

# MILNOR FIBRATIONS OF ARRANGEMENTS WITH TRIVIAL ALGEBRAIC MONODROMY

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Each complex hyperplane arrangement gives rise to a Milnor fibration of its complement. Although the Betti numbers of the Milnor fiber  $F$  can be expressed in terms of the jump loci for rank 1 local systems on the complement, explicit formulas are still lacking in full generality, even for  $b_1(F)$ . We study here the “generic” case (in which  $b_1(F)$  is as small as possible), and look deeper into the algebraic topology of such Milnor fibrations with trivial algebraic monodromy. Our main focus is on the cohomology jump loci and the lower central series quotients of  $\pi_1(F)$ . In the process, we produce a pair of arrangements for which the respective Milnor fibers have the same Betti numbers, yet non-isomorphic fundamental groups: the difference is picked by the higher-depth characteristic varieties and by the Schur multipliers of the second nilpotent quotients.

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## 1. INTRODUCTION

### 1.1. The Milnor fibration

In a seminal book [43], Milnor introduced a fibration which soon became the central object of study in singularity theory. In its simplest form, the construction associates to a homogeneous polynomial  $f \in \mathbb{C}[z_0, \dots, z_d]$  a smooth fibration over  $\mathbb{C}^*$ , defined by restricting the map  $f: \mathbb{C}^{d+1} \rightarrow \mathbb{C}$  to the complement of its zero-set. The Milnor fiber,  $F = f^{-1}(1)$ , is a smooth complex affine variety of complex dimension  $d$ . The monodromy of the fibration,  $h: F \rightarrow F$ , is given by  $h(z) = e^{2\pi i/n}z$ , where  $n = \deg f$ . A key question is to compute the characteristic polynomials of the induced homomorphisms in homology,  $h_q: H_q(F; \mathbb{C}) \rightarrow H_q(F; \mathbb{C})$ .

We are mainly interested in the case when  $f$  has singularities in codimension 1. Arguably, the simplest situation in this regard is when the polynomial  $f$

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completely factors into distinct linear forms. This situation is neatly described by a hyperplane arrangement, that is, a finite collection  $\mathcal{A}$  of codimension-1 linear subspaces in  $\mathbb{C}^{d+1}$ . Choosing a linear form  $f_H$  with kernel  $H$  for each hyperplane  $H \in \mathcal{A}$ , we obtain a homogeneous polynomial,  $f = \prod_{H \in \mathcal{A}} f_H$ , which in turn defines the Milnor fibration of the complement of the arrangement,  $M = M(\mathcal{A})$ , with fiber  $F = F(\mathcal{A})$ . More generally, if  $\mathbf{m}: \mathcal{A} \rightarrow \mathbb{N}$ ,  $H \mapsto m_H$  is a choice of multiplicities for the hyperplanes comprising  $\mathcal{A}$ , we may consider the polynomial  $f_{\mathbf{m}} = \prod_{H \in \mathcal{A}} f_H^{m_H}$  and the corresponding Milnor fibration, with fiber  $F_{\mathbf{m}}$ .

To analyze these fibrations, it is most natural to use the rich combinatorial structure encoded in the intersection lattice of  $\mathcal{A}$ , that is, the poset of all intersections of hyperplanes in  $\mathcal{A}$ , ordered by reverse inclusion and ranked by codimension. A much-studied question in the subject asks: Is the characteristic polynomial of the algebraic monodromy of the (usual) Milnor fibration,  $\Delta_{\mathcal{A},q}(t) = \det(tI - h_q)$ , determined by the intersection sub-lattice  $L_{\leq q+1}(\mathcal{A})$ ? Despite much effort—and some progress—over the past 30–40 years, the problem is still open, even in degree  $q = 1$ .

In this paper, we take a different tack, and focus instead on the “generic” situation, to wit, on those hyperplane arrangements for which the monodromy of the Milnor fibration acts trivially on the homology of the Milnor fiber, either with  $\mathbb{Z}$  or with  $\mathbb{C}$  coefficients.

## 1.2. Cohomology jump loci

We start by analyzing the structure of the *characteristic varieties* (the jump loci for homology in rank 1 local systems) and the *resonance varieties* (the jump loci of the Koszul complex associated to the cohomology algebra) of the Milnor fiber of a multi-arrangement in the trivial algebraic monodromy setting.

Let  $U = \mathbb{P}(M)$  be the projectivization of the complement  $M = M(\mathcal{A})$ . Since  $U$  is a smooth, connected, quasi-projective variety, its characteristic varieties,  $\mathcal{V}_s^q(U)$ , are finite unions of torsion-translates of algebraic subtori of the character group,  $\text{Hom}(\pi_1(U), \mathbb{C}^*) = H^1(U; \mathbb{C}^*)$ , see [2, 6]. Since  $U$  is also a formal space, its resonance varieties,  $\mathcal{R}_s^q(U)$ , coincide with the tangent cone at the trivial character to  $\mathcal{V}_s^q(U)$ , see [15, 22, 21]. As shown in [28], the varieties  $\mathcal{R}_s^1(U)$  may be described solely in terms of multinets on sub-arrangements of  $\mathcal{A}$ . In general, though, the varieties  $\mathcal{V}_s^1(U)$  may contain components which do not pass through the origin, see [61, 10, 16]. We explain in detail the relationship between the cohomology jump loci of  $M$  and  $U$  in Proposition 3.3 and Corollary 6.13.

Now let  $(\mathcal{A}, \mathbf{m})$  be a multi-arrangement in  $\mathbb{C}^{d+1}$  and let  $F_{\mathbf{m}} \rightarrow M \rightarrow \mathbb{C}^*$  be the Milnor fibration of the complement, with monodromy  $h: F_{\mathbf{m}} \rightarrow F_{\mathbf{m}}$ . We then have a regular  $\mathbb{Z}_N$ -cover,  $\sigma_{\mathbf{m}}: F_{\mathbf{m}} \rightarrow U$ , where  $N = \sum_{H \in \mathcal{A}} m_H$ . In Theorem 5.7, we prove the following result, which relates the degree 1 cohomology jump loci of  $F_{\mathbf{m}}$  to those of  $U = \mathbb{P}(M)$ , under a trivial algebraic monodromy assumption.

**THEOREM 1.1.** *Suppose the map  $h: F_{\mathbf{m}} \rightarrow F_{\mathbf{m}}$  induces the identity on  $H_1(F_{\mathbf{m}}; \mathbb{Q})$ . Then,*

1. *The induced homomorphism  $\sigma_{\mathbf{m}}^*: H^1(U; \mathbb{C}) \rightarrow H^1(F_{\mathbf{m}}; \mathbb{C})$  is an isomorphism that identifies  $\mathcal{R}_s^1(U)$  with  $\mathcal{R}_s^1(F_{\mathbf{m}})$ , for all  $s \geq 1$ .*
2. *The induced homomorphism  $\sigma_{\mathbf{m}}^*: H^1(U; \mathbb{C}^*) \rightarrow H^1(F_{\mathbf{m}}; \mathbb{C}^*)^0$  is a surjection with kernel isomorphic to  $\mathbb{Z}_N$ . Moreover,*
  - (a) *For each  $s \geq 1$ , the map  $\sigma_{\mathbf{m}}^*$  establishes a bijection between the sets of irreducible components of  $\mathcal{V}_s^1(U)$  and  $\mathcal{W}_s^1(F_{\mathbf{m}})$  that pass through the identity.*
  - (b) *The map  $\sigma_{\mathbf{m}}^*: \mathcal{V}_1^1(U) \rightarrow \mathcal{W}_1^1(F_{\mathbf{m}})$  is a surjection.*

In the above,  $H^1(F_{\mathbf{m}}; \mathbb{C}^*)^0$  denotes the identity component of the character group  $H^1(F_{\mathbf{m}}; \mathbb{C}^*)$ , while  $\mathcal{W}_s^1(F_{\mathbf{m}})$  denotes its intersection with  $\mathcal{V}_s^1(F_{\mathbf{m}})$ . The theorem builds on and sharpens results of Dimca and Papadima from [20].

### 1.3. Abelian duality and propagation

It has long been recognized that complements of complex hyperplane arrangements satisfy certain vanishing properties for homology with coefficients in local systems. In [18, 19], we revisited this subject, in a more general framework.

Given a connected, finite-type CW-complex  $X$  with fundamental group  $G$ , we say that  $X$  is an *ab-duality space* of dimension  $m$  if  $H^q(X; \mathbb{Z}G^{\text{ab}}) = 0$  for  $q \neq m$  and  $H^m(X; \mathbb{Z}G^{\text{ab}})$  is non-zero and torsion-free. Replacing the abelianization of  $G$  by the torsion-free abelianization,  $G_{\text{abf}} = G_{\text{ab}}/\text{Tors}$ , we obtain the analogous notion of *abf-duality space* (of dimension  $m$ ).

These properties impose stringent conditions on the cohomological invariants of the space  $X$ . Most notably, as shown in [19], if  $X$  is an ab-duality space of dimension  $n$ , then the characteristic varieties of  $X$  *propagate*, that is,  $\{\mathbf{1}\} = \mathcal{V}_1^0(X) \subseteq \mathcal{V}_1^1(X) \subseteq \dots \subseteq \mathcal{V}_1^m(X)$ .

It was shown in [18, 19] that complements of hyperplane arrangements are ab-duality spaces; see also [17, 37] for generalizations of this result. Moreover,

it was shown in [19] that the ab-duality property behaves well under a certain type of “ab-exact” fibrations. Making use of these results, together with their adaptations in the abf-duality/abf-exact context, we establish in Theorem 6.15 and Corollary 6.16 the following:

**THEOREM 1.2.** *Let  $\mathcal{A}$  be a central arrangement of rank  $r$  and let fiber  $F_{\mathbf{m}} = F_{\mathbf{m}}(\mathcal{A})$  be the Milnor fiber associated to a multiplicity vector  $\mathbf{m}: \mathcal{A} \rightarrow \mathbb{N}$ .*

1. *Suppose the monodromy action on  $H_1(F_{\mathbf{m}}; \mathbb{Z})$  is trivial. Then,*

(a)  *$F_{\mathbf{m}}$  is an ab-duality space of dimension  $r - 1$ .*

(b) *The characteristic varieties of  $F_{\mathbf{m}}$  propagate; that is,*

$$\mathcal{V}_1^1(F_{\mathbf{m}}) \subseteq \mathcal{V}_1^2(F_{\mathbf{m}}) \subseteq \cdots \subseteq \mathcal{V}_1^{r-1}(F_{\mathbf{m}}).$$

2. *Suppose the monodromy action on  $H_1(F_{\mathbf{m}}; \mathbb{Q})$  is trivial. Then.*

(a)  *$F_{\mathbf{m}}$  is an abf-duality space of dimension  $r - 1$ .*

(b) *The restricted characteristic varieties of  $F_{\mathbf{m}}$  propagate; that is,*

$$\mathcal{W}_1^1(F_{\mathbf{m}}) \subseteq \mathcal{W}_1^2(F_{\mathbf{m}}) \subseteq \cdots \subseteq \mathcal{W}_1^{r-1}(F_{\mathbf{m}}).$$

This result strengthens [19, Theorem 6.7], where only part (1) is proved (in the particular case when  $F = F(\mathcal{A})$  is the usual Milnor fiber of an essential arrangement), but not part (2). We also show: If the monodromy action on  $H_i(F_{\mathbf{m}}; \mathbb{Q})$  is trivial for  $i \leq q$ , then the resonance varieties of  $F_{\mathbf{m}}$  propagate in that range; that is,  $\mathcal{R}_1^q(F_{\mathbf{m}}) \subseteq \cdots \subseteq \mathcal{R}_1^q(F_{\mathbf{m}})$ .

#### 1.4. Associated graded Lie algebras

The lower central series (LCS) of a group  $G$  is defined inductively by setting  $\gamma_1(G) = G$  and  $\gamma_{k+1}(G) = [G, \gamma_k(G)]$  for  $k \geq 1$ . This series is both normal and central; therefore, its successive quotients,  $\text{gr}_k(G) = \gamma_k(G)/\gamma_{k+1}(G)$ , are abelian groups. The first such quotient coincides with the abelianization,  $G_{\text{ab}} = H_1(G; \mathbb{Z})$ . The associated graded Lie algebra of the group,  $\text{gr}(G)$ , is the direct sum of the groups  $\text{gr}_k(G)$ , with Lie bracket (compatible with the grading) induced from the group commutator. Important in this context is also the Chen Lie algebra of  $G$ , that is, the associated graded Lie algebra  $\text{gr}(G/G'')$  of the maximal metabelian quotient of  $G$ .

When the group  $G$  is finitely generated, the LCS quotients of  $G$  are also finitely generated. We let  $\phi_k(G) := \text{rank}(\text{gr}_k(G))$  be the ranks of these abelian groups and we let  $\theta_k(G) := \text{rank}(\text{gr}_k(G/G''))$  be the Chen ranks of  $G$ . Quite a bit is known about the LCS ranks and the Chen ranks of arrangement groups,

though almost nothing is known about the corresponding ranks for the Milnor fiber groups. As a first step in this direction, we show that the former determine the latter when the algebraic monodromy is trivial. More precisely, we prove in Theorems 7.1 and 7.2 the following statements.

**THEOREM 1.3.** *Let  $(\mathcal{A}, \mathbf{m})$  be a multi-arrangement and let  $F_{\mathbf{m}}$  be the corresponding Milnor fiber, with monodromy  $h: F_{\mathbf{m}} \rightarrow F_{\mathbf{m}}$ . Set  $G = \pi_1(M(\mathcal{A}))$  and  $K = \pi_1(F_{\mathbf{m}})$ .*

1. *If  $h_*: H_1(F_{\mathbf{m}}; \mathbb{Z}) \rightarrow H_1(F_{\mathbf{m}}; \mathbb{Z})$  is the identity map, then  $\text{gr}_{\geq 2}(K) \cong \text{gr}_{\geq 2}(G)$  and  $\text{gr}_{\geq 2}(K/K'') \cong \text{gr}_{\geq 2}(G/G'')$ , as graded Lie algebras.*
2. *If  $h_*: H_1(F_{\mathbf{m}}; \mathbb{Q}) \rightarrow H_1(F_{\mathbf{m}}; \mathbb{Q})$  is the identity map, then  $\text{gr}_{\geq 2}(K) \otimes \mathbb{Q} \cong \text{gr}_{\geq 2}(G) \otimes \mathbb{Q}$  and  $\text{gr}_{\geq 2}(K/K'') \otimes \mathbb{Q} \cong \text{gr}_{\geq 2}(G/G'') \otimes \mathbb{Q}$ , as graded Lie algebras.*

*In either case,  $\phi_k(K) = \phi_k(G)$  and  $\theta_k(K) = \theta_k(G)$  for all  $k \geq 2$ .*

Consequently, if the algebraic monodromy is trivial, both the LCS ranks and the Chen ranks of  $\pi_1(F_{\mathbf{m}})$  are combinatorially determined.

## 1.5. Constructions and examples

In Section 8, we describe several classes of hyperplane arrangements for which the Milnor fibration has trivial algebraic monodromy. The simplest are the Boolean arrangements, followed by the generic arrangements. In both cases, complete answers regarding the homology of the Milnor fiber are known. We review these classical topics, in the more general context of arrangements with multiplicities.

Next, we consider the class of decomposable arrangements. Following [48], we say that an arrangement  $\mathcal{A}$  is *decomposable* (over  $\mathbb{Q}$ ) if there are no elements in  $\text{gr}_3(\pi_1(M(\mathcal{A})) \otimes \mathbb{Q})$  besides those coming from the rank 2 flats; that is, if  $\phi_3(\pi_1(M(\mathcal{A})))$  is equal to  $\sum_{X \in L_2(\mathcal{A})} \binom{\mu(X)}{2}$ , where  $\mu: L(\mathcal{A}) \rightarrow \mathbb{Z}$  is the Möbius function. As shown in [70], for any choice of multiplicities  $\mathbf{m}$  on such an arrangement, the algebraic monodromy of the Milnor fibration,  $h_*: H_1(F_{\mathbf{m}}; \mathbb{Q}) \rightarrow H_1(F_{\mathbf{m}}; \mathbb{Q})$ , is trivial, provided a certain technical condition is satisfied. Other classes of arrangements for which this conclusion holds are those for which certain multiplicities conditions are satisfied (see [10, 33, 74, 75, 39]), or the associated double point graph is connected and satisfies some additional requirements (see [4, 58, 73]).

In [25], Falk constructed a pair of rank-3 arrangements that have non-isomorphic intersection lattices, yet whose complements are homotopy equiva-

lent. In Section 9, we analyze in detail the Milnor fibrations of these arrangements. In both cases, the monodromy acts as the identity on first integral homology of the Milnor fiber. Nevertheless, the respective Milnor fibers are not homotopy equivalent. The difference is picked by both the degree-1, depth-2 characteristic varieties, and by the Schur multipliers of the second nilpotent quotients of their fundamental groups.

As shown in [61], deleting a suitable hyperplane from the  $B_3$  reflection arrangement yields an arrangement  $\mathcal{A}$  of 8 hyperplanes for which the variety  $\mathcal{V}_1^1(M(\mathcal{A}))$  has an irreducible component (a subtorus translated by a character of order 2) that does not pass through the identity of the character torus. As a consequence, there is a choice of multiplicities  $\mathbf{m}$  on  $\mathcal{A}$  such that the monodromy  $h: F_{\mathbf{m}} \rightarrow F_{\mathbf{m}}$  acts trivially on  $H_1(F_{\mathbf{m}}; \mathbb{Q}) = \mathbb{Q}^7$  but not on  $H_1(F_{\mathbf{m}}; \mathbb{Z}) = \mathbb{Z}^7 \oplus \mathbb{Z}_2^2$ , see [9, 16]. We illustrate our techniques in Section 10 with a computation of the degree-1 characteristic varieties of  $F_{\mathbf{m}}$  and the low-degree LCS quotients and Chen groups of  $\pi_1(F_{\mathbf{m}})$ . Using a different approach, Yoshinaga constructed in [76] an arrangement  $\mathcal{A}$  of 16 hyperplanes such that the usual Milnor fiber itself,  $F = F(\mathcal{A})$ , has non-trivial 2-torsion. We summarize in Section 11 the information our techniques yield in this case regarding the LCS quotients and the Chen groups of  $\pi_1(F)$ .

## 1.6. Organization of the paper

Roughly speaking, the paper is divided into three parts. The first one deals with some basic notions regarding hyperplane arrangements. In Section 2, we discuss the combinatorics of an arrangement  $\mathcal{A}$ , as it relates to the topology of the complement  $M(\mathcal{A})$ , while in Section 3, we review the resonance and characteristic varieties of  $\mathcal{A}$ .

The second part covers the Milnor fibration of a multi-arrangement  $(\mathcal{A}, \mathbf{m})$ . In Section 4, we discuss the homology of the Milnor fiber  $F_{\mathbf{m}}$  and the monodromy action in homology. Under the assumption that this action is trivial, we investigate several topological invariants of the Milnor fiber: the cohomology jump loci in Section 5, abelian duality and propagation of cohomology jump loci in Section 6, and the lower central series of  $\pi_1(F_{\mathbf{m}})$  in Section 7.

The third part starts with Section 8, where we describe ways to construct arrangements with trivial algebraic monodromy. The techniques developed in this work are illustrated with several examples worked out in detail: the pair of Falk arrangements in Section 9, the deleted  $B_3$  arrangement in Section 10, and Yoshinaga's icosidodecahedral arrangement in Section 11.

## 2. COMPLEMENTS OF HYPERPLANE ARRANGEMENTS

### 2.1. Hyperplane arrangements

An *arrangement of hyperplanes* is a finite set  $\mathcal{A}$  of codimension-1 linear subspaces in a finite-dimensional complex vector space  $\mathbb{C}^{d+1}$ . The combinatorics of the arrangement is encoded in its *intersection lattice*,  $L(\mathcal{A})$ , that is, the poset of all intersections of hyperplanes in  $\mathcal{A}$  (also known as flats), ordered by reverse inclusion, and ranked by codimension.

Without much loss of generality, we will assume throughout that the arrangement is *central*, that is, all the hyperplanes pass through the origin. For each hyperplane  $H \in \mathcal{A}$ , let  $f_H: \mathbb{C}^{d+1} \rightarrow \mathbb{C}$  be a linear form with kernel  $H$ . The product  $f = \prod_{H \in \mathcal{A}} f_H$ , then, is a defining polynomial for the arrangement, unique up to a non-zero constant factor. Notice that  $f$  is a homogeneous polynomial of degree equal to  $n = |\mathcal{A}|$ , the number of hyperplanes comprising  $\mathcal{A}$ .

The complement of the arrangement,  $M(\mathcal{A}) = \mathbb{C}^{d+1} \setminus \bigcup_{H \in \mathcal{A}} H$ , is a connected, smooth, complex quasi-projective variety. Moreover,  $M = M(\mathcal{A})$  is a Stein manifold, and thus it has the homotopy type of a CW-complex  $K$  of dimension at most  $d + 1$ . In fact,  $M$  splits off the complex linear subspace  $\bigcap_{H \in \mathcal{A}} H$ , whose dimension we call the *corank* of  $\mathcal{A}$ . Thus, setting  $\text{rank}(\mathcal{A}) := d + 1 - \text{corank}(\mathcal{A})$ , we have that  $\dim(K) \leq \text{rank}(\mathcal{A})$ . If  $\text{corank}(\mathcal{A}) = 0$ , we will say that  $\mathcal{A}$  is *essential*.

The group  $\mathbb{C}^*$  acts freely on  $\mathbb{C}^{d+1} \setminus \{0\}$  via  $\zeta \cdot (z_0, \dots, z_d) = (\zeta z_0, \dots, \zeta z_d)$ . The orbit space is the complex projective space of dimension  $d$ , while the orbit map,  $\pi: \mathbb{C}^{d+1} \setminus \{0\} \rightarrow \mathbb{C}\mathbb{P}^d$ ,  $z \mapsto [z]$ , is the Hopf fibration. The set  $\mathbb{P}(\mathcal{A}) = \{\pi(H) : H \in \mathcal{A}\}$  is an arrangement of codimension 1 projective subspaces in  $\mathbb{C}\mathbb{P}^d$ . Its complement,  $U = U(\mathcal{A})$ , coincides with the quotient  $\mathbb{P}(M) = M/\mathbb{C}^*$ . The Hopf map restricts to a bundle map,  $\pi: M \rightarrow U$ , with fiber  $\mathbb{C}^*$ . Fixing a hyperplane  $H_0 \in \mathcal{A}$ , we see that  $\pi$  is also the restriction to  $M$  of the bundle map  $\mathbb{C}^{d+1} \setminus H_0 \rightarrow \mathbb{C}\mathbb{P}^d \setminus \pi(H_0) \cong \mathbb{C}^d$ . This latter bundle is trivial, and so we have a diffeomorphism  $M \cong U \times \mathbb{C}^*$ .

### 2.2. Fundamental group

Fix a basepoint  $x_0$  in the complement of  $\mathcal{A}$ , and consider the fundamental group  $G(\mathcal{A}) = \pi_1(M(\mathcal{A}), x_0)$ . For each hyperplane  $H \in \mathcal{A}$ , pick a meridian curve about  $H$ , oriented compatibly with the complex orientations on  $\mathbb{C}^{d+1}$  and  $H$ , and let  $\gamma_H$  denote the based homotopy class of this curve, joined to the basepoint by a path in  $M$ . By the van Kampen theorem, then, the arrangement

group,  $G = G(\mathcal{A})$ , is generated by the set  $\{\gamma_H : H \in \mathcal{A}\}$ . Using the braid monodromy algorithm from [13], one may obtain a finite presentation of the form  $G = F_n/R$ , where  $F_n$  is the rank  $n$  free group on the set of meridians and the relators in  $R$  belong to the commutator subgroup  $F'_n$ . Consequently, the abelianization of the arrangement group,  $G_{\text{ab}} = H_1(G; \mathbb{Z})$ , is isomorphic to  $\mathbb{Z}^n$ .

*Example 2.1.* The reflection arrangement of type  $A_{n-1}$ , also known as the braid arrangement, consists of the diagonal hyperplanes  $H_{ij} = \{z_i - z_j = 0\}$  in  $\mathbb{C}^n$ . The intersection lattice is the lattice of partitions of the set  $\{1, \dots, n\}$ , ordered by refinement. The complement  $M$  is the configuration space of  $n$  ordered points in  $\mathbb{C}$ , which is a classifying space for the Artin pure braid group on  $n$  strings,  $P_n$ .  $\diamond$

Under the diffeomorphism  $M \cong U \times \mathbb{C}^*$ , the arrangement group splits as  $\pi_1(M) \cong \pi_1(U) \times \pi_1(\mathbb{C}^*)$ , where the central subgroup  $\pi_1(\mathbb{C}^*) = \mathbb{Z}$  corresponds to the subgroup of  $\pi_1(M)$  generated by the product of the meridional curves  $\gamma_H$  (taken in the order given by an ordering of the hyperplanes). We shall denote by  $\bar{\gamma}_H = \pi_{\#}(\gamma_H)$  the image of  $\gamma_H$  under the induced homomorphism  $\pi_{\#}: \pi_1(M) \rightarrow \pi_1(U) = \pi_1(M)/\mathbb{Z}$ .

For the purpose of computing the group  $G(\mathcal{A}) = \pi_1(M(\mathcal{A}))$ , it is enough to assume that the arrangement  $\mathcal{A}$  lives in  $\mathbb{C}^3$ , in which case  $\bar{\mathcal{A}} = \mathbb{P}(\mathcal{A})$  is an arrangement of (projective) lines in  $\mathbb{C}\mathbb{P}^2$ . This is clear when the rank of  $\mathcal{A}$  is at most 2, and may be achieved otherwise by taking a generic 3-slice, an operation which does not change either the poset  $L_{\leq 2}(\mathcal{A})$  or the group  $G(\mathcal{A})$ . For a rank-3 arrangement, the set  $L_1(\mathcal{A})$  is in 1-to-1 correspondence with the lines of  $\bar{\mathcal{A}}$ , while  $L_2(\mathcal{A})$  is in 1-to-1 correspondence with the intersection points of  $\bar{\mathcal{A}}$ . Moreover, the poset structure of  $L_{\leq 2}(\mathcal{A})$  mirrors the incidence structure of the point-line configuration  $\bar{\mathcal{A}}$ .

The *localization* of an arrangement  $\mathcal{A}$  at a flat  $X \in L(\mathcal{A})$  is defined as the sub-arrangement  $\mathcal{A}_X := \{H \in \mathcal{A} \mid H \supset X\}$ . The inclusion  $\mathcal{A}_X \subset \mathcal{A}$  gives rise to an inclusion of complements,  $j_X: M(\mathcal{A}) \hookrightarrow M(\mathcal{A}_X)$ . Choosing a point  $x_0$  sufficiently close to  $\mathbf{0} \in \mathbb{C}^{d+1}$ , we can make it a common basepoint for both  $M(\mathcal{A})$  and all the local complements  $M(\mathcal{A}_X)$ . As shown in [18], there exist basepoint-preserving maps  $r_X: M(\mathcal{A}_X) \rightarrow M(\mathcal{A})$  such that  $j_X \circ r_X \simeq \text{id}$  relative to  $x_0$ ; moreover, if  $H \in \mathcal{A}$  and  $H \not\supset X$ , then the map  $r_X \circ j_X \circ r_H$  is null-homotopic. Consequently, the induced homomorphisms  $(r_X)_{\#}: G(\mathcal{A}_X) \rightarrow G(\mathcal{A})$  are all injective.

