Dedicated to Laurenţiu Păunescu and Alexandru Suciu on their 70th anniversary

q-DEFORMATION OF AOMOTO COMPLEX

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A degree one element of the Orlik–Solomon algebra of a hyperplane arrangement defines a cochain complex known as the Aomoto complex. The Aomoto complex can be considered as the "linear approximation" of the twisted cochain complex with coefficients in a complex rank one local system.

In this paper, we discuss q-deformations of the Aomoto complex. The q-deformation is defined by replacing the entries of representation matrices of the coboundary maps with their q-analogues. While the resulting maps do not generally define cochain complexes, for certain special basis derived from real structures, the q-deformation becomes again a cochain complex. Moreover, it exhibits universality in the sense that any specialization of q to a complex number yields the cochain complex computing the corresponding local system cohomology group.

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1. INTRODUCTION

Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^\ell$. We denote the complement by $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H\in\mathcal{A}} H$. These spaces hold significant importance in topology and various areas of mathematics. One of the central themes in the study of hyperplane arrangements is the relationship between the topological structures of $M(\mathcal{A})$ and the combinatorial structures of $\mathcal{A}$.

The cohomology ring of $M(\mathcal{A})$ has an expression as the Orlik–Solomon algebra ([9]) which is defined in a purely combinatorial way by the intersection poset $L(\mathcal{A})$ of $\mathcal{A}$. In some cases, the local system cohomology groups and Betti numbers of the covering spaces are also described in combinatorial way. Nevertheless, these problems remain still open in general (see Section 3 for further details).

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In this paper, we focus on complex rank 1 local systems. One of the key notions in exploring local systems is the *Aomoto complex* $(A^\bullet_{\mathbb{Z}}(\mathcal{A}), \omega)$, originally introduced in the context of higher-dimensional generalizations of hypergeometric integrals [1]. The Aomoto complex is defined by using the Orlik–Solomon algebra, hence it is described by the intersection poset. It is known that the cohomology of the Aomoto complex approximates the local system cohomology groups, namely, it can be considered as the "linearlized version" of local system cochain complex.

The purpose of this paper is to appropriately deform the Aomoto complex in such a way that it recovers arbitrary local system cohomology groups as its specializations ("universality"). The idea is to construct the universal cochain complex as a "q-deformation of the Aomoto complex" (Figure 1) which:

- is a cochain complex over the ring of Laurent polynomials invariant under the involution $\mathcal{Z} = \mathbb{Z}[q^{1/2}, q^{-1/2}]^{\langle \iota \rangle}$ (see Section 2 for details),
- reverts to the Aomoto complex as $q \to 1$,
- can compute the local system cohomology group by specializing q to a complex number $q_0 \neq 0, 1$.

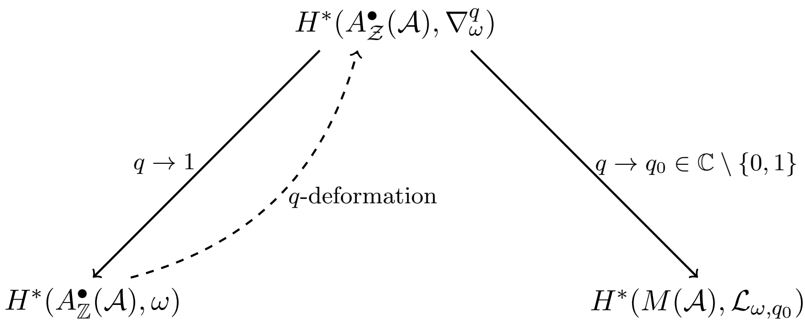

Figure 1 – q-deformation and specializations of Aomoto complex.

The paper is organized as follows. In Section 2, we recall the notion of q-integers and the relations among them. In Section 3, we recall basic facts on hyperplane arrangements and motivating problems related to local system cohomology groups and covering spaces. In Section 4, we consider q-deformations of the Aomoto complex. First, we see that a naive q-deformation does not maintain being a cochain complex in general. Therefore, we need to choose carefully the basis of the Aomoto complex. We observe that, for some bases, the canonical q-deformation becomes again a cochain complex. Then, we formulate the main result that complexified real line arrangements have an ideal basis such that the canonical q-deformation becomes the universal cochain complex. The proof relies on previous works. In Section 5, we exhibit examples. We compute local system cohomology groups for certain local systems on the deleted B_3-arrangement. We also discuss further problems. See [3] for another type of deformation.

