A CONSTRUCTION VIA FORCING OF A HEREDITARILY WEAKLY KOSZMIDER SPACE

ANDRÉ SANTOLERI VILLA BARBEIRO and ROGÉRIO AUGUSTO DOS SANTOS FAJARDO

Communicated by Dan Timotin

We present an alternative construction – using forcing, instead of diamond principle – of a Hausdorff compactum K such that $C(L)$ has few operators, for every closed $L \subset K$. Moreover, we prove a new result about the density of $C(K)$ with few operators and present some open problems.

AMS 2020 Subject Classification: 47A53, 03E65, 54D80.

Key words: Banach spaces $C(K)$, indecomposable Banach spaces, forcing, few operators, Efimov's problem.

1. INTRODUCTION

In [10], Koszmider constructed an example of a Banach space of the form $C(K)$ with few operators in the sense that every bounded linear operator $T : C(K) \longrightarrow C(K)$ satisfies $T(f_n)(x_n) \to 0$, whenever $(f_n)_{n\in\mathbb{N}}$ is a bounded pairwise disjoint (i.e., $f_n \cdot f_m = 0$, for $n \neq m$) sequence in $C(K)$ and $(x_n)_{n\in\mathbb{N}}$ is a sequence in K such that $f_n(x_n) = 0$, for every $n \in \mathbb{N}$. Such operators are called *weak multipliers* and we say that K is a *weakly Koszmider space* if all bounded linear operators on $C(K)$ are weak multipliers. Also in [10], Koszmider constructed a Banach space of the form $C(K)$ with few operators in the sense that every operator on $C(K)$ is a multiplication by a continuous functions plus a strictly singular operator. Such operators are called *weak multiplications* and we say that K is a *Koszmider space* if all bounded linear operators on $C(K)$ are weak multiplications. It is clear that a weak multiplication is a weak multiplier and therefore, every Koszmider space is a weakly Koszmider space.

When K is a connected Koszmider space (as it is constructed in [10], using CH, and in [17], in ZFC), $C(K)$ is an indecomposable Banach space, which

The first author was supported by a grant from CAPES and a grant from CNPQ, number 142058/2016-5. The second author was supported by Fapesp, project 2016/25574-8 and individual grant 2018/10254-3.

doi: 10.59277/RRMPA.2024.61.76

means that it cannot be written as the direct sum of two infinite-dimensional subspaces. This is related to the result of Gowers and Maurey (see [7] and [8]) about general Banach spaces: a construction of a Banach space where all operators have the form $\lambda I + S$, where λ is a scalar (real or complex), I is the identity operator and S is strictly singular. Such space is *hereditarily indecomposable*, i.e., all its closed subspaces are indecomposable. Banach spaces of the form $C(K)$ cannot be hereditarily indecomposable, since it always contains c_0 as a subspace.

Two of the most important questions arisen from this subject, the first one asked in the original paper [10]: Is there a compactum K such that every closed $L \subset K$ is a weakly Koszmider space? Given a cardinal κ, is it possible to build a (connected) Koszmider space with weight κ?

The first question was answered positively, under axiom $\diamondsuit$, in [4]. Since neither $\beta\mathbb{N}$ neither a convergent sequence is a weakly Koszmider space, this provides another consistent counter-example for the Efimov's problem (the first one was given by [6], assuming CH, which is weaker than $\diamondsuit$). We will refer to infinite compact spaces with the property that every infinite closed subspace is weakly Koszmider as *hereditarily weakly Koszmider space.*

The hereditarily indecomposable Banach space constructed by Gowers and Maurey is separable. It is proved in [18] that the maximum density of an hereditarily indecomposable Banach space is 2^ω. This is the density of $C(K)$ constructed in [10], as well as in [4], regarding that the density of $C(K)$ is the weight of K. Koszmider spaces have uncountable weight and the authors in [14] showed that there is no bound on the weight of connected Koszmider spaces and, consequently, there is no bound on sizes of indecomposable Banach spaces.

A consistent construction of a Koszmider space with weight $\omega_1 < 2^\omega$ was made in [3]. Other examples of Koszmider spaces with weight $2^{\omega_1} > 2^\omega$ were constructed in [11] (where K is 0-dimensional) and in [13] (where K is connected). All the three constructions use forcing.

In this paper, we present a new method of construction of a hereditarily weakly Koszmider space using forcing. The notion of the forcing is given in Section 3 and the main result is proved in Section 4, where we work in the generic extension. Section 2 shows the basic results and terminology used in the paper.

Using forcing instead of the diamond principle simplifies the notation in several aspects, providing a cleaner proof for whom is acquainted with the forcing technique.

In Section 5, we provide a new result about the density of $C(K)$ with few operators. We show that the classical constructions of $C(K)$ with few operators imposes an upper bound for the density of $C(K)$. Finally, Section 6 discusses some open problems concerning the theory of hereditarily weakly Koszmider spaces.

2. COMPLETE EXTENSIONS BY CONTINUOUS FUNCTIONS

In this section, we state some basic terminology used in [4] and [10].

If f is a real continuous function on a compact space K, we denote by $supp(f)$ the closure of $\{x \in K : f(x) \neq 0\}$ in K. We say that two real functions f and g on the same domain are *disjoint* if $f(x) \cdot g(x) = 0$, for every x in the domain of f and g.

If K is a compact Hausdorff space, we denote by $C(K)$ the Banach space of all continuous function from K into $\mathbb{R}$, normed by the supremum. We denote by $M(K)$ the Banach space of the Radon measures on K, normed by the variation of the measure.

All topological spaces appearing in this paper are Hausdorff.

Definition 2.1. Let K be a compact space and let $(f_n)_{n\in\mathbb{N}}$ be a bounded pairwise disjoint sequence in $C(K)$. We define

$$D((f_n)_{n\in\mathbb{N}}) = \bigcup\{U : U \text{ is open and } \{n : U \cap supp(f_n) \neq \emptyset\} \text{ is finite}\}$$

and

$$\Delta((f_n)_{n\in\mathbb{N}}) = K \setminus D((f_n)_{n\in\mathbb{N}}).$$

The following lemma is immediate from the definition.