For an arrangement  $\mathcal{A}$  in  $\mathbb{C}^3$ , we will say that a rank-2 flat  $X$  has multiplicity  $q = q_X$  if  $|\mathcal{A}_X| = q$ , or, equivalently, if the point  $\mathbb{P}(X)$  has exactly  $q$  lines from  $\bar{\mathcal{A}}$  passing through it. In this case, the localized sub-arrangement  $\mathcal{A}_X$  is a pencil of  $q$  planes. Consequently,  $M(\mathcal{A}_X)$  is homeomor-



phic to  $(\mathbb{C} \setminus \{q-1 \text{ points}\}) \times \mathbb{C}^* \times \mathbb{C}$ , and thus it is a classifying space for the group  $G(\mathcal{A}_X) \cong F_{q-1} \times \mathbb{Z}$ .

### 2.3. Cohomology ring

The cohomology ring of a hyperplane arrangement complement  $M = M(\mathcal{A})$  was computed by Brieskorn [5], building on the work of Arnol'd on the cohomology ring of the pure braid group. In [46], Orlik and Solomon gave a simple description of this ring, solely in terms of the intersection lattice  $L(\mathcal{A})$ , as follows. Fix a linear order on  $\mathcal{A}$ , and let  $E$  be the exterior algebra over  $\mathbb{Z}$  with generators  $\{e_H \mid H \in \mathcal{A}\}$  in degree 1. Next, define a differential  $\partial: E \rightarrow E$  of degree  $-1$ , starting from  $\partial(1) = 0$  and  $\partial(e_H) = 1$ , and extending  $\partial$  to a linear operator on  $E$ , using the graded Leibniz rule. Finally, let  $I(\mathcal{A})$  be the ideal of  $E$  generated by  $\partial e_{\mathcal{B}}$ , for all  $\mathcal{B} \subset \mathcal{A}$  such that  $\text{codim} \bigcap_{H \in \mathcal{B}} H < |\mathcal{B}|$ , where  $e_{\mathcal{B}} := \prod_{H \in \mathcal{B}} e_H$ . Then

$$(2.1) \quad H^*(M(\mathcal{A}); \mathbb{Z}) \cong E/I(\mathcal{A}).$$

The inclusions  $\{j_X\}_{X \in L(\mathcal{A})}$  assemble into a map  $j: M \rightarrow \prod_{X \in L(\mathcal{A})} M(\mathcal{A}_X)$ . The work of Brieskorn [5] insures that the homomorphism induced by  $j$  in cohomology is an isomorphism in all positive degrees. By the Künneth formula, then, we have that  $H^k(M; \mathbb{Z}) \cong \bigoplus_{X \in L_k(\mathcal{A})} H^k(M(\mathcal{A}_X); \mathbb{Z})$ , for all  $k \geq 1$ . It follows that the homology groups of the complement of  $\mathcal{A}$  are torsion-free, with ranks given by

$$(2.2) \quad b_k(M) = \sum_{X \in L_k(\mathcal{A})} (-1)^k \mu(X),$$

where  $\mu: L(\mathcal{A}) \rightarrow \mathbb{Z}$  is the Möbius function, defined inductively by  $\mu(\mathbb{C}^{d+1}) = 1$  and  $\mu(X) = -\sum_{Y \supseteq X} \mu(Y)$ . The homology groups of the projectivized complement,  $U = \mathbb{P}(M)$ , are also torsion free, with ranks computed inductively from the formulas  $b_0(U) = 1$  and  $b_k(U) + b_{k-1}(U) = b_k(M)$  for  $k \geq 1$ .

In particular, we have that  $H_1(M; \mathbb{Z}) \cong \mathbb{Z}^n$ , with basis  $\{x_H : H \in \mathcal{A}\}$ , where  $x_H$  is the homology class represented by the meridional curve  $\gamma_H$ . Moreover,  $H_1(U; \mathbb{Z}) = H_1(M; \mathbb{Z}) / (\sum_{H \in \mathcal{A}} x_H) \cong \mathbb{Z}^{n-1}$ . We will denote by  $\bar{x}_H = [\bar{\gamma}_H]$  the image of  $x_H$  in  $H_1(U; \mathbb{Z})$ .

### 2.4. Formality

A connected, finite-type CW-complex  $X$  is said to be *formal* if its rational cohomology algebra,  $H^*(X; \mathbb{Q})$ , can be connected by a zig-zag of quasi-isomorphisms to  $A_{\text{PL}}^*(X)$ , the algebra of polynomial differential forms on  $X$

defined by Sullivan in [72]. The notion of  $q$ -formality is defined similarly, with the cdga morphisms in the zig-zag only being required to induce isomorphisms in degrees up to  $q$  and monomorphisms in degree  $q + 1$ . It is known that a  $q$ -formal CW-complex of dimension  $q + 1$  is actually formal. Moreover, if  $Y \rightarrow X$  is a finite, regular cover and  $Y$  is  $q$ -formal, then  $X$  is also  $q$ -formal. For more on this topic, we refer to [71, 67] and references therein.

For an arrangement  $\mathcal{A}$  in  $\mathbb{C}^{d+1}$ , the complement  $M$  is formal, in a very strong sense. Indeed, for each  $H \in \mathcal{A}$ , the 1-form  $\omega_H = \frac{1}{2\pi i} d \log f_H$  on  $\mathbb{C}^{d+1}$  restricts to a 1-form on  $M$ . As shown by Brieskorn [5], if  $\mathcal{D}$  denotes the subalgebra of the de Rham algebra  $\Omega_{\text{dR}}^*(M)$  generated over  $\mathbb{R}$  by these 1-forms, the correspondence  $\omega_H \mapsto [\omega_H]$  induces an isomorphism  $\mathcal{D} \rightarrow H^*(M; \mathbb{R})$ . Sullivan’s machinery from [72] then implies that  $M$  is formal. Alternatively, it is known that the mixed Hodge structure on  $H^*(M; \mathbb{Q})$  is pure; thus, the “purity implies formality” results of Dupont [23] and Chataur–Cirici [7] yield another proof of the formality of  $M$ .

### 3. COHOMOLOGY JUMP LOCI OF ARRANGEMENTS

#### 3.1. Resonance varieties

Let  $A$  be a graded, graded-commutative algebra over  $\mathbb{C}$ . We will assume that each graded piece  $A^q$  is finite-dimensional and  $A^0 = \mathbb{C}$ . For each element  $a \in A^1$ , we turn the algebra  $A$  into a cochain complex,  $(A, \delta_a)$ , with differentials  $\delta_a^q: A^q \rightarrow A^{q+1}$ ,  $u \mapsto au$ . The fact that  $\delta_a^{q+1} \circ \delta_a^q = 0$  follows at once from the observation that  $a^2 = -a^2$  (by graded-commutativity of multiplication in  $A$ ), which implies  $a^2 = 0$ . By definition, the (degree  $q$ , depth  $s$ ) *resonance varieties* of  $A$  are the jump loci for the cohomology of this complex,

$$(3.1) \quad \mathcal{R}_s^q(A) = \{a \in A^1 \mid \dim_{\mathbb{C}} H^q(A, \delta_a) \geq s\}.$$

These sets are Zariski-closed, homogeneous subsets of the affine space  $A^1$ ; More specifically, they are either empty or they contain the zero-vector  $\mathbf{0} \in A^1$ . Setting  $b_q(A, a) := \dim_{\mathbb{C}} H^q(A, \delta_a)$  for the Betti numbers of the cochain complex  $(A, \delta_a)$ , we see that  $b_q(A, \mathbf{0})$  is equal to the usual Betti number  $b_q(A) = \dim_{\mathbb{C}} A^q$ . Therefore, the point  $\mathbf{0} \in A^1$  belongs to  $\mathcal{R}_s^q(A)$  if and only if  $b_q(A) \geq s$ . In particular, since  $A^0 = \mathbb{C}$ , we have that  $\mathcal{R}_1^0(A) = \{\mathbf{0}\}$  and  $\mathcal{R}_s^0(A) = \emptyset$  if  $s > 1$ .

We will mostly consider the degree one resonance varieties. Clearly, these varieties depend only on the truncated algebra  $A^{\leq 2}$ . More explicitly,  $\mathcal{R}_s^1(A)$  consists of  $\mathbf{0}$ , together with all elements  $a \in A^1$  for which there exist  $u_1, \dots, u_s \in A^1$  such that the span of  $\{a, u_1, \dots, u_s\}$  has dimension  $s + 1$  and

$au_1 = \cdots = au_s = 0$  in  $A^2$ . Finally, if  $\varphi: A \rightarrow B$  is a morphism of commutative graded algebras, and  $\varphi$  is injective in degree 1, then the linear map  $\varphi^1: A^1 \rightarrow B^1$  embeds  $\mathcal{R}_s^1(A)$  into  $\mathcal{R}_s^1(B)$ , for each  $s \geq 1$ .

Completely analogous definitions work for algebras  $A$  over a field  $\mathbb{k}$  of characteristic different from 2. When  $\text{char}(\mathbb{k}) = 2$ , special care needs to be taken, to account for the fact that the square of an element  $a \in A^1$  may not vanish in this case; we refer to [66] for details.

Now let  $X$  be a connected, finite-type CW-complex. Its cohomology algebra,  $A = H^*(X; \mathbb{C})$ , with multiplication given by the cup-product, satisfies the properties listed at the start of this section. Therefore, we may define the resonance varieties of the space  $X$  to be the sets  $\mathcal{R}_s^q(X) := \mathcal{R}_s^q(H^*(X; \mathbb{C}))$ , viewed as homogeneous subsets of the affine space  $H^1(X; \mathbb{C})$ , and likewise for  $\mathcal{R}_s^q(X, \mathbb{k}) \subseteq H^1(X; \mathbb{k})$ . When  $M = M(\mathcal{A})$  is an arrangement complement, the fact that  $H_1(M; \mathbb{Z})$  is torsion-free implies that  $a^2 = 0$  for all  $a \in H^1(M; \mathbb{k})$ , even when  $\text{char}(\mathbb{k}) = 2$ ; thus, the usual definition of resonance works for all fields.

### 3.2. Multinets and pencils

The resonance varieties of complements of hyperplane arrangements were introduced in the mid-1990s by Falk [26] and further studied in the ensuing decade in papers such as [15, 41, 34, 60, 61]. The work of Falk and Yuzvinsky [28] greatly clarified the nature of the degree 1 resonance varieties of arrangements. Let us briefly review their construction.

A *multinet*  $\mathcal{N}$  on an arrangement  $\mathcal{A}$  consists of a partition  $\mathcal{A}_1 \sqcup \cdots \sqcup \mathcal{A}_k$  of  $\mathcal{A}$  into  $k \geq 3$  subsets; an assignment of multiplicities  $\mathbf{m} = \{m_H\}_{H \in \mathcal{A}}$ ; and a subset  $\mathcal{X} \subseteq L_2(\mathcal{A})$ , called the base locus, such that the following conditions hold:

1. There is an integer  $\ell$  such that  $\sum_{H \in \mathcal{A}_i} m_H = \ell$ , for all  $i \in [k]$ .
2. For any two hyperplanes  $H$  and  $K$  in different classes,  $H \cap K \in \mathcal{X}$ .
3. For each  $X \in \mathcal{X}$ , the sum  $n_X := \sum_{H \in \mathcal{A}_i: H \supset X} m_H$  is independent of  $i$ .
4. For each  $1 \leq i \leq k$  and  $H, K \in \mathcal{A}_i$ , there is a sequence  $H = H_0, \dots, H_r = K$  such that  $H_{j-1} \cap H_j \notin \mathcal{X}$  for  $1 \leq j \leq r$ .

We say that a multinet  $\mathcal{N}$  as above is a  $(k, \ell)$ -multinet, or simply a  $k$ -multinet. Without essential loss of generality, we may assume that  $\text{gcd}(\mathbf{m}) = 1$ . If all the multiplicities are equal to 1, the multinet is said to be *reduced*. If, furthermore, every flat in  $\mathcal{X}$  is contained in precisely one hyperplane from each class, the multinet is called a  $(k, \ell)$ -net.

For instance, a 3-net on  $\mathcal{A}$  is a partition into 3 non-empty subsets with the property that, for each pair of hyperplanes  $H, K \in \mathcal{A}$  in different classes, we have  $H \cap K = H \cap K \cap L$ , for some hyperplane  $L$  in the third class. As another example, if  $X \in L_2(\mathcal{A})$  is a 2-flat of multiplicity at least 3, we may form a net on  $\mathcal{A}_X$  by assigning to each hyperplane  $H \supset X$  the multiplicity 1, putting one hyperplane in each class, and setting  $\mathcal{X} = \{X\}$ .

Now let  $f = \prod_{H \in \mathcal{A}} f_H$  be a defining polynomial for  $\mathcal{A}$ . Given a  $k$ -multinet  $\mathcal{N}$  on  $\mathcal{A}$ , with parts  $\mathcal{A}_i$  and multiplicity vector  $\mathbf{m}$ , write  $f_i = \prod_{H \in \mathcal{A}_i} f_H^{m_H}$  and define a rational map  $\psi: \mathbb{C}^3 \rightarrow \mathbb{C}\mathbb{P}^1$  by  $\psi(x) = [f_1(x) : f_2(x)]$ . There is then a set  $D = \{[a_1 : b_1], \dots, [a_k : b_k]\}$  of  $k$  distinct points in  $\mathbb{C}\mathbb{P}^1$  such that each of the degree  $d$  polynomials  $f_1, \dots, f_k$  can be written as  $f_i = a_i f_2 - b_i f_1$ , and, furthermore, the image of  $\psi: M(\mathcal{A}) \rightarrow \mathbb{C}\mathbb{P}^1$  misses  $D$ , see [28]. The corestriction  $\psi: M(\mathcal{A}) \rightarrow \mathbb{C}\mathbb{P}^1 \setminus D$ , then, is the *pencil* associated to the multinet  $\mathcal{N}$ . Following [54, 64], we may describe the homomorphism induced in homology by this pencil, as follows. Let  $\alpha_1, \dots, \alpha_k$  be compatibly oriented simple closed curves on  $S = \mathbb{C}\mathbb{P}^1 \setminus D$ , going around the points of  $D$ , so that  $H_1(S; \mathbb{Z})$  is generated by the homology classes  $c_i = [\alpha_i]$ , subject to the single relation  $\sum_{i=1}^k c_i = 0$ . Further, the induced homomorphism  $\psi_*: H_1(M; \mathbb{Z}) \rightarrow H_1(S; \mathbb{Z})$  is given by  $\psi_*(x_H) = m_H c_i$  for  $H \in \mathcal{A}_i$ , and thus  $\psi^*: H^1(S; \mathbb{Z}) \rightarrow H^1(M; \mathbb{Z})$  is given by  $\psi^*(c_i^\vee) = u_i$ , where  $c_i^\vee$  is the Kronecker dual of  $c_i$  and  $u_i = \sum_{H \in \mathcal{A}_i} m_H e_H$ .

It follows that the map  $\psi^*: H^1(S; \mathbb{C}) \rightarrow H^1(M; \mathbb{C})$  is injective, and thus sends  $\mathcal{R}_1^1(S)$  to  $\mathcal{R}_1^1(M)$ . Let us identify  $\mathcal{R}_1^1(S)$  with  $H^1(S; \mathbb{C}) = \mathbb{C}^{k-1}$ , and view  $P_{\mathcal{N}} := \psi^*(H^1(S; \mathbb{C}))$  as lying inside  $\mathcal{R}_1^1(\mathcal{A}) := \mathcal{R}_1^1(M)$ . Then  $P_{\mathcal{N}}$  is the  $(k-1)$ -dimensional linear subspace spanned by the vectors  $u_2 - u_1, \dots, u_k - u_1$ . Moreover, as shown in [28, Theorems 2.4–2.5], this subspace is an essential component of  $\mathcal{R}_1^1(\mathcal{A})$ ; that is,  $P_{\mathcal{N}}$  is not contained in any proper coordinate subspace of  $H^1(M; \mathbb{C})$ . More generally, suppose there is a sub-arrangement  $\mathcal{B} \subseteq \mathcal{A}$  supporting a multinet  $\mathcal{N}$ . In this case, the inclusion  $M(\mathcal{A}) \hookrightarrow M(\mathcal{B})$  induces a monomorphism  $H^1(M(\mathcal{B}); \mathbb{C}) \hookrightarrow H^1(M(\mathcal{A}); \mathbb{C})$ , which restricts to an embedding  $\mathcal{R}_1^1(\mathcal{B}) \hookrightarrow \mathcal{R}_1^1(\mathcal{A})$ . The linear space  $P_{\mathcal{N}}$ , then, lies inside  $\mathcal{R}_1^1(\mathcal{B})$ , and thus, inside  $\mathcal{R}_1^1(\mathcal{A})$ . Conversely, as shown in [28, Theorem 2.5] all (positive-dimensional) irreducible components of  $\mathcal{R}_1^1(\mathcal{A})$  arise in this fashion.

### 3.3. Characteristic varieties

Let  $X$  be a connected, finite-type CW-complex. Fix a basepoint  $x_0$  at a 0-cell; then the fundamental group  $G = \pi_1(X, x_0)$  is a finitely generated (in fact, finitely presented) group. Therefore, the group  $\mathbb{T}_G = \text{Hom}(G, \mathbb{C}^*)$  of  $\mathbb{C}$ -valued, multiplicative characters on  $G$  is an affine, commutative algebraic group, which

we will identify with  $H^1(X; \mathbb{C}^*)$ . Its identity  $\mathbf{1}$  is the trivial representation  $g \mapsto 1 \in \mathbb{C}^*$ ; the connected component of  $G$  containing the identity,  $\mathbb{T}_G^0$ , is an algebraic torus isomorphic to  $(\mathbb{C}^*)^n$ , where  $n = b_1(G)$ . Moreover,  $\mathbb{T}_G/\mathbb{T}_G^0$  is in bijection with the finite abelian group  $\text{Tors}(G_{\text{ab}})$ .

The *characteristic varieties* of  $X$  (in degree  $q$  and depth  $s$ , where  $q, s \geq 0$ ) are the jump loci for homology with coefficients in rank-1 local systems on  $X$ :

$$(3.2) \quad \mathcal{V}_s^q(X) = \{\rho \in H^1(X; \mathbb{C}^*) \mid \dim_{\mathbb{C}} H_q(X; \mathbb{C}_\rho) \geq s\}.$$

Here,  $\mathbb{C}_\rho = \mathbb{C}$  with  $\mathbb{C}[G]$ -module structure defined by the character  $\rho: G \rightarrow \mathbb{C}^*$  by setting  $g \cdot z := \rho(g)z$  for  $g \in G$  and  $z \in \mathbb{C}$ , while  $H_*(X; \mathbb{C}_\rho)$  denotes the homology of the chain complex  $C_*(\tilde{X}; \mathbb{C}) \otimes_{\mathbb{C}[G]} \mathbb{C}_\rho$ , where  $C_*(\tilde{X}; \mathbb{C})$  is the  $G$ -equivariant chain complex of the universal cover of  $X$ , with coefficients in  $\mathbb{C}$ .

The sets  $\mathcal{V}_s^q(X)$  are Zariski-closed subsets of the character group. We will denote by  $\mathcal{W}_s^q(X)$  the intersection of  $\mathcal{V}_s^q(X)$  with  $\mathbb{T}_G^0$ . Observe that the (degree  $q$ ) depth of a character  $\rho$ , defined as  $\text{depth}_q(\rho) := \dim_{\mathbb{C}} H_q(X; \mathbb{C}_\rho)$ , is equal to  $\max\{s \mid \rho \in \mathcal{V}_s^q(X)\}$ ; in particular,  $\text{depth}_q(\mathbf{1}) = b_q(X)$ , the  $q$ -th Betti number of  $X$ . Note also that  $\mathcal{V}_1^0(X) = \{\mathbf{1}\}$  and  $\mathcal{V}_s^0(X) = \emptyset$  if  $s > 1$ , while  $\mathcal{V}_0^q(X) = H^1(X; \mathbb{C}^*)$  for all  $q \geq 0$ .

Completely analogous definitions work for the characteristic varieties as  $\mathcal{V}_s^q(X, \mathbb{k})$ , viewed as subsets of  $H^1(X; \mathbb{k}^*)$ , for any field  $\mathbb{k}$ .

*Example 3.1.* Let  $\Sigma_{g,n}$  be a Riemann surface of genus  $g$  with  $n$  punctures ( $g, n \geq 0$ ), and let  $\chi := \chi(\Sigma_{g,n}) = 2 - 2g - n$  be its Euler characteristic. Then  $\mathcal{V}_s^1(\Sigma_{g,n})$  is equal to  $H^1(\Sigma_{g,n}; \mathbb{C}^*)$  if  $s \leq -\chi$  and it is contained in  $\{\mathbf{1}\}$ , otherwise.  $\diamond$

The characteristic varieties  $\mathcal{V}_s^1(X)$  depend only on the fundamental group  $G = \pi_1(X)$ ; thus, we will often denote them by  $\mathcal{V}_s^1(G)$ . At least away from the trivial character,  $\mathcal{V}_s^1(G)$  is the zero set of the ideal  $\text{ann}(\bigwedge^s G'/G'' \otimes \mathbb{C})$ , where the  $\mathbb{Z}G_{\text{ab}}$ -module structure on the group  $G'/G''$  arises from the short exact sequence  $1 \rightarrow G'/G'' \rightarrow G/G'' \rightarrow G'/G'' \rightarrow 1$ ; see, e.g., [69] and references therein. Therefore, the characteristic varieties  $\mathcal{V}_s^1(G)$  of a finitely generated group  $G$  depend only on its maximal metabelian quotient,  $G/G''$ .

### 3.4. Homology of finite abelian covers

The characteristic varieties control the Betti numbers of regular, connected, finite abelian covers  $p: Y \rightarrow X$ . For instance, suppose that the deck-transformation group is cyclic of order  $N$ . Then the cover is determined by an epimorphism  $\chi: G \twoheadrightarrow \mathbb{Z}_N$ , so that  $\ker(\chi) = \text{im}(p_{\#})$ . Fix an inclusion

$\iota: \mathbb{Z}_N \hookrightarrow \mathbb{C}^*$ , by sending 1 to  $e^{2\pi i/N}$ . With this choice, the map  $\chi$  yields a torsion character,  $\rho = \iota \circ \chi: G \rightarrow \mathbb{C}^*$ . Since  $\chi$  is surjective, the induced morphism between character groups,  $\chi^*: \mathbb{T}_{\mathbb{Z}_N} \rightarrow \mathbb{T}_G$ , is injective, and so  $\text{im}(\chi^*) \cong \mathbb{Z}_N$ . Furthermore, if  $\xi: G \rightarrow \mathbb{C}^*$  is a non-trivial character belonging to  $\text{im}(\chi^*)$ , then  $\xi = \rho^{N/k}$  for some positive integer  $k$  dividing  $N$ .

Now view the homology groups  $H_q(Y; \mathbb{C})$  as modules over the group algebra  $\mathbb{C}[\mathbb{Z}_N] \cong \mathbb{C}[t]/(t^N - 1)$ . By a transfer argument, the invariant submodule,  $H_q(Y; \mathbb{C})^{\mathbb{Z}_N}$ , is isomorphic to the trivial module  $H_q(X; \mathbb{C}) \cong (\mathbb{C}[t]/(t-1))^{b_q(X)}$ . In fact, a result proved in various levels of generality in [32, 57, 30, 42, 16] yields isomorphisms of  $\mathbb{C}[\mathbb{Z}_N]$ -modules,

$$\begin{aligned}
 (3.3) \quad H_q(Y; \mathbb{C}) &\cong \bigoplus_{s \geq 1} \bigoplus_{\xi \in \text{im}(\chi^*) \cap \mathcal{V}_s^q(X)} \mathbb{C}_\xi \\
 &\cong H_q(X; \mathbb{C}) \oplus \bigoplus_{1 < k | N} (\mathbb{C}[t]/\Phi_k(t))^{\text{depth}_q(\rho^{N/k})},
 \end{aligned}$$

where  $\Phi_k(t)$  is the  $k$ -th cyclotomic polynomial. Consequently,

$$\begin{aligned}
 (3.4) \quad b_q(Y) &= \sum_{s \geq 1} |\text{im}(\chi^*) \cap \mathcal{V}_s^q(X)| \\
 &= b_q(X) + \sum_{1 < k | N} \varphi(k) \cdot \text{depth}_q(\rho^{N/k}),
 \end{aligned}$$

where  $\varphi(k) = \deg \Phi_k(t)$  is the Euler totient function. Moreover, if  $h: Y \rightarrow Y$  is the deck transformation corresponding to the generator  $1 \in \mathbb{Z}_N$ , then the characteristic polynomial  $\Delta_q(t) = \det(t \cdot \text{id} - h_*)$  of the induced automorphism  $h_*: H_q(Y; \mathbb{C}) \rightarrow H_q(Y; \mathbb{C})$  is given by

$$(3.5) \quad \Delta_q(t) = (t - 1)^{b_q(X)} \cdot \prod_{1 < k | N} \Phi_k(t)^{\text{depth}_q(\rho^{N/k})}.$$

### 3.5. Characteristic varieties of arrangements

Let  $M$  be a smooth, quasi-projective variety. A general result of Arapura [2] (as refined in [6]), insures that the characteristic varieties  $\mathcal{V}_s^q(M)$  are finite unions of torsion-translated subtori of the character torus. In degree  $q = 1$ , these varieties can be described more precisely, as follows.

Let  $S = (\Sigma_{g,r}, \mu)$  be a Riemann surface of genus  $g \geq 0$ , with  $r \geq 0$  points removed (so that  $\Sigma_{g,0} = \Sigma_g$ ), and with  $h \geq 0$  marked points,  $(p_1, \mu_1), \dots, (p_h, \mu_h)$ , with  $\mu_i \geq 2$ . A surjective, holomorphic map  $\psi: M \rightarrow \Sigma_{g,n}$  is called an *orbifold fibration* (or, a *pencil*) if the fiber over any non-marked point is connected, the multiplicity of each fiber  $\psi^{-1}(p_i)$  is equal to  $\mu_i$ , and  $\psi$  has an

extension to the respective compactifications,  $\bar{\psi}: \bar{M} \rightarrow \Sigma_g$ , which is also a surjective, holomorphic map with connected generic fibers. Then each positive-dimensional component of  $\mathcal{V}_1^1(M)$  is of the form  $T = \psi^*(H^1(S; \mathbb{C}^*))$ , for some pencil  $\psi: M \rightarrow S$  for which the orbifold Euler characteristic of the surface,  $\chi^{\text{orb}}(\Sigma_{g,r}, \mu) := \chi(\Sigma_{g,r}) - \sum_{i=1}^h (1 - 1/\mu_i)$ , is negative.