2. q-INTEGERS

2.1. q-analogue of integers

The Laurent polynomial in the variable $q^{1/2}$

$$[n]_q := \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = q^{\frac{n-1}{2}} + q^{\frac{n-3}{2}} + \cdots + q^{\frac{1-n}{2}}$$

is called the *q-analogue of an integer* $n \in \mathbb{Z}$ (or the *canonical q-deformation* of n). (For instance, $[0]_q = 0$, $[1]_q = 1$, $[2]_q = q^{\frac{1}{2}} + q^{-\frac{1}{2}}$, $[3]_q = q^1 + 1 + q^{-1}$, $[4]_q = q^{\frac{3}{2}} + q^{\frac{1}{2}} + q^{-\frac{1}{2}} + q^{-\frac{3}{2}}$). Note that the assignment $n \longmapsto [n]_q$ is not a homomorphism.

Define the involution $\iota : \mathbb{Z}[q^{1/2}, q^{-1/2}] \to \mathbb{Z}[q^{1/2}, q^{-1/2}]$ by $\iota(q^{1/2}) = q^{-1/2}$. Clearly, $[n]_q$ is ι-invariant, hence $[n]_q \in \mathbb{Z}[q^{1/2}, q^{-1/2}]^{\langle \iota \rangle}$. Conversely, we have the following.

Proposition 2.1. *The invariant submodule $\mathbb{Z}[q^{1/2}, q^{-1/2}]^{\langle \iota \rangle}$ is freely generated by $[n]_q$ $(n \in \mathbb{Z}_{>0})$ as $\mathbb{Z}$-module, hence*

$$\mathbb{Z}[q^{1/2}, q^{-1/2}]^{\langle \iota \rangle} = \bigoplus_{n=1}^{\infty} \mathbb{Z} \cdot [n]_q. \tag{1}$$

2.2. Clebsch–Gordan relation

Proposition 2.2. *The q-analogues of integers satisfy the following.*

1. *For any $n \in \mathbb{Z}$,*

$$[-n]_q = -[n]_q. \tag{2}$$

2. *Let $m, n \in \mathbb{Z}$. Suppose $n > 0$. Then*

$$[m]_q[n]_q = [m+n-1]_q + [m+n-3]_q + \cdots + [m-n+1]_q. \tag{3}$$

The second assertion is called the Clebsch–Gordan relation. Since it is elementary, we omit the proof.

Remark 2.3. Let us briefly describe the relation between the formula (3) and representations of the quantum group $U_q(\mathfrak{sl}_2)$. First, $U_q(\mathfrak{sl}_2)$ is defined, as the $\mathbb{C}$-algebra, generated by the four variables E, F, K, K^{-1} with the relations ([5, Definition VI. 1.1])

$$\begin{aligned} &KK^{-1} = K^{-1}K = 1, \\ &KEK^{-1} = q^2E,\ KFK^{-1} = q^{-2}F, \\ &EF - FE = \frac{K - K^{-1}}{q - q^{-1}}. \end{aligned}$$

There exists an $(n+1)$-dimensional irreducible representation $\rho_n : U_q(\mathfrak{sl}_2) \longrightarrow \mathrm{End}(V_n)$ which is called the highest weight representation with the highest weight q^n ([5, Theorem VI. 3.5]). The trace of $\rho_n(K)$ can be easily computed as ([5, p. 129])

$$\mathrm{Tr}(\rho_n(K)) = q^n + q^{n-2} + \cdots + q^{-n} = [n+1]_{q^2}.$$

The tensor products of these representations are not irreducible. The irreducible decomposition is described by the following Clebsch–Gordan formula ([5, Theorem VII. 7.2])

$$V_n \otimes V_m \simeq V_{n+m} \oplus V_{n+m-2} \oplus \cdots \oplus V_{n-m}, \tag{4}$$

for $n \geq m \geq 0$. By comparing the trace of K for both sides, we obtain the formula (3).