LEMMA 2.2. *Let K be a compact space and let $(f_n)_{n\in\mathbb{N}}$ be a bounded pairwise disjoint sequence in $C(K)$. Then $\Delta((f_n)_{n\in a}) \subset \Delta((f_n)_{n\in b})$, for all infinite subsets $a \subset b \subset \mathbb{N}$.*

LEMMA 2.3 ([10, Lemma 4.1]). *Let K be a compact space and let $(f_n)_{n\in\mathbb{N}}$ be a bounded pairwise disjoint sequence in $C(K)$. Then:*

(i) *$f \in C(K)$ is $\sup\{f_n : n \in \mathbb{N}\}$ in the lattice $C(K)$ if, and only if,*

$$\{x \in K : \Sigma_{n\in\mathbb{N}} f_n(x) \neq f(x)\}$$

is nowhere dense in K;

(ii) *$D((f_n)_{n\in\mathbb{N}})$ is an open dense subset of K and $\Sigma_{n\in\mathbb{N}} f_n$ is continuous on $D((f_n)_{n\in\mathbb{N}})$.*

Definition 2.4. Suppose that K is compact, $L \subseteq K \times [0,1]$ and $(f_n)_{n\in\mathbb{N}}$ is a pairwise disjoint sequence of continuous functions from K into $[0,1]$. Let

$\pi : L \longrightarrow K$ be the standard projection. We say that L is an *extension* of K by $(f_n)_{n\in\mathbb{N}}$ – and we will denote by $K((f_n)_{n\in\mathbb{N}})$ – if L is the closure of the graph of $\Sigma_{n\in\mathbb{N}} f_n|D((f_n)_{n\in\mathbb{N}})$. We say that L is a *complete extension* if, moreover, for every $x \in K$, $\pi^{-1}[\{x\}]$ is either a singleton or $\{x\} \times [0,1]$.

LEMMA 2.5 ([10, Lemmas 4.2 and 4.4] and [4, Lemma 3.6]). *Let $(f_n)_{n\in\mathbb{N}}$ be a pairwise disjoint sequence of continuous functions from K into $[0,1]$, $L = K((f_n)_{n\in\mathbb{N}})$ and $\pi : L \longrightarrow K$ be the standard projection. Then:*

(i) *The function $f : L \to [0,1]$ defined by $f(x,t) = t$ is the supremum of $(f_n \circ \pi)_{n\in\mathbb{N}}$ in $C(L)$;*

(ii) *If M is a nowhere dense set in K, then $\pi^{-1}[M]$ is a nowhere dense set in L. In particular, if $(g_n)_{n\in\mathbb{N}}$ has supremum in $C(K)$, then $(g_n \circ \pi)_{n\in\mathbb{N}}$ has supremum in $C(L)$;*

(iii) *If K is connected and L is a complete extension of K, then L is also connected.*

3. DEFINING THE FORCING

We suppose CH holds in the ground model V. Let κ be an uncountable cardinal.

For $I \subseteq J \subseteq \kappa$, we denote by $\pi_{J,I}$ the standard projection from $[0,1]^J$ onto $[0,1]^I$, given by $\pi_{J,I}(x) = x|_I$. When J is clear from the context, we will denote $\pi_{J,I}$ simply by π_I. In particular, we will denote $\pi_{\kappa,I}$ by π_I.

For a an infinite subset of $\mathbb{N}$ and $(F_n)_{n\in a}$ a sequence of closed subsets of a topological space K, we say that $(F_n)_{n\in a}$ *converges to* $x \in K$ if the set $\{n \in a : F_n \not\subseteq U\}$ is finite, for every open neighbourhood U of x.

We define the forcing $\langle \mathbb{P}, \leq \rangle$ taking $\mathbb{P}$ the set of all 4-uples $p = (I_p, K_p, \mathcal{P}_p, \mathcal{B}_p)$ which satisfy the following conditions:

P1 $I_p \subseteq \kappa$ and $|I_p| \leq \omega$;

P2 $K_p \subseteq [0,1]^{I_p}$ is compact and connected;

P3 $\mathcal{P}_p$ is a countable set of 4-uples $((F_n)_{n\in\mathbb{N}}, a, b, z)$ such that

- $(F_n)_{n\in\mathbb{N}}$ is a pairwise disjoint sequence of closed subsets of K_p;
- a, b are disjoint infinite subsets of $\mathbb{N}$;
- $z \in K_p$;
- $(F_n)_{n\in a}$ and $(F_n)_{n\in b}$ converge to z.

P4 $\mathcal{B}_p = \{(x^p_\xi, y^p_\xi) : \xi \in I_p\}$, where $x^p_\xi, y^p_\xi \in K_p$ and $x^p_\xi|_{I_p \cap \xi} = y^p_\xi|_{I_p \cap \xi}$.

The order $\leq$ is defined by taking $q \leq p$ if, and only if,

O1 $I_q \supseteq I_p$;

O2 $\pi_{I_q, I_p}[K_q] = K_p$;

O3 If M is nowhere dense in K_p, then $\pi^{-1}_{I_q, I_p}[M]$ is nowhere dense in K_q;

O4 For every $((F_n)_{n\in\mathbb{N}}, a, b, z) \in \mathcal{P}_p$, there exists $((F'_n)_{n\in\mathbb{N}}, a', b', z') \in \mathcal{P}_q$ such that

- $|b' \setminus b \cup a' \setminus a| < \omega$;
- $z'|_{I_p} = z$;
- $F'_n = \pi^{-1}_{I_q, I_p}[F_n]$, for every $n \in \mathbb{N}$.

O5 For every $\xi \in I_p$, we have $x^q_\xi|_{I_p} = x^p_\xi$ and $y^q_\xi|_{I_p} = y^p_\xi$.

Lemma 3.1. *$(\mathbb{P}, \leq)$ is σ-closed.*

Proof. Let $(p_n)_{n\in\mathbb{N}}$ be a decreasing sequence in $\mathbb{P}$. We have to define $p = (I_p, K_p, \mathcal{P}_p, \mathcal{B}_p)$ such that $p \in \mathbb{P}$ and $p \leq p_n$, for all $n \in \mathbb{N}$.

Let $I_p = \bigcup_{n\in\mathbb{N}} I_{p_n}$, K_p the inverse limit of $(K_{p_n})_{n\in\mathbb{N}}$ and $\mathcal{B}_p = \{(x^p_\xi, y^p_\xi) : \xi \in I_p\}$, where $x^p_\xi = \bigcup_{n\geq n_0} x^{p_n}_\xi$, $y^p_\xi = \bigcup_{n\geq n_0} y^{p_n}_\xi$ and n_0 is the least integer such that $\xi \in I_{p_{n_0}}$. Clearly I_p and $\mathcal{B}_p$ satisfy the conditions P1 and P4. Moreover, since inverse limits preserve compactness and connectedness, K_p satisfies P2.

In order to define $\mathcal{P}_p$, we will define a function ϕ whose domain is $\bigcup \mathcal{P}_{p_n}$ and $\mathcal{P}_p$ will be taken as the range of ϕ.

Let $Q \in \mathcal{P}_{n_0}$, for some n_0. Fix a sequence $((F^n_m)_{m\in\mathbb{N}}, a_n, b_n, z_n)_{n\geq n_0}$ such that $Q = ((F^{n_0}_m)_{m\in\mathbb{N}}, a_{n_0}, b_{n_0}, z_{n_0})$ and, for every $n > n_0$, the 4-uple $((F^n_m)_{m\in\mathbb{N}}, a_n, b_n, z_n) \in \mathcal{P}_{p_n}$ is obtained from $((F^{n-1}_m)_{m\in\mathbb{N}}, a_{n-1}, b_{n-1}, z_{n-1})$ using condition O4 of the order $\leq$.