The following result of Artal, Cogolludo, and Matei ([3, Proposition 6.9]) helps locate characters that lie in the higher-depth characteristic varieties.

**THEOREM 3.2** ([3]). *Let  $M$  be a smooth, quasi-projective variety. Suppose  $T_1$  and  $T_2$  are two distinct, positive-dimensional irreducible components of  $\mathcal{V}_r^1(M)$  and  $\mathcal{V}_s^1(M)$ , respectively. If  $\xi \in T_1 \cap T_2$  is a torsion character, then  $\xi \in \mathcal{V}_{r+s}^1(M)$ .*

Now let  $\mathcal{A}$  be an arrangement of  $n$  hyperplanes in  $\mathbb{C}^{d+1}$ , with complement  $M = M(\mathcal{A})$ . The characteristic varieties  $\mathcal{V}_s^q(M)$  are subsets of the character torus  $H^1(M; \mathbb{C}^*) = (\mathbb{C}^*)^n$ . Moreover, the tangent cone at the identity  $\mathbf{1}$  to  $\mathcal{V}_s^q(M)$  coincides with the resonance variety  $\mathcal{R}_s^q(M)$ , for each  $q, s \geq 1$ . This ‘‘Tangent Cone Theorem’’ (which does not hold for all quasi-projective manifolds) relies in an essential way on the formality of the arrangement complement, and was proved in [15, 34, 22, 21] in various levels of generality. Let  $\exp: H^1(M; \mathbb{C}) \rightarrow H^1(M; \mathbb{C}^*)$  be the coefficient homomorphism induced by the exponential map  $\mathbb{C} \rightarrow \mathbb{C}^*$ . Then, if  $P \subset H^1(M; \mathbb{C})$  is one of the linear subspaces comprising  $\mathcal{R}_s^q(M)$ , its image under the exponential map,  $\exp(P) \subset H^1(M; \mathbb{C}^*)$ , is one of the subtori comprising  $\mathcal{V}_s^q(M)$ . Furthermore, the correspondence  $P \rightsquigarrow T = \exp(P)$  gives a bijection between the components of  $\mathcal{R}_s^q(M)$  and the components of  $\mathcal{V}_s^q(M)$  passing through  $\mathbf{1}$ , which in turn yields an identification  $\text{TC}_{\mathbf{1}}(\mathcal{V}_s^q(M)) = \mathcal{R}_s^q(M)$  for each  $q, s \geq 1$ .

Next, in degree  $q = 1$ , each positive-dimensional component of  $\mathcal{V}_1^1(M)$  that passes through  $\mathbf{1}$  is of the form  $T = \psi^*(H^1(S; \mathbb{C}^*))$ , for some pencil  $\psi: M \rightarrow S = \mathbb{CP}^1 \setminus \{k \text{ points}\}$  with  $k \geq 3$ . An easy computation shows that  $\mathcal{V}_s^1(S) = H^1(S; \mathbb{C}^*) = (\mathbb{C}^*)^{k-1}$  for all  $s \leq k - 2$ . Hence, the subtorus  $T$  is a  $(k - 1)$ -dimensional component of  $\mathcal{V}_1^1(M)$  that contains  $\mathbf{1}$  and lies inside  $\mathcal{V}_{k-2}^1(M)$ .

### 3.6. Torsion-translated subtori

Let  $(\Sigma_{g,r}, \mu)$  be a 2-dimensional orbifold as above. For our purposes here, we may assume  $r \geq 1$ , in which case the orbifold fundamental group  $\Gamma := \pi_1^{\text{orb}}(\Sigma_{g,r}, \mu)$ , is isomorphic to the free product  $F_n * \mathbb{Z}_{\mu_1} * \cdots * \mathbb{Z}_{\mu_h}$ , where  $n = 2g + r - 1$ . Note that  $\Gamma_{\text{ab}} = \mathbb{Z}^n \oplus \Lambda$ , where  $\Lambda = \mathbb{Z}_{\mu_1} \oplus \cdots \oplus \mathbb{Z}_{\mu_\ell}$  is the torsion subgroup, and each component of the character group  $\mathbb{T}_\Gamma = \mathbb{T}_\Gamma^0 \times \mathbb{T}_\Lambda$

is of the form  $\lambda \cdot \mathbb{T}_\Gamma^0$  for some  $\lambda = (\lambda_1 \dots, \lambda_h) \in \mathbb{T}_\Lambda$ . Let  $\ell(\lambda) = |\{i: \lambda_i \neq 1\}|$ . A computation detailed in [3, Proposition 2.10] shows that

$$(3.6) \quad \mathcal{V}_s^1(\Gamma) = \begin{cases} \mathbb{T}_\Gamma & \text{if } s \leq n-1, \\ (\mathbb{T}_\Gamma \setminus \mathbb{T}_\Gamma^0) \cup \{\mathbf{1}\} & \text{if } s = n, \\ \bigcup_{\ell(\lambda) \geq n-s+1} \lambda \cdot \mathbb{T}_\Gamma^0 & \text{if } n < s < n+h, \end{cases}$$

and is empty if  $s \geq n+h$ .

Suppose  $M$  is a smooth, quasi-projective variety, and  $\psi: M \rightarrow (\Sigma_{g,r}, \mu)$  is an orbifold pencil with either  $n \geq 2$ , or  $n = 1$  and  $h > 0$ . Since the generic fiber of  $\psi$  is connected, the induced homomorphism on orbifold fundamental groups,  $\psi_\#: G = \pi_1(M) \rightarrow \Gamma = \pi_1^{\text{orb}}(\Sigma_{g,r}, \mu)$ , is surjective. Therefore, the induced morphism  $\psi_\#^*: \mathbb{T}_\Gamma \rightarrow \mathbb{T}_G$  embeds  $\mathcal{V}_s^1(\Gamma)$ , as computed in (3.6), into  $\mathcal{V}_s^1(M)$ , for all  $s \geq 1$ . In particular, if  $\psi: M \rightarrow (\mathbb{C}^*, m)$  is an orbifold pencil with a single multiple fiber of multiplicity  $m \geq 2$ , then there is a 1-dimensional algebraic subtorus  $T \subset H^1(M; \mathbb{C}^*)$  and a torsion character  $\rho \notin T$  such that  $\mathcal{V}_1^1(M)$  contains the translated tori  $\rho T, \dots, \rho^{m-1}T$ .

As shown in [61], the (degree 1, depth 1) characteristic variety of an arrangement complement may have irreducible components that do not pass through the origin (see Section 10.2). A combinatorial machine for producing translated subtori in the characteristic varieties of certain arrangements was given in [16]. Namely, suppose  $\mathcal{A}$  admits a *pointed multinet*, that is, a multinet  $\mathcal{N}$  and a hyperplane  $H \in \mathcal{A}$  for which  $m_H > 1$ , and  $m_H \mid n_X$  for each flat  $X$  in the base locus such that  $X \subset H$ . Letting  $\mathcal{A}' = \mathcal{A} \setminus \{H\}$  be the deletion of  $\mathcal{A}$  with respect to  $H$ , it turns out that  $\mathcal{V}_1^1(M(\mathcal{A}'))$  has a component which is a 1-dimensional subtorus of  $H^1(M(\mathcal{A}'); \mathbb{C}^*)$ , translated by a character of order  $m_H$ . Whether all positive-dimensional translated subtori in the (degree 1, depth 1) characteristic varieties of arrangements occur in this fashion is an open problem. It is also an open problem whether the isolated (torsion) points in the characteristic varieties of an arrangement are combinatorially determined.

### 3.7. Cohomology jump loci of the projectivized complement

Once again, let  $\mathcal{A}$  be a (central) hyperplane arrangement in  $\mathbb{C}^{d+1}$ . The next result relates the cohomology jump loci of the complement  $M = M(\mathcal{A})$  to those of the projectivized complement,  $U = \mathbb{P}(M)$ . A more precise relationship in degrees  $q > 1$  will be given in Corollary 6.13.

**PROPOSITION 3.3.** *Let  $\pi: M \rightarrow U$  be the restriction of the Hopf map,  $\pi: \mathbb{C}^{d+1} \setminus \{\mathbf{0}\} \rightarrow \mathbb{C}\mathbb{P}^d$ , to the complement of  $\mathcal{A}$ , and set  $n = |\mathcal{A}|$ . Then,*



1. The induced homomorphism  $\pi^*: H^1(U; \mathbb{C}) \hookrightarrow H^1(M; \mathbb{C})$  restricts to isomorphisms  $\mathcal{R}_s^1(U) \xrightarrow{\cong} \mathcal{R}_s^1(M)$  for all  $1 \leq s < n$  and  $\mathcal{R}_1^q(U) \cup \mathcal{R}_1^{q-1}(U) \xrightarrow{\cong} \mathcal{R}_1^q(M)$  for all  $q \geq 1$ .
2. The induced morphism  $\pi^*: H^1(U; \mathbb{C}^*) \hookrightarrow H^1(M; \mathbb{C}^*)$  restricts to isomorphisms  $\mathcal{V}_s^1(U) \xrightarrow{\cong} \mathcal{V}_s^1(M)$  for all  $1 \leq s < n$  and  $\mathcal{V}_1^q(U) \cup \mathcal{V}_1^{q-1}(U) \xrightarrow{\cong} \mathcal{V}_1^q(M)$  for all  $q \geq 1$ .

*Proof.* As noted previously, upon fixing a hyperplane  $H_0 \in \mathcal{A}$ , the restriction to  $M = M(\mathcal{A})$  of the (trivial) bundle map  $\pi: \mathbb{C}^{d+1} \setminus H_0 \rightarrow \mathbb{C}\mathbb{P}^d \setminus \pi(H_0)$  yields a diffeomorphism  $M \xrightarrow{\cong} U \times \mathbb{C}^*$  so that the following diagram commutes,

$$(3.7) \quad \begin{array}{ccc} M & \xrightarrow{\cong} & U \times \mathbb{C}^* \\ & \searrow \pi & \swarrow \text{pr}_1 \\ & U & \end{array}$$

Thus, we may replace in the argument the map  $\pi: M \rightarrow U$  by the first-coordinate projection map  $\text{pr}_1: U \times \mathbb{C}^* \rightarrow U$ . At this stage, the claims in depth  $s = 1$  follow from the product formulas for cohomology jump loci from [52, Proposition 13.1]. For completeness, we provide a full argument, which works in all cases.

For part (3.3), consider the cohomology algebras  $A = H^*(U \times \mathbb{C}^*; \mathbb{C})$ ,  $A_1 = H^*(U; \mathbb{C})$ , and  $A_2 = H^*(\mathbb{C}^*; \mathbb{C})$ , and let  $a = (a_1, a_2)$  be an element in  $A^1 = A_1^1 \oplus A_2^1$ . By the Künneth formula, the cochain complex  $(A, \delta_a)$  splits as a tensor product of cochain complexes,  $(A_1, \delta_{a_1}) \otimes_{\mathbb{C}} (A_2, \delta_{a_2})$ . Therefore,

$$(3.8) \quad b_q(A, a) = \sum_{i+j=q} b_i(A_1, a_1) b_j(A_2, a_2).$$

Clearly,  $b_0(A_2, 0) = b_1(A_2, 0) = 1$  and  $b_j(A_2, a_2) = 0$  otherwise. Therefore,

$$(3.9) \quad b_q(A, (a_1, a_2)) = \begin{cases} b_q(A_1, a_1) + b_{q-1}(A_1, a_1) & \text{if } a_2 = 0, \\ 0 & \text{if } a_2 \neq 0. \end{cases}$$

In particular,  $b_1(A, (a_1, 0)) = b_1(A_1, a_1)$  if  $a_1 \neq 0$  and  $b_1(A, \mathbf{0}) = b_1(A_1, \mathbf{0}) + 1$ . The first claim follows at once from these formulas.

For part (3.3), let us identify  $G = \pi_1(U \times \mathbb{C}^*)$  with  $\pi_1(U) \times \mathbb{Z}$  and the universal cover of  $U \times \mathbb{C}^*$  with  $\tilde{U} \times \mathbb{C}$ . We then have a  $G$ -equivariant isomorphism of chain complexes,  $C_*(\widetilde{U \times \mathbb{C}^*}) \cong C_*(\tilde{U}) \otimes_{\mathbb{C}} C_*(\mathbb{C})$ . Given a character  $\rho = (\rho_1, \rho_2)$  in  $\text{Hom}(G, \mathbb{C}^*) \cong \text{Hom}(\pi_1(U), \mathbb{C}^*) \times \mathbb{C}^*$ , we obtain an isomorphism  $C_*(U \times \mathbb{C}^*, \mathbb{C}_\rho) \cong C_*(U, \mathbb{C}_{\rho_1}) \otimes_{\mathbb{C}} C_*(\mathbb{C}^*, \mathbb{C}_{\rho_2})$ . Therefore,

$$(3.10) \quad H_q(U \times \mathbb{C}^*; \mathbb{C}_\rho) \cong \bigoplus_{i+j=q} H_i(U; \mathbb{C}_{\rho_1}) \otimes_{\mathbb{C}} H_j(\mathbb{C}^*; \mathbb{C}_{\rho_2}),$$

and the second claim follows from the fact that  $H_0(\mathbb{C}^*; \mathbb{C}) = H_1(\mathbb{C}^*; \mathbb{C}) = \mathbb{C}$  and  $H_j(\mathbb{C}^*; \mathbb{C}_{\rho_2}) = 0$ , otherwise.  $\square$

Now fix an ordering  $H_1, \dots, H_n$  of the hyperplanes in  $\mathcal{A}$  and set  $H_0 = H_n$ . Then  $H^1(M; \mathbb{C}^*)$  may be identified with  $(\mathbb{C}^*)^n$ , with coordinates  $t = (t_1, \dots, t_n)$  and  $H^1(U; \mathbb{C}^*)$  may be identified with  $(\mathbb{C}^*)^{n-1}$ , with coordinates  $(t_1, \dots, t_{n-1})$ . The characteristic varieties of  $U$  are then given by

$$(3.11) \quad \mathcal{V}_s^q(U) = \{t \in (\mathbb{C}^*)^n \mid t \in \mathcal{V}_s^q(M) \text{ and } t_1 \cdots t_n = 1\};$$

that is,  $\mathcal{V}_s^q(U)$  is the subvariety of  $(\mathbb{C}^*)^n$  obtained by intersecting  $\mathcal{V}_s^q(M)$  with the subtorus  $(\mathbb{C}^*)^{n-1} = \{t : t_1 \cdots t_n = 1\}$ . Furthermore, the induced homomorphism  $\pi^* : H^1(U; \mathbb{C}^*) \hookrightarrow H^1(M; \mathbb{C}^*)$  may be identified with the monomial map

$$(3.12) \quad (\mathbb{C}^*)^{n-1} \hookrightarrow (\mathbb{C}^*)^n, \quad (t_1, \dots, t_{n-1}) \mapsto (t_1, \dots, t_{n-1}, t_1^{-1} \cdots t_{n-1}^{-1}).$$

In turn, this map restricts to isomorphisms  $\mathcal{V}_s^1(U) \xrightarrow{\cong} \mathcal{V}_s^1(M)$  for all  $1 \leq s < n$  and  $\mathcal{V}_1^q(U) \cup \mathcal{V}_1^{q-1}(U) \xrightarrow{\cong} \mathcal{V}_1^q(M)$  for all  $q \geq 1$ , where, in fact,  $\mathcal{V}_1^q(U) \cup \mathcal{V}_1^{q-1}(U) = \mathcal{V}_1^q(U)$ , as we shall see in Corollary 6.12.

Similar considerations apply to the resonance varieties of  $M$  and  $U$ , with the induced homomorphism  $\pi^* : H^1(U; \mathbb{C}) \hookrightarrow H^1(M; \mathbb{C})$  being identified with the linear map  $\mathbb{C}^{n-1} \hookrightarrow \mathbb{C}^n, (x_1, \dots, x_{n-1}) \mapsto (x_1, \dots, x_{n-1}, -(x_1 + \cdots + x_{n-1}))$ .

## 4. MILNOR FIBRATIONS OF ARRANGEMENTS

### 4.1. The Milnor fibration of a multi-arrangement

Let  $\mathcal{A}$  be a central arrangement of  $n$  hyperplanes in  $\mathbb{C}^{d+1}$ , and fix an ordering on  $\mathcal{A}$ . To each hyperplane  $H \in \mathcal{A}$ , we may associate a multiplicity  $m_H \in \mathbb{N}$ . This yields a multi-arrangement  $(\mathcal{A}, \mathbf{m})$ , where  $\mathbf{m} = (m_H)_{H \in \mathcal{A}} \in \mathbb{N}^n$  is the resulting multiplicity vector, and a homogeneous polynomial,

$$(4.1) \quad f_{\mathbf{m}} = \prod_{H \in \mathcal{A}} f_H^{m_H}$$

of degree  $N = \sum_{H \in \mathcal{A}} m_H$ . Note that  $f_{\mathbf{m}}$  is a proper power if and only if  $\gcd(\mathbf{m}) > 1$ , where  $\gcd(\mathbf{m}) = \gcd(m_H : H \in \mathcal{A})$ .

The polynomial map  $f_{\mathbf{m}} : \mathbb{C}^{d+1} \rightarrow \mathbb{C}$  restricts to a map  $f_{\mathbf{m}} : M(\mathcal{A}) \rightarrow \mathbb{C}^*$ . As shown by Milnor [43] in a much more general context,  $f_{\mathbf{m}}$  is the projection map of a smooth, locally trivial bundle, known as the (global) Milnor fibration of the multi-arrangement  $(\mathcal{A}, m)$ ,

$$(4.2) \quad F_{\mathbf{m}} \longrightarrow M \xrightarrow{f_{\mathbf{m}}} \mathbb{C}^*.$$

The typical fiber of this fibration,  $f_{\mathbf{m}}^{-1}(1)$ , is a smooth manifold of dimension  $2d$ , called the *Milnor fiber* of the multi-arrangement, denoted by  $F_{\mathbf{m}} = F_{\mathbf{m}}(\mathcal{A})$ . It is readily seen that  $F_{\mathbf{m}}$  is a Stein domain of complex dimension  $d$ , and thus has the homotopy type of a finite CW-complex of dimension at most  $d$ , in fact, of dimension at most  $\text{rank}(\mathcal{A}) - 1$ . Moreover,  $F_{\mathbf{m}}$  is connected if and only if  $\text{gcd}(\mathbf{m}) = 1$ , a condition we will assume henceforth. As shown in [64], the homomorphism  $(f_{\mathbf{m}})_{\sharp}: \pi_1(M) \rightarrow \pi_1(\mathbb{C}^*)$  induced on fundamental groups by  $f_{\mathbf{m}}$  is the map  $\mu_{\mathbf{m}}: \pi_1(M) \rightarrow \mathbb{Z}$  given by  $x_H \mapsto m_H$ . In the case when all the multiplicities  $m_H$  are equal to 1, the polynomial  $f = f_{\mathbf{m}}$  is the usual defining polynomial and  $F = F_{\mathbf{m}}$  is the usual Milnor fiber of  $\mathcal{A}$ .

For each  $\theta \in [0, 1]$ , let  $F_{\theta} = f_{\mathbf{m}}^{-1}(e^{2\pi i \theta})$  be the fiber over the point  $e^{2\pi i \theta} \in \mathbb{C}^*$ . For each  $z \in M$ , the path  $\gamma_{\theta}: [0, 1] \rightarrow \mathbb{C}^*$ ,  $t \mapsto e^{2\pi i t \theta}$  lifts to a path  $\tilde{\gamma}_{\theta,z}: [0, 1] \rightarrow M$ ,  $t \mapsto e^{2\pi i t \theta / N} z$  which satisfies  $\tilde{\gamma}_{\theta,z}(0) = z$ . Notice that  $f_{\mathbf{m}}(\tilde{\gamma}_{\theta,z}(1)) = e^{2\pi i \theta} f_{\mathbf{m}}(z)$ ; thus, if  $z \in F_0 = F_{\mathbf{m}}$ , then  $\tilde{\gamma}_{\theta,z}(1) \in F_{\theta}$ . By definition, the *monodromy* of the Milnor fibration is the diffeomorphism  $h: F_0 \rightarrow F_1$  given by  $h(z) = \tilde{\gamma}_{1,z}(1)$ . In view of these observations, we may interpret  $h$  as the self-diffeomorphism  $h: F_{\mathbf{m}} \rightarrow F_{\mathbf{m}}$  of order  $N$  given by  $z \mapsto e^{2\pi i / N} z$ , and identify the complement  $M$  with the mapping torus of  $h$ .

### 4.2. The Milnor fiber as a finite cyclic cover

The monodromy diffeomorphism  $h: F_{\mathbf{m}} \rightarrow F_{\mathbf{m}}$  generates a cyclic group of order  $N = \sum_{H \in \mathcal{A}} m_H$  which acts freely on  $F_{\mathbf{m}}$ . The quotient space,  $F_{\mathbf{m}}/\mathbb{Z}_N$ , may be identified with the projective complement,  $U = \mathbb{P}(M)$ , in a manner such that the projection map,  $\sigma_{\mathbf{m}}: F_{\mathbf{m}} \rightarrow F_{\mathbf{m}}/\mathbb{Z}_N$ , coincides with the restriction of the Hopf fibration map,  $\pi: M \rightarrow U$ , to the subspace  $F_{\mathbf{m}}$ . Thus, letting  $\iota_{\mathbf{m}}: F_{\mathbf{m}} \rightarrow M$  denote the inclusion map, all this information may be summarized in the diagram

$$(4.3) \quad \begin{array}{ccccc} & & \mathbb{C}^* & & \\ & & \downarrow v & \searrow z \mapsto z^N & \\ & & M & \xrightarrow{f_{\mathbf{m}}} & \mathbb{C}^* \\ F_{\mathbf{m}} & \xrightarrow{\iota_{\mathbf{m}}} & & & \\ & \searrow \sigma_{\mathbf{m}} & \downarrow \pi & & \\ & & U & & \end{array}$$

where both the row and the column are fibrations and the diagonal arrows are  $N$ -fold cyclic covers. Consequently, the Euler characteristic of the Milnor fiber is given by  $\chi(F_{\mathbf{m}}) = N \cdot \chi(U)$ . Taking fundamental groups in (4.3), we obtain

the diagram

$$(4.4) \quad \begin{array}{ccccccc} & & & \mathbb{Z} & & & \\ & & & \downarrow v_{\#} & \searrow \cdot N & & \\ & & & \mathbb{Z} & & & \\ 1 & \longrightarrow & \pi_1(F_{\mathbf{m}}) & \xrightarrow{(\iota_{\mathbf{m}})_{\#}} & \pi_1(M) & \xrightarrow{\mu_{\mathbf{m}}} & \mathbb{Z} \longrightarrow 1, \\ & & \searrow (\sigma_{\mathbf{m}})_{\#} & & \downarrow \pi_{\#} & & \\ & & & & \pi_1(U) & & \end{array}$$

with exact row and column. By construction,  $\sigma_{\mathbf{m}} = \pi \circ \iota_{\mathbf{m}}$ , and so the lower triangle commutes. The upper triangle in (4.4) also commutes, since  $v_{\#}(1)$  is the product of the meridians  $\gamma_H$  (taken in the order given by an ordering of the hyperplanes), and since  $N = \sum_{H \in \mathcal{A}} m_H$ . Hence, the homomorphism  $\mu_{\mathbf{m}}: \pi_1(M) \rightarrow \mathbb{Z}$  descends to an epimorphism,

$$(4.5) \quad \chi_{\mathbf{m}}: \pi_1(U) \twoheadrightarrow \mathbb{Z}_N,$$

given by  $\bar{\gamma}_H \mapsto m_H \bmod N$ . As shown in [12, 9, 62, 64], the regular,  $N$ -fold cyclic cover  $\sigma_{\mathbf{m}}: F_{\mathbf{m}} \rightarrow U$  is classified by this epimorphism. In particular, the usual Milnor fiber  $F = F(\mathcal{A})$  is classified by the “diagonal” homomorphism,  $\chi: \pi_1(U) \rightarrow \mathbb{Z}_n$ , given by  $\chi(\bar{\gamma}_H) = 1$ , for all  $H \in \mathcal{A}$ .

### 4.3. The characteristic polynomial of the algebraic monodromy

We now fix an ordering on the  $n$  hyperplanes of  $\mathcal{A}$ , and identify the character group  $H^1(U; \mathbb{C}^*)$  with  $(\mathbb{C}^*)^{n-1}$ . Recall that we also fixed an embedding  $j: \mathbb{Z}_N \hookrightarrow \mathbb{C}^*$ ,  $1 \mapsto e^{2\pi i/N}$ . By (4.5), the character  $\rho_{\mathbf{m}} = j \circ \chi_{\mathbf{m}}: \pi_1(U) \rightarrow \mathbb{C}^*$  is given by  $\bar{\gamma}_H \mapsto e^{2\pi i m_H/N}$ ; hence, for each divisor  $k$  of  $N$ , the character  $\rho_{\mathbf{m}}^{N/k}$  takes  $\bar{\gamma}_H$  to  $e^{2\pi i/k}$ . By formula (3.4), the Betti numbers of the Milnor fiber  $F_{\mathbf{m}} = F_{\mathbf{m}}(\mathcal{A})$  are given by

$$(4.6) \quad b_q(F_{\mathbf{m}}) = b_q(U) + \sum_{1 < k | N} \varphi(k) \text{depth}_q(\rho_{\mathbf{m}}^{N/k}).$$

Likewise, formula (3.5) implies that the characteristic polynomial of the algebraic monodromy  $h_*: H_q(F_{\mathbf{m}}; \mathbb{C}) \rightarrow H_q(F_{\mathbf{m}}; \mathbb{C})$  is given by

$$(4.7) \quad \Delta_q(t) = (t - 1)^{b_q(U)} \cdot \prod_{1 < k | N} \Phi_k(t)^{\text{depth}_q(\rho_{\mathbf{m}}^{N/k})}.$$

In the above expressions, the crucial quantities are the (non-negative) depths of the characters  $\rho_{\mathbf{m}}^{N/k} \in H^1(U; \mathbb{C}^*)$ , which depend on the position of

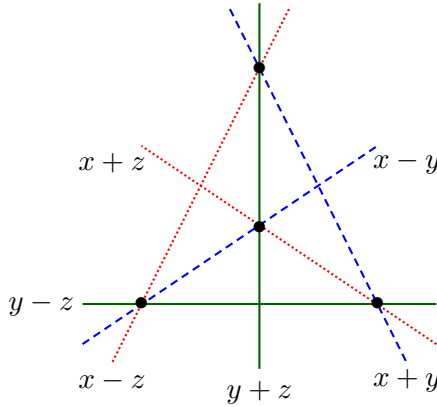


Figure 1 – A (3,2)-net on the braid arrangement.

these characters with respect to the characteristic varieties  $\mathcal{V}_s^q(U)$ . Here are some basic (well-known) examples of how such a computation goes.