Next, we prove that relations among q-integers are generated by the relations (2) and (3) together with $[0]_q = 0, [1]_q = 1$. More precisely, we have the following.

Proposition 2.4. *Let $\mathbb{Z}[x_n; n \in \mathbb{Z}]$ be the polynomial ring generated by $x_n (n \in \mathbb{Z})$, and let I be the ideal generated by the following elements.*

$$\begin{cases} \text{(i) } x_0, \\ \text{(ii) } x_1 - 1, \\ \text{(iii) } x_n + x_{-n}, \quad (n \in \mathbb{Z}) \\ \text{(iv) } x_m x_n - (x_{m+n-1} + x_{m+n-3} + \cdots + x_{m-n+1}), \quad (m, n \in \mathbb{Z}, m \geq n > 0). \end{cases}$$

Then, the assignment $\varphi(x_n) = [n]_q$ induces an isomorphism of $\mathbb{Z}$-algebras

$$\varphi : \mathbb{Z}[x_n; n \in \mathbb{Z}]/I \xrightarrow{\simeq} \mathbb{Z}[q^{1/2}, q^{-1/2}]^{\langle \iota \rangle}.$$

Proof. By Proposition 2.2 and Proposition 2.1, the map $\varphi : \mathbb{Z}[x_n; n \in \mathbb{Z}]/I \to \mathbb{Z}[q^{1/2}, q^{-1/2}]^{\langle \iota \rangle}$ is a well-defined surjective homomorphism.

It remains to show that φ is injective. Suppose $g(x_i) \in \mathrm{Ker}(\varphi)$. Using the generators of the ideal I, we may assume that $g(x_i)$ is of the form

$$g(x_i) = \sum_{i\geq 2} a_i x_i + b,$$

$a_i, b \in \mathbb{Z}$ (a_i is zero except for finitely many i). By the assumption, we have $\sum_{i\geq 2} a_i[i]_q + b = 0$ in $\mathbb{Z}[q^{1/2}, q^{-1/2}]^{\langle\iota\rangle}$. Since $[n]_q$ ($n \geq 1$) are linearly independent over $\mathbb{Z}$, we have $a_i = b = 0$. $\square$

Denote the algebra in Proposition 2.4 by

$$\mathcal{Z} := \mathbb{Z}[x_n; n \in \mathbb{Z}]/I \simeq \mathbb{Z}[q^{1/2}, q^{-1/2}]^{\langle\iota\rangle}. \tag{5}$$

There is a natural ring homomorphism

$$\gamma : \mathcal{Z} \longrightarrow \mathbb{Z} \tag{6}$$

defined by $x_n \longmapsto n$, which is equivalent to $q \longmapsto 1$.

Definition 2.5. 1. A *q-deformation* of an integer $n \in \mathbb{Z}$ is an element $\widetilde{n} \in \mathcal{Z}$ such that $\gamma(\widetilde{n}) = n$.

2. Among others, the q-deformation $n \longmapsto n \cdot [1]_q$ is called the *trivial q-deformation*, and $n \longmapsto [n]_q$ is called the *canonical q-deformation.*

3. A q-deformation of an integer matrix $A = (a_{ij})_{ij} \in M_{m,n}(\mathbb{Z})$ is a matrix $\widetilde{A} = (\widetilde{a_{ij}})_{ij} \in M_{m,n}(\mathcal{Z})$ such that $\gamma(\widetilde{a_{ij}}) = a_{ij}$.