For each $m \in \mathbb{N}$, take F_m the inverse limit of $(F^n_m)_{n\geq n_0}$. Let a be an infinite pseudo-intersection of $(a_n)_{n\geq n_0}$, i.e., $a \setminus a_n$ is finite, for every $n \geq n_0$ (the existence of a follows from [2, Theorem 3.1] and the fact that $a_{n+1} \setminus a_n$ is finite, for every $n \in \mathbb{N}$).

Let b' be an infinite pseudo-intersection of $(b_n)_{n\geq n_0}$ and take $b = b' \setminus a$. Clearly, $a \cap b'$ is finite and, hence, b is still a pseudo-intersection of $(b_n)_{n\geq n_0}$.

Finally, take $z = \bigcup_{n\geq n_0} z_n$ and define $\phi(Q) = ((F_m)_{m\in\mathbb{N}}, a, b, z)$. Now we will prove P3. Disjointness of a and b follows immediately from the definition. Clearly, $z \in K_p$ and $(F_m)_{m\in\mathbb{N}}$ is a pairwise disjoint sequence of closed subsets of K_p. Suppose that $(F_m)_{m\in a}$ does not converge to z. It means that there exist a basic open neighbourhood U of z, an infinite $c \subset a$ and a sequence $(x_m)_{m\in c}$ such that $x_m \in F_m \setminus U$, for every $m \in c$. Taking $n \in \mathbb{N}$ such that U depends only on coordinates in I_{p_n}, we have that $\pi_{I_{p_n}}[U]$ is an open neighbourhood of z_n, in K_{p_n}, and $x_m|I_{p_n} \in F^n_m \setminus \pi_{I_{p_n}}[U]$, for every $m \in c$. Since $a \setminus a_n$ is finite, $a_n \cap c$ is an infinite subset of a_n, which leads us to a contradiction with condition P3 of p_n. We prove analogously that $(F_m)_{m\in b}$ converges to z.

It remains to prove that $p \leq p_n$, for every $n \in \mathbb{N}$. Condition O1, O2 and O5 are trivial, and O4 follows from the definition of ϕ. Let us verify O3. Let M be a nowhere dense subset of K_{p_n}, for some n, and suppose that $\overline{\pi^{-1}_{I_p,I_{p_n}}[M]}$ is not nowhere dense in K_p. Let $V \subseteq \overline{\pi^{-1}_{I_p,I_{p_n}}[M]}$ be a basic non-empty open set and take $m > n$ such that the coordinates which determine V belong to I_{p_m}. We notice that $\pi^{-1}_{I_{p_m},I_{p_n}}[\overline{M}]$ is nowhere dense in K_{p_m}, because $p_m \leq p_n$ and $\overline{M}$ is nowhere dense in K_{p_n}. Since $\overline{\pi^{-1}_{I_p,I_{p_n}}[M]} \subseteq \pi^{-1}_{I_p,I_{p_n}}[\overline{M}]$, we have

$$\pi_{I_p,I_{p_m}}[V] \subseteq \pi_{I_p,I_{p_m}}[\pi^{-1}_{I_p,I_{p_n}}[\overline{M}]] = \pi_{I_p,I_{p_m}}[\pi^{-1}_{I_p,I_{p_m}}[\pi^{-1}_{I_{p_m},I_{p_n}}[\overline{M}]]] = \pi^{-1}_{I_{p_m},I_{p_n}}[\overline{M}]$$

and $\pi_{I_p,I_{p_m}}[V]$ is open in K_{p_m}, which contradicts the fact that $\pi^{-1}_{I_{p_m},I_{p_n}}[\overline{M}]$ is nowhere dense in K_{p_m}. □

Lemma 3.2. *Let $p \in \mathbb{P}$. Given*

(a) *a pairwise disjoint sequence $(f_n : n \in \mathbb{N})$ from K_p into $[0,1]$;*

(b) *a relatively discrete sequence $(x_n)_{n\in\mathbb{N}}$ of distinct points of K_p such that $x_n \notin supp(f_m)$, for every $n, m \in \mathbb{N}$;*

(c) *an $\varepsilon > 0$;*

(d) *a bounded sequence $(\mu_n : n \in \mathbb{N})$ of Radon measures on K such that $|\int f_n d\mu_n| > \varepsilon$, for every $n \in \mathbb{N}$;*

there exist $q \leq p$, $\delta > 0$, infinite $b \subset a \subset \mathbb{N}$, $z' \in K_q$ and continuous functions $f'_n : K_p \longrightarrow [0,1]$ such that $supp(f'_n) \subset supp(f_n)$ and

(e) *$|\int f'_n d\mu_n| > \delta$ and $\Sigma\{\int f'_m d|\mu_n| : m \neq n, m \in a\} < \delta/3$, for every $n \in a$;*

(f) *$K_q = K_p((f'_n)_{n\in b})$ is a complete extension;*

(g) *$(f'_n \circ \pi_{I_q,I_p})_{n\in b}$ has supremum in $C(K_q)$;*

(h) *$((\pi^{-1}_{I_q,I_p}[\{x_n\}])_{n\in\mathbb{N}}, b, a \setminus b, z') \in \mathcal{P}_q$.*

Proof. It follows mostly from [1, Lemma 2.10] and the definition of $\mathbb{P}$, taking $I_q = I_p \cup \{\alpha\}$, for α any upper boundary of I_p in κ. However, we have to define $\mathcal{B}_q$ and prove condition O5 to guarantee that $q \leq p$. In fact, it is enough to define x^q_ξ and y^q_ξ as any extensions of x^p_ξ and y^p_ξ, respectively, in K_q, for $\xi \in I_p$, and $x^q_\alpha = y^q_\alpha$ as any element of K_q. Since $\alpha > \xi$, for every $\xi \in I_p$, condition O5 of $q \leq p$ clearly holds. □

Lemma 3.3. *Given $p \in \mathbb{P}$ and $\alpha < \kappa$, there exist $q \leq p$ and $\xi > \alpha$ such that $\xi \in I_q$ and $x^q_\xi \neq y^q_\xi$.*

Proof. Fix $z \in K_p$ and $(z_n)_{n\in\mathbb{N}}$ a sequence in K_p converging to z. Let V_n open neighbourhoods of z_n pairwise disjoint whose diameters converge to zero. Assume $z \notin V_n$, for every n. Fix continuous functions $f_n : K_p \longrightarrow [0,1]$ such that $f_n(z_n) = 1$ and $supp(f_n) \subset V_n$. Define $\mu_n = \delta_{z_n}$ and $\varepsilon = \frac{1}{2}$. Take $(x_n)_{n\in\mathbb{N}}$ any relatively discrete sequence in K_p.

