$, and any such $u \in \mathrm{U}(\mathfrak{A})$ yields $\alpha \in \mathcal{R}_*(\mathbb{U}_n,\mathfrak{A})$ by $\alpha(\omega^k) = u^k$, $0 \leq k \leq n-1$. Elementary spectral theory implies that $\mathrm{spec}(u) \subseteq \mathbb{U}_n$. Consequently, there exists a projection n-partition $\mathfrak{p} = (p_1, p_2, \ldots, p_n)$ of $1_{\mathfrak{A}}$ such that

$$u = p_1 + \omega\, p_2 + \omega^2 p_3 + \cdots + \omega^{n-1} p_n. \tag{3.8}$$

The process works both ways. If $\mathfrak{p} = (p_1, p_2, \ldots, p_n)$ is a projection n-partition of $1_{\mathfrak{A}}$ and u is defined by (3.8), then $u \in \mathrm{U}(\mathfrak{A})$ and $u^n = 1_{\mathfrak{A}}$. Using the n-fold product algebra environment $(\mathfrak{E}_n(\mathfrak{A}), \Pi, \mathfrak{A})$ introduced in Subsection 1.3 and a result proved there, we conclude that $\mathcal{R}_*(\mathbb{U}_n,\mathfrak{A})$ and the structure manifold $\mathcal{S}_*(\mathfrak{E}_n(\mathfrak{A}))$ of $(\mathfrak{E}_n(\mathfrak{A}), \Pi, \mathfrak{A})$ are difeomorphic real analytic Banach manifolds, and $\mathcal{R}_*(\mathbb{U}_n,\mathfrak{A})$ has $2^{n(n-1)/2}$ complex structures.

4. **CONCLUDING COMMENTS**

Over the years, a great deal of developments made apparent the important part played by non-commutative conditional expectations in operator algebra theory. The article by Takesaki [63] and all the references therein deserve a special mention in this regard. Kadison [26] is yet another highly recommended source of information, with a particular emphasis on the role of group algebras. The comprehensive monograph by Strătilă [62] on modular theory in operator algebras covers the intricate transition from basic requisites and early results to recent discoveries with far-reaching consequences.

Structure manifolds of group with two-cocycle algebra environments and the requirements in Definition 1.7 were introduced in Martin [35], as means of studying spaces of projective compact group representations in C^*-algebras. The group of order two yields Grassmann manifolds, an object investigated by Porta, Recht [54] and Salinas [61], two early articles that prompted us to search for a concept adapted to more general settings including, for instance, cyclic groups of higher order that generate flag manifolds of C^*-algebras.

Notewothy contributions to the development of lines of research centered on a multitude of themes implicitly related to Banach algebra environments and spaces of group representations are due to Andruchow, Stojanoff [4], Andruchow, Recht [5], Beltita [8], Beltita, Gale [9, 10], Corach, Porta, Recht [19], Lubotzky, Magid [32], Magid [33], Wilkins [65].

C^*-algebra environments were defined and studied in Martin, Salinas [47]. Sections 1 and 2 include new and simplified proofs of results from this article, although many issues such as normalized lifts of continuous curves on structure manifolds to the groups of invertible or unitary elements of the base algebra, or linear connections on structure manifolds have been left out. The two sequels, Martin, Salinas [48, 49] analyze flag manifolds of C^*-algebras and generalize results from Cowen, Douglas [20] in a C^*-algebra framework. Holomorphic mappings of several complex variables with values in Grassmann manifolds, hermitian holomorphic vector bundles of finite or infinite rank, and n-tuples of Hilbert space operators in Cowen–Douglas classes, which all have a role in developing holomorphic spectral theory, are investigated in Martin [46].

An incipient use of the symbol and conjugation operators Σ_α and Γ_α in connection with geometric structures and derivations on smooth vector bundles is illustrated in Martin [34]. Their significance becomes apparent in Martin [44], an article that provides explicit ways of finding the linear connections on Clifford vector bundles used to define Dirac and Laplace operators, and to set up Bochner–Weitzenböck and Bochner–Kodaira–Nakano curvature identities.