4. A q-deformation of a homomorphism $f : M_1 \longrightarrow M_2$ of $\mathbb{Z}$-modules $M_i (i = 1, 2)$ is a homomorphsim $\widetilde{f} : \widetilde{M_1} \longrightarrow \widetilde{M_2}$ of $\mathcal{Z}$-modules together with isomorphisms $\widetilde{M_i} \otimes_{\mathcal{Z}} \mathbb{Z} \simeq M_i (i = 1, 2)$ which commute the following diagram.

$$\begin{array}{ccc} \widetilde{M_1} \otimes_{\mathcal{Z}} \mathbb{Z} & \xrightarrow{\widetilde{f} \otimes_{\mathcal{Z}} \mathbb{Z}} & \widetilde{M_2} \otimes_{\mathcal{Z}} \mathbb{Z} \\ \simeq \downarrow & & \downarrow \simeq \\ M_1 & \xrightarrow[f]{} & M_2 \end{array}$$

5. Let $f : \mathbb{Z}^m \longrightarrow \mathbb{Z}^n$ be a homomorphism of free abelian groups represented by a matrix $A = (a_{ij})_{ij}$. The *canonical deformation* of f is $\widetilde{f} : \mathcal{Z}^m \longrightarrow \mathcal{Z}^n$ defined by the matrix $\widetilde{A} = ([a_{ij}]_q)_{ij}$.

Remark 2.6. There are many q-deformations other than the trivial and the canonical deformations, e.g., $n \longmapsto [n-2]_q + [2]_q$.

3. PROBLEMS RELATED TO LOCAL SYSTEM COHOMOLOGY GROUPS

3.1. Hyperplane arrangements and Aomoto complex

Let $V = \mathbb{C}^\ell$. An arrangement of hyperplanes is a set $\mathcal{A} = \{H_1, \dots, H_n\}$ consisting of finitely many affine hyperplanes in V. We denote its complement by $M(\mathcal{A}) = \mathbb{C}^\ell \smallsetminus \bigcup_{H\in\mathcal{A}} H$. Let α_H be a defining linear equation of the hyperplane H. Then, the first cohomology group $H^1(M(\mathcal{A}), \mathbb{Z})$ is a free abelian group generated by

$$e_i = \frac{1}{2\pi\sqrt{-1}} d\log \alpha_{H_i} \in H^1(M(\mathcal{A}), \mathbb{Z}).$$

Dually, the first homology group $H_1(M(\mathcal{A}), \mathbb{Z})$ is a free abelian group generated by the meridian cycles γ_i of each hyperplane H_i.

The cohomology ring $H^*(M(\mathcal{A}), \mathbb{Z})$ is generated by $e_1, \dots, e_n$. Furthermore, the following presentation (by Orlik and Solomon) is known [9]. Let $E = \bigoplus_{i=1}^n \mathbb{Z}e_i$ be the free abelian group generated by $e_1, \dots, e_n$ and $\wedge E$ be its exterior algebra. There are two types of algebraic relations among $e_1, \dots, e_n$ in $H^*(M(\mathcal{A}), \mathbb{Z})$. The first one is

$$e_{i_1} e_{i_2} \dots e_{i_p} = 0, \tag{7}$$

for $H_{i_1} \cap \dots \cap H_{i_p} = \emptyset$. The second one is

$$\partial(e_{i_1} e_{i_2} \dots e_{i_p}) := \sum_{\alpha=1}^{p} (-1)^{\alpha-1} e_{i_1} \dots \widehat{e_{i_\alpha}} \dots e_{i_p} = 0, \tag{8}$$

for dependent subset $\{H_{i_1}, \dots, H_{i_p}\}$, namely, the subset of $\mathcal{A}$ that satisfies $(H_{i_1} \cap \dots \cap H_{i_p} \neq \emptyset)$ $\operatorname{codim} H_{i_1} \cap \dots \cap H_{i_p} < p$.

Let $I_\mathcal{A}$ be the graded ideal of the exterior algebra $\wedge E$ generated by

$$\begin{cases} e_{i_1} e_{i_2} \dots e_{i_p} & \text{for } H_{i_1} \cap \dots \cap H_{i_p} = \emptyset, \\ \partial(e_{i_1} e_{i_2} \dots e_{i_p}) & \text{for } \operatorname{codim} H_{i_1} \cap \dots \cap H_{i_p} < p. \end{cases}$$