Take $\xi > max\{\alpha, \sup I_p\}$ in κ and $q \leq p$ as in Lemma 3.2, taking $I_q = I_p \cup \{\xi\}$. By item (e), we have $f'_n(z_n) > \delta$. Therefore, taking t a limit point of $\{f'_n(z_n) : n \in \mathbb{N}\}$, we have $t \geq \delta$ and $(z,t) \in K_q$. Define $x^q_\xi = (z,0)$ and $y^q_\xi = (z,t)$. □

Lemma 3.4. *Let $p \in \mathbb{P}$. Take $(f_n)_{n\in\mathbb{N}}$ a pairwise disjoint sequence of continuous functions from K_p into $[0,1]$ and suppose that $(f_n)_{n\in\mathbb{N}}$ has supremum f in $C(K_p)$. Hence, for every $q \leq p$, $f \circ \pi_{I_q,I_p}$ is the supremum of $(f_n \circ \pi_{I_q,I_p})_{n\in\mathbb{N}}$ in $C(K_q)$.*

Proof. By [10, Lemma 4.1], f is the supremum of $(f_n)_{n\in\mathbb{N}}$ in $C(K_p)$ if, and only if, the set

$$\Delta(f,(f_n)_{n\in\mathbb{N}}) = \{x \in K_p : \Sigma_{n\in\mathbb{N}} f_n(x) \neq f(x)\}$$

is nowhere dense in K_p.

Since $\Delta(f \circ \pi_{I_q,I_p}, (f_n \circ \pi_{I_q,I_p})_{n\in\mathbb{N}}) = \pi^{-1}_{I_q,I_p}[\Delta(f,(f_n)_{n\in\mathbb{N}})]$, by condition P3 and [10, Lemma 4.3] it follows that $\Delta(f \circ \pi_{I_q,I_p}, (f_n \circ \pi_{I_q,I_p})_{n\in\mathbb{N}})$ is nowhere dense in K_q, proving that $f \circ \pi_{I_q,I_p}$ is the supremum of $f_n \circ \pi_{I_q,I_p}$ in $C(K_q)$. □

4. GENERIC EXTENSION

Since $\mathbb{P}$ is σ-closed, $\mathbb{P}$ does not add countable sets (see [15]). In particular, $\mathbb{P}$ does not add real numbers and, hence, $[0,1]^V = [0,1]^{V[G]}$ in any generic extension $V[G]$.

Let G be a $\mathbb{P}$-generic over V. In $V[G]$, define $I_G = \cup\{I_p : p \in G\}$. Let $\dot{I}_G$ be a $\mathbb{P}$-name for I_G in V.

Lemma 4.1. *I_G is unbounded in $(\kappa)^{V[G]}$.*

Proof. Immediate consequence of Lemma 3.3, which proves that the set $\{p \in \mathbb{P} : \exists \xi > \alpha(\xi \in I_p)\}$ is dense in $\mathbb{P}$, for every $\alpha < \kappa$. $\square$

Lemma 4.2. *For every $p \in \mathbb{P}$, $C(K_p)^{V[G]} = C(K_p)^V$ and $M(K_p)^{V[G]} = M(K_p)^V$.*

Proof. Since $\mathbb{P}$ does not add countable subsets of sets in the ground model, we have $[0,1]^{V[G]} = [0,1]^V$. Let $f : K_p \longrightarrow \mathbb{R}$ be a continuous function in $V[G]$. We have to prove that $f \in V$ and f is continuous in V. Since $K_p \subseteq [0,1]^{I_p}$, K_p is metrizable and separable. Let E be a countable dense subset of K_p and take $g = f|_E$. Clearly, g is a countable subset of $K_p \times \mathbb{R}$ and, hence, $g \in V$.

For each $x \in K_p$, using metrizability of K_p and density of E, we find a sequence $(x_n)_{n\in\mathbb{N}}$ in E which converges to x, in V. By the fact that the forcing does not add real numbers and, consequently, it does not add basic open sets of $[0,1]^{I_p}$, the notion of convergence is absolute for V and $V[G]$, and, therefore, x_n converges to x in $V[G]$. By continuity of f in $V[G]$ it implies that $f(x_n)$ converges $f(x)$ and, in particular, $g(x_n) = f(x_n)$ also converges to $f(x)$ in V. Hence, we may define a function $h : K_p \longrightarrow \mathbb{R}$ in V given by

$$h(x) = \lim_{n\to\infty} g(x_n),$$

where $(x_n)_{n\in\mathbb{N}}$ is a sequence in E converging to x. By the below observations, the above limit exists and does not depend on the choice of $(x_n)_{n\in\mathbb{N}}$. Moreover, h is continuous in V and $h(x) = f(x)$, for every x, proving that $f \in C(K_p)^V$.

We concluded that $C(K_p)^{V[G]} \subseteq C(K_p)^V$. Conversely, we proceed analogously to prove that a continuous real function defined on K_p in V is also continuous in $V[G]$.

Finally, since we may identify Radon measures on K_p with functions on a countable basis of K_p with range in $\mathbb{R}$, the proof that $M(K_p)^{V[G]} = M(K_p)^V$ is analogous. $\square$

Lemma 4.3. *If $p \Vdash (\dot{I} \subseteq \dot{I}_G) \wedge (|\dot{I}| \leq \check{\omega})$, then there exists $q \leq p$ such that $q \Vdash \dot{I} \subseteq \check{I}_q$.*

Proof. Let $\dot{\alpha}_n$ names for elements of κ such that

$$p \Vdash \dot{I} = \{\dot{\alpha}_n : n \in \check{\mathbb{N}}\}.$$

Take $p_{-1} = p$. Suppose we have defined p_{n-1}, for some $n \geq 0$. Let $p'_n, q_n \in \mathbb{P}$ and $\alpha_n \in \kappa$ such that $p'_n \leq p_{n-1}$ and

$$p'_n \Vdash \dot{\alpha}_n = \check{\alpha}_n, \check{\alpha}_n \in \check{I}_{q_n}, \check{q}_n \in \dot{G}.$$

Since $p'_n \Vdash \check{p}'_n, \check{q}_n \in \dot{G}$, there exist $p''_n \leq p'_n$ and $p_n \in \mathbb{P}$ such that

$$p''_n \Vdash \check{p}_n \leq \check{p}'_n, \check{q}_n.$$

By absoluteness of the order of $\mathbb{P}$, we have

$$p_n \Vdash \check{p}_n \leq \check{p}'_n, \check{q}_n.$$

Hence, $p_n \Vdash \check{I}_{q_n} \subseteq \check{I}_{p_n}$ and

$$p_n \Vdash \dot{\alpha}_n = \check{\alpha}_n, \check{\alpha}_n \in \check{I}_{p_n}.$$

Since $(p_n)_{n\in\mathbb{N}}$ is decreasing and $\mathbb{P}$ is σ-closed, there exists $q \in \mathbb{P}$ such that $q \leq p_n$, for every $n \in \mathbb{N}$. Therefore,

$$q \Vdash \forall n \in \check{\mathbb{N}}\ \dot{\alpha}_n = \check{\alpha}_n,\ \check{\alpha}_n \in \check{I}_q.$$

I.e., $q \Vdash \dot{I} \subseteq \check{I}_q$, proving the lemma. □

In $V[G]$ we define

$$K = \{x \in [0,1]^{I_G} \ :\ \forall p \in G(x|_{I_p} \in K_p)\}$$

as a topological subspace of $[0,1]^{I_G}$.