Clifford algebra environments provide a framework for developing Clifford analysis and spin geometry, two research areas related to the study of Dirac and Laplace operators in appropriate topological settings. Algebraic K-theory, Euclidean harmonic analysis, and in particular the theory of singular integral operators, are also worth mentioning in this regard. For specific details, we refer to Anglés [6], Berline, Getzler, Vergne [11], Brackx, Delange, Sommen [16], Colombo, Sabadini, Sommen, Struppa [18], Gilbert, Murray [23], Gürlebeck, Sprössig [24], Karoubi [27], Lawson, Michelsohn [30], Mitrea [51], Rocha–Chavez, Shapiro, Sommen [56]. The volumes edited by Ablamowicz [1], Bernstein, Kähler, Sabadini, Sommen [12, 13], Qian, Hempfling, McIntosh, Sommen [55], Ryan [57], Ryan, Sprössing [58], Sabadini, Shapiro, Sommen [58] include proceedings of international conferences organized by several academic institutions, attended by scientists interested in Clifford analysis and its applications. The volume edited by Alpay [2] as part of an ongoing project provides an excellent illustration of the full scope of past and current developments in quaternion and Clifford analysis. The recent article by Alpay, Cerejeiras, Kaehler [3] introduces generalized Clifford $\mathbb{Z}_n$-graded algebras, $n = 3$, $n = 6$, that we expect to trigger new inquiries.

For more applications of algebra environments related to harmonic analysis and multivariable operator theory, we direct the attention of our reader to two groups of articles, Martin [36–39], and Martin [40–43, 45], Martin, Salinas [50]. The list of specific issues includes Dirac operators with coefficients in a C^*-algebra, Cauchy–Pompeiu and Bochner–Martinelli–Koppelman representation formulas in a Banach algebra setting, maximal and fractional integral operators in Clifford analysis, generalizations of Ahlfors–Beurling and Alexander inequalities, quantitative Hartogs–Rosenthal theorems, Bochner–Weitzenböck and Bochner–Kodaira–Nakano self-commutator identities, extensions of Putnam inequality and singular integral models of seminormal systems of operators using Riesz transforms.

REFERENCES

[1] R. Ablamowicz (Ed.), *Clifford Algebras Applications to Mathematics, Physics, and Engineering*, Progress in Mathematical Physics Series **34**, Birkhäuser Verlag, 2004.

[2] D. Alpay (Ed.), *Operator Theory.* Springer Verlag, 2015.

[3] D. Alpay, P. Cerejeiras, and U. Kaehler, *Generalized Clifford algebras and applications to stochastic processes.* Math. Methods Appl. Sci. **45** (2022), 383–401.

[4] E. Andruchow and D. Stojanoff, *Geometry of unitary orbits.* J. Operator Theory **26** (1991), 25–41.

[5] E. Andruchow and L. Recht, *Grassmannians of a finite algebra in the strong operator topology.* Internat. J. Math. **17** (2006), 477–491.

[6] P. Anglès, *Conformal Groups in Geometry and Spin Structures.* Progress in Mathematical Physics Series **50**, Birkhäuser Verlag, 2008.

[7] M. F. Atiyah, R. Bott, and A. Shapiro, *Clifford modules.* Topology **3** (1964), 3–38.

[8] D. Beltita, *Smooth Homogeneous Structures in Operator Theory.* Chapman and Hall/CRC Monographs and Surveys in Pure and Applied Mathematics **137**, CRC Press Taylor and Francis Group, 2006.

[9] D. Beltita and J.E. Gale, *Holomorphic geometric models for representations of C^*-algebras.* J. Funct. Anal. **205** (2008), *10*, 2888–2932.

[10] D. Beltita and J.E. Gale, *On complex infinite dimensional Grassmann manifolds.* Complex Anal. Oper. Theory **3-4** (2009), 739–758.

[11] N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators.* Grundlehren der mathematischen Wissenschaften **298**, Springer Verlag, 1992.

[12] S. Bernstein, U. Kähler, I. Sabadini, and F. Sommen (Eds.), *Hypercomplex Analysis: New Perspectives and Applications.* Trends in Mathematics Series, Birkhäuser Verlag, 2014.

[13] S. Bernstein, U. Kähler, I. Sabadini, and F. Sommen (Eds.), *Modern Trends in Hypercomplex Analysis.* Trends in Mathematics Series, Birkhäuser Verlag, Basel, 2016.

[14] A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces.* Amer. J. Math. **80** (1958), *2*, 458–538.

[15] N. Bourbaki, *Variété Différentielles et Analytiques, Fascicule de results*. Hermann, 1967.

[16] F. Brackx, R. Delanghe, and F. Sommen, *Clifford Analysis*, Pitman Research Notes in Mathematics Series **76**, 1982.