*Example 4.1.* Let  $\mathcal{A}$  be a pencil of  $n \geq 3$  lines through the origin of  $\mathbb{C}^2$  defined by the polynomial  $f = x^n - y^n$ . Then  $U$  is homeomorphic to  $\Sigma_{0,n} = \mathbb{C} \setminus \{n - 1 \text{ points}\}$ , and so its characteristic varieties are  $\mathcal{V}_1^1(U) = \dots = \mathcal{V}_{n-2}^1(U) \cong (\mathbb{C}^*)^{n-1}$  and  $\mathcal{V}_{n-1}^1(U) = \{\mathbf{1}\}$  (see Example 3.1). It follows that  $b_1(F) = n - 1 + (n - 2)(n - 1) = (n - 1)^2$  and  $\Delta_1(t) = (t - 1)(t^n - 1)^{n-2}$ . In turn, either this computation or an Euler characteristic argument shows that  $F = \Sigma_{g,n}$ , a Riemann surface of genus  $g = \binom{n-1}{2}$  with  $n$  punctures.  $\diamond$

*Example 4.2.* Let  $\mathcal{A}$  be the braid arrangement in  $\mathbb{C}^3$ , defined by the polynomial  $f = (x + y)(x - y)(x + z)(x - z)(y + z)(y - z)$ . Its complement  $M$  is, up to a  $\mathbb{C}$  factor, homeomorphic to the complement of the reflection arrangement of type  $A_3$  in  $\mathbb{C}^4$ ; thus,  $\pi_1(M) = P_4$ . Labeling the hyperplanes of  $\mathcal{A}$  as the factors of  $f$ , the flats in  $L_2(\mathcal{A})$  may be labeled as 136, 145, 235, and 246. The braid arrangement supports a (3,2)-net, corresponding to the partition (12|34|56) depicted in Figure 1. This net defines a rational map,  $\psi: \mathbb{C}P^2 \dashrightarrow \mathbb{C}P^1$ , sending  $[x, y, z] \mapsto [x^2 - y^2, x^2 - z^2]$ . In turn, this map restricts to a pencil,  $\psi: U \rightarrow \Sigma_{0,3} = \mathbb{C}P^1 \setminus \{[0, 1], [1, 0], [1, 1]\}$ , which yields by pullback a 2-dimensional essential component of  $\mathcal{V}_1^1(U)$ , namely, the subtorus

$$(4.8) \quad T = \{(s, s, t, t, (st)^{-1}) : s, t \in \mathbb{C}^*\}.$$

Letting  $\rho: \pi_1(U) \rightarrow \mathbb{C}^*$ ,  $\bar{\gamma}_H \mapsto e^{2\pi i/6}$  be the diagonal character which defines the  $\mathbb{Z}_6$ -cover  $\sigma: F \rightarrow U$ , we have that  $\rho^2 \in T$ , yet  $\rho \notin T$ . Since  $\mathcal{V}_2^1(U) = \{\mathbf{1}\}$ , it follows that  $b_1(F) = 5 + \varphi(3) \cdot \text{depth}_1(\rho^2) = 5 + 2 \cdot 1 = 7$  and  $\Delta_1(t) = (t - 1)^5(t^2 + t + 1)$ .  $\diamond$

More generally, as shown in [54, Thm. 1.6], if an arrangement of projective lines in  $\mathbb{C}\mathbb{P}^2$  has only double or triple points, then the characteristic polynomial of the algebraic monodromy of the Milnor fibration is given by a completely combinatorial formula.

For an arrangement  $\mathcal{A}$  and a prime  $p$ , define

$$(4.9) \quad \beta_p(\mathcal{A}) := \max\{s : \omega \in \mathcal{R}_s^1(M(\mathcal{A}); \mathbb{Z}_p)\},$$

where  $\omega = \sum_{H \in \mathcal{A}} e_H \in H^1(M(\mathcal{A}); \mathbb{Z}_p)$ . Clearly, the non-negative integer  $\beta_p(\mathcal{A})$  depends only on  $L_{\leq 2}(\mathcal{A})$  and  $p$ .

**THEOREM 4.3** ([54]). *Suppose  $L_2(\mathcal{A})$  has only flats of multiplicity 2 and 3. Then  $\beta_3(\mathcal{A}) \in \{0, 1, 2\}$  and*

$$\Delta_1(t) = (t-1)^{|\mathcal{A}|-1} \cdot (t^2 + t + 1)^{\beta_3(\mathcal{A})}.$$

*Moreover,  $\beta_3(\mathcal{A}) \neq 0$  if and only if  $\mathcal{A}$  supports a 3-net.*

#### 4.4. Trivial algebraic monodromy

Henceforth, we will concentrate mainly on the case when the algebraic monodromy of the Milnor fibration is trivial. More precisely, suppose that  $F_{\mathbf{m}} \rightarrow M \rightarrow \mathbb{C}^*$  is the Milnor fibration of a multi-arrangement  $(\mathcal{A}, \mathbf{m})$ , with monodromy diffeomorphism  $h: F_{\mathbf{m}} \rightarrow F_{\mathbf{m}}$ . We say that  $(\mathcal{A}, \mathbf{m})$  has *trivial algebraic monodromy over  $\mathbb{k}$*  (where  $\mathbb{k}$  is either  $\mathbb{Z}$  or a field) if  $h_*: H_*(F_{\mathbf{m}}; \mathbb{k}) \rightarrow H_*(F_{\mathbf{m}}; \mathbb{k})$  is the identity. Clearly, when  $\mathbb{k}$  a field, this condition only depends on the characteristic of  $\mathbb{k}$ .

The condition that  $h_*: H_q(F_{\mathbf{m}}; \mathbb{Q}) \rightarrow H_q(F_{\mathbf{m}}; \mathbb{Q})$  be the identity is equivalent to  $\Delta_q(t) = (t-1)^{b_q(F_{\mathbf{m}})}$ . Thus, in view of formulas (4.6) and (4.7), the condition is equivalent to  $b_q(F_{\mathbf{m}}) = b_q(U)$ , where  $U = \mathbb{P}(M)$ . Therefore,  $(\mathcal{A}, \mathbf{m})$  has trivial algebraic monodromy over  $\mathbb{Q}$  if and only if  $H_*(F_{\mathbf{m}}; \mathbb{Q}) \cong H_*(U; \mathbb{Q})$ . In fact, more is true. As noted previously, the homology groups of both  $U$  and  $M$  are torsion-free. Making use of the Künneth formula for  $M \cong U \times \mathbb{C}^*$  and the Wang exact sequence for the fibration  $F_{\mathbf{m}} \rightarrow M \rightarrow \mathbb{C}^*$ , we conclude that  $(\mathcal{A}, \mathbf{m})$  has trivial algebraic monodromy over  $\mathbb{k}$  (where  $\mathbb{k} = \mathbb{Z}$  or  $\mathbb{k}$  a field) if and only if  $H_*(F_{\mathbf{m}}; \mathbb{k}) \cong H_*(U; \mathbb{k})$ . Likewise,  $h_*: H_1(F_{\mathbf{m}}; \mathbb{Z}) \rightarrow H_*(F_{\mathbf{m}}; \mathbb{Z})$  is the identity if and only if  $H_1(F_{\mathbf{m}}; \mathbb{Z}) = \mathbb{Z}^{n-1}$ , where  $n = |\mathcal{A}|$ .

*Remark 4.4.* Triviality of the algebraic monodromy in degree  $q = 1$  does not imply triviality of the action in higher degrees. For instance, if  $\mathcal{A}$  is a graphic arrangement, that is, a sub-arrangement of the braid arrangement of type  $A_{n-1}$  from Example 2.1, then  $h_*$  always acts trivially on  $H_1(F(\mathcal{A}); \mathbb{Q})$ , except when  $\mathcal{A}$  is a reflection arrangement of type  $A_2$  or  $A_3$ , see [40, Theorem B]. On the other hand, if  $\mathcal{A}$  is the braid arrangement of type  $A_{n-1}$ , then

$h_*$  always acts non-trivially on the top homology group,  $H_{n-2}(F(\mathcal{A}); \mathbb{Q})$ , see [14, §7].

Unlike the homology groups of the complement  $M$ , examples from [9, 16, 76] show that the homology groups of the Milnor fiber  $F_{\mathbf{m}}$  may have non-trivial torsion. Therefore, if the monodromy  $h: F_{\mathbf{m}} \rightarrow F_{\mathbf{m}}$  acts as the identity on  $H_q(F_{\mathbf{m}}; \mathbb{Q})$ , for some  $q \geq 1$ , we cannot conclude that it also acts as the identity on  $H_q(F_{\mathbf{m}}; \mathbb{Z})$ . Indeed, if  $H_q(F_{\mathbf{m}}; \mathbb{Z})$  has torsion, then the Wang sequence of the fibration  $F_{\mathbf{m}} \rightarrow M \rightarrow \mathbb{C}^*$  shows that  $h_*: H_q(F_{\mathbf{m}}; \mathbb{Z}) \rightarrow H_q(F_{\mathbf{m}}; \mathbb{Z})$  cannot be equal to the identity. We will illustrate this point in Sections 10–11.

#### 4.5. Triviality tests

Let  $\mathcal{A}$  be a central arrangement of  $n$  hyperplanes in  $\mathbb{C}^3$ . For the usual Milnor fiber  $F = F(\mathcal{A})$ , there are two useful tests informing on whether the algebraic monodromy  $h_*: H_1(F; \mathbb{C}) \rightarrow H_1(F; \mathbb{C})$  is equal to the identity. Both of these tests are based on the nature of the multinets supported by  $L(\mathcal{A})$  and of the characteristic varieties of the complement  $M = M(\mathcal{A})$ .

We start with a criterion insuring the triviality of the algebraic monodromy. We will say that a subvariety of the algebraic torus  $(\mathbb{C}^*)^n$  is *essential* if it is not contained in any proper coordinate subtorus.

**PROPOSITION 4.5.** *If the characteristic variety  $\mathcal{V}_1^1(M)$  has no essential irreducible components, then the algebraic monodromy  $h_*: H_1(F; \mathbb{C}) \rightarrow H_1(F; \mathbb{C})$  is trivial.*

*Proof.* Set  $n = |\mathcal{A}|$ . By formulas (3.4) and (4.5), the first Betti number of  $F$  is given by

$$(4.10) \quad b_1(F) = \sum_{s \geq 1} |\mathrm{im}(\chi^*) \cap \mathcal{V}_s^1(U)|,$$

where  $U = \mathbb{P}(M)$  and  $\chi: \pi_1(U) \rightarrow \mathbb{Z}_n$  is the homomorphism that sends each meridian curve  $\bar{\gamma}_H$  to 1. The cyclic subgroup  $\mathrm{im}(\chi^*) \subset H^1(U; \mathbb{C}^*) \cong (\mathbb{C}^*)^{n-1}$  is generated by the character  $\rho = (\zeta, \dots, \zeta)$ , where  $\zeta = e^{2\pi i/n}$ .

Recall that the Hopf map  $\pi: M \rightarrow U$  induces the following morphism  $\pi^*: H^1(U; \mathbb{C}^*) \rightarrow H^1(M; \mathbb{C}^*)$  which restricts to an isomorphism  $\mathcal{V}_1^1(U) \xrightarrow{\cong} \mathcal{V}_1^1(M)$ . Recall also that the map  $\pi^*: (\mathbb{C}^*)^{n-1} \rightarrow (\mathbb{C}^*)^n$  is given in coordinates by formula (3.12). Since  $\zeta^n = 1$ , it follows that  $\pi^*(\mathrm{im}(\chi^*))$  is the cyclic subgroup of  $(\mathbb{C}^*)^n$  generated by  $\tilde{\rho} = (\zeta, \dots, \zeta, \zeta)$ . Therefore,  $\pi^*(\mathrm{im}(\chi^*))$  is contained in the diagonal subtorus  $T_{\Delta} = \{(z, \dots, z) \mid z \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^n$ .

Now let  $C$  be an irreducible component of  $\mathcal{V}_1^1(M)$ . By our assumption,  $C$  lies in a proper coordinate subtorus of  $H^1(M; \mathbb{C}^*) = (\mathbb{C}^*)^n$ ; hence,  $C$  intersects

intersects  $T_\Delta$  only at the identity. It follows that  $\pi^*(\text{im}(\chi^*)) \cap \mathcal{V}_1^1(M) = \{\mathbf{1}\}$ , and therefore,  $\text{im}(\chi^*) \cap \mathcal{V}_1^1(U) = \{\mathbf{1}\}$ . In view of formula (4.10), this shows that  $b_1(F) = n - 1$ , and the proof is complete.  $\square$

The following criterion for non-triviality of the algebraic monodromy is proved in [54, Theorem 8.3], based on results from [20] and [28].

**PROPOSITION 4.6** ([54]). *Let  $\mathcal{A}$  be a central arrangement in  $\mathbb{C}^3$ . If  $\mathcal{A}$  admits a reduced multinet, then the algebraic monodromy (in degree 1) over  $\mathbb{C}$  is non-trivial.*

If an arrangement supports essential multinets, but none of those multinets is reduced, then the algebraic monodromy (over  $\mathbb{C}$ ) may still be trivial, as illustrated by the  $B_3$  reflection arrangement from Section 10.1, though it may also be non-trivial, as illustrated by the complex reflection arrangements of type  $G(3d + 1, 1, 3)$  with  $d > 0$  from [54, Example 8.11].

## 5. COHOMOLOGY JUMP LOCI OF MILNOR FIBERS

In this section, we analyze the resonance and characteristic varieties of the Milnor fibers of a hyperplane arrangement, under the assumption that the algebraic monodromy of the Milnor fibration is trivial.

### 5.1. Cohomology jump loci in finite regular covers

We start with some general results regarding the behavior of jump loci in finite regular covers. These results were proved by Dimca and Papadima in [20, Proposition 2.1, Corollary 2.2, Theorem 2.8]. In the next two propositions, we state them in a slightly modified form, that is better adapted to our context.

**PROPOSITION 5.1** ([20]). *Let  $p: Y \rightarrow X$  be a finite regular cover. Then,*

1. *The induced homomorphism  $p^*: H^1(X; \mathbb{C}) \rightarrow H^1(Y; \mathbb{C})$  is an injection which restricts to maps  $p^*: \mathcal{R}_s^q(X) \rightarrow \mathcal{R}_s^q(Y)$ , for all  $q \geq 0$  and  $s \geq 1$ .*
2. *The morphism  $p^*: H^1(X; \mathbb{C}^*) \rightarrow H^1(Y; \mathbb{C}^*)$  restricts to the maps  $p^*: \mathcal{V}_s^q(X) \rightarrow \mathcal{V}_s^q(Y)$ , for all  $q \geq 0$  and  $s \geq 1$ .*

When the action of the group of deck transformations of the cover is homologically trivial (in degree 1), more can be said.

**PROPOSITION 5.2** ([20]). *Let  $p: Y \rightarrow X$  be a finite regular cover. Suppose the group of deck transformations acts trivially on  $H_1(Y; \mathbb{Q})$ . Then,*



1. The map  $p^*: H^1(X; \mathbb{C}) \rightarrow H^1(Y; \mathbb{C})$  is an isomorphism that identifies  $\mathcal{R}_s^1(X)$  with  $\mathcal{R}_s^1(Y)$ , for all  $s \geq 1$ .
2. The map  $p^*: H^1(X; \mathbb{C}^*)^0 \rightarrow H^1(Y; \mathbb{C}^*)^0$  is a surjection with finite kernel. Moreover, if  $X$  is 1-formal, this map establishes a bijection between the sets of irreducible components of  $\mathcal{W}_s^1(X)$  and  $\mathcal{W}_s^1(Y)$  that pass through the identity, for all  $s \geq 1$ .

Let us note that the homological triviality hypothesis of this proposition is definitely needed. For instance, if  $X$  is a wedge of  $n$  circles ( $n \geq 2$ ), and  $p: Y \rightarrow X$  is a  $k$ -fold cover ( $k \geq 2$ ), then  $\mathcal{R}_1^1(X) = \mathbb{C}^n$ , whereas  $\mathcal{R}_1^1(Y) = \mathbb{C}^{k(n-1)+1}$ , and so the map  $p^*: \mathcal{R}_1^1(X) \rightarrow \mathcal{R}_1^1(Y)$  is not surjective.

### 5.2. Cohomology jump loci in extensions

Next, we recall some general results relating cohomology jump loci in group extensions. In [69], we made a detailed analysis of how the characteristic and resonance varieties behave under certain split extensions with trivial monodromy action in homology. We summarize those results in the form that is needed here.

**THEOREM 5.3** ([69]). *Let  $1 \rightarrow K \xrightarrow{\iota} G \rightarrow Q \rightarrow 1$  be a split exact sequence of finitely generated groups. Assume  $Q$  is abelian. Then,*

1. If  $Q$  acts trivially on  $H_1(K; \mathbb{Z})$ , then the induced homomorphism  $\iota^*: H^1(G; \mathbb{C}^*) \rightarrow H^1(K; \mathbb{C}^*)$  restricts to maps  $\iota^*: \mathcal{V}_s^1(G) \rightarrow \mathcal{V}_s^1(K)$  for all  $s \geq 1$ ; furthermore,  $\iota^*: \mathcal{V}_1^1(G) \rightarrow \mathcal{V}_1^1(K)$  is a surjection.
2. If  $Q$  is torsion-free and acts trivially on  $H_1(K; \mathbb{Q})$ , then the map  $\iota^*: H^1(G; \mathbb{C}^*)^0 \rightarrow H^1(K; \mathbb{C}^*)^0$  restricts to maps  $\iota^*: \mathcal{W}_s^1(G) \rightarrow \mathcal{W}_s^1(K)$  for all  $s \geq 1$ ; furthermore,  $\iota^*: \mathcal{W}_1^1(G) \rightarrow \mathcal{W}_1^1(K)$  is a surjection.
3. If  $Q$  acts trivially on  $H_1(K; \mathbb{Q})$  and  $G$  is 1-formal, then the map  $\iota^*: H^1(G; \mathbb{C}) \rightarrow H^1(K; \mathbb{C})$  restricts to maps  $\iota^*: \mathcal{R}_s^1(G) \rightarrow \mathcal{R}_s^1(K)$  for all  $s \geq 1$ ; furthermore,  $\iota^*: \mathcal{R}_1^1(G) \rightarrow \mathcal{R}_1^1(K)$  is a surjection.

All these results are sharp. For instance, regarding part (3), we make the following observation: In depth  $s > 1$ , the map  $\iota^*: \mathcal{R}_s^1(G) \rightarrow \mathcal{R}_s^1(K)$  is not necessarily a surjection, while in depth  $s = 1$  it is not necessarily an isomorphism. We illustrate both assertions with an example (see [50, 51] for the necessary background).

*Example 5.4.* Let  $G = \langle a_1, \dots, a_4 \mid [a_1, a_2] = [a_2, a_3] = [a_3, a_4] = 1 \rangle$  be the right-angled Artin group associated to a path  $\Gamma$  on 4 vertices, and let  $K$  be the corresponding Bestvina–Brady group. We then have an exact sequence  $1 \rightarrow K \xrightarrow{\iota} G \xrightarrow{\nu} \mathbb{Z} \rightarrow 1$ , where  $\nu$  is the homomorphism sending each generator  $a_i$  to 1. Since  $\Gamma$  is a tree, the group  $K$  is free (of rank 3), and so  $\mathcal{R}_1^1(K) = \mathcal{R}_2^1(K) = \mathbb{C}^3$ . On the other hand,  $\mathcal{R}_1^1(G) = \{x_2 = 0\} \cup \{x_3 = 0\}$  and  $\mathcal{R}_2^1(G) = \{x_2 = x_3 = x_4 = 0\} \cup \{x_1 = x_2 = x_3 = 0\}$ . Thus, the map  $\iota^*: \mathcal{R}_s^1(G) \rightarrow \mathcal{R}_s^1(K)$  is not a surjection for  $s = 2$  and is not an isomorphism for  $s = 1$ .  $\diamond$

### 5.3. Cohomology jump loci of Milnor fibers

As before, let  $(\mathcal{A}, \mathbf{m})$  be a multi-arrangement. Denote by  $\iota_{\mathbf{m}}: F_{\mathbf{m}} \hookrightarrow M$  the inclusion map of the Milnor fiber  $F_{\mathbf{m}} = F_{\mathbf{m}}(\mathcal{A})$  into the complement  $M = M(\mathcal{A})$  and by  $\sigma_{\mathbf{m}} = \pi \circ \iota_{\mathbf{m}}: F_{\mathbf{m}} \rightarrow U$  the restriction of the Hopf map  $\pi: M \rightarrow U = \mathbb{P}(M)$  to  $F_{\mathbf{m}}$ . Applying Proposition 5.1 to the finite, regular cover  $\sigma_{\mathbf{m}}: F_{\mathbf{m}} \rightarrow U$ , we obtain the following immediate corollary.

**COROLLARY 5.5.** *For all  $q, s \geq 1$ , the following hold.*

1. *The induced homomorphism  $\sigma_{\mathbf{m}}^*: H^1(U; \mathbb{C}) \hookrightarrow H^1(F_{\mathbf{m}}; \mathbb{C})$  restricts to maps  $\mathcal{R}_s^q(U) \hookrightarrow \mathcal{R}_s^q(F_{\mathbf{m}})$ .*
2. *The morphism  $\sigma_{\mathbf{m}}^*: H^1(U; \mathbb{C}^*) \rightarrow H^1(F_{\mathbf{m}}; \mathbb{C}^*)$  restricts to maps  $\mathcal{V}_s^q(U) \rightarrow \mathcal{V}_s^q(F_{\mathbf{m}})$ .*

Consider now the usual Milnor fiber,  $F = F(\mathcal{A})$ , and the finite cyclic cover  $\sigma: F \rightarrow U$ . In general, the morphism  $\sigma^*: \mathcal{V}_1^1(U) \rightarrow \mathcal{V}_1^1(F)$  from Corollary 5.5, part (2) is not surjective. For instance, suppose  $\mathcal{A}$  admits a non-trivial, reduced multinet, and let  $T$  be the corresponding component of  $\mathcal{V}_1^1(U)$ . It is then shown in [20, Corollary 3.3] that  $\mathcal{V}_1^1(F)$  has a component  $W$  passing through the identity and containing  $\sigma^*(T)$  as a proper subset. We illustrate this phenomenon with a concrete example.

*Example 5.6.* Let  $\mathcal{A}$  be the braid arrangement from Example 4.2. Recall that  $\mathcal{V}_1^1(U) \subset (\mathbb{C}^*)^5$  has four local components,  $T_1, \dots, T_4$ , corresponding to the four triple points of  $\vec{\mathcal{A}}$ , and an essential, 2-dimensional component  $T$ , corresponding to the (3, 2)-net depicted in Figure 1. Let  $\psi: U \rightarrow S = \Sigma_{0,3}$  be the pencil defined by this net, so that  $T = \psi^*(H^1(S; \mathbb{C}^*))$ . Note that  $S = U(\mathcal{B})$ , where  $\mathcal{B}$  is the arrangement in  $\mathbb{C}^2$  defined by the polynomial  $xy(x - y)$ ; therefore, the Milnor fiber of this arrangement,  $\hat{S} = F(\mathcal{B})$ , may be identified with  $\Sigma_{1,3} = S^1 \times S^1 \setminus \{3 \text{ points}\}$  (see Example 4.1). Let  $\nu: \hat{S} \rightarrow S$

be the corresponding  $\mathbb{Z}_3$ -cover, and consider the pull-back diagram,

$$(5.1) \quad \begin{array}{ccc} \hat{U} & \xrightarrow{\hat{\psi}} & \hat{S} \\ \downarrow \tau & & \downarrow \nu \\ U & \xrightarrow{\psi} & S. \end{array}$$

In the above,  $\tau: \hat{U} \rightarrow U$  is the pull-back along  $\psi$  of the cover  $\nu: \hat{S} \rightarrow S$ . By construction,  $\tau$  is the  $\mathbb{Z}_3$ -cover defined by the diagonal homomorphism  $\pi_1(U) \rightarrow \mathbb{Z}_3$ . It is readily seen that  $H_1(\hat{U}; \mathbb{Z}) = \mathbb{Z}^7$ . By [77, Proposition 2], the map  $\hat{\psi}$  is an (irrational) pencil on  $\hat{U}$ ; therefore, the 4-dimensional torus  $W_0 = \hat{\psi}^*(H^1(\hat{S}; \mathbb{C}^*))$  is a component of the characteristic variety  $\mathcal{V}_1^1(\hat{U}) \subset (\mathbb{C}^*)^7$ .

Finally, let  $F = F(\mathcal{A})$  be the Milnor fiber of  $\mathcal{A}$ . Note that the  $\mathbb{Z}_6$ -cover  $\sigma: F \rightarrow U$  factors as the composite  $F \xrightarrow{\kappa} \hat{U} \xrightarrow{\tau} U$ , where  $\kappa$  is a 2-fold cover. Therefore, the characteristic variety  $\mathcal{V}_1^1(F)$  has four 2-dimensional components,  $\sigma^*(T_1), \dots, \sigma^*(T_4)$ , as well as a 4-dimensional component,  $W = \kappa^*(W_0)$ , which strictly contains  $\sigma^*(T)$ . Direct computation shows that  $\mathcal{V}_1^1(F)$  has no other irreducible components.  $\diamond$

#### 5.4. Arrangements with trivial algebraic monodromy

We return now to the general case of a multi-arrangement  $(\mathcal{A}, \mathbf{m})$ . As usual, let  $F_{\mathbf{m}}$  be the Milnor fiber of the multi-arrangement, and let  $\sigma_{\mathbf{m}}: F \rightarrow U$  be the corresponding  $\mathbb{Z}_N$ -cover, where  $N = \sum_{H \in \mathcal{A}} m_H$ . Using the machinery developed above, we obtain the following theorem, which sharpens results from [20] in a way that is needed later on.