Let $\dot{K}$ and $\dot{G}$ $\mathbb{P}$-names for K and G, respectively.

THEOREM 4.4. *In $V[G]$, let K be as above.*

(A) *K is compact and connected;*

(B) *Given*

- (a) *a pairwise disjoint sequence of continuous functions $(f_n : n \in \mathbb{N})$ from K into $[0,1]$;*
- (b) *a relatively discrete sequence $(x_n : n \in \mathbb{N})$ of points of K such that $f_n(x_m) = 0$, for every $n, m \in \mathbb{N}$;*
- (c) *an $\varepsilon > 0$;*
- (d) *a bounded sequence $(\mu_n)_{n\in\mathbb{N}}$ on $M(K)$ such that $|\int f_n d\mu_n| > \varepsilon$, for every $n \in \mathbb{N}$;*

there exist $\delta > 0$, infinite $b \subseteq a \subseteq \mathbb{N}$ and continuous functions f'_n from K into $[0,1]$ such that

- (e) *$supp(f'_n) \subseteq supp(f_n)$, for every $n \in \mathbb{N}$;*
- (f) *$|\int f'_n d\mu_n| > \delta$ and $\Sigma\{\int f'_m d|\mu_n| : m \neq n, m \in a\} < \delta/3$, for every $n \in a$;*
- (g) *$\{f'_n : n \in b\}$ has supremum in $C(K)$;*

(h) $\overline{\{x_n : n \in b\}} \cap \overline{\{x_n : n \in a \setminus b\}} \neq \emptyset$.

(C) *For every closed $L \subseteq K$, every operator in $C(L)$ is a weak multiplier.*

Proof. To prove (A), we notice that $K = \bigcap_{p \in G} \pi^{-1}_{I_G, I_p}[K_p]$ and, by compactness of each K_p, K is closed and compact in $[0,1]^{I_G}$. Suppose by contradiction that K is not connected. Let V and W be disjoint open sets of K such that $K \setminus V \cup W = \emptyset$. By compactness, we may assume that V and W are finite unions of basic open sets and, therefore, they are determined by finite coordinates. Hence, there exists $p \in G$ such that I_p contains all the coordinates which determine V and W. This implies that $\pi_{I_p}[V]$ and $\pi_{I_p}[W]$ are disjoint open sets whose union is K_p, contradicting connectedness of K_p.

Now we will prove part (B). Let $(f_n)_{n\in\mathbb{N}}$, $(\mu_n)_{n\in\mathbb{N}}$, $(x_n)_{n\in\mathbb{N}}$ and ε as in the items from (a) to (d) of part (B).

By Tietze's theorem there exist continuous functions $\tilde{f}_n : [0,1]^{I_G} \longrightarrow [0,1]$ such that $\tilde{f}_n|_K = f_n$. By a theorem of Mibu (see [16]) there exists a countable $I \subseteq I_G$ such that $\tilde{f}_n(x) = \tilde{f}_n(y)$, whenever $x|I = y|I$ and $n \in \mathbb{N}$.

Let $\mu'_n \in M(K_p)$ defined as $\mu'_n(A) = \mu_n(\pi^{-1}_{I_p}[A])$ for every Borel set $A \subset K_p$. We notice that $\int g d\mu'_n = \int g \circ \pi_{I_p} d\mu_n$, for every $g \in C(K_p)$.

Let $\dot{f}_n$, $\dot{\mu}'_n$, $\dot{x}_n$ and $\dot{I}$ be $\mathbb{P}$-names for the objects described above. By Maximum Principle (see [15]) we may assume that $\mathbb{P}$ forces items (a) to (d) and $\dot{I}$ satisfies the condition of Mibu's Theorem. We have to prove that for every $p \in \mathbb{P}$ there exists $q \leq p$ which forces items (e) to (h).

Fix $p \in \mathbb{P}$. By Lemma 4.3 there exists $r \leq p$ such that $r \Vdash \dot{I} \subseteq \check{I}_r$. By Mibu's Theorem and $\pi_{I_r}[K] = K_r$ there exist $\mathbb{P}$-names $\dot{g}_n$ for continuous functions from K_r into $[0,1]$ such that

$$r \Vdash \dot{g}_n(\dot{x}|\check{I}_r) = \dot{f}_n(\dot{x}), \text{ for every } \dot{x} \in \dot{K}.$$

Clearly, r forces that $(\dot{g}_n)_{n\in\mathbb{N}}$ are pairwise disjoint. We have $r \Vdash \dot{J} \supseteq \check{I}_r$ and

$$r \Vdash \dot{f}_n \circ \pi_{\dot{J}} = \dot{g}_n \circ \pi_{\dot{J}, \check{I}_r},$$

for every $n \in \mathbb{N}$.

Taking $g_n = \mathrm{val}_G(\dot{g}_n)$, by Lemma 4.2 we have $g_n \in V$.

Let $\nu_n \in M(K_r)$ defined by $\nu_n(A) = \mu_n(\pi^{-1}_{I_r}[A])$, for every Borel set $A \subset K_r$.

We may also assume that there exist $z_n \in K_r$, for $n \in \mathbb{N}$, such that

$$(*) \qquad r \Vdash \check{z}_n = \dot{x}_n|_{\check{I}_r}.$$

By Lemma 3.2 there exist $q \leq r$, $\delta > 0$, infinite $b \subset a \subset \mathbb{N}$, $z' \in K_q$ and continuous functions $g'_n : K_r \longrightarrow [0,1]$ such that $supp(g'_n) \subset supp(g_n)$ and

(i) $|\int g'_n d\nu_n| > \delta$ and $\Sigma\{\int g'_m d|\nu_n| : m \neq n, m \in a\} < \delta/3$, for every $n \in a$;

(ii) $K_q = K_r((g'_n)_{n\in b})$ is a complete extension;

(iii) $(g'_n \circ \pi_{I_q,I_r})_{n\in b}$ has supremum in $C(K_q)$;

(iv) $((\pi^{-1}_{I_q,I_r}[\{z_n\}])_{n\in\mathbb{N}}, b, a \setminus b, z') \in \mathcal{P}_q$.

Let $\dot{g}$ be a $\mathbb{P}$-name for the supremum of $(\check{g}'_n \circ \pi_{I_q,I_r})_{n\in b}$ in $C(K_q)$. We may assume that such condition is forced by q.