Denote by $A^\bullet_\mathbb{Z}(\mathcal{A}) = E/I_\mathcal{A}$ the quotient, which is a graded algebra called the *Orlik–Solomon algebra* of $\mathcal{A}$. Note that the Orlik–Solomon algebra is defined by only using the poset structure of the non-empty intersections

$$L(\mathcal{A}) = \Big\{ \bigcap_{H\in\mathcal{B}} H \neq \emptyset \mid \mathcal{B} \subset \mathcal{A} \Big\}.$$

The Orlik–Solomon algebra is known to be isomorphic to the cohomology ring, namely, we have the algebra isomorphism

$$H^\bullet(M(\mathcal{A}), \mathbb{Z}) \simeq A^\bullet_\mathbb{Z}(\mathcal{A}).$$

Let $\omega = \sum_{i=1}^{n} a_i e_i \in A^1_{\mathbb{Z}}(\mathcal{A})$. Then, since $\omega \wedge \omega = 0$,

$$(A^{\bullet}_{\mathbb{Z}}(\mathcal{A}), \omega): \quad \cdots \xrightarrow{\omega\wedge} A^k_{\mathbb{Z}}(\mathcal{A}) \xrightarrow{\omega\wedge} A^{k+1}_{\mathbb{Z}}(\mathcal{A}) \xrightarrow{\omega\wedge} \cdots$$

forms a cochain complex, which is called the *Aomoto complex*.

3.2. **Local system cohomology group**

Let

$$\rho : \pi_1(M(\mathcal{A})) \longrightarrow \mathbb{C}^{\times}$$

be a group homomorphism. Since $\mathbb{C}^{\times}$ is an abelian group, the map ρ factors through the abelianization $H_1(M(\mathcal{A}), \mathbb{Z})$. Hence, ρ is determined by $q_i = \rho(\gamma_i) \in \mathbb{C}^{\times}$ $(i = 1, \dots, n)$, where γ_i is the meridian cycle around the hyperplane H_i. Denote the associated complex rank one local system by $\mathcal{L}_\rho$. The computation of the twisted cohomology group $H^k(M(\mathcal{A}), \mathcal{L}_\rho)$ is one of the central problems in the topology of hyperplane arrangements. The following problem is still open.

Problem 3.1. Can one describe $\dim H^k(M(\mathcal{A}), \mathcal{L}_\rho)$ in terms of the intersection poset $L(\mathcal{A})$ of $\mathcal{A}$ and $(q_1, \dots, q_n) \in (\mathbb{C}^{\times})^n$?

Roughly speaking, if ρ is close to the trivial local system (equivalently, $(q_1, \dots, q_n)$ is close to $(1, 1, \dots, 1)$ in $(\mathbb{C}^{\times})^n$), $H^k(M(\mathcal{A}), \mathcal{L}_\rho)$ can be described combinatorially. More precisely, suppose that $q_i = \exp(2\pi\sqrt{-1}\lambda_i)$, and let $\omega_\lambda = \lambda_1 e_1 + \cdots + \lambda_n e_n \in A^1_{\mathbb{C}}(\mathcal{A})$. Then, there exists the following open subset $(0, \dots, 0) \in U \subseteq \mathbb{C}^n$ such that if $\lambda = (\lambda_1, \dots, \lambda_n) \in U$, the Aomoto complex $(A^{\bullet}_{\mathbb{C}}(\mathcal{A}), \omega_\lambda)$ is quasi-isomorphic to the twisted de Rham complex ([2, 13]). In particular,

$$H^k(M(\mathcal{A}), \mathcal{L}_\rho) \simeq H^k(A^{\bullet}_{\mathbb{C}}(\mathcal{A}), \omega_\lambda),$$

which is also known as the tangent cone theorem [17].

However, it is also known that the Aomoto complex $(A^{\bullet}_{\mathbb{C}}(\mathcal{A}), \omega_\lambda)$ with complex coefficients cannot describe even $H^1(M(\mathcal{A}), \mathcal{L}_\rho)$ in general. In fact, the characteristic variety

$$V_1(\mathcal{A}) = \{(q_1, \dots, q_n) \in (\mathbb{C}^{\times})^n \mid \dim H^1(M(\mathcal{A}), \mathcal{L}_\rho) \geq 1\}$$

may have tosion translated components that cannot be detected by the Aomoto complex $(A^{\bullet}_{\mathbb{C}}(\mathcal{A}), \omega_\lambda)$ ([15]).