**THEOREM 5.7.** *Suppose the monodromy map  $h: F_{\mathbf{m}} \rightarrow F_{\mathbf{m}}$  induces the identity on  $H_1(F_{\mathbf{m}}; \mathbb{Q})$ . Then,*

1. *The induced homomorphism  $\sigma_{\mathbf{m}}^*: H^1(U; \mathbb{C}) \rightarrow H^1(F_{\mathbf{m}}; \mathbb{C})$  is an isomorphism that identifies  $\mathcal{R}_s^1(U)$  with  $\mathcal{R}_s^1(F_{\mathbf{m}})$ , for all  $s \geq 1$ .*
2. *The induced homomorphism  $\sigma_{\mathbf{m}}^*: H^1(U; \mathbb{C}^*) \rightarrow H^1(F_{\mathbf{m}}; \mathbb{C}^*)^0$  is a surjection with kernel isomorphic to  $\mathbb{Z}_N$ . Moreover,*

- (a) *For each  $s \geq 1$ , the map  $\sigma_{\mathbf{m}}^*$  establishes a bijection between the sets of irreducible components of  $\mathcal{V}_s^1(U)$  and  $\mathcal{W}_s^1(F_{\mathbf{m}})$  that pass through the identity.*
- (b) *The map  $\sigma_{\mathbf{m}}^*: \mathcal{V}_1^1(U) \rightarrow \mathcal{W}_1^1(F_{\mathbf{m}})$  is a surjection.*

*Proof.* We start with some preliminary observations. From the discussion in Section 4.2, we know that the map  $\sigma_{\mathbf{m}}: F_{\mathbf{m}} \rightarrow U$  is a regular  $\mathbb{Z}_N$ -cover, corresponding to the exact sequence

$$(5.2) \quad 1 \longrightarrow \pi_1(F_{\mathbf{m}}) \xrightarrow{(\sigma_{\mathbf{m}})_{\sharp}} \pi_1(U) \xrightarrow{\chi_{\mathbf{m}}} \mathbb{Z}_N \longrightarrow 1.$$

As noted in Section 4.4, the assumption that  $h: F_{\mathbf{m}} \rightarrow F_{\mathbf{m}}$  induces the identity on  $H_1(F_{\mathbf{m}}; \mathbb{Q})$  is equivalent to  $H_1(F_{\mathbf{m}}; \mathbb{Q}) \cong H_1(U; \mathbb{Q})$ . It follows that we have an exact sequence,

$$(5.3) \quad 0 \longrightarrow H_1(F_{\mathbf{m}}; \mathbb{Z})/\text{Tors} \xrightarrow{(\sigma_{\mathbf{m}})_*} H_1(U; \mathbb{Z}) \xrightarrow{(\chi_{\mathbf{m}})^*} \mathbb{Z}_N \longrightarrow 0.$$

We now proceed with the proof. Claim (1) follows directly from Proposition 5.2, part (1). To prove the first assertion of Claim (2), we apply the functor  $H^1(-; \mathbb{C}^*) = \text{Hom}(-, \mathbb{C}^*)$  to the exact sequence (5.3). Since the abelian group  $\mathbb{C}^*$  is divisible, and thus an injective  $\mathbb{Z}$ -module, we obtain an exact sequence,

$$(5.4) \quad 0 \longleftarrow H^1(F_{\mathbf{m}}; \mathbb{C}^*)^0 \xleftarrow{\sigma_{\mathbf{m}}^*} H^1(U; \mathbb{C}^*) \xleftarrow{\chi_{\mathbf{m}}^*} H^1(\mathbb{Z}_N; \mathbb{C}^*) \longleftarrow 0.$$

Identifying the group  $H^1(\mathbb{Z}_N; \mathbb{C}^*)$  with its Pontryagin dual,  $\mathbb{Z}_N$ , completes the proof of the first part of Claim (2).

Since the space  $U$  is formal, Claim (2a) follows from Proposition 5.2, part (2).

Finally, recall from diagram (4.4) that we have a (split) exact sequence,

$$(5.5) \quad 1 \longrightarrow \pi_1(F_{\mathbf{m}}) \xrightarrow{(\iota_{\mathbf{m}})_{\sharp}} \pi_1(M) \xrightarrow{\mu_{\mathbf{m}}} \mathbb{Z} \longrightarrow 1.$$

Our hypothesis on the monodromy  $h$  says that  $\mathbb{Z}$  acts trivially on  $H_1(F_{\mathbf{m}}; \mathbb{Q})$ . Thus, we may apply Theorem 5.3 and conclude that the morphism

$$\iota_{\mathbf{m}}^*: H^1(M; \mathbb{C}^*) \rightarrow H^1(F_{\mathbf{m}}; \mathbb{C}^*)^0$$

restricts to a surjection,  $\iota_{\mathbf{m}}^*: \mathcal{V}_1^1(M) \twoheadrightarrow \mathcal{V}_1^1(F_{\mathbf{m}})$ . On the other hand, as shown in Proposition 3.3, part (3.3), the map  $\pi^*: H^1(U; \mathbb{C}^*) \rightarrow H^1(M; \mathbb{C}^*)$  restricts to an isomorphism,  $\pi^*: \mathcal{V}_1^1(U) \xrightarrow{\cong} \mathcal{V}_1^1(M)$ . Since  $\sigma_{\mathbf{m}} = \pi \circ \iota_{\mathbf{m}}$ , Claim (2b) follows, and the proof is complete.  $\square$

## 6. ABELIAN DUALITY AND PROPAGATION OF COHOMOLOGY JUMP LOCI

### 6.1. Abelian duality spaces

Let  $X$  be a space having the homotopy type of a connected, finite CW-complex of dimension  $m$ . Without loss of generality, we may assume  $X$  has a

single 0-cell, say,  $x_0$ . Letting  $G = \pi_1(X, x_0)$  be the fundamental group of  $X$ , the group ring of its abelianization,  $R = \mathbb{Z}[G_{\text{ab}}]$ , may be viewed as a module over  $\mathbb{Z}G$  via extension of scalars. Inspired by the classical notion of “duality group” due to Bieri and Eckmann [19], the following concept was introduced.

We say that  $X$  is an *abelian duality space* (for short, *ab-duality space*) of dimension  $m$  if  $H^q(X; R) = 0$  for  $q \neq m$  and  $H^m(X, R)$  is non-zero and torsion-free. In that case, for all (left)  $R$ -modules  $A$  and all  $q \geq 0$ , we have isomorphisms

$$(6.1) \quad H^q(X; A) \cong \text{Tor}_{m-q}^R(D; A) \cong H_{m-q}(G_{\text{ab}}; D \otimes_{\mathbb{Z}} A),$$

where  $D = H^m(X; R)$ , viewed as an  $R$ -module. Thus, if  $Y \rightarrow X$  is a connected, regular abelian cover, classified by an epimorphism  $G \xrightarrow{\text{ab}} G_{\text{ab}} \xrightarrow{\chi} H$ , where  $H$  is a (finitely generated) abelian group, then  $H_q(Y; \mathbb{Z}) \cong \text{Ext}_R^{m-q}(D, H)$ , for all  $q \geq 0$ .

Motivated by our work in [69], we adapt this definition to a related context. Let  $G_{\text{abf}} = G_{\text{ab}}/\text{Tors}$  be the maximal torsion-free abelian quotient of  $G$ . We say that  $X$  is a *torsion-free abelian duality space* (for short, *abf-duality space*) of dimension  $m$  if the above conditions are satisfied with  $R = \mathbb{Z}[G_{\text{ab}}]$  replaced by  $\mathbb{Z}[G_{\text{abf}}]$ . Clearly, if  $X$  is an abelian duality space and  $G_{\text{ab}} = H_1(X; \mathbb{Z})$  is torsion-free, then  $X$  is a torsion-free abelian duality space.

### 6.2. Formality

Recall that both an arrangement complement,  $M = M(\mathcal{A})$ , and its projectivization,  $U = \mathbb{P}(M)$ , are (rationally) formal spaces. Moreover, for every choice of multiplicities  $\mathbf{m}$  on  $\mathcal{A}$ , the Milnor fiber  $F_{\mathbf{m}}$  is a cyclic, regular cover of  $U$ . This raises the question of whether these Milnor fibers are also formal spaces or, at least  $q$ -formal, for some  $q \geq 1$ . The following lemma gives a sufficient condition for this to happen.

LEMMA 6.1 ([20]). *Let  $Y \rightarrow X$  be a finite, regular cover, and suppose the group of deck-transformations acts trivially on  $H_i(Y; \mathbb{Q})$ , for all  $i \leq q$ . Then  $Y$  is  $q$ -formal if and only if  $X$  is  $q$ -formal.*

COROLLARY 6.2. *Let  $(\mathcal{A}, \mathbf{m})$  be a multi-arrangement of rank  $r$ , with Milnor fiber  $F_{\mathbf{m}}$  and monodromy  $h: F_{\mathbf{m}} \rightarrow F_{\mathbf{m}}$ .*

1. *If the algebraic monodromy  $h_*: H_i(F_{\mathbf{m}}; \mathbb{Q}) \rightarrow H_i(F_{\mathbf{m}}; \mathbb{Q})$  is the identity for all  $i \leq q$ , for some  $q \geq 1$ , then  $F_{\mathbf{m}}$  is  $q$ -formal.*
2. *If  $h_*: H_i(F_{\mathbf{m}}; \mathbb{Q}) \rightarrow H_i(F_{\mathbf{m}}; \mathbb{Q})$  is the identity for all  $i \leq r - 2$ , then  $F_{\mathbf{m}}$  is formal.*

*Proof.* Part (1) follows directly from the above lemma. For part (2), first recall that  $F_{\mathbf{m}}$  has the homotopy type of a finite CW-complex of dimension at most  $r - 1$ . Thus, the claim follows from part (1) and the discussion in Section 2.4.  $\square$

In general, though, Milnor fibers may be non-formal, as illustrated by the following example of Zuber [77].

*Example 6.3.* Let  $\mathcal{A} = \mathcal{A}(3, 3, 3)$  be the monomial arrangement in  $\mathbb{C}^3$  defined by the polynomial  $f = (x^3 - y^3)(y^3 - z^3)(x^3 - z^3)$ . There are four  $(3, 3)$ -nets on  $\mathcal{A}$ , associated with the partitions  $(123|456|789)$ ,  $(147|258|369)$ ,  $(159|267|348)$ , and  $(168|249|357)$  in a suitable ordering of the hyperplanes. The first of these nets defines a rational map,  $\psi: \mathbb{C}\mathbb{P}^2 \dashrightarrow \mathbb{C}\mathbb{P}^1$ ,  $[x : y : z] \mapsto [x^3 - y^3 : x^3 - z^3]$ , which in turn restricts to a pencil  $\psi: U \rightarrow S$  from  $U = U(\mathcal{A})$  to  $S = \mathbb{C}\mathbb{P}^1 \setminus \{[1 : 0], [0 : 1], [1 : 1]\}$ . Let  $T = \psi^*(H^1(S; \mathbb{C}^*))$  be the essential 2-dimensional component of  $\mathcal{V}_1^1(U)$  obtained by pullback along this pencil. The subgroup generated by the diagonal character  $\rho: \pi_1(U) \rightarrow \mathbb{C}^*$  intersects  $\mathcal{V}_2^1(U)$  at the identity  $\mathbf{1}$  and two other points, both lying on  $T$ , and both of order 3. Hence,  $\Delta_1(t) = (t - 1)^8(1 + t + t^2)^2$ .

Next, let  $\mathcal{B}$  be the arrangement in  $\mathbb{C}^2$  defined by the polynomial  $xy(x - y)$ , and let  $\nu: \hat{S} = F(\mathcal{B}) \rightarrow S = U(\mathcal{B})$  be the corresponding 3-fold cover. As shown in [77, Proposition 2], the rational pencil  $\psi: U \rightarrow S = \Sigma_{0,3}$  lifts to an irrational pencil,  $\hat{\psi}: \hat{U} \rightarrow \hat{S} = \Sigma_{1,3}$ , as in diagram (5.1). Here,  $\tau: \hat{U} \rightarrow U$  is the pull-back of  $\nu$  along  $\psi$ , that is, the  $\mathbb{Z}_3$ -cover defined by the diagonal homomorphism  $\pi_1(U) \rightarrow \mathbb{Z}_3$ . It is readily seen that  $H_1(\hat{U}; \mathbb{Z}) = \mathbb{Z}^{12}$ ; therefore, the 4-dimensional torus  $W_0 = \hat{\psi}^*(H^1(\hat{S}; \mathbb{C}^*))$  is a component of the characteristic variety  $\mathcal{V}_1^1(\hat{U}) \subset (\mathbb{C}^*)^{12}$ .

Finally, let  $F = F(\mathcal{A})$  be the Milnor fiber of  $\mathcal{A}$ . Then the  $\mathbb{Z}_9$ -cover  $\sigma: F \rightarrow U$  factors as the composite  $F \xrightarrow{\kappa} \hat{U} \xrightarrow{\tau} U$ , where  $\kappa$  is a 3-fold cover. Therefore, the characteristic variety  $\mathcal{V}_1^1(F)$  has a 4-dimensional component,  $W = \kappa^*(W_0)$ , which strictly contains the 2-dimensional subtorus  $\sigma^*(T)$ . Write  $W = \exp(L)$ , for some linear subspace  $L \subset H^1(F; \mathbb{C})$ . Using the mixed Hodge structure on  $H^*(F; \mathbb{C})$ , Zuber showed in [77] that  $L$  cannot be a component of the resonance variety  $\mathcal{R}_1^1(F)$ . Therefore,  $\text{TC}_1(\mathcal{V}_1^1(F)) \subsetneq \mathcal{R}_1^1(F)$ , and so, by the Tangent Cone theorem of [22],  $F$  is not 1-formal.  $\diamond$

### 6.3. Ab- and abf-exactness

Let  $F \xrightarrow{\iota} E \xrightarrow{\pi} B$  be a fibration sequence of connected CW-complexes. Setting  $K = \pi_1(F)$ ,  $G = \pi_1(E)$ , and  $Q = \pi_1(B)$ , we have an exact sequence

$K \xrightarrow{\iota_{\sharp}} G \xrightarrow{\pi_{\sharp}} Q \rightarrow 1$ . Moreover, the exact sequence of low-degree terms in the Serre spectral sequence of the fibration takes the form

(6.2)

$$H_2(E; \mathbb{Z}) \xrightarrow{\pi_*} H_2(B; \mathbb{Z}) \xrightarrow{\delta} H_1(F; \mathbb{Z})_Q \xrightarrow{\iota_*} H_1(E; \mathbb{Z}) \xrightarrow{\pi_*} H_1(B; \mathbb{Z}) \longrightarrow 0,$$

where  $H_1(F; \mathbb{Z})_Q$  denotes the coinvariants of  $K_{\text{ab}} = H_1(F; \mathbb{Z})$  under the action of  $Q$ .

Following [19], we say that the fibration is *ab-exact* if (1)  $Q$  acts trivially on  $K_{\text{ab}}$ ; and (2) the homomorphism  $\delta$  is zero. In the presence of the first condition, the second condition is equivalent to the exactness of the sequence  $0 \rightarrow K_{\text{ab}} \rightarrow G_{\text{ab}} \rightarrow Q_{\text{ab}} \rightarrow 0$ . Finally, as shown in [69, Proposition 8.4], if  $K \triangleleft G$  and the sequence  $1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$  is exact and admits a splitting, then the fibration is ab-exact if and only if  $Q$  acts trivially on  $K_{\text{ab}}$ .

As shown in [19, Proposition 4.13], the notion of ab-duality behaves well with respect to ab-exact fibrations: if any two of the spaces have the abelian duality property, then the third one does, too. In particular, the product of two ab-exact exact spaces is again ab-exact. We record here the part of this result that will be needed later on.

PROPOSITION 6.4 ([19]). *Suppose  $F \rightarrow E \rightarrow B$  is an ab-exact fibration of connected, finite-type CW-complexes. If  $E$  and  $B$  are ab-duality spaces of dimensions  $m$  and  $n$ , respectively, and if  $\dim F = m - n$ , then  $F$  is an ab-duality space of dimension  $m - n$ .*

By analogy with the above, we say that a fibration  $F \rightarrow E \rightarrow B$  is *abf-exact* if  $Q$  acts trivially on  $K_{\text{abf}}$  and the composite  $H_2(B; \mathbb{Z}) \xrightarrow{\delta} H_1(F; \mathbb{Z})_Q \rightarrow K_{\text{abf}}$  is zero. In the presence of the first condition, the second condition is equivalent to the exactness of the sequence  $0 \rightarrow K_{\text{abf}} \rightarrow G_{\text{abf}} \rightarrow Q_{\text{abf}} \rightarrow 0$ . Alternatively, let  $\delta_{\mathbb{Q}}: H_2(B; \mathbb{Q}) \rightarrow H_1(F; \mathbb{Q})_Q$  be the analog of the map  $\delta$  in the exact sequence (6.2) with  $\mathbb{Q}$ -coefficients. Since  $K_{\text{abf}}$  is finitely generated, an argument similar to the one used in [69, Lemma 9.2] shows that the fibration is abf-exact if and only if  $Q$  acts trivially on  $H_1(F; \mathbb{Q})$  and  $\delta_{\mathbb{Q}}$  is the zero map. Finally, as shown in [69, Proposition 9.4], if  $K \triangleleft G$  and the sequence  $1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$  is split exact, then the fibration is abf-exact if and only if  $Q$  acts trivially on  $H_1(F; \mathbb{Q})$ .

The same argument as in [19], using now the Serre spectral sequence of the fibration  $F \rightarrow E \rightarrow B$  with coefficients in  $\mathbb{Z}[G_{\text{abf}}]$  instead of  $\mathbb{Z}[G_{\text{ab}}]$ , shows the following: if any two of the spaces have the torsion-free abelian duality property, then the third one does, too. In particular, the product of two abf-exact exact spaces is again abf-exact. We record here only the result that we need later in this section.

PROPOSITION 6.5. *Suppose  $F \rightarrow E \rightarrow B$  is an abf-exact fibration of connected, finite-type CW-complexes. If  $E$  and  $B$  are abf-duality spaces of dimensions  $m$  and  $n$ , respectively, and if  $\dim F = m - n$ , then  $F$  is an abf-duality space of dimension  $m - n$ .*

### 6.4. Propagation of jump loci

One of the main motivations for the study of the abelian duality properties of spaces is the implications these properties have on the nature of the cohomology jump loci and the Betti numbers of those spaces. We start with a result relating ab-duality to propagation of characteristic varieties.

THEOREM 6.6 ([19]). *Let  $X$  be an abelian duality space of dimension  $m$ . Then the characteristic varieties of  $X$  propagate; that is, for any character  $\rho \in H^1(X; \mathbb{C}^*)$  such that  $H^p(X; \mathbb{C}_\rho) \neq 0$ , it follows that  $H^q(X; \mathbb{C}_\rho) \neq 0$  for all  $p \leq q \leq m$ . Equivalently,*

$$(6.3) \quad \{\mathbf{1}\} = \mathcal{V}_1^0(X) \subseteq \mathcal{V}_1^1(X) \subseteq \mathcal{V}_1^2(X) \subseteq \dots \subseteq \mathcal{V}_1^m(X).$$

Applying this theorem to the trivial character  $\rho = \mathbf{1}$ , it follows at once that  $b_q(X) > 0$  for  $0 \leq q \leq m$ . Moreover, as shown in [19, Proposition 5.9], we also have  $b_1(X) \geq m$ . Finally, as noted in [37, Theorem 1.8], the above result implies that the “signed Euler characteristic” of an  $m$ -dimensional ab-duality space,  $(-1)^m \chi(X)$ , is non-negative. A similar argument, using [19, Proposition 2.8], applied to the  $\mathbb{C}[G_{\text{abf}}]$ -chain complex  $C_*(X; \mathbb{C}[G_{\text{abf}}])$ , yields the following result.

THEOREM 6.7. *Let  $X$  be an abf-duality space of dimension  $m$ . Then*

$$(6.4) \quad \{\mathbf{1}\} = \mathcal{W}_1^0(X) \subseteq \mathcal{W}_1^1(X) \subseteq \mathcal{W}_1^2(X) \subseteq \dots \subseteq \mathcal{W}_1^m(X).$$

Now suppose  $X$  is formal. Then, the Tangent Cone theorem of [22, 21], allows us to identify the tangent cone at  $\mathbf{1}$  to  $\mathcal{V}_1^q(X)$  with  $\mathcal{R}_1^q(X)$  for all  $q \leq m$ . Applying Theorem 6.6, we obtain the following immediate corollary.

COROLLARY 6.8. *Let  $X$  is an abelian duality space of dimension  $m$ . If  $X$  is  $q$ -formal, for some  $q \leq m$ , then  $\mathcal{R}_1^1(X) \subseteq \dots \subseteq \mathcal{R}_1^q(X)$ . In particular, if  $X$  is formal, then the resonance varieties of  $X$  propagate; that is,  $\mathcal{R}_1^1(X) \subseteq \dots \subseteq \mathcal{R}_1^m(X)$ .*

Remark 6.9 ([19]). If  $X$  is a connected, finite, 2-dimensional CW-complex with  $\chi(X) \geq 0$  and  $b_1(X) > 0$ , then both the characteristic and the resonance varieties of  $X$  propagate (that is,  $\mathcal{V}_1^1(X) \subseteq \mathcal{V}_1^2(X)$  and  $\mathcal{R}_1^1(X) \subseteq \mathcal{R}_1^2(X)$ ), even though  $X$  may be neither an abelian duality space nor a formal space. On the



other hand, if  $X$  is a closed, orientable 3-manifold with  $b_1(X)$  even and non-zero, then the resonance varieties do not propagate, since  $\mathcal{R}_1^1(X) = H^1(X; \mathbb{C})$ , whereas  $\mathcal{R}_1^3(X) = \{\mathbf{0}\}$ .

### 6.5. Abelian duality and propagation for arrangements

A basic topological property of arrangement complements is provided by the following result, which is proved in [18, Theorem 5.6] (see also [19, Theorem 6.1]).

**THEOREM 6.10** ([18, 19]). *Let  $\mathcal{A}$  be a central arrangement of rank  $r$ . Then the complement  $M = M(\mathcal{A})$  is an abelian duality space of dimension  $r$  and the projectivized complement  $U = \mathbb{P}(M)$  is an abelian duality space of dimension  $r - 1$ .*

In particular, if  $\mathcal{A}$  is a central, essential arrangement of hyperplanes in  $\mathbb{C}^{d+1}$ , then  $M(\mathcal{A})$  is an abelian duality space of dimension  $d + 1$  and  $U(\mathcal{A})$  is an abelian duality space of dimension  $d$ .

*Remark 6.11.* More generally, let  $M$  be a connected, smooth, complex quasi-projective variety of dimension  $m$ . Suppose  $M$  has a smooth compactification  $\overline{M}$  for which the components of  $\overline{M} \setminus M$  form a non-empty arrangement of hypersurfaces,  $\mathcal{A}$ , such that, for each submanifold  $X$  in the intersection poset  $L(\mathcal{A})$ , the complement of the restriction of  $\mathcal{A}$  to  $X$  is either empty or a Stein manifold. Then, by [17, Theorem 1.1],  $M$  is an abelian duality space of dimension  $m$ . Another generalization of Theorem 6.10 is given in [37, Theorem 1.10]: If  $M$  has a smooth compactification  $\overline{M}$  with  $b_1(\overline{M}) = 0$  and  $M$  admits a proper, semi-small map to a complex algebraic torus, then the same conclusion holds.

Recall now that arrangement complements are also formal. It follows from Theorem 6.10 and Corollary 6.8 that both their characteristic and resonance varieties propagate. More precisely, we have the following corollary.

**COROLLARY 6.12.** *Let  $\mathcal{A}$  be a central arrangement of rank  $r$ , with complement  $M = M(\mathcal{A})$  and projectivized complement  $U = \mathbb{P}(M)$ . Then*

1.  $\mathcal{V}_1^1(M) \subseteq \cdots \subseteq \mathcal{V}_1^r(M)$  and  $\mathcal{R}_1^1(M) \subseteq \cdots \subseteq \mathcal{R}_1^r(M)$ .
2.  $\mathcal{V}_1^1(U) \subseteq \cdots \subseteq \mathcal{V}_1^{r-1}(U)$  and  $\mathcal{R}_1^1(U) \subseteq \cdots \subseteq \mathcal{R}_1^{r-1}(U)$ .

In view of part (2) of this result, Proposition 3.3 yields the following immediate corollary.

**COROLLARY 6.13.** *Let  $\pi: M \rightarrow U$  be the restriction of the Hopf map. Then,*

1. The induced homomorphism  $\pi^*: H^1(U; \mathbb{C}) \hookrightarrow H^1(M; \mathbb{C})$  restricts to isomorphisms  $\mathcal{R}_1^q(U) \xrightarrow{\cong} \mathcal{R}_1^q(M)$  for all  $q \geq 1$ .
2. The induced morphism  $\pi^*: H^1(U; \mathbb{C}^*) \hookrightarrow H^1(M; \mathbb{C}^*)$  restricts to isomorphisms  $\mathcal{V}_1^q(U) \xrightarrow{\cong} \mathcal{V}_1^q(M)$  for all  $q \geq 1$ .

### 6.6. Abelian duality and propagation for Milnor fibers

We now turn to the Milnor fibration  $F_{\mathbf{m}} \rightarrow M \rightarrow \mathbb{C}^*$  of a multi-arrangement  $(\mathcal{A}, \mathbf{m})$ . To start with, let us note that Corollary 5.5, when used in conjunction with Proposition 3.3 and Corollary 6.13, has the following consequence.

**COROLLARY 6.14.** *Let  $\iota_{\mathbf{m}}: F_{\mathbf{m}} \hookrightarrow M$  be the inclusion map of the Milnor fiber into the complement of  $\mathcal{A}$ .*

1. The epimorphism  $\iota_{\mathbf{m}}^*: H^1(M; \mathbb{C}) \twoheadrightarrow H^1(F_{\mathbf{m}}; \mathbb{C})$  restricts to the maps  $\mathcal{R}_s^1(M) \rightarrow \mathcal{R}_s^1(F_{\mathbf{m}})$ , for all  $s \geq 1$ , and  $\mathcal{R}_1^q(M) \rightarrow \mathcal{R}_1^q(F_{\mathbf{m}})$ , for all  $q \geq 1$ .
2. The epimorphism  $\iota_{\mathbf{m}}^*: H^1(M; \mathbb{C}^*) \twoheadrightarrow H^1(F_{\mathbf{m}}; \mathbb{C}^*)$  restricts to the maps  $\mathcal{V}_s^1(M) \rightarrow \mathcal{V}_s^1(F_{\mathbf{m}})$ , for all  $s \geq 1$ , and  $\mathcal{V}_1^q(M) \rightarrow \mathcal{V}_1^q(F_{\mathbf{m}})$ , for all  $q \geq 1$ .

The next result strengthens [19, Theorem 6.7], where only part (1) is proved (in the particular case when  $F = F(\mathcal{A})$  is the usual Milnor fiber of an essential arrangement), but not part (2).

**THEOREM 6.15.** *Let  $\mathcal{A}$  be a central arrangement of rank  $r$  and let  $F_{\mathbf{m}} = F_{\mathbf{m}}(\mathcal{A})$  be the Milnor fiber associated to a multiplicity vector  $\mathbf{m}: \mathcal{A} \rightarrow \mathbb{N}$ .*

1. *If the monodromy action on  $H_1(F_{\mathbf{m}}; \mathbb{Z})$  is trivial, then  $F_{\mathbf{m}}$  is an ab-duality space of dimension  $r - 1$ .*
2. *If the monodromy action on  $H_1(F_{\mathbf{m}}; \mathbb{Q})$  is trivial, then  $F_{\mathbf{m}}$  is an abf-duality space of dimension  $r - 1$ .*

*Proof.* From Theorem 6.10, we know that the total space of the Milnor fibration,  $M = M(\mathcal{A})$ , is an ab-duality space of dimension  $r$ . Thus,  $M$  is also and abf-duality space of the same dimension, since  $H_1(M; \mathbb{Z}) = \mathbb{Z}^{|\mathcal{A}|}$  is torsion-free. Clearly, the base of the fibration,  $B = \mathbb{C}^*$ , is both an ab- and abf-duality space of dimension 1. In view of our hypothesis on the monodromy of the fibration, the two claims regarding the fiber  $F_{\mathbf{m}}$  now follow directly from Propositions 6.4 and 6.5, respectively.  $\square$

Applying this theorem, we obtain the following corollary regarding propagation of cohomology jump loci of Milnor fibers of arrangements with trivial algebraic monodromy.