In $V[G]$ define $f'_n = g'_n \circ \pi_{I_q}$. Clearly (e) holds. Item (f) follows from (i). In order to prove (g) and (h) and conclude the proof of part (B), it is sufficient to prove that:

(v) $q \Vdash \dot{g} \circ \pi_{I_q}$ is the supremum of $(\dot{f}'_n)_{n\in b}$ in $C(\dot{K})$ and $\overline{\{\dot{x}_n : n \in \check{b}\}} \cap \overline{\{\dot{x}_n : n \in \check{a} \setminus \check{b}\}} \neq \emptyset$.

Suppose that $q \not\Vdash \dot{g} \circ \pi_{I_q} = \sup(\dot{f}'_n)_{n\in b}$. We have

$$q \Vdash \dot{f}'_n(x) = \dot{g}'_n(x|_{\check{I}_r}) = \dot{g}'_n \circ \pi_{I_p,I_r}(x) \leq \dot{g}(x|_{\check{I}_q}) = \dot{g} \circ \pi_{I_q}(x)$$

and, so, there exist $s \leq q$ and a $\mathbb{P}$-name $\dot{h}$ for a continuous function from K into $[0,1]$ such that $s \Vdash \dot{f}'_n \leq \dot{h} < \dot{g} \circ \pi_{\check{I}_q}$, for every $n \in b$. By Mibu's theorem there exist $s' \leq s$ and $\dot{J}$ such that

$$s' \Vdash \dot{J} \subseteq \dot{I}_G,\ |\dot{J}| = \check{\mathbb{N}} \text{ and, if } x|_{\dot{J}} = y|_{\dot{J}} \text{ then } \dot{h}(x) = \dot{h}(y).$$

By Lemma 4.3 there exists $t \leq s'$ such that $t \Vdash \dot{J} \subseteq \check{I}_t$. Hence, there exists a $\mathbb{P}$-name $\dot{h}'$ for a continuous function from K_t into $[0,1]$ such that

$$t \Vdash \dot{h} = \dot{h}' \circ \pi_{\check{I}_t},\ \dot{g}'_n \circ \pi_{\check{I}_r} \leq \dot{h}.$$

Since $t \leq s$, there exists a $\mathbb{P}$-name $\dot{x}$ for an element of K such that

$$t \Vdash \dot{h}'(\dot{x}|_{\check{I}_t}) = \dot{h}(\dot{x}) < \dot{g} \circ \pi_{\check{I}_r}(\dot{x}) = \dot{g} \circ \pi_{\check{I}_t,\check{I}_q}(\dot{x}|_{\check{I}_q}).$$

Therefore, t forces that $\dot{g} \circ \pi_{\check{I}_t,\check{I}_q}$ is not the supremum of $(\dot{g}'_n \circ \pi_{\check{I}_r})_{n\in b}$ in K_r, because

$$t \Vdash \dot{g}'_n \circ \pi_{\check{I}_t,\check{I}_r} \leq \dot{h}' < \dot{g} \circ \pi_{\check{I}_t,\check{I}_r}.$$

This contradicts Lemma 3.4.

To prove the second part of (v), suppose that

$$q \not\Vdash \overline{\{\dot{x}_n : n \in \check{b}\}} \cap \overline{\{\dot{x}_n : n \in \check{a} \setminus \check{b}\}} \neq \emptyset.$$

By compactness of K, there exist $s \leq q$ and $\mathbb{P}$-names $\dot{V}_1$ and $\dot{V}_2$ of basic open sets of K such that

$$s \Vdash \dot{V}_1 \cap \dot{V}_2 = \emptyset,\ \{\dot{x}_n : n \in \check{b}\} \subseteq \dot{V}_1,\ \{\dot{x}_n : n \in \check{a} \setminus \check{b}\} \subseteq \dot{V}_2.$$

Let $\dot{I}$ be a $\mathbb{P}$-name such that $s \Vdash \dot{I} = \{\alpha \in \check{I_G} : \pi_{\{\alpha\}}[\dot{V}_1] \neq \pi_{\{\alpha\}}[\dot{K}]$ or $\pi_{\{\alpha\}}[\dot{V}_2] \neq \pi_{\{\alpha\}}[\dot{K}]\}$.

Since $\dot{V}_1$ and $\dot{V}_2$ are $\mathbb{P}$-names for basic open sets, we may assume that $s \Vdash |\dot{I}| < \omega$. By Lemma 4.3 there exists $t \leq s$ such that

$$t \Vdash \dot{I} \subseteq \check{I}_t.$$

Hence, t forces that $\pi_{\check{I}_t}[\dot{V}_1]$ and $\pi_{\check{I}_t}[\dot{V}_2]$ separate $\{\dot{x}_n|_{\check{I}_t} : n \in \check{b}\}$ and $\{\dot{x}_n|_{\check{I}_t} : n \in \check{a} \setminus \check{b}\}$ in K_t. I.e.,

$$(**) \qquad t \Vdash \overline{\{\dot{x}_n|_{\check{I}_t} : n \in \check{b}\}} \cap \overline{\{\dot{x}_n|_{\check{I}_t} : n \in \check{a} \setminus \check{b}\}} = \emptyset \text{ in } \check{\mathrm{K}}_{\mathrm{t}}.$$

This contradicts (iv) and conditions P3 and O4 of the definition of the forcing. In fact, by $(*)$ we have $x_n|_{I_t} \in \pi^{-1}_{I_t,I_r}[\{z_n\}]$, for every $n \in \mathbb{N}$. Moreover, by (iv) and O4 there exist $z'' \in K_t$ and infinite disjoint $a', b' \subseteq \mathbb{N}$ such that $((\pi^{-1}_{I_t,I_r}[\{z_n\}])_{n\in\mathbb{N}}, a', b', z'') \in \mathcal{P}_t$. By P3 this implies that $(\pi^{-1}_{I_t,I_r}[\{z_n\}])_{n\in a'}$ and $(\pi^{-1}_{I_t,I_r}[\{z_n\}])_{n\in b'}$ converge to z'' and, since $a' \setminus b$ and $b' \setminus (a \setminus b)$ are finite, we get a contradiction with $(**)$.

Part (C) follows immediately from (B) and [5, Theorem 2.2].