3.3. Motivating problems on covering spaces

One of the motivations to study twisted cohomology groups is the topology of covering spaces. Although the following problems are subcases of Problem 3.1, it is worth noting separately by their importance in relations to the topology of covering spaces.

Problem 3.2. Let $\rho : \pi_1(M(\mathcal{A})) \longrightarrow \{\pm 1\}$ be a surjective homomorphism. Can one describe $\dim H^k(M(\mathcal{A}), \mathcal{L}_\rho)$ in terms of $L(\mathcal{A})$ and ρ?

Problem 3.2 is related to double coverings as follows. The index 2 subgroup $\mathrm{Ker}(\rho) \subset \pi_1(M(\mathcal{A}))$ determines a double covering $p : M(\mathcal{A})^\rho \longrightarrow M(\mathcal{A})$. It is known that

$$H^k(M(\mathcal{A})^\rho, \mathbb{C}) \simeq H^k(M(\mathcal{A}), \mathbb{C}) \oplus H^k(M(\mathcal{A}), \mathcal{L}_\rho).$$

Hence, Problem 3.2 is equivalent to describe the Betti numbers of the double covering in terms of combinatorial information. Related topics are recently actively studied by several researchers [4, 6, 7, 8, 18, 19, 23, 24].

Problem 3.3. Define $\rho : \pi_1(M(\mathcal{A})) \longrightarrow \mathbb{C}^\times$ by $\rho(\gamma_i) = \exp\left(\frac{2\pi\sqrt{-1}}{n+1}\right)$. Can one describe $\dim H^k(M(\mathcal{A}), \mathcal{L}_{\rho^{\otimes i}})$ $(i = 0, 1, \dots, n)$ in terms of $L(\mathcal{A})$?

The kernel $\mathrm{Ker}(\rho)$ is a subgroup of $\pi_1(M(\mathcal{A}))$ of index $(n+1)$, and determines a $\mathbb{Z}_{n+1}$-cyclic covering $p : F \longrightarrow M(\mathcal{A})$. The space F is known to be homeomorphic to the Milnor fiber of the coning of $\mathcal{A}$. It is known that the cohomology group of the Milnor fiber is decomposed as

$$H^k(F, \mathbb{C}) \simeq \bigoplus_{i=0}^{n} H^k(M(\mathcal{A}), \mathcal{L}_{\rho^{\otimes i}}),$$

which is equivalent to the decomposition of the cohomology group into the monodromy eigenspaces. Hence, Problem 3.3 is equivalent to the combinatorial description of the Betti numbers (more precisely, the dimension of monodromy eigenspaces) of the Milnor fiber of (a central) hyperplane arrangement. See [16] for more on Milnor fibers.

Recently, Papadima and Suciu [10, 11] discovered that Problem 3.3 is deeply related to the Aomoto complex $(A^\bullet_{\mathbb{F}_q}(\mathcal{A}), \omega = e_1 + \cdots + e_n)$ over the finite field $\mathbb{F}_q$, where q is a prime power. Actually, if $3|(n+1)$, and the multiplicities of the intersections of corresponding projective lines are not contained in $\{6, 9, 12, \dots, 3n, \dots\}$, then, the dimension of the nontrivial part is

$$\sum_{i=1}^{n} \dim H^1(M(\mathcal{A}), \mathcal{L}_{\rho^{\otimes i}}) = 2 \cdot \dim_{\mathbb{F}_3} H^1(A^\bullet_{\mathbb{F}_3}(\mathcal{A}), e_1 + \cdots + e_n).$$

However, the rank of the cohomology group of the Aomoto complex over finite fields generally provides only an upper bound ([24]). Instead, in the next section, we will investigate the q-deformation of the integral Aomoto complex $A^\bullet_{\mathbb{Z}}(\mathcal{A})$.