**COROLLARY 6.16.** *Let  $\mathcal{A}$  be a central arrangement of rank  $r$ , and let  $\mathbf{m}: \mathcal{A} \rightarrow \mathbb{N}$  be a choice of multiplicities.*

1. *If the monodromy action on  $H_1(F_{\mathbf{m}}; \mathbb{Z})$  is trivial, then the characteristic varieties of  $F_{\mathbf{m}}$  propagate; that is,  $\mathcal{V}_1^1(F_{\mathbf{m}}) \subseteq \mathcal{V}_1^2(F_{\mathbf{m}}) \subseteq \dots \subseteq \mathcal{V}_1^{r-1}(F_{\mathbf{m}})$ .*
2. *If the monodromy action on  $H_1(F_{\mathbf{m}}; \mathbb{Q})$  is trivial, then the restricted characteristic varieties of  $F_{\mathbf{m}}$  propagate; that is,  $\mathcal{W}_1^1(F_{\mathbf{m}}) \subseteq \mathcal{W}_1^2(F_{\mathbf{m}}) \subseteq \dots \subseteq \mathcal{W}_1^{r-1}(F_{\mathbf{m}})$ .*
3. *If the monodromy action on  $H_i(F_{\mathbf{m}}; \mathbb{Q})$  is trivial for  $i \leq q$ , then the resonance varieties of  $F_{\mathbf{m}}$  propagate in that range; that is,  $\mathcal{R}_1^1(F_{\mathbf{m}}) \subseteq \dots \subseteq \mathcal{R}_1^q(F_{\mathbf{m}})$ .*
4. *If the monodromy action on  $H_i(F_{\mathbf{m}}; \mathbb{Q})$  is trivial for  $i \leq r - 2$ , then the resonance varieties of  $F_{\mathbf{m}}$  propagate; that is,  $\mathcal{R}_1^1(F_{\mathbf{m}}) \subseteq \dots \subseteq \mathcal{R}_1^{r-1}(F_{\mathbf{m}})$ .*

*Proof.* Claim (1) follows from Theorem 6.6 and Theorem 6.15, part (1), while Claim (2) follows from Theorem 6.7 and Theorem 6.15, part (2).

Claims (3) and (4) follow from claim (2) and the Tangent Cone theorem, using Corollary 6.2, parts (1) and (2), respectively.  $\square$

In particular, if  $\mathcal{A}$  is a central, essential arrangement in  $\mathbb{C}^3$  and the monodromy action on  $H_1(F_{\mathbf{m}}; \mathbb{Q})$  is trivial, then  $\mathcal{W}_1^1(F_{\mathbf{m}}) \subseteq \mathcal{W}_1^2(F_{\mathbf{m}})$  and  $\mathcal{R}_1^1(F_{\mathbf{m}}) \subseteq \mathcal{R}_1^2(F_{\mathbf{m}})$ .

*Remark 6.17.* More generally, let  $f \in \mathbb{C}[z_0, \dots, z_d]$  be a homogeneous polynomial of degree  $n$ , and set  $M = \mathbb{C}^{d+1} \setminus \{f = 0\}$ . We then have a (global) Milnor fibration,  $f: M \rightarrow \mathbb{C}^*$ , with fiber  $F = f^{-1}(1)$  and monodromy  $h: F \rightarrow F$  given by  $h(z) = e^{2\pi i/n}z$ . Now suppose  $M$  satisfies one of the conditions laid out in Remark 6.11, so that  $M$  is an abelian duality space of dimension  $d + 1$ , and suppose further that  $h_*: H_1(F; \mathbb{Q}) \rightarrow H_1(F; \mathbb{Q})$  is the identity. Then similar proofs show that  $F$  is an abf-duality space of dimension  $d$  and the restricted characteristic varieties of  $F$  propagate, that is,  $\mathcal{W}_1^1(F) \subseteq \dots \subseteq \mathcal{W}_1^d(F)$ .

## 7. TRIVIAL ALGEBRAIC MONODROMY AND LOWER CENTRAL SERIES

In this section, we investigate the lower central series ranks and the Chen ranks of the fundamental groups of Milnor fibers of arrangements for which the algebraic monodromy is trivial.

## 7.1. Lower central series and nilpotent quotients

The lower central series (LCS) of a group  $G$  is defined inductively by setting  $\gamma_1(G) = G$  and  $\gamma_{k+1}(G) = [G, \gamma_k(G)]$  for all  $k \geq 1$ . This is a central series (i.e.,  $[G, \gamma_k(G)] \subseteq \gamma_{k+1}(G)$  for all  $k \geq 1$ ), and thus, a normal series (i.e.,  $\gamma_k(G) \triangleleft G$  for all  $k \geq 1$ ). Consequently, each LCS quotient,

$$(7.1) \quad \text{gr}_k(G) := \gamma_k(G)/\gamma_{k+1}(G),$$

lies in the center of  $G/\gamma_{k+1}(G)$ , and thus is an abelian group. The first such quotient,  $\text{gr}_1(G) = G/\gamma_2(G)$ , coincides with the abelianization  $G_{\text{ab}} = H_1(G; \mathbb{Z})$ . The associated graded Lie algebra of  $G$  is the direct sum  $\text{gr}(G) = \bigoplus_{k \geq 1} \text{gr}_k(G)$ ; the addition in  $\text{gr}(G)$  is induced from the group multiplication, while the Lie bracket (which is compatible with the grading) is induced from the group commutator. By construction, the Lie algebra  $\text{gr}(G)$  is generated by its degree 1 piece. Thus, if  $G_{\text{ab}}$  is finitely generated, then so are the LCS quotients of  $G$ ; we let  $\phi_k(G) := \text{rank gr}_k(G)$  be ranks of those quotients.

Replacing in this construction the group  $G$  by its maximal metabelian quotient,  $G/G''$ , leads to the Chen Lie algebra  $\text{gr}(G/G'')$ , and, in the case when  $G_{\text{ab}}$  is finitely generated, the Chen ranks  $\theta_k(G) := \text{rank gr}_k(G/G'')$ . It is readily seen that  $\theta_k(G) \leq \phi_k(G)$  for all  $k \geq 1$ , with equality for  $k \leq 3$ .

For each  $k \geq 1$ , the group  $G/\gamma_{k+1}(G)$  is nilpotent, and in fact, the maximal  $k$ -step nilpotent quotient of  $G$ . Letting  $q_k: G/\gamma_{k+1}(G) \rightarrow G/\gamma_k(G)$  be the projection maps, we obtain a tower of nilpotent groups, starting at  $G/\gamma_2(G) = G_{\text{ab}}$ . Moreover, at each stage in the tower, there is a central extension,

$$(7.2) \quad 0 \longrightarrow \text{gr}_k(G) \longrightarrow G/\gamma_{k+1}(G) \xrightarrow{q_k} G/\gamma_k(G) \longrightarrow 0,$$

which is classified by an extension class (or,  $k$ -invariant),  $\chi_k: H_2(G/\gamma_k(G); \mathbb{Z}) \rightarrow \text{gr}_k(G)$ .

## 7.2. Lower central series of arrangement groups

The LCS ranks, the Chen ranks, and the nilpotent quotients of arrangement groups have been much studied. The most basic example is that of the free group,  $F_n = \pi_1(\mathbb{C} \setminus \{n \text{ points}\})$ , of rank  $n \geq 2$ . Work of P. Hall, W. Magnus, and E. Witt from the 1930s shows that, for each  $k \geq 1$ , the abelian group  $\text{gr}_k(F_n)$  is torsion-free, of rank equal to

$$(7.3) \quad \phi_k(F_n) = \frac{1}{k} \sum_{d|k} \mu(d) n^{k/d},$$

where  $\mu: \mathbb{N} \rightarrow \{0, \pm 1\}$  denotes the Möbius function. Furthermore, work of K. T. Chen from 1951 shows that the group  $\mathrm{gr}_k(F_n/F_n'')$  are also torsion-free, of rank equal to

$$(7.4) \quad \theta_k(F_n) = (k-1) \binom{n+k-2}{k} \text{ for } k \geq 2.$$

Now let  $M = M(\mathcal{A})$  be any arrangement complement, and let  $G = \pi_1(M)$  be its fundamental group. As mentioned previously,  $M$  is formal, and hence the group  $G$  is 1-formal. Classical results of Quillen and Sullivan in rational homotopy theory insure that the LCS ranks  $\phi_k(G)$  are determined by the (truncated) cohomology algebra  $H^{\leq 2}(M; \mathbb{Q})$ . Since this algebra is determined by the (truncated) intersection lattice  $L_{\leq 2}(\mathcal{A})$ , it follows that the LCS ranks of  $G$  are combinatorially determined. Explicit combinatorial formulas for these ranks are known in a few cases, e.g., when  $\mathcal{A}$  is either supersolvable [27] or decomposable [49], but no such formula is known in general, even for  $\phi_3(G)$ . As shown in [48], the Chen ranks  $\theta_k(G)$  are also combinatorially determined. An explicit combinatorial formula was conjectured in [60], expressing those ranks in terms of the dimensions of the irreducible components of  $\mathcal{B}_1^1(M)$ , at least for  $k$  large enough. This formula has been verified by Cohen and Schenck in [11] (see also [1] for a more general setting).

Turning to the nilpotent quotients of an arrangement group  $G = G(\mathcal{A})$ , it was shown in [55] that all the quotients  $G/\gamma_k(G)$  are combinatorially determined when  $\mathcal{A}$  is decomposable (see Section 8.3 below for more on this). On the other hand, Rybnikov [56] showed that the third nilpotent quotient,  $G/\gamma_4(G)$ , is *not* combinatorially determined, in general. Nevertheless, the second nilpotent quotient,  $G/\gamma_3(G)$ , is always determined by  $L_{\leq 2}(\mathcal{A})$ . To see why, recall from Section 2.3 that  $H^*(M; \mathbb{Z}) = E/I$ , where  $E = \bigwedge G_{\mathrm{ab}}$  and  $I = I(\mathcal{A})$  is the Orlik–Solomon ideal associated to  $L(\mathcal{A})$ . As shown in [41, Proposition 1.14], the abelian group  $\mathrm{gr}_2(G)$  is the  $\mathbb{Z}$ -dual of  $I^2$  (and thus, it is torsion-free), and the exact sequence (7.2) with  $k = 2$  is classified by the homomorphism  $\chi_2: H_2(G_{\mathrm{ab}}; \mathbb{Z}) \rightarrow \mathrm{gr}_2(G)$  dual to the inclusion map  $I^2 \hookrightarrow E^2$ . Set  $n = |\mathcal{A}|$  and let  $F_n$  be the free group on generators  $\{x_H : H \in \mathcal{A}\}$ . It follows that  $G/\gamma_3(G)$  is the quotient of the free, 2-step nilpotent group  $F_n/\gamma_3(F_n)$  by the normal subgroup generated by all commutation relations of the form

$$(7.5) \quad \left[ x_H, \prod_{\substack{K \in \mathcal{A} \\ K \supset X}} x_K \right],$$

indexed by all pairs of hyperplanes  $H \in \mathcal{A}$  and flats  $X \in L_2(\mathcal{A})$  such that  $H \supset X$ . From this description, it is apparent that the second nilpotent quotient of an arrangement group is combinatorially determined; that is, if  $L_2(\mathcal{A}) \cong L_2(\mathcal{B})$ , then  $G(\mathcal{A})/\gamma_3(G(\mathcal{A})) \cong G(\mathcal{B})/\gamma_3(G(\mathcal{B}))$ .

### 7.3. LCS and Chen ranks of Milnor fibers

Let  $(\mathcal{A}, \mathbf{m})$  be a multi-arrangement, with complement  $M = M(\mathcal{A})$ . Let  $F_{\mathbf{m}} = F_{\mathbf{m}}(\mathcal{A})$  be the Milnor fiber and let  $h: F_{\mathbf{m}} \rightarrow F_{\mathbf{m}}$  be the monodromy of the corresponding Milnor fibration.

Denoting by  $G = \pi_1(M)$  and  $K = \pi_1(F_{\mathbf{m}})$  the fundamental groups of the respective spaces, we have a (split) exact sequence,  $1 \rightarrow K \rightarrow G \rightarrow \mathbb{Z} \rightarrow 1$ , so that the arrangement group splits as the semidirect product  $G = K \rtimes_{\varphi} \mathbb{Z}$ , where  $\varphi = h_{\sharp} \in \text{Aut}(K)$  is the automorphism of  $K = \pi_1(F_{\mathbf{m}})$  induced by  $h$ . Note that  $\varphi_{\text{ab}}: K_{\text{ab}} \rightarrow K_{\text{ab}}$  may be identified with the (integral) algebraic monodromy,  $h_*: H_1(F_{\mathbf{m}}; \mathbb{Z}) \rightarrow H_1(F_{\mathbf{m}}; \mathbb{Z})$ .

**THEOREM 7.1.** *Suppose  $h_*: H_1(F_{\mathbf{m}}; \mathbb{Z}) \rightarrow H_1(F_{\mathbf{m}}; \mathbb{Z})$  is the identity map. We then have the following isomorphisms of graded Lie algebras.*

1.  $\text{gr}(G) \cong \text{gr}(K) \rtimes_{\bar{\varphi}} \mathbb{Z}$ , where  $\bar{\varphi}: \mathbb{Z} \rightarrow \text{Der}(\text{gr}(K))$  is the morphism of Lie algebras induced by the homomorphism  $\varphi: \mathbb{Z} \rightarrow \text{Aut}(K)$  sending 1 to  $h_{\sharp}$ .
2.  $\text{gr}_{\geq 2}(K) \cong \text{gr}_{\geq 2}(G)$ .
3.  $\text{gr}_{\geq 2}(K/K'') \cong \text{gr}_{\geq 2}(G/G'')$ .

*Proof.* Part (1) follows from a well-known result of Falk and Randell [27, Theorem 3.1], as refined in [68, Corollary 6.7]. Part (2) is a direct consequence of part (1). Finally, part (3) follows from [69, Corollary 8.10].  $\square$

**THEOREM 7.2.** *Suppose  $h_*: H_1(F_{\mathbf{m}}; \mathbb{Q}) \rightarrow H_1(F_{\mathbf{m}}; \mathbb{Q})$  is the identity map. We then have the following isomorphisms of graded Lie algebras.*

1.  $\text{gr}(G) \otimes \mathbb{Q} \cong (\text{gr}(K) \rtimes_{\bar{\varphi}} \mathbb{Z}) \otimes \mathbb{Q}$ .
2.  $\text{gr}_{\geq 2}(K) \otimes \mathbb{Q} \cong \text{gr}_{\geq 2}(G) \otimes \mathbb{Q}$ .
3.  $\text{gr}_{\geq 2}(K/K'') \otimes \mathbb{Q} \cong \text{gr}_{\geq 2}(G/G'') \otimes \mathbb{Q}$ .

Consequently,  $\phi_k(\pi_1(F_{\mathbf{m}})) = \phi_k(\pi_1(M))$  and  $\theta_k(\pi_1(F_{\mathbf{m}})) = \theta_k(\pi_1(M))$  for all  $k \geq 2$ .

*Proof.* Parts (1) and (2) follow from Proposition 7.5 and Theorem 9.5 from [68], while part (3) follows from [69, Corollary 8.10]. The equality between the respective LCS and Chen ranks follows at once from parts (2) and (3).  $\square$

Consequently, if the algebraic monodromy  $h_*: H_1(F_{\mathbf{m}}; \mathbb{Q}) \rightarrow H_1(F_{\mathbf{m}}; \mathbb{Q})$  is trivial, then both the LCS ranks and the Chen ranks of  $\pi_1(F_{\mathbf{m}})$  are determined by  $L_{\leq 2}(\mathcal{A})$ . Moreover, letting  $U = \mathbb{P}(M)$ , we have that  $\phi_k(\pi_1(F_{\mathbf{m}})) = \phi_k(\pi_1(U))$  and  $\theta_k(\pi_1(F_{\mathbf{m}})) = \theta_k(\pi_1(U))$  for all  $k \geq 1$ .

## 8. CONSTRUCTIONS OF ARRANGEMENTS WITH TRIVIAL ALGEBRAIC MONODROMY

In this section, we describe several classes of hyperplane arrangements for which the Milnor fibration has trivial algebraic monodromy (in some range).

### 8.1. Boolean arrangements

Arguably the simplest kind of arrangements are the *Boolean arrangements*,  $\mathcal{B}_n$ , consisting of the coordinate hyperplanes  $\{z_i = 0\}$  in  $\mathbb{C}^n$ . The intersection lattice  $L(\mathcal{B}_n)$  is the Boolean lattice of subsets of  $\{0, 1\}^n$ , while the complement  $M(\mathcal{B}_n)$  is the complex algebraic torus  $(\mathbb{C}^*)^n$ .

Given a multiplicity function  $\mathbf{m}: \mathcal{B}_n \rightarrow \mathbb{N}$ , the map  $f_{\mathbf{m}}: (\mathbb{C}^*)^n \rightarrow \mathbb{C}^*$ ,  $z \mapsto z_1^{m_1} \cdots z_n^{m_n}$  is a morphism of complex algebraic groups. Hence, the Milnor fiber  $F_{\mathbf{m}} = \ker(f_{\mathbf{m}})$  is an algebraic subgroup, realized as the disjoint union of  $\gcd(\mathbf{m})$  copies of  $(\mathbb{C}^*)^{n-1}$ , with the monodromy automorphism,  $h: F_{\mathbf{m}} \rightarrow F_{\mathbf{m}}$ , permuting those copies in a circular fashion.

Now suppose  $\gcd(\mathbf{m}) = 1$ . Then  $F_{\mathbf{m}}$  is an algebraic  $(n - 1)$ -torus and  $h$  is isotopic to the identity, through the isotopy  $h_t(z) = e^{2\pi i t/N} z$ . Thus, the bundle  $F_{\mathbf{m}} \rightarrow M(\mathcal{B}_n) \rightarrow \mathbb{C}^*$  is trivial, and the algebraic monodromy,  $h_*: H_*(F_{\mathbf{m}}; \mathbb{Z}) \rightarrow H_*(F_{\mathbf{m}}; \mathbb{Z})$ , is equal to the identity map. Consequently, the characteristic polynomial of the algebraic monodromy is given by  $\Delta_q(t) = (t - 1)^{\binom{n-1}{q}}$  for  $0 < q < n$ .

### 8.2. Generic arrangements

Let  $\mathcal{A}$  be a central arrangement of  $n$  hyperplanes in  $\mathbb{C}^{d+1}$ , where  $n > d + 1 > 2$ . We say  $\mathcal{A}$  is *generic* if the intersection of every subset of  $d + 1$  distinct hyperplanes is the origin, in which case,  $\mathcal{A}$  is the cone over an affine, general position arrangement  $\mathcal{A}'$  of  $n - 1$  hyperplanes in  $\mathbb{C}^d$ , see [45, 47].

By a classical result of Hattori ([29, Theorem 1]), the complement of  $\mathcal{A}'$  is homotopy equivalent to the  $d$ -skeleton of the real,  $(n - 1)$ -dimensional torus  $T^{n-1}$ . Since  $U(\mathcal{A}) \cong M(\mathcal{A}')$ , it follows that  $\pi_1(U(\mathcal{A})) = \mathbb{Z}^{n-1}$  and  $b_q(U(\mathcal{A})) = \binom{n-1}{q}$  for  $q \leq d$ . Moreover, if  $\rho: \pi_1(U(\mathcal{A})) \rightarrow \mathbb{C}^*$  is a non-trivial character, then [29, Theorem 4] insures that  $H_q(U(\mathcal{A}); \mathbb{C}_\rho) = 0$  for  $q \neq d$  and  $\dim_{\mathbb{C}} H_d(U(\mathcal{A}); \mathbb{C}_\rho) = \binom{n-2}{d}$ . It follows that the characteristic varieties of  $U(\mathcal{A})$  are given by

$$(8.1) \quad \mathcal{V}_s^q(U(\mathcal{A})) = \begin{cases} \{\mathbf{1}\} & \text{for } q < d \text{ and } 1 \leq s \leq \binom{n-1}{q}, \\ \mathbb{C}^{n-1} & \text{for } q = d \text{ and } 1 \leq s \leq \binom{n-2}{d} \end{cases}$$

and are empty otherwise.

Now let  $\mathbf{m}: \mathcal{A} \rightarrow \mathbb{N}$  be a choice of multiplicities, and let  $F_{\mathbf{m}}$  be the corresponding Milnor fiber. Applying formula (3.4), we find that  $b_q(F_{\mathbf{m}}) = \binom{n-1}{q}$  for  $q \leq d-1$  and  $b_d(F_{\mathbf{m}}) = \binom{n-1}{d} + (n-1)\binom{n-2}{d}$ . Consequently, the algebraic monodromy  $h_q: H_q(F_{\mathbf{m}}; \mathbb{Q}) \rightarrow H_q(F_{\mathbf{m}}; \mathbb{Q})$  is equal to the identity if  $q < d$ , and the characteristic polynomial of  $h_q$  takes the form

$$(8.2) \quad \Delta_q(t) = \begin{cases} (t-1)^{\binom{n-1}{q}} & \text{if } q \leq d-1, \\ (t-1)^{\binom{n-2}{d-1}}(t^n-1)^{\binom{n-2}{d}} & \text{if } q = d. \end{cases}$$

In the case when  $F_{\mathbf{m}} = F(\mathcal{A})$  is the usual Milnor fiber, this recovers a result of Orlik and Randell [45] (see also [47, 12]).

### 8.3. Decomposable arrangements

Recall from Section 2.2 that every flat  $X \in L_2(\mathcal{A})$  gives rise to a “localized” sub-arrangement,  $\mathcal{A}_X$ , which consists of all hyperplanes  $H \in \mathcal{A}$  that contain  $X$ . Furthermore, the inclusions  $\mathcal{A}_X \subset \mathcal{A}$  yield inclusions of complements,  $j_X: M(\mathcal{A}) \hookrightarrow M(\mathcal{A}_X)$ , which assemble into a map

$$(8.3) \quad j = (j_X): M \longrightarrow \prod_{X \in L_2(\mathcal{A})} M(\mathcal{A}_X).$$

Let  $j_{\sharp}: G(\mathcal{A}) \rightarrow \prod_{X \in L_2(\mathcal{A})} G(\mathcal{A}_X)$  be the induced homomorphism on fundamental groups. It was shown in [24, 49] that the morphism

$$(8.4) \quad \text{gr}(j_{\sharp}): \text{gr}(G(\mathcal{A})) \longrightarrow \prod_{X \in L_2(\mathcal{A})} \text{gr}(G(\mathcal{A}_X))$$

between the respective associated graded Lie algebras is an isomorphism in degree 2 and, after tensoring with  $\mathbb{Q}$ , becomes surjective in all degrees greater than 2. Since each of the groups  $G(\mathcal{A}_X)$  is isomorphic to  $F_{\mu(X)} \times \mathbb{Z}$ , it follows that the LCS ranks of  $G(\mathcal{A})$  admit the lower bounds

$$(8.5) \quad \phi_k(G(\mathcal{A})) \geq \sum_{X \in L_2(\mathcal{A})} \phi_k(F_{\mu(X)})$$

for all  $k \geq 2$ , with equality for  $k = 2$ .

Following [49], we say that a hyperplane arrangement  $\mathcal{A}$  is *decomposable* (over  $\mathbb{Q}$ ) if the third LCS rank of the group  $G(\mathcal{A})$  attains the lower bound from (8.5); that is,

$$(8.6) \quad \phi_3(G(\mathcal{A})) = \sum_{X \in L_2(\mathcal{A})} \binom{\mu(X)}{2}.$$



It is shown in [49] that once this condition is satisfied, equality is attained in (8.5) for all  $k \geq 2$ ; in fact, the morphism  $\mathrm{gr}(j_{\#}) \otimes \mathbb{Q}$  restricts to an isomorphism of graded Lie algebras in degrees  $\geq 2$ .

More generally, let  $\mathfrak{h}(\mathcal{A})$  be the holonomy Lie algebra of  $\mathcal{A}$ , that is, the quotient of the free Lie algebra on generators  $\{x_H : H \in \mathcal{A}\}$  by the ideal generated by the Lie brackets of the form

$$(8.7) \quad \left[ x_H, \sum_{\substack{K \in \mathcal{A} \\ K \supset X}} x_K \right],$$

for all hyperplanes  $H \in \mathcal{A}$  and 2-flats  $X \in L_2(\mathcal{A})$  such that  $H \supset X$ . There is then an epimorphism  $\mathfrak{h}(\mathcal{A}) \rightarrow \mathrm{gr}(G(\mathcal{A}))$  that becomes an isomorphism upon tensoring with  $\mathbb{Q}$  (due to the 1-formality of the arrangement group). The arrangement  $\mathcal{A}$  is said to be decomposable over  $\mathbb{k}$  (where  $\mathbb{k}$  is either  $\mathbb{Z}$  or a field) if  $\mathfrak{h}_3(\mathcal{A}) \otimes \mathbb{k}$  decomposes as the direct sum  $\bigoplus_{X \in L_2(\mathcal{A})} \mathfrak{h}_3(\mathcal{A}_X) \otimes \mathbb{k}$ . It is shown in [49] that once this condition is satisfied, a similar decomposition holds in all degrees  $k \geq 2$ . Furthermore, the following is shown in [55, Theorem 8.8]: If  $\mathcal{A}$  is decomposable over  $\mathbb{Z}$ , then all the nilpotent quotients  $G(\mathcal{A})/\gamma_k(G(\mathcal{A}))$  are determined by  $L_{\leq 2}(\mathcal{A})$ . The same proof works if  $\mathcal{A}$  is decomposable over  $\mathbb{Q}$ , with the nilpotent quotients replaced by their rationalizations.

Let  $B(\mathcal{A}) = G(\mathcal{A})'/G(\mathcal{A})''$  be the Alexander invariant of an arrangement  $\mathcal{A}$ , viewed as module over the group ring  $\mathbb{Z}[G(\mathcal{A})_{\mathrm{ab}}]$ , and endowed with the filtration by the powers of the augmentation ideal. An in-depth study of the Alexander invariant and of the Milnor fibrations of a decomposable arrangement is done in [70]. We record in the next theorem one of the main results of this study.

**THEOREM 8.1** ([70]). *Let  $\mathcal{A}$  be an arrangement of rank 3 or higher. Suppose  $\mathcal{A}$  is decomposable over  $\mathbb{Q}$  and  $B(\mathcal{A}) \otimes \mathbb{Q}$  is separated in the  $I$ -adic topology. Then, for any choice of multiplicities  $\mathbf{m}: \mathcal{A} \rightarrow \mathbb{N}$ , the algebraic monodromy of the Milnor fibration,  $h_*: H_1(F_{\mathbf{m}}; \mathbb{Q}) \rightarrow H_1(F_{\mathbf{m}}; \mathbb{Q})$ , is trivial.*

A large supply of decomposable arrangements may be constructed by taking suitable sections of products of (central) arrangements in  $\mathbb{C}^2$ . For such an arrangement  $\mathcal{A}$ , the group  $G(\mathcal{A})$  is a finite direct product of finitely generated free groups (see [8] for a detailed study of such arrangements). We shall encounter two concrete examples of arrangements from this class in Section 9.