□

Corollary 4.5. *The infinite compact topological space K is an Efimov space.*

Proof. Let $L \subseteq K$ be any infinite closed subspace. By the above theorem, every operator on $C(L)$ is a weak multiplier and therefore $C(L)$ is not isomorphic to its hyperplanes (see [10]). Thus, if L is a nontrivial convergent sequence with its limit or if L is homeomorphic to $\beta\mathbb{N}$, then we would have $C(L) \cong c_0 \cong c_0 \oplus \mathbb{R}$ or $C(L) \cong l_\infty \cong l_\infty \oplus \mathbb{R}$, which is a contradiction. □

5. ON THE DENSITY OF $C(K)$ WITH FEW OPERATORS

In [12], Koszmider showed that a sufficient condition on K to guarantee that $C(K)$ have few operators is the following: there exists a dense subset $E \subseteq K$ such that for every sequence $(x_n)_{n\in\mathbb{N}}$ in E and every sequence of open sets $(U_n)_{n\in\mathbb{N}} \subseteq K$ such that $x_n \notin U_m$ for all m and n, there exists $M \subseteq \mathbb{N}$ infinite and coinfinite satisfying $\overline{\{x_n : n \in M\}} \cap \overline{\{x_n : n \in \mathbb{N} \setminus M\}} \neq \emptyset$ and $\overline{\bigcup\{U_n : n \in M\}} \cap \overline{\bigcup\{U_n : n \in \mathbb{N} \setminus M\}} = \emptyset$. We will prove that this condition, commonly used in the constructions of Banach spaces $C(K)$ with few operators, imposes an upper bound for the density of $C(K)$ or, equivalently, for the weight of K.

LEMMA 5.1. *If K is a compact topological space with density κ and $E \subseteq K$ is a dense subset, then there exists a sequence $(X_\alpha)_{\alpha\in\kappa}$ of subsets of E such that each $X_\alpha = \{x_n^\alpha : n \in \mathbb{N}\}$, where $(x_n^\alpha)_{n\in\mathbb{N}}$ is an infinite relatively discrete sequence, and $\overline{X_\beta} \cap X_\alpha = \emptyset\ \forall \beta < \alpha$.*

Proof. We will construct the sequence $(X_\alpha)_{\alpha\in\kappa}$ by transfinite induction. For this purpose, let us fix $\alpha < \kappa$ and suppose that we have already constructed $X_\beta \subseteq E$, for $\beta < \alpha$. Now, define $X = \bigcup_{\beta<\alpha} X_\beta = \bigcup_{\beta<\alpha} \{x_n^\beta : n \in \mathbb{N}\}$. Since $|X| < d(K) = \kappa$ we have that $K \setminus \overline{X}$ is an infinite open subset and, using the fact that E is dense, we conclude that $(K \setminus \overline{X}) \cap E$ is also infinite. Hence, we can take $(x_n^\alpha)_{n\in\mathbb{N}}$ an infinite relatively discrete sequence in $E \setminus \overline{X}$ and define $X_\alpha = \{x_n^\alpha : n \in \mathbb{N}\}$. To see that the sequence $(X_\alpha)_{\alpha\in\kappa}$ satisfies the condition of the statement, take any $\beta < \alpha$. Since $X_\alpha \subseteq E \setminus \overline{X_\beta} \subseteq K \setminus \overline{X_\beta}$, we have that $\overline{X_\beta} \cap X_\alpha = \emptyset$.
□

LEMMA 5.2 ([9, Theorem 2.7]). *Let K be a compact topological space* [1]. *Then $w(K) \leq 2^{d(K)}$.*

THEOREM 5.3. *Let K be a compact topological space and suppose that there exists a dense set $E \subseteq K$ such that for every sequence $(x_n)_{n\in\mathbb{N}}$ in E and every sequence of open sets $(U_n)_{n\in\mathbb{N}} \subseteq K$ such that $x_n \notin U_m$ for all m and n, there exists $M \subseteq \mathbb{N}$ infinite and coinfinite satisfying $\overline{\{x_n : n \in M\}} \cap \overline{\{x_n : n \in \mathbb{N}\setminus M\}} \neq \emptyset$ and $\overline{\bigcup\{U_n : n \in M\}} \cap \overline{\bigcup\{U_n : n \in \mathbb{N}\setminus M\}} = \emptyset$. Then $w(K) \leq 2^{2^{2^\omega}}$.*

Proof. Let $d(K) = \kappa$ and $E \subseteq K$ be a dense subset. Take $(x_n^\alpha)_{n\in\mathbb{N},\alpha\in\kappa}$ a sequence satisfying the statement of Lemma 5.1 and consider the function $\varphi : \kappa \longrightarrow \mathcal{P}(\mathcal{P}(\mathbb{N}))$ given by $\varphi(\alpha) = \{M \subseteq \mathbb{N}$ infinite and coinfinite : $\overline{\{x_n^\alpha : n \in M\}} \cap \overline{\{x_n^\alpha : n \in \mathbb{N}\setminus M\}} \neq \emptyset\}$. Suppose by contradiction that $\kappa > 2^{2^\omega}$. Hence there exist $\alpha, \beta \in \kappa$ such that $\beta < \alpha$ and $\varphi(\beta) = \varphi(\alpha)$. Using the normality of K and the fact that $\overline{\{x_n^\beta : n \in \mathbb{N}\}} \cap \{x_n^\alpha : n \in \mathbb{N}\} = \emptyset$, we can take $(U_n)_{n\in\mathbb{N}} \subseteq K$ a pairwise disjoint sequence of open nonempty subsets satisfying $x_n^\alpha \in U_n$ and $x_n^\beta \notin U_m$ for all m and n. By hypothesis, there exists $M \subseteq \mathbb{N}$ infinite and coinfinite such that $\overline{\{x_n^\beta : n \in M\}} \cap \overline{\{x_n^\beta : n \in \mathbb{N}\setminus M\}} \neq \emptyset$ and $\overline{\bigcup\{U_n : n \in M\}} \cap \overline{\bigcup\{U_n : n \in \mathbb{N}\setminus M\}} = \emptyset$. In particular, $M \in \varphi(\beta)$ and then $M \in \varphi(\alpha)$. Hence, we have $\overline{\{x_n^\alpha : n \in M\}} \cap \overline{\{x_n^\alpha : n \in \mathbb{N}\setminus M\}} \neq \emptyset$, which is a contradiction because $\overline{\{x_n^\alpha : n \in M\}} \cap \overline{\{x_n^\alpha : n \in \mathbb{N}\setminus M\}} \subseteq \overline{\bigcup\{U_n : n \in M\}} \cap \overline{\bigcup\{U_n : n \in \mathbb{N}\setminus M\}} = \emptyset$. Therefore, we conclude that $\kappa \leq 2^{2^\omega}$ and, by Lemma 5.2, we have that $\omega(K) \leq 2^{d(K)} \leq 2^{2^{2^\omega}}$. □

[1]The same result holds for regular topological spaces in general.