[17] C. Chevalley, *The Algebraic Theory of Spinors*, Springer Verlag, 1997.

[18] F. Colombo, I. Sabadini, F. Sommen, and D.C. Struppa, *Analysis of Dirac Systems and Computational Algebra*. Progress in Mathematical Physics Series **39**, Birkhäuser Verlag, 2004.

[19] G. Corach, H. Porta, and L. Recht, *The geometry of spaces of projections in C^*-algebras*. Adv. Math. **101** (1993), 59–77.

[20] M.J. Cowen and R.G. Douglas, *Complex geometry and operator theory*. Acta Math. **141** (1978), 187–261.

[21] R.G. Douglas, V. Paulsen, and K. Yan, *Operator theory and algebraic geometry*. Bull. Amer. Math. Soc. **20** (1988), 67–71.

[22] C. Ehresmann, *Les connexions infinitésimales dans un espace fibré différentiable*. Seminare N. Bourbaki **24** (1952), 153–168.

[23] J.E Gilbert and M.A.M. Murray, *Clifford Algebras and Dirac Operators in Harmonic Analysis*. Cambridge Studies in Advanced Mathematics **26**, Cambridge Univ. Press, 1991.

[24] K. Gürlebeck and W. Sprössig, *Quaternionic and Clifford Calculus for Physicists and Engineers*. John Wiley and Sons, 1997.

[25] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*. Academic Press, 1978.

[26] R.V Kadison, *Non-commutative conditional expectations and their applications*. Contemporary Mathematics **365** (2004), 143–179.

[27] M. Karoubi, *K-Theory: An Introduction*. Grundlehren der mathematischen Wissenschften, Comprehensive Studies in Mathematics **226**, Springer Verlag, 1978.

[28] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry* I, II. Willey–Interscience, 1996.

[29] S. Lang, *Fundamentals of Differential Geometry*. Springer Verlag, 1999.

[30] H.B. Lawson, M.-L. Michelsohn, *Spin Geometry*. Princeton Mathematical Series **38**, Princeton Univ. Press, 1989.

[31] P. Louenesto, *Clifford Algebras and Spinors*. Cambridge Univ. Press, 2001.

[32] A. Lubotzky and A.R. Magid, *Varietes of representations and finitely generated groups*. Mem. Amer. Math. Soc. **336**, 1985.

[33] A.R. Magid, *Deformations of representations of discrete groups*. Classical Groups and Related Topics, Contemporary Mathematics **82** (1989), 79–87.

[34] M. Martin, *Geometric structures on vector bundles* II. *Geometric structures and derivations*. Stud. Cerc. Mat./Math. Reports **36** (1984), 171–192.

[35] M. Martin, *Projective representations of compact groups in C^*-algebras*. Operator Theory: Advances and Applications **43**, Birkhäuser Verlag, 1990, 237–253.

[36] M. Martin, *Joint seminormality and Dirac operators*. Integral Equations Operator Theory **30** (1998), *1*, 101–121.

[37] M. Martin, *Self-commutator inequalities in higher dimension.* Proc. Amer. Math. Soc. **130** (2002), 2971–2983.

[38] M. Martin, *Spin geometry, Clifford analysis, and joint seminormality.* Advances in Analysis and Geometry, Proceedings of ICM 2002 Satellite Conference on Clifford Analysis, University of Macau, Macau, Trends in Mathematics Series **1**, Birkhäuser Verlag, 2004, 227–255.

[39] M. Martin, *Uniform approximation by solutions of elliptic equations and seminormality in higher dimension.* Current Trends in Operator Theory and Its Applications, Proceedings of IWOTA 2002, Virginia Tech, Virginia, USA, Operator Theory: Advances and Applications **149**, Birkhäuser Verlag, 2004, 387–406.

[40] M. Martin, *Deconstructing Dirac operators* I: *Quantitative Hartogs–Rosenthal theorems.* More Progress in Analysis, Proceedings of ISAAC 2005, University of Catania, Italy, World Scientific, 2009, 1065–1074.

[41] M. Martin, *Deconstructing Dirac operators* III: *Dirac and semi-Dirac pairs.* Topics in Operator Theory, Proceedings of IWOTA 2008, College of William and Mary, Virginia, USA, Operator Theory: Advances and Applications **203**, Birkhäuser Verlag, 2010, 347–362.