In general, though, there are decomposable arrangements for which the arrangement group is much more complicated. For instance, let  $\mathcal{A}$  be the arrangement in  $\mathbb{C}^3$  defined by the polynomial  $f = xyz(x+y)(x-z)(2z+y)$ . It is readily checked that  $\mathcal{A}$  is decomposable (over  $\mathbb{Z}$ ). Nevertheless, the group

$G(\mathcal{A})$  does not even have a finite-dimensional classifying space  $K(G(\mathcal{A}), 1)$ , see [63, Remark 12.4].

### 8.4. Multiplicity conditions

If  $F = F(\mathcal{A})$  is the Milnor fiber of a central arrangement  $\mathcal{A}$  in  $\mathbb{C}^{d+1}$ ,  $d > 1$ , there are various combinatorial conditions insuring that the algebraic monodromy  $h_*: H_1(F; \mathbb{k}) \rightarrow H_1(F; \mathbb{k})$  over  $\mathbb{k} = \mathbb{Z}$  or  $\mathbb{k}$  a field is the identity, such as the ones given in [10, 33, 74, 75, 39].

In [74], Williams gave a very nice combinatorial upper bound on the first Betti number of  $F$  and a criterion for triviality of the algebraic monodromy over  $\mathbb{Z}$ , stated in the case when  $\mathcal{A}$  is the complexification of a real arrangement. A partial generalization was obtained in [75], and the result was recently proved by Liu and Xie [39] in full generality. We summarize these results, as follows.

**THEOREM 8.2** ([74, 75, 39]). *Let  $\mathcal{A}$  be a central arrangement of  $n$  hyperplanes. For each hyperplane  $H \in \mathcal{A}$ , set*

$$s_H = \sum_{\substack{X \in L_2(\mathcal{A}) \\ X \subset H}} (q_X - 2)(\gcd(q_X, n) - 1),$$

where  $q_X = |\mathcal{A}_X|$ . Then,

1.  $\dim_{\mathbb{k}} H_1(F; \mathbb{k}) \leq n - 1 + \min \{s_H : H \in \mathcal{A}\}$ , for all fields  $\mathbb{k}$ .
2.  $\Delta_1(t) = (t - 1)^{n-1}p(t)$ , for some  $p(t) \in \mathbb{C}[t]$  dividing the polynomials

$$\left( \frac{t^{\gcd(q_X, n)} - 1}{t - 1} \right)^{q_X - 2}$$

for all  $X \in L_2(\mathcal{A})$ .

3. If there is a hyperplane  $H \in \mathcal{A}$  such that  $\gcd(q_X, n) = 1$  for all 2-flats  $X$  with  $q_X > 2$  (for instance, if  $n$  is a prime), then  $H_1(F; \mathbb{Z}) = \mathbb{Z}^{n-1}$ .

### 8.5. The double point graph

Let  $\mathcal{A}$  be a central arrangement of planes in  $\mathbb{C}^3$ , and let  $\bar{\mathcal{A}} = \mathbb{P}(\mathcal{A})$  be the corresponding arrangement of projective lines in  $\mathbb{C}P^2$ . The *double point graph* associated to  $\mathcal{A}$  is the graph  $\Gamma$  with vertex set  $\mathcal{A}$  and with an edge joining two hyperplanes  $H, K \in \mathcal{A}$  if  $\bar{H} \cap \bar{K}$  is a double point (see [4, 58]). The components of  $\Gamma$  define a partition of  $\mathcal{A}$  which is a refinement of all partitions induced by multinets on  $\mathcal{A}$ .

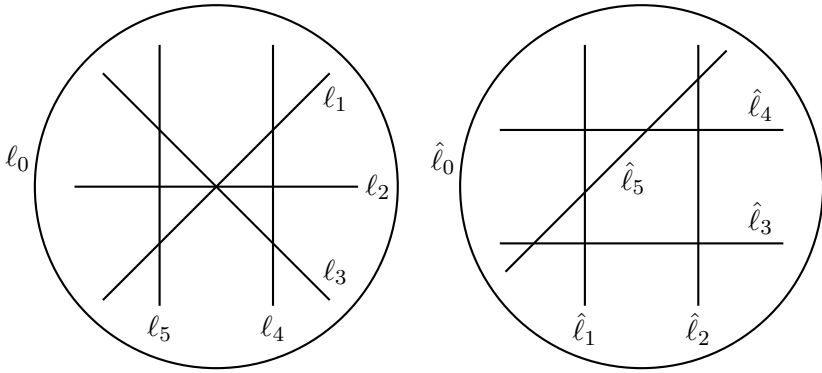


Figure 2 – The Falk arrangements  $\mathcal{A}$  and  $\hat{\mathcal{A}}$ .

Now suppose  $\Gamma$  is connected. Using results from [54], Bailet showed in [4] that the algebraic monodromy of the Milnor fibration,  $h_*: H_1(F; \mathbb{C}) \rightarrow H_1(F; \mathbb{C})$ , is the identity map, provided  $|\mathcal{A}_X| \leq 9$  for all  $X \in L_2(\mathcal{A})$  and either  $6 \nmid |\mathcal{A}|$ , or there exists a hyperplane  $H \in A$  such that  $|\mathcal{A}_X| \neq 6$ , for all  $X \subset H$ . Under the same connectivity assumption on  $\Gamma$ , Salvetti and Serventi [58] show that  $\mathcal{A}$  admits no multinet. Furthermore, they show that  $h_* = \text{id}$  if  $\Gamma$  admits a “good” spanning tree, and conjecture that this holds for arbitrary connected graphs. In [73], Venturelli establishes this conjecture under the assumption that  $\hat{\mathcal{A}}$  has two multiple points,  $P_1$  and  $P_2$ , such that every line in  $\hat{\mathcal{A}}$  passes through either  $P_1$  or  $P_2$ ; in [70], we give another proof of this result, in a more general setting.

## 9. THE FALK ARRANGEMENTS

### 9.1. A pair of arrangements and their complements

In this section, we analyze in detail a pair of hyperplane arrangements introduced by Falk in [25] and further studied in [65]. The two arrangements,  $\mathcal{A}$  and  $\hat{\mathcal{A}}$ , are central arrangements of 6 planes in  $\mathbb{C}^3$ , defined by the polynomials

$$(9.1) \quad \begin{aligned} f &= z(x - y)y(x + y)(x - z)(x + z), \\ \hat{f} &= z(x + z)(x - z)(y + z)(y - z)(x - y + z). \end{aligned}$$

The projectivizations of  $\mathcal{A}$  and  $\hat{\mathcal{A}}$  are depicted in Figure 2; the numbering of the lines corresponds to the ordering of the linear factors in the respective defining polynomials. Both  $\mathbb{P}(\mathcal{A})$  and  $\mathbb{P}(\hat{\mathcal{A}})$  have 2 triple points and 9 double points, yet the two intersection lattices are non-isomorphic: the two triple points of  $\mathbb{P}(\mathcal{A})$  do not lie on a common line, whereas the triple points of  $\mathbb{P}(\hat{\mathcal{A}})$

lie on a common line (namely,  $\hat{\ell}_0$ ). Nevertheless, as shown by Falk in [25], the two projective complements,  $U = \mathbb{P}(M)$  and  $\hat{U} = \mathbb{P}(\hat{M})$ , are homotopy equivalent. Let us note that  $\mathbb{P}(\hat{\mathcal{A}})$  has a line (namely,  $\hat{\ell}_5$ ) in general position with the others. A well-known result of Oka and Sakamoto [44] then implies that  $\pi_1(\hat{U})$  splits off a  $\mathbb{Z}$  factor; it easily follows that both groups are isomorphic to  $F_2 \times F_2 \times \mathbb{Z}$ .

The cohomology rings  $A = H^*(U; \mathbb{Z})$  and  $\hat{A} = H^*(\hat{U}; \mathbb{Z})$  are the quotients of the exterior algebra  $E = \bigwedge(e_1, \dots, e_5)$  by the ideals  $I = (\partial e_{123}, e_{45})$  and  $\hat{I} = (e_{12}, e_{34})$ , respectively. The automorphism  $E \xrightarrow{\sim} E$  given by  $e_1 \mapsto e_1 - e_3$ ,  $e_2 \mapsto e_2 - e_3$ ,  $e_3 \mapsto e_4$ ,  $e_4 \mapsto e_5$ , and  $e_5 \mapsto e_1 + e_2 + e_3$  induces an isomorphism  $\hat{A} \xrightarrow{\sim} A$ . It is readily verified that the only multinets supported on subarrangements of either  $\mathcal{A}$  or  $\hat{\mathcal{A}}$  are those coming from the triple points, and that the respective characteristic varieties are given by

$$(9.2) \quad \begin{aligned} \mathcal{V}_1^1(U) &= \{t_1 t_2 t_3 = t_4 = t_5 = 1\} \cup \{t_1 = t_2 = t_3 = 1\}, \\ \mathcal{V}_1^1(\hat{U}) &= \{t_3 = t_4 = t_5 = 1\} \cup \{t_1 = t_2 = t_5 = 1\}. \end{aligned}$$

## 9.2. The Milnor fibers of the Falk arrangements

Let  $F = F(\mathcal{A})$  and  $\hat{F} = F(\hat{\mathcal{A}})$  be the fibers of the Milnor fibrations  $f: M \rightarrow \mathbb{C}^*$  and  $\hat{f}: \hat{M} \rightarrow \mathbb{C}^*$ . Since both  $\mathbb{P}(\mathcal{A})$  and  $\mathbb{P}(\hat{\mathcal{A}})$  have only double and triple points, and since neither of the two arrangements supports a 3-net, Theorem 4.3 shows that the characteristic polynomial of the algebraic monodromy acting on either  $H_1(F; \mathbb{Q})$  or  $H_1(\hat{F}; \mathbb{Q})$  is  $(t-1)^5$ . Alternatively, it is easily verified that both arrangements are decomposable (over  $\mathbb{Z}$ ); therefore, Theorem 8.1 shows once again that their algebraic monodromy is trivial in degree 1. It now follows from Corollary 6.2 that both  $F$  and  $\hat{F}$  are formal spaces.

Since  $\hat{\mathcal{A}}$  contains a line meeting the other ones only in double points, Theorem 8.2, part (3) implies that  $H_1(\hat{F}; \mathbb{Z}) = \mathbb{Z}^5$ . Direct computation shows that  $H_1(F; \mathbb{Z}) = \mathbb{Z}^5$ , too, and so the monodromy action on both these groups is trivial. Moreover, both Milnor fibers have Euler characteristic  $6 \cdot 4 = 24$ , and thus  $H_2(F; \mathbb{Z}) = H_2(\hat{F}; \mathbb{Z}) = \mathbb{Z}^{28}$ . Let  $\zeta$  be a primitive 6th root of unity, and let  $\mathcal{H}_k$  be the  $\zeta^k$ -eigenspace of the monodromy action on  $H_2(F; \mathbb{C})$ . Then, by [12], we have that  $\dim_{\mathbb{C}} \mathcal{H}_k = 4$  for  $1 \leq k \leq 5$  and  $\dim_{\mathbb{C}} \mathcal{H}_0 = 8$ .

Let  $K = \pi_1(F)$  and  $\hat{K} = \pi_1(\hat{F})$  be the fundamental groups of the two Milnor fibers and let  $G = \pi_1(M) \cong \pi_1(\hat{M})$ . Applying Theorem 7.1, we find that the associated graded Lie algebras, respectively, the Chen Lie algebras of

all these groups are isomorphic in degrees 2 and more:

$$(9.3) \quad \begin{aligned} \text{gr}_{\geq 2}(K) &\cong \text{gr}_{\geq 2}(\hat{K}) \cong \text{gr}_{\geq 2}(G), \\ \text{gr}_{\geq 2}(K/K'') &\cong \text{gr}_{\geq 2}(\hat{K}/\hat{K}'') \cong \text{gr}_{\geq 2}(G/G''). \end{aligned}$$

From the discussion in Section 9.1, we have that  $G \cong F_2^2 \times \mathbb{Z}^2$ . Therefore, all the LCS quotients and Chen groups of  $K$  and  $\hat{K}$  are torsion-free, with ranks in degrees  $k \geq 2$  given by

$$(9.4) \quad \begin{aligned} \phi_k(K) &= \phi_k(\hat{K}) = \frac{2}{k} \sum_{d|k} \mu(d) 2^{k/d}, \\ \theta_k(K) &= \theta_k(\hat{K}) = 2(k - 1). \end{aligned}$$

Although all these homological and group-theoretic invariants of  $F$  and  $\hat{F}$  agree, the two Milnor fibers are *not* homotopy equivalent, as the next result shows.

**PROPOSITION 9.1.** *Let  $F$  and  $\hat{F}$  be the Milnor fibers of the two Falk arrangements, and let  $K$  and  $\hat{K}$  be their fundamental groups. Then,*

1.  $K/K'' \not\cong \hat{K}/\hat{K}''$ .
  2.  $K/\gamma_3(K) \not\cong \hat{K}/\gamma_3(\hat{K})$ .
- Consequently,  $\pi_1(F) \not\cong \pi_1(\hat{F})$ .

A proof of this proposition will be given in the next two subsections.

### 9.3. The characteristic varieties of $F$ and $\hat{F}$

The (degree 1) characteristic varieties of the Milnor fibers of the two Falk arrangements were first computed in [65]. Since that computation was based on a machine calculation, we redo it here by hand, using a method which works for any arrangement with trivial algebraic monodromy in degree 1.

We start with the Milnor fiber  $F = F(\mathcal{A})$ . As remarked above,  $H_1(F; \mathbb{Z}) = \mathbb{Z}^5$ . The inclusion map  $\iota: F \rightarrow M$  induces a morphism  $\iota^*: H^1(M; \mathbb{C}^*) \rightarrow H^1(F; \mathbb{C}^*)$  on character tori, given in coordinates by

$$(9.5) \quad \iota^*(z_1, \dots, z_6) = (z_1/z_2, z_2/z_3, z_3/z_4, z_4/z_5, z_5/z_6).$$

It follows from Theorem 5.7, part (2b), that the first characteristic variety  $\mathcal{V}_1^1(F) \subset H^1(F; \mathbb{C}^*)$  is the image under the map  $\iota^*$  of  $\mathcal{V}_1^1(M) \subset H^1(M; \mathbb{C}^*)$ . Therefore,

$$\begin{aligned} \mathcal{V}_1^1(F) &= \iota^* \left( \left\{ (z_1, z_2, \frac{1}{z_1 z_2}, 1, 1, 1) \mid z_1, z_2 \in \mathbb{C}^* \right\} \cup \right. \\ &\quad \left. \iota^* \left( \left\{ (1, 1, 1, z_4, z_5, \frac{1}{z_4 z_5}) \mid z_4, z_5 \in \mathbb{C}^* \right\} \right) \right) \end{aligned}$$

$$= \left\{ \left( \frac{z_1}{z_2}, z_1 z_2^2, \frac{1}{z_1 z_2}, 1, 1 \right) \mid z_1, z_2 \in \mathbb{C}^* \right\} \cup \left\{ \left( 1, 1, \frac{1}{z_4}, \frac{z_4}{z_5}, z_4 z_5^2 \right) \mid z_4, z_5 \in \mathbb{C}^* \right\},$$

and so  $\mathcal{V}_1^1(F) \subset (\mathbb{C}^*)^5$  is the union of two 2-dimensional subtori,  $T_1 = \{u \in (\mathbb{C}^*)^5 \mid u_1 u_2^2 u_3^3 = u_4 = u_5 = 1\}$  and  $T_2 = \{u \in (\mathbb{C}^*)^5 \mid u_1 = u_2 = u_3^3 u_4^2 u_5 = 1\}$ . Notice that

$$(9.6) \quad T_1 \cap T_2 = \{\mathbf{1}, (1, 1, \omega, 1, 1), (1, 1, \omega^2, 1, 1)\},$$

where  $\omega = \exp(2\pi i/3)$ . By Theorem 3.2, the torsion characters comprising  $T_1 \cap T_2$  lie in  $\mathcal{V}_2^1(F)$ . In fact, direct computation reveals that  $\mathcal{V}_2^1(F) = T_1 \cap T_2$ .

Proceeding in the same manner with the Milnor fiber of the second Falk arrangement,  $\hat{F} = F(\hat{\mathcal{A}})$ , we obtain:

$$\begin{aligned} \mathcal{V}_1^1(\hat{F}) &= \iota^* \left( \left\{ \left( z_1, z_2, 1, 1, 1, \frac{1}{z_1 z_2} \right) \mid z_1, z_2 \in \mathbb{C}^* \right\} \right) \cup \\ &\quad \iota^* \left( \left\{ \left( 1, 1, z_3, z_4, 1, \frac{1}{z_3 z_4} \right) \mid z_3, z_4 \in \mathbb{C}^* \right\} \right) \\ &= \left\{ \left( \frac{z_1}{z_2}, z_2, 1, 1, z_1 z_2 \right) \mid z_1, z_2 \in \mathbb{C}^* \right\} \cup \left\{ \left( 1, \frac{1}{z_3}, \frac{z_3}{z_4}, z_4, z_3 z_4 \right) \mid z_3, z_4 \in \mathbb{C}^* \right\}, \end{aligned}$$

and so  $\mathcal{V}_1^1(\hat{F}) = \hat{T}_1 \cup \hat{T}_2$ , where  $\hat{T}_1 = \{u \in (\mathbb{C}^*)^5 \mid u_1 u_2^2 u_5^{-1} = u_3 = u_4 = 1\}$  and  $\hat{T}_2 = \{u \in (\mathbb{C}^*)^5 \mid u_1 = u_2 u_3 u_4 = u_2 u_4^{-1} u_5 = 1\}$ . Notice that these two subtori intersect only at the origin; in fact, direct computation shows that

$$(9.7) \quad \mathcal{V}_2^1(\hat{F}) = \hat{T}_1 \cap \hat{T}_2 = \{\mathbf{1}\}.$$

The above computations show that  $\mathcal{V}_2^1(F) \not\cong \mathcal{V}_2^1(\hat{F})$ : the first variety consists of 3 points, while the second consists of a single point. Finally, recall from Section 3.3 that the characteristic varieties  $\mathcal{V}_s^1(G)$  of a (finitely generated) group  $G$  depend only on its maximal metabelian quotient,  $G/G''$ . Therefore, we have shown that  $K/K'' \not\cong \hat{K}/\hat{K}''$ , thereby completing the proof of part (1) of Proposition 9.1.  $\square$

*Remark 9.2.* Since both Milnor fibers are formal, the tangent cones to their first characteristic varieties coincide with the first resonance varieties. Using either this observation, together with the computations from above, or Theorem 5.7, part (1), we find that

$$\begin{aligned} \mathcal{R}_1^1(F) &= \{x_1 + 2x_2 + 3x_3 = x_4 = x_5 = 0\} \cup \{x_1 = x_2 = 3x_3 + 2x_4 + x_5 = 0\}, \\ \mathcal{R}_1^1(\hat{F}) &= \{x_1 + 2x_2 - x_5 = x_3 = x_4 = 0\} \cup \\ &\quad \{x_1 = x_2 + x_3 + x_4 = x_2 - x_4 + x_5 = 0\}, \end{aligned}$$

while  $\mathcal{R}_2^1(F) = \mathcal{R}_2^1(\hat{F}) = \{\mathbf{0}\}$ . Thus, the resonance varieties do not distinguish between  $\pi_1(F)$  and  $\pi_1(\hat{F})$ .

### 9.4. The second nilpotent quotients of $K$ and $\hat{K}$

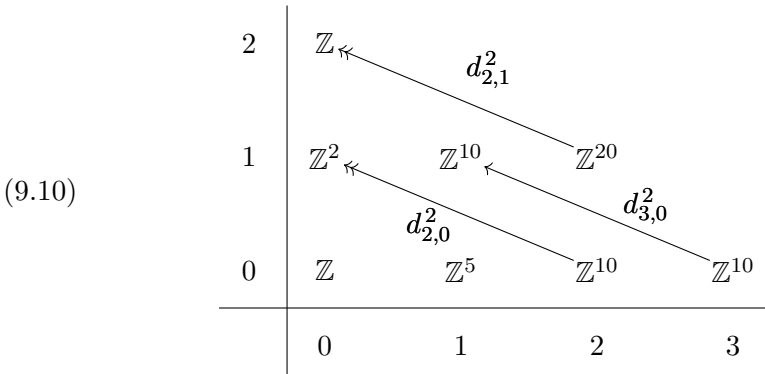
We now give a proof of Proposition 9.1, part (2). First consider the projectivized complement  $U = U(\mathcal{A})$  and its fundamental group,  $\bar{G} = \pi_1(U)$ . Recall that  $H^*(U; \mathbb{Z}) = E^*/I^*$ , where  $E = \bigwedge(e_1, \dots, e_5)$  and  $I = (\partial e_{123}, e_{45})$ . Writing  $E_r = (E^r)^\vee$  and  $I_r = (I^r)^\vee$  for the  $\mathbb{Z}$ -dual groups, the second nilpotent quotient  $\bar{G}/\gamma_3(\bar{G})$  is the central extension of  $\text{gr}_1(\bar{G}) = E_1 \cong \mathbb{Z}^5$  by  $\text{gr}_2(\bar{G}) = I_2 \cong \mathbb{Z}^2$  classified by the cocycle  $\chi_2: E_2 \rightarrow I_2$  given by the matrix

$$(9.8) \quad \chi_2^\top = \begin{pmatrix} & 12 & 13 & 23 & 14 & 24 & 34 & 15 & 25 & 35 & 45 \\ \begin{matrix} 1 \\ 0 \end{matrix} & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \begin{matrix} 0 \\ 1 \end{matrix} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

To compute the Schur multiplier  $H_2(\bar{G}/\gamma_3(\bar{G}); \mathbb{Z})$ , we use an approach similar to the one used in the proof of [55, Theorem 4.1]. Consider the homology spectral sequence of the central extension  $0 \rightarrow I_2 \rightarrow \bar{G}/\gamma_3(\bar{G}) \xrightarrow{\text{ab}} E_1 \rightarrow 0$ ,

$$(9.9) \quad E_{p,q}^2 = H_p(E_1; H_q(I_2; \mathbb{Z})) \Rightarrow H_{p+q}(\bar{G}/\gamma_3(\bar{G}); \mathbb{Z}).$$

Since the  $(E_2, d^2)$  page of the cohomology spectral sequence is a CDGA, and since its  $\mathbb{Z}$ -dual is  $(E^2, d_2)$ , due to lack of torsion on either of these two pages, the differentials  $d_2: E_{p,q}^2 \rightarrow E_{p-2,q+1}^2$  in diagram (9.10) are determined by the map  $d_{2,0}^2 = \chi_2$ .



Clearly,  $E_{2,0}^3 = \ker(d_{2,0}^2) = \mathbb{Z}^8$ . The differential  $d_{3,0}^2$  is dual to the composite  $E^1 \otimes I^2 \hookrightarrow E^1 \otimes E^2 \rightarrow E^3$ , whose kernel is generated by the elements  $u_1 = (e_1 - e_2) \otimes \partial e_{123}$ ,  $u_2 = (e_2 - e_3) \otimes \partial e_{123}$ ,  $u_3 = e_4 \otimes e_{45}$ , and  $u_4 = e_5 \otimes e_{45}$ . Taking transposes, we see that  $E_{1,1}^3 = \text{coker}(d_{3,0}^2)$  is equal to  $\mathbb{Z}^4$ , generated by the duals  $u_i^\vee$  of those elements (written in terms of the duals  $\varepsilon_i = e_i^\vee$ ). Finally, note that the map  $d_{2,1}^2: E_2 \otimes I_2 \rightarrow I_2 \wedge I_2$  is surjective, since it sends  $\partial \varepsilon_{123} \otimes \varepsilon_{45}$  to the generator  $\partial \varepsilon_{123} \wedge \varepsilon_{45}$  of  $I_2 \wedge I_2 = \mathbb{Z}$ ; hence,  $E_{0,2}^3 = 0$ . Looking at the

domains and ranges of the higher-order differentials in the spectral sequence, we see that  $E_{p,q}^3 = E_{p,q}^\infty$  for  $p + q \leq 2$ . Therefore,

$$(9.11) \quad H_2(\overline{G}/\gamma_3(\overline{G}); \mathbb{Z}) = E_{2,0}^3 \oplus E_{1,1}^3 = \mathbb{Z}^8 \oplus \mathbb{Z}^4 = \mathbb{Z}^{12}.$$

Consider next the Milnor fiber  $F = F(\mathcal{A})$ . The inclusion map  $\iota: F(\mathcal{A}) \hookrightarrow M(\mathcal{A})$  induces a monomorphism  $\iota_*: H_1(F; \mathbb{Z}) \hookrightarrow H_1(M; \mathbb{Z})$  given in suitable bases by  $\alpha_i \mapsto x_i - x_{i+1}$  for  $1 \leq i \leq 5$ . Letting  $\alpha_i = a_i^\vee$ , the ring morphism  $\sigma^*: H^*(U; \mathbb{Z}) \rightarrow H^*(F; \mathbb{Z})$  is given in degree 1 by  $e_1 \mapsto a_1 + a_5$ ,  $e_2 \mapsto -a_1 + a_2 + a_5$ ,  $e_3 \mapsto -a_2 + a_3 + a_5$ ,  $e_4 \mapsto -a_3 + a_4 + a_5$ ,  $e_5 \mapsto -a_4 + 2a_5$ . It follows that the group  $J^2 := \sigma^*(I^2)$  is free abelian, with basis  $\sigma^*(\partial e_{123}) = 3a_{12} - 2a_{13} + a_{23}$  and  $\sigma^*(e_{45}) = a_{34} - 2a_{35} + 3a_{45}$ .

The second nilpotent quotient of the group  $K = \pi_1(F)$  fits into the central extension  $0 \rightarrow J_2 \rightarrow K/\gamma_3(K) \xrightarrow{\text{ab}} H \rightarrow 0$ , where  $H = K_{\text{ab}} \cong \mathbb{Z}^5$  and  $J_2 = (J^2)^\vee \cong \mathbb{Z}^2$ . Furthermore, the extension is classified by the cocycle  $\chi_2: \bigwedge^2 H \rightarrow J_2$  given by the matrix

$$(9.12) \quad \chi_2^\top = \begin{pmatrix} & 12 & 13 & 23 & 14 & 24 & 34 & 15 & 25 & 35 & 45 \\ 3 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 3 \end{pmatrix}.$$

The spectral sequence of the extension has the same entries in the  $E^2$  page as in display (9.10). The differentials  $d_{2,0}^2$  and  $d_{2,1}^2$  are still surjective, giving  $E_{2,0}^3 = \mathbb{Z}^8$  and  $E_{0,2}^3 = 0$ . The difference, though, lies with the differential  $d_{3,0}^2$ : the elements  $\sigma^*(u_i)^\vee$  are still in  $\text{coker}(d_{3,0}^2)$ , generating a  $\mathbb{Z}^4$ -summand, but now there is an extra element of order 3 in that cokernel, namely, the element  $a_4 \otimes (3a_{12} - 2a_{13} + a_{23})$ . Therefore,  $E_{1,1}^3 = \mathbb{Z}^4 \oplus \mathbb{Z}_3$ . Proceeding as before, we find that  $H_2(K/\gamma_3(K); \mathbb{Z}) = \mathbb{Z}^{12} \oplus \mathbb{Z}_3$ .