6. FINAL REMARKS AND OPEN PROBLEMS

The space K constructed using the technique of forcing has weight at least 2^ω and therefore $C(K)$ has density at least 2^ω since the weight of K is equal to the density of $C(K)$. In fact, K has weight at least 2^ω because every weakly Koszmider space has uncountable weight (see [10]). We started the construction assuming that CH occurs in the ground model V and since the forcing is σ-closed, we have that CH also occurs in the extension. Moreover, we have the following results:

Lemma 6.1. *If $\mathbb{P}$ preserves cardinals and κ is regular, then K has weight κ.*

Proof. Let λ be the weight of K. Suppose that $\lambda < \kappa$ and we will get a contradiction. Let $\mathcal{B}$ be a basis for K with cardinality λ. By compactness of K we may assume, without loss of generality, that each element of $\mathcal{B}$ depends on a finite number of coordinates and, therefore, by regularity of κ, there exists $\alpha < \kappa$ such that every element of $\mathcal{B}$ depends on coordinates below α. By Lemma 3.3, we have that

$$D_\alpha = \{p \in \mathbb{P} : \exists \beta > \alpha\, (\beta \in I_p \wedge x^p_\beta \neq y^p_\beta)\}$$

is dense in $\mathbb{P}$ and, hence, $G \cap D_\alpha \neq \emptyset$. Let $\tilde{p} \in G \cap D_\alpha$ and fix $\beta > \alpha$ such that $\beta \in I_{\tilde{p}} \wedge x^{\tilde{p}}_\beta \neq y^{\tilde{p}}_\beta$. Define $x_\beta = \bigcup_{p\in G} x^p_\beta$ and $y_\beta = \bigcup_{p\in G} y^p_\beta$. Therefore, $x_\beta \neq y_\beta$ and there exist elements of $\mathcal{B}$ which separate x_β and y_β. By hypothesis, the separation happens below α, which gets to a contradiction, since by P4 and O5, we have $x_\beta|\alpha = y_\beta|\alpha$. □

Theorem 6.2. *Suppose $\kappa = \omega_2 = 2^{\omega_1}$ in the ground model V. If $\mathbb{P}$ is ω_2-c.c then:*

(a) $(2^{\omega_1})^{V[G]} = (2^{\omega_1})^V$;

(b) *K has weight $\kappa = \omega_2 = 2^{\omega_1} > 2^\omega$.*

Proof. Since $\mathbb{P}$ preserves cardinals, to prove item (a) we have to prove that $(2^{\omega_1})^{V[G]} = \kappa$. Let X be a subset of ω_1 in $V[G]$. Let σ be a nice name for X, i.e.,

$$\sigma = \cup_{\xi\in\omega_1}\{\check{\xi}\} \times A_\xi,$$

where each A_ξ is an antichain in $\mathbb{P}$. Since $\mathbb{P}$ is ω_2-c.c, we have $|A_\xi| \leq \omega_1$. Using CH in the ground model, we have $|\mathbb{P}| = \kappa = 2^{\omega_1}$. Therefore, there exist $\omega_1 \times (2^{\omega_1})^{\omega_1} = 2^{\omega_1} = \kappa$ nice names for subsets of ω_1,proving that $2^{\omega_1} = \kappa$ in $V[G]$.

Part (b) follows immediately from (a) and Lemma 6.1.

□

Theorem 6.2 shows us that if $\kappa = \omega_2 = 2^{\omega_1}$ and $\mathbb{P}$ is ω_2-c.c then K will have weight bigger than 2^ω, responding affirmatively to the following problem:

Problem 6.3. Is it relatively consistent with ZFC that there exists a connected hereditarily weakly Koszmider space of weight bigger than 2^ω?

In [14], the authors prove that there is no bound on size of Banach spaces of continuous functions with few operators. The technique used is quite different from the usual techniques and it is based on the ideas contained in [20] and [21]. The following question is probably the most interesting question in the context of hereditarily weakly Koszmider spaces:

Problem 6.4. Is there any bound on the weight of hereditarily weakly Koszmider spaces?

REFERENCES

[1] A. Barbeiro and R. Fajardo, *Non homeomorphic hereditarily weakly Koszmider spaces.* Topology Appl. **265** (2019), 1–15.

[2] E.K. Douwen, *The integers and topology.* In: K. Kunen and J.E. Vaughan (Eds.), *Handbook of set-theoretic topology*, pp. 111–167. North Holland, Amsterdam, 1984.

[3] R. Fajardo, *An indecomposable Banach space of continuous functions which has small density.* Fund. Math. **202** (2009), *1*, 43–63.

[4] R. Fajardo, *Quotients of indecomposable Banach spaces of continuous functions.* Studia Math. **212** (2012), *3*, 259–283.

[5] R. Fajardo, A. Gómez, and L. Pellegrini, *Decompositions of Banach spaces $C(K)$ with few operators.* São Paulo J. Math. Sci. **13** (2019), 305–319.

[6] V.V. Fedorčuk, *A bicompactum whose infinite closed subsets are all n-dimensional.* Math. USSR Sbornik. **25** (1975), *1*, 37–57.

[7] W.T. Gowers and B. Maurey, *The unconditional basic sequence problem.* J. Amer. Math. Soc. **6** (1993), 851–874.

[8] W.T. Gowers and B. Maurey, *Banach spaces with small spaces of operators.* Math. Ann. **307** (1997), 543–568.

[9] Juhász, I. *Cardinal functions in topology: ten years later.* Mathematisch Centrum Tracts **123**, Amsterdam, 1980.

[10] P. Koszmider, *Banach spaces of continuous functions with few operators.* Math. Ann. **330** (2004), 151–183.

[11] P. Koszmider, *A space $C(K)$ where all nontrivial complemented subspaces have big densities.* Studia Math. **168** (2005), *2*, 109–127.

[12] P. Koszmider, *A survey on Banach spaces $C(K)$ with few operators.* Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **104** (2010), *2*, 309–326.

[13] P. Koszmider, *On large indecomposable Banach spaces.* J. Funct. Anal. **264** (2013), 1779–1805.

[14] P. Koszmider, S. Shelah, and M. Świętek, *There is no bound on sizes of indecomposable Banach spaces*. Adv. Math. **323** (2018), 745–783.

[15] K. Kunen, *Set Theory: An Introduction to Independence Proofs*. North Holland, Amsterdam, 1980.

[16] Y. Mibu, *On Baire functions on infinite product spaces*. Proc. Imperial Acad. Tokyo **20** (1944), *9*, 661–663.

[17] G. Plebanek, *A construction of a Banach space $C(K)$ with few operators*. Topology Appl. **143** (2004), 217–239.

[18] A. Plichko and D. Yost, *Complemented and Uncomplemented subspaces of Banach spaces*. Extracta Math. **15** (2000), *2*, 335–371.

[19] I. Schlackow, *Centripetal operators and Koszmider spaces*. Topology Appl. **155** (2008), 1227–1236.

[20] S. Shelah, *Existence of endo-rigid Boolean algebras*. In: S. Shelah (Ed.), *Around Classification Theory of Models*. pp. 91–119. Springer, Berlin, 1986.

[21] S. Shelah, *Existence of endo-rigid Boolean algebras*. Preprint (2011). Available at https://arxiv.org/abs/1105.3777.

Received 4 July 2023

Universidade de São Paulo
Instituto de Matemática e Estatística
Rua do Matão, 1010 – São Paulo – SP – Brazil
CEP 05508-900

andre.santoleri@gmail.com
fajardo@ime.usp.br