4. MAIN RESULTS

4.1. *q*-deformable universal basis of Aomoto complex

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a line arrangement in $\mathbb{C}^2$. Our approach is to consider local system cohomology groups via q-deformations of the Aomoto complexes. Let $\omega = \sum_{i=1}^n a_i e_i \in A^1_{\mathbb{Z}}(\mathcal{A})$. Let $q_0 \in \mathbb{C}^\times \setminus \{1\}$. Define the homomorphism $\rho(\omega, q_0) : \pi_1(M(\mathcal{A})) \longmapsto \mathbb{C}^\times$ as

$$\rho(\omega, q_0)(\gamma_i) = q_0^{a_i}.$$

We denote the associated local system by $\mathcal{L}_{\rho(\omega,q_0)}$. The main result asserts that there exists a q-deformation of the Aomoto complex whose specialization to $q = q_0$ computes the local system cohomology group.

Let

$$\begin{cases} 1 & \in A^0_{\mathbb{Z}}(\mathcal{A}), \\ \eta_1, \dots, \eta_n & \in A^1_{\mathbb{Z}}(\mathcal{A}), \\ \theta_1, \dots, \theta_b & \in A^2_{\mathbb{Z}}(\mathcal{A}) \end{cases} \tag{9}$$

be a $\mathbb{Z}$-basis of the Aomoto complex ($b = \operatorname{rank} A^2_{\mathbb{Z}}(\mathcal{A})$). Let $\omega \in A^1_{\mathbb{Z}}(\mathcal{A})$. The coboundary map of the Aomoto complex is expressed as

$$\omega \wedge 1 = s_1\eta_1 + \cdots + s_n\eta_n$$
$$\omega \wedge \eta_i = t_{1i}\theta_1 + \cdots + t_{bi}\theta_b,$$

with $s_i, t_{ij} \in \mathbb{Z}$. Let us denote the coefficient matrices by $S(\omega) = (s_i)$ and $T(\omega) = (t_{ij})$. Since they are matrix representations of coboundary maps, we have the following relation

$$T(\omega)S(\omega) = \begin{pmatrix} t_{11} & t_{12} & \dots & t_{1n} \\ t_{21} & t_{22} & \dots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{b1} & t_{b2} & \dots & t_{bn} \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix} = 0. \tag{10}$$

Generally, the canonical q-deformation breaks the relation (10) (as seen in Example 4.2).

Definition 4.1. The basis (9) is called *canonically q-deformable* if

$$[T(\omega)]_q[S(\omega)]_q := \begin{pmatrix} [t_{11}]_q & [t_{12}]_q & \dots & [t_{1n}]_q \\ [t_{21}]_q & [t_{22}]_q & \dots & [t_{2n}]_q \\ \vdots & \vdots & \ddots & \vdots \\ [t_{b1}]_q & [t_{b2}]_q & \dots & [t_{bn}]_q \end{pmatrix} \begin{pmatrix} [s_1]_q \\ [s_2]_q \\ \vdots \\ [s_n]_q \end{pmatrix} = 0. \tag{11}$$

Then, we denote the q-deformed Aomoto complex

$$0 \longrightarrow A^0_{\mathcal{Z}}(\mathcal{A}) \xrightarrow{[S(\omega)]_q} A^1_{\mathcal{Z}}(\mathcal{A}) \xrightarrow{[T(\omega)]_q} A^2_{\mathcal{Z}}(\mathcal{A}) \longrightarrow 0 \tag{12}$$

by $(A^\bullet_{\mathcal{Z}}(\mathcal{A}), \nabla^q_\omega)$.

Example 4.2. Let $\mathcal{A} = \{H_1, H_2, H_3\}$ be three lines as in Figure 2. Note that by the definition of the Orlik–Solomon algebra, $e_1e_3 = e_1e_2 + e_2e_3$. Let $\omega = e_1 + e_2 + e_3$ and consider the q-deformation of the Aomoto complex $(A^\bullet_{\mathbb{Z}}(\mathcal{A}), \omega)$.

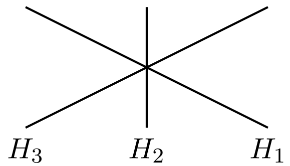

Figure 2 – $\mathcal{A} = \{H_1, H_2, H_3\}$.