For the group  $\hat{K} = \pi_1(\hat{F})$ , an entirely similar computation shows that the cokernel of  $d_{3,0}^2$  is equal to  $\mathbb{Z}^4$ , and hence  $H_2(\hat{K}/\gamma_3(\hat{K}); \mathbb{Z}) = \mathbb{Z}^{12}$ . Therefore,  $K/\gamma_3(K) \not\cong \hat{K}/\gamma_3(\hat{K})$ , thereby completing the proof of Proposition 9.1, part (2).  $\square$

## 10. THE $B_3$ ARRANGEMENT AND ITS DELETION

### 10.1. The $B_3$ arrangement

Let  $\mathcal{A}$  be the rank-3 reflection arrangement of type  $B_3$ , defined by the polynomial

$$(10.1) \quad f = xyz(x - y)(x + y)(x - z)(x + z)(y - z)(y + z).$$



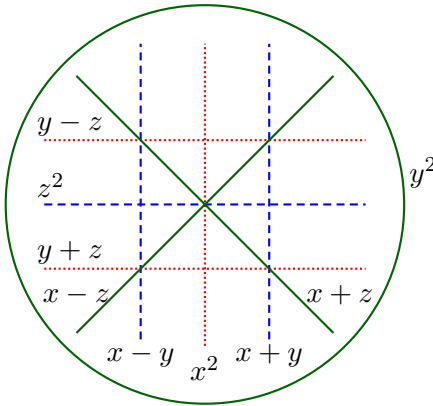


Figure 3 – The  $B_3$  reflection arrangement, with  $(3, 4)$ -multinet.

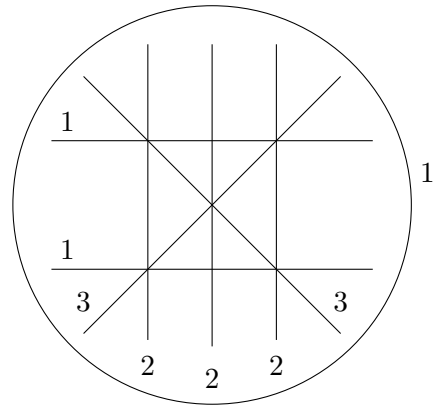


Figure 4 – The deleted  $B_3$  arrangement, with multiplicities.

Figure 3 shows a plane section of  $\mathcal{A}$ . The  $B_3$  arrangement is of fiber-type, with exponents  $\{1, 3, 5\}$ . Thus, the complement  $M = M(\mathcal{A})$  is aspherical and its projectivization,  $U = \mathbb{P}(M)$ , has fundamental group which decomposes as a semidirect product of free groups,  $\pi_1(U) = F_5 \rtimes_{\alpha} F_3$ . The braid monodromy algorithm from [13] shows that the monodromy map  $\alpha: F_3 \rightarrow \text{Aut}(F_5)$  takes values in the pure braid group  $P_5$ , viewed as a subgroup of  $\text{Aut}(F_5)$  via the Artin representation. Denoting by  $u_i$  the generators of  $F_3$  and by  $A_{ij}$  the standard generators of the pure braid group (corresponding to the meridians around the hyperplanes  $H_{ij}$  of the braid arrangement), the monodromy map is given by

$$(10.2) \quad \alpha(u_1) = A_{23}A_{24}A_{34}, \quad \alpha(u_2) = A_{14}^{A_{24}A_{34}}A_{25}, \quad \alpha(u_3) = A_{35}^{A_{23}A_{25}},$$

where  $a^b = b^{-1}ab$ , see [61, Example 10.8]. Since pure braid automorphisms act trivially in homology, the extension  $1 \rightarrow F_5 \rightarrow \pi_1(U) \rightarrow F_3 \rightarrow 1$  is ab-exact. Thus, by the aforementioned result of Falk and Randell [27], the LCS quotients  $\text{gr}_k(\pi_1(U))$  are isomorphic to  $\text{gr}_k(F_5) \oplus \text{gr}_k(F_3)$ , for all  $k \geq 1$ . Moreover, the Chen ranks are given by  $\theta_k(\pi_1(U)) = (k - 1)(3k + 19)$  for  $k \geq 4$ , see [60, 11],

We now turn to the cohomology jump loci of the  $B_3$  arrangement (see [15, Remark 6.4] and [28, Example 3.6]). Notably,  $\mathcal{A}$  supports a (non-reduced) multinet  $\mathcal{N}$ , depicted in Figure 3; ordering the hyperplanes as the factors of the defining polynomial (10.1), this multinet has associated partition  $(189|267|345)$ . The resonance variety  $\mathcal{R}_1^1(M) \subset H^1(M; \mathbb{C}) = \mathbb{C}^9$  has 7 local components, corresponding to the 4 triple points and 3 quadruple points, 11 components corresponding to braid sub-arrangements, and one essential, 2-dimensional component,  $P = P_{\mathcal{N}}$ . All the components of the characteristic variety  $\mathcal{V}_1^1(M) \subset$

$H^1(M; \mathbb{C}^*) = (\mathbb{C}^*)^9$  pass through the origin, and thus are obtained by exponentiating the linear subspaces comprising  $\mathcal{R}_1^1(M)$ . In particular, there is a single essential component,  $T = \exp(P)$ . More explicitly, the multinet  $\mathcal{N}$  determines a pencil,

$$(10.3) \quad \psi: M \longrightarrow S = \mathbb{CP}^1 \setminus \{[0 : 1], [1 : 0], [1 : 1]\},$$

which is given by  $\psi(x, y, z) = [x^2(y^2 - z^2) : y^2(x^2 - z^2)]$ . In turn, the induced homomorphism  $\psi^*: H^1(S; \mathbb{Z}) \rightarrow H^1(M; \mathbb{Z})$  is given by  $c_1^\vee \mapsto 2e_1 + e_8 + e_9$ ,  $c_2^\vee \mapsto 2e_2 + e_6 + e_7$ ,  $c_3^\vee \mapsto 2e_3 + e_4 + e_5$ , where  $c_i = [\gamma_i]$  are the homology classes of standard loops around the punctures of  $S$  (see Section 3.2). Hence,

$$(10.4) \quad T = \psi^*(H^1(S; \mathbb{C}^*)) = \{(t^2, s^2, (st)^{-2}, s, s, (st)^{-1}, (st)^{-1}, t, t) : s, t \in \mathbb{C}^*\}.$$

Finally, let  $F = F(\mathcal{A})$  be the Milnor fiber of the  $B_3$  arrangement; then none of the aforementioned components of  $\mathcal{V}_1^1(M)$  contributes to a jump in  $b_1(F)$ . In fact, as first shown in [12], the monodromy  $h: F \rightarrow F$  acts trivially on  $H_1(F; \mathbb{Q})$ ; analyzing more carefully that computation shows that  $h$  acts trivially on  $H_1(F; \mathbb{Z})$ . Applying Theorem 7.1, we conclude that  $\text{gr}_k(\pi_1(F)) \cong \text{gr}_k(F_5) \oplus \text{gr}_k(F_3)$ , and  $\theta_k(\pi_1(F)) = \theta_k(\pi_1(U))$  for all  $k \geq 1$ .

### 10.2. The deleted $B_3$ arrangement

Consider now the arrangement  $\mathcal{A}'$  obtained from  $\mathcal{A}$  by deleting the hyperplane  $\{z = 0\}$ , as shown in Figure 4. This is the *deleted  $B_3$  arrangement*, defined by the polynomial

$$(10.5) \quad f' = xy(x - y)(x + y)(x - z)(x + z)(y - z)(y + z).$$

This is again a fiber-type arrangement, with exponents  $\{1, 3, 4\}$ . Thus, the complement  $M' = M(\mathcal{A}')$  is aspherical and its projectivization,  $U' = \mathbb{P}(M')$ , has fundamental group  $\pi_1(U') = F_4 \rtimes_{\alpha'} F_3$ , where, as noted in [60, Example 10.6], the monodromy automorphism  $\alpha'$  is given by the pure braids  $A_{23}$ ,  $A_{13}^{A_{23}} A_{24}$ , and  $A_{14}^{A_{24}}$ .

The cohomology jump loci of  $M'$  were computed in [61]. Briefly, the resonance variety  $\mathcal{R}_1^1(M') \subset H^1(M'; \mathbb{C}) = \mathbb{C}^8$  contains 7 local components, corresponding to the 6 triple points and 1 quadruple point, and 5 non-local components, corresponding to braid sub-arrangements. In addition to the 12 subtori obtained by exponentiating these linear subspaces, the characteristic variety  $\mathcal{V}_1^1(M') \subset H^1(M'; \mathbb{C}^*) = (\mathbb{C}^*)^8$  also contains a component of the form  $\rho \cdot T'$ , where  $T'$  is a 1-dimensional algebraic subtorus and  $\rho$  is a root of unity

of order 2, given by

$$(10.6) \quad \begin{aligned} T' &= \{(t^2, t^{-2}, t^{-1}, t^{-1}, 1, 1, t, t) : t \in \mathbb{C}^*\}, \\ \rho &= (1, 1, -1, -1, -1, -1, 1, 1). \end{aligned}$$

As explained in [16, Example 5.7], this translated subtorus arises from the pencil  $\psi$  from (10.3), as follows. The point  $[0 : 1]$  is not in the image of  $\psi$ ; however, extending the domain of  $\psi$  to  $M' = M \cup \{z = 0\}$  defines a map

$$(10.7) \quad \psi' : M' \longrightarrow \mathbb{C}^* = \mathbb{CP}^1 \setminus \{[0 : 1], [1 : 0]\}.$$

Note that  $\psi'(x, y, 0) = [x^2y^2 : x^2y^2]$ , so the fiber over  $[1 : 1]$  has multiplicity 2. Therefore, we may view the map  $\psi' : M' \rightarrow (\mathbb{C}^*, (2))$  as an orbifold pencil, with one multiple fiber of multiplicity 2. The orbifold fundamental group  $\Gamma = \pi_1^{\text{orb}}(\mathbb{C}^*, (2))$  may be identified with the free product  $\mathbb{Z} * \mathbb{Z}_2$ , while the character group  $H^1(\Gamma; \mathbb{C}^*)$  may be identified with  $\mathbb{C}^* \times \{\pm 1\}$ . It follows from (3.6) that  $\mathcal{V}_1^1(\Gamma) = \mathbb{C}^* \times \{-1\}$ . The map  $\psi'$  induces an epimorphism  $\psi'_\# : \pi_1(M') \rightarrow \Gamma$ , which in turn induces a monomorphism  $(\psi'_\#)^* : H^1(\Gamma; \mathbb{C}^*) \hookrightarrow H^1(\pi_1(M'); \mathbb{C}^*)$ . The image of  $\mathcal{V}_1^1(\Gamma)$  under this morphism is precisely the translated torus  $\rho T' \subset \mathcal{V}_1^1(M')$ . Moreover, if we let  $j : M \hookrightarrow M'$  be the inclusion map between the respective complements, then the induced homomorphism,  $j^* : H^1(M'; \mathbb{C}^*) \hookrightarrow H^1(M; \mathbb{C}^*)$ , embeds  $\rho T'$  into the torus  $T$  from (10.4). In fact,  $T \cap \{t \in (\mathbb{C}^*)^9 : t_3 = 1\} = T' \cup \rho T'$ .

### 10.3. Milnor fibrations of the deleted $B_3$ arrangement

It follows from the above discussion that the deleted  $B_3$  arrangement  $\mathcal{A}'$  supports no essential, reduced multinet. It is readily verified that none of aforementioned components of  $\mathcal{V}_1^1(M')$  contributes to a jump in the first Betti number of  $F' = F(\mathcal{A}')$ . Direct computation shows that, in fact,  $H_1(F'; \mathbb{Z}) = \mathbb{Z}^8$ , and so the monodromy acts trivially on  $H_1(F'; \mathbb{Z})$ . For suitable choices of multiplicities, though, the Milnor fiber of the multi-arrangement acquires non-trivial 2-torsion. We treat in detail one such choice.

Let  $F'_m = F_m(\mathcal{A}')$  be the Milnor fiber of the multi-arrangement  $(\mathcal{A}', \mathbf{m})$  with multiplicity vector  $\mathbf{m} = (2, 1, 2, 2, 3, 3, 1, 1)$ . As noted in [9, 16], the monodromy of the Milnor fibration acts trivially on  $H_1(F'_m; \mathbb{Q})$ , but not on  $H_1(F'_m; \mathbb{Z})$ , which has torsion subgroup  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$  on which the monodromy acts as  $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ .

Let  $U' = U(\mathcal{A}')$ , and consider the pullback square on the right side of

the following diagram

$$(10.8) \quad \begin{array}{ccccc} F'_m & & & & \\ & \searrow \kappa & & & \\ & & \hat{U}' & \xrightarrow{\hat{\psi}'} & \hat{S} = (\mathbb{C}^*, (2, 2, 2)) \\ & \searrow \sigma_m & \downarrow \tau & & \downarrow \nu \\ & & U' & \xrightarrow{\psi'} & S = (\mathbb{C}^*, (2)). \end{array}$$

where  $\psi'$  is the (projectivized) orbifold pencil from Section 10.2 and  $\nu$  is the orbifold 3-fold cover corresponding to the epimorphism  $\pi_1^{\text{orb}}(S) = \mathbb{Z} * \mathbb{Z}_2 \twoheadrightarrow \mathbb{Z}_3$  that sends the (meridional) generator of  $\pi_1(\mathbb{C}^*) = \mathbb{Z}$  to 1 and the generator of  $\mathbb{Z}_2$  to 0. The orbifold fundamental group  $\Gamma = \pi_1^{\text{orb}}(\hat{S})$  is isomorphic to  $\mathbb{Z} * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ , and so  $\mathbb{T}_\Gamma = \mathbb{T}_\Gamma^0 \times \{(\pm 1, \pm 1, \pm 1)\}$ , where  $\mathbb{T}_\Gamma^0 = \mathbb{C}^*$ . It follows from (3.6) that

$$(10.9) \quad \begin{aligned} \mathcal{V}_1^1(\Gamma) &= \{\mathbf{1}\} \cup (\mathbb{T}_\Gamma \setminus \mathbb{T}_\Gamma^0), \\ \mathcal{V}_2^1(\Gamma) &= (1, -1, -1)\mathbb{T}_\Gamma^0 \cup (-1, 1, -1)\mathbb{T}_\Gamma^0 \cup (-1, -1, 1)\mathbb{T}_\Gamma^0 \cup \\ &\quad (-1, -1, -1)\mathbb{T}_\Gamma^0, \\ \mathcal{V}_3^1(\Gamma) &= (-1, -1, -1)\mathbb{T}_\Gamma^0. \end{aligned}$$

Moreover, the lift  $\hat{\psi}': \hat{U}' \rightarrow \hat{S}$  is again an orbifold pencil.

The  $\mathbb{Z}_{15}$ -cover  $\sigma_m: F'_m \rightarrow U'$  factors as the composite  $F'_m \xrightarrow{\kappa} \hat{U}' \xrightarrow{\tau} U'$ , where  $\kappa$  is a 5-fold cover. By Theorem 5.7, part (2a), the subvariety  $\mathcal{W}_1^1(F'_m)$  has 12 components passing through the identity of  $H^1(F'_m; \mathbb{C}^*)^0 = (\mathbb{C}^*)^7$ : eleven subtori of dimension 2 and one subtorus of dimension 3 (which in fact is a component of  $\mathcal{W}_2^1(F'_m)$ ), all obtained by pullback along  $\sigma_m$ . By Theorem 5.7, part (2b), there is also a 1-dimensional component of  $\mathcal{W}_1^1(F'_m)$  of the form  $\sigma_m^*(\rho T')$ , where  $\rho T'$  is the translated subtorus in  $\mathcal{V}_1^1(U')$  from (10.6). Pulling back along the map  $(\hat{\psi}' \circ \kappa)^*: H^1(\hat{S}; \mathbb{C}^*) \rightarrow H^1(F'_m; \mathbb{C}^*)$  the translated tori comprising  $\mathcal{V}_1^1(\Gamma)$  yields seven 1-dimensional components of  $\mathcal{V}_1^1(F'_m)$ , of the form  $\rho' \sigma_m^*(T')$ , for certain order 2 characters  $\rho'$ . Of those, 4 are also components of  $\mathcal{V}_2^1(F'_m)$ , while one of those, namely,

$$(\hat{\psi}' \circ \kappa)^*((-1, -1, -1)\mathbb{T}_\Gamma^0) = \sigma_m^*(\rho T'),$$

is the unique component of  $\mathcal{V}_3^1(F'_m)$ .

Finally, since  $\mathcal{A}'$  is fiber-type with exponents  $\{1, 3, 4\}$ , the lower central series quotients  $\text{gr}_k(\pi_1(U'))$  are isomorphic to  $\text{gr}_k(F_4) \oplus \text{gr}_k(F_3)$  for  $k \geq 2$ , while, by [60, 11], the Chen ranks  $\theta_k(\pi_1(U'))$  are equal to  $(k - 1)(k + 12)$  for  $k \geq 4$ . By Theorem 7.2, the group  $K = \pi_1(F'_m)$  has the same LCS and Chen ranks as  $\pi_1(U')$ . In fact, it can be shown that  $\text{gr}_k(K) \otimes \mathbb{Z}_p \cong \text{gr}_k(\pi_1(U')) \otimes \mathbb{Z}_p$

for all primes  $p \neq 2$ , and likewise for the Chen groups of  $K$ . Direct computation shows that the first few lower central series quotients of  $K$  and  $K/K''$  are as in the following table.

$k$	1	2	3	4	5
$\text{gr}_k(K)$	$\mathbb{Z}^7 \oplus \mathbb{Z}_2^2$	$\mathbb{Z}^9 \oplus \mathbb{Z}_2^5$	$\mathbb{Z}^{28} \oplus \mathbb{Z}_2^{15}$	$\mathbb{Z}^{78} \oplus \mathbb{Z}_2^{41}$	$\mathbb{Z}^{252} \oplus \mathbb{Z}_2^{117}$
$\text{gr}_k(K/K'')$	$\mathbb{Z}^7 \oplus \mathbb{Z}_2^2$	$\mathbb{Z}^9 \oplus \mathbb{Z}_2^5$	$\mathbb{Z}^{28} \oplus \mathbb{Z}_2^{15}$	$\mathbb{Z}^{48} \oplus \mathbb{Z}_2^?$	$\mathbb{Z}^{68} \oplus \mathbb{Z}_2^?$

## 11. YOSHINAGA'S ICOSIDODECAHEDRAL ARRANGEMENT

In this final section, we describe an arrangement, introduced by Yoshinaga in [76], which exhibits 2-torsion in the first homology of its (usual) Milnor fiber.

### 11.1. Mod-2 Betti numbers of 2-fold covers

Before proceeding with the example, we return to the general setup for computing the homology of finite abelian covers treated in Section 3.4, approached this time from a different angle.

Let  $p: Y \rightarrow X$  be a regular  $\mathbb{Z}_N$ -cover, classified by a homomorphism  $\alpha: \pi_1(X) \rightarrow \mathbb{Z}_N$ . Alternatively, we may view  $\alpha$  as a cohomology class in  $H^1(X; \mathbb{Z}_N)$ , called the characteristic class of the cover. The covering space  $Y = X^\alpha$  is connected if and only if the homomorphism  $\alpha$  is surjective, in which case  $\pi_1(Y) = \ker(\alpha)$ . In the case when  $N = 2$ , more can be said. The next two results were first proved in [76] and then strengthened in [66].

LEMMA 11.1 ([76, 66]). *Let  $p: Y \rightarrow X$  be a connected  $\mathbb{Z}_2$ -cover, with characteristic class  $\alpha \in H^1(X; \mathbb{Z}_2)$ . Then  $p$  lifts to a connected, regular  $\mathbb{Z}_4$ -cover  $\bar{p}: \bar{Y} \rightarrow X$  if and only if  $\alpha^2 = 0$ .*

PROPOSITION 11.2 ([76, 66]). *Let  $p: Y \rightarrow X$  be a 2-fold cover, classified by a non-zero class  $\alpha \in H^1(X; \mathbb{Z}_2)$ . Suppose that  $\alpha^2 = 0$ . Then, for all  $q \geq 1$ ,*

$$(11.1) \quad b_q(Y, \mathbb{Z}_2) = b_q(X, \mathbb{Z}_2) + \dim_{\mathbb{Z}_2} H^q(H^*(X; \mathbb{Z}_2), \delta_\alpha),$$

where the differential  $\delta_\alpha: H^*(X; \mathbb{Z}_2) \rightarrow H^{*+1}(X; \mathbb{Z}_2)$  is given by  $\delta_\alpha(u) = \alpha u$ . In particular,  $b_q(Y, \mathbb{Z}_2) \geq b_q(X, \mathbb{Z}_2)$ .

Further work on the integral homology groups of double covers, and how this homology relates to the homology with coefficients in rank 1 integral local systems on the base of the cover can be found in [59, 36, 38].

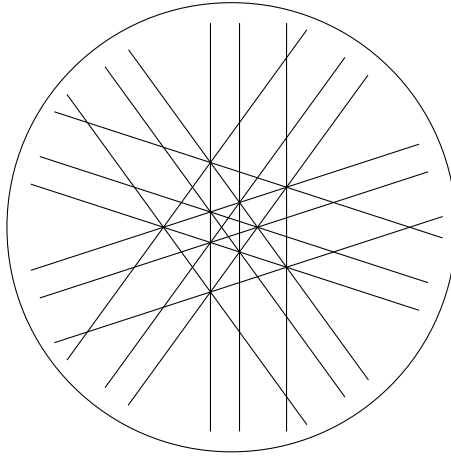


Figure 5 – The icosidodecahedral arrangement.

### 11.2. Modular inequalities

Once again, let  $Y \rightarrow X$  be a connected  $\mathbb{Z}_2$ -cover with characteristic class  $\alpha \in H^1(X; \mathbb{Z}_2)$ . Assuming  $H_*(X; \mathbb{Z})$  is torsion-free, it follows from [53, Theorem C] that

$$(11.2) \quad b_q(Y) \leq b_q(X) + \dim_{\mathbb{Z}_2} H^q(H^*(X; \mathbb{Z}_2), \delta_\alpha).$$

When  $U = U(\mathcal{A})$  is the projectivized complement of a hyperplane arrangement  $\mathcal{A}$ , an explicit formula was proposed in [54, Conjecture 1.9], expressing the first Betti number  $b_1(F)$  of the Milnor fiber of the arrangement in terms of the resonance varieties  $\mathcal{R}_s^1(U, \mathbb{Z}_p)$ , for  $p = 2$  and  $3$ , generalizing the formula from Theorem 4.3. At the prime  $p = 2$ , the conjecture is equivalent to the inequality (11.2) holding as equality in degree  $q = 1$  for the 2-fold cover  $U^\alpha \rightarrow U$  corresponding to the class  $\alpha \in H^1(U; \mathbb{Z}_2)$  which evaluates to 1 on each meridional generator of  $H_1(U; \mathbb{Z}_2)$ .

In recent work [31], Ishibashi, Sugawara, and Yoshinaga revisit this topic. In [31, Corollary 2.5], they prove that equality holds in (11.2) if and only if  $H_1(Y; \mathbb{Z})$  has no non-trivial 2-torsion. Therefore, the formula conjectured in [54] fails at the prime  $p = 2$  precisely when  $H_1(U^\alpha; \mathbb{Z})$  has non-trivial 2-torsion. An explicit example where this happens is given next.

### 11.3. The icosidodecahedral arrangement

In [76], Yoshinaga constructed an arrangement of 16 hyperplanes in  $\mathbb{C}^3$  with some remarkable properties. The construction is based on the symmetries

of a polyhedron in  $\mathbb{R}^3$ , called the *icosidodecahedron*. This is a quasiregular polyhedron with 20 triangular and 12 pentagonal faces that has 30 vertices (each one at the intersection of 2 triangles and 2 pentagons), and 60 edges (each one separating a triangle from a pentagon). Letting  $\phi = (1 + \sqrt{5})/2$  denote the golden ratio, the vertices of an icosidodecahedron with edges of unit length are given by the even permutations of  $(0, 0, \pm 1)$  and  $\frac{1}{2}(\pm 1, \pm \phi, \pm \phi^2)$ .

One can choose 10 edges to form a decagon, corresponding to great circles in the spherical tiling; there are 6 ways to choose these decagons, thereby giving 6 planes. Each pentagonal face has five diagonals, and there are 60 such diagonals in all, which partition in 10 disjoint sets of coplanar ones, thereby giving 10 planes, each containing 6 diagonals. These 16 planes form an arrangement  $\mathcal{A}_{\mathbb{R}}$  in  $\mathbb{R}^3$ , whose complexification is the icosidodecahedral arrangement  $\mathcal{A}$  depicted in Figure 5.

The projective line arrangement  $\mathbb{P}(\mathcal{A})$  has 15 quadruple points and 30 double points. The projective complement  $U = U(\mathcal{A})$  is aspherical [35], and has Poincaré polynomial  $P(t) = 1 + 15t + 60t^2$ . Let  $F = F(\mathcal{A})$  be the Milnor fiber of is arrangement. As shown in [76], we have that  $H_1(F; \mathbb{Z}) = \mathbb{Z}^{15} \oplus \mathbb{Z}_2$ . Thus, the algebraic monodromy of the Milnor fibration is trivial over  $\mathbb{Q}$ , but not over  $\mathbb{Z}$ .

Since the monodromy of the Milnor fibration acts trivially on  $H_1(F; \mathbb{k})$  for every field  $\mathbb{k}$  of characteristic different from 2, the results of [68] show that  $\text{gr}(\pi_1(F)) \otimes \mathbb{k} \cong \text{gr}(\pi_1(U)) \otimes \mathbb{k}$  for such fields  $\mathbb{k}$ . Direct computation shows that the first few lower central series quotients of the group  $K = \pi_1(F)$  and of its maximal metabelian quotient are given by

$k$	1	2	3	4
$\text{gr}_k(K)$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{45} \oplus \mathbb{Z}_2^7$	$\mathbb{Z}^{250} \oplus \mathbb{Z}_2^{43}$	$\mathbb{Z}^{1,405} \oplus T$
$\text{gr}_k(K/K'')$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{45} \oplus \mathbb{Z}_2^7$	$\mathbb{Z}^{250} \oplus \mathbb{Z}_2^{43}$	$\mathbb{Z}^{530} \oplus \bar{T}$

where  $T$  is a finite abelian 2-group and  $\bar{T}$  is a quotient of  $T$ .

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