[42] M. Martin, *Deconstructing Dirac operators* II: *Integral representation formulas.* Hypercomplex Analysis and Applications, Proceedings of ISAAC 2009, Imperial College, England, Birkhäuser Verlag, 2011, 195–211.

[43] M. Martin, *Function spaces in quaternionic and Clifford analysis.* Operator Theory, Springer Verlag, 2015, 1393–1422.

[44] M. Martin, *Derivations and linear connections on Clifford vector bundles.* Adv. Appl. Clifford Algebr. **27** (2017), 2585–2605.

[45] M. Martin, *Seminormal systems of operators in Clifford environments.* Opuscula Math. **37** (2017), *1*, 81–107.

[46] M. Martin, *Holomorphic spectral theory. A geometric approach.* Complex Anal. Oper. Theory **16** (2022), *1*, 1–39.

[47] M. Martin and N. Salinas, *Differential geometry of generalized Grassmann manifolds in C^*-algebras.* Operator Theory and Boundary Eigenvalue Problems, Operator Theory: Advances and Applications **80**, Birkhäuser Verlag, 1995, 206–243.

[48] M. Martin and N. Salinas, *Flag manifolds and the Cowen–Douglas theory.* J. Operator Theory **38** (1997), *2*, 329–365.

[49] M. Martin and N. Salinas, *The canonical complex structure of flag manifolds in a C^*-algebra.* Non–Selfadjoint Operator Algebras, Operator Theory, and Related Topics, Operator Theory: Advances and Applications, **104**, Birkhäuser Verlag, 1998, 173–187.

[50] M. Martin and N. Salinas, *Weitzenböck type formulas and joint seminormality.* Operator Theory for Complex and Hypercomplex Analysis, Contemporary Mathematics **212** (1998), 157–167.

[51] M. Mitrea, *Singular Integrals, Hardy Spaces, and Clifford Wavelets.* Lecture Notes in Math. **1575**, Springer Verlag, 1994.

[52] S.A. Morris, *Pontryagin Duality and the Structure of Locally Compact Abelian Groups.* Cambridge Univ. Press, 1977.

[53] D. Mumford, *Algebraic Geometry* I: *Complex Projective Varieties.* Springer Verlag, 1976.

[54] H. Porta and L. Recht, *Minimality of geodesics in Grassmann manifolds*. Proc. Amer. Math. Soc. **100** (1987), 464–466.

[55] T. Qian, T. Hempfling, A. McIntosh, and F. Sommen (Eds.), *Advances in Analysis and Geometry. New Developments Using Clifford Algebras*. Trends in Math. Series **14**, Birkhäuser Verlag, 2004.

[56] R. Rocha-Chavez, M. Shapiro, and F. Sommen, *Integral Theorems for Functions and Differential Forms in* $\mathbb{C}^m$. Research Notes in Math. **428**, Chapman and Hall, 2002.

[57] J. Ryan (Ed.), *Clifford Algebras in Analysis and Related Topics*. CRC Press, 1995.

[58] J. Ryan and W. Sprössig (Eds.), *Clifford Algebras and Their Applications in Mathematical Physics. Clifford Analysis*. Progress in Mathematical Physics **19**, Birkhäuser Verlag, 2000.

[59] I. Sabadini, M. Shapiro, and F. Sommen (Eds.), *Hypercomplex Analysis*. Trends in Mathematics Series **6**, Birkhäuser Verlag, 2009.

[60] S. Sakai, *On a conjecture of Kaplansky*. Tôhoku Math. J. **12** (1960), 31–33.

[61] N. Salinas, *The Grassmann manifold of a* C^**-algebra and hermitian holomorphic bundles*. Operator Theory: Advances and Applications, **28**, Birkhäuser Verlag, 1988, 267–289.

[62] Ş. Strătilă, *Modular Theory in Operator Algebras*. Cambridge Univ. Press, 2020.

[63] M. Takesaki, *Conditional expectations in von Neumann algebras*. J. Funct. Anal. **9** (1972), 306–321.

[64] H. Upmeier, *Symmetric Banach Manifolds and Jordan* C^**-Algebras*. North Holland Mathematics Studies **104**, 1985.

[65] D.R. Wilkins, *The Grassmann manifold of a* C^**-algebra*. Proc. Royal Irish Acad. **90** (1990), *1*, 99–116.

Received 15 May 2023

Baker University
Baldwin City, KS 66006, USA
mircea.martin@bakeru.edu