1. Consider the basis $1 \in A^0_{\mathbb{Z}}(\mathcal{A}), e_1, e_2, e_3 \in A^1_{\mathbb{Z}}(\mathcal{A})$, and $e_1e_2, e_2e_3 \in A^2_{\mathbb{Z}}(\mathcal{A})$. Then, the coefficients matrices are

$$T(\omega) = \begin{pmatrix} -2 & 1 & 1 \\ -1 & -1 & 2 \end{pmatrix}, \text{ and } S(\omega) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Since $[2]_q \neq [1]_q + [1]_q$, we have $[T(\omega)]_q[S(\omega)]_q \neq 0$. Therefore, the basis is not canonically q-deformable.

2. Consider the basis $1 \in A^0_{\mathbb{Z}}(\mathcal{A}), \eta_1 = e_1, \eta_2 = e_1 - e_2, \eta_3 = e_2 - e_3 \in A^1_{\mathbb{Z}}(\mathcal{A})$, and $e_1e_2, e_2e_3 \in A^2_{\mathbb{Z}}(\mathcal{A})$. Then, the coefficients matrices are

$$T(\omega) = \begin{pmatrix} -2 & -3 & 0 \\ -1 & 0 & -3 \end{pmatrix}, \text{ and } S(\omega) = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}.$$

In this case, we have $[T(\omega)]_q[S(\omega)]_q = 0$. Actually, the basis is a canonically q-deformable basis.

Thus, the canonical q-deformability depends on the choice of the basis. We also make the following definition.

Definition 4.3. A canonically q-deformable basis (9) is called *universal* if the q-deformation computes the local system cohomology group. Namely, for any $\omega \in A^1_{\mathbb{Z}}(\mathcal{A})$ and $q_0 \in \mathbb{C}^\times \setminus \{1\}$, the cohomology

$$\frac{\mathrm{Ker}([T(\omega)]_{q_0} : A^1_{\mathbb{C}}(\mathcal{A}) \longrightarrow A^2_{\mathbb{C}}(\mathcal{A}))}{\mathrm{Im}([S(\omega)]_{q_0} : A^0_{\mathbb{C}}(\mathcal{A}) \longrightarrow A^1_{\mathbb{C}}(\mathcal{A}))}$$

is isomorphic to $H^1(M(\mathcal{A}), \mathcal{L}_{\rho(\omega,q_0)})$.

Our main result is the following.

THEOREM 4.4. *Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a complexified real line arrangement. Then, there exists a universal canonically q-deformable basis of the Aomoto complex $1 \in A^0_{\mathbb{Z}}(\mathcal{A}), \eta_1, \dots, \eta_n \in A^1_{\mathbb{Z}}(\mathcal{A}), \theta_1, \dots, \theta_b \in A^2_{\mathbb{Z}}(\mathcal{A})$.*

4.2. **Proof of Theorem 4.4**

The proof is based on previous results [21, 22, 20]. We first recall the general result [21, Theorem 4.1] concerning the relation between the Aomoto complex and the twisted cohomology group for complexified real arrangements.

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an affine hyperplane arrangement in $\mathbb{R}^\ell$. Let $M(\mathcal{A})$ be the complexified complement $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H \otimes \mathbb{C}$. We describe the twisted cohomology $H^k(M(\mathcal{A}), \mathcal{L})$ by using chambers.

Recall that a connected component of $\mathbb{R}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$ is called a *chamber*. Denote by $\mathrm{ch}(\mathcal{A})$ the set of all chambers. Let

$$\mathcal{F} : \ F^0 \subset F^1 \subset \cdots \subset F^\ell = \mathbb{R}^\ell$$

be a generic flag, that is, F^k is a generic k-dimensional affine subspace. For a given generic flag $\mathcal{F}$, we decompose the set of chambers. Let

$$\mathrm{ch}^k_{\mathcal{F}}(\mathcal{A}) = \{C \in \mathrm{ch}(\mathcal{A}) \mid C \cap F^k \neq \emptyset, C \cap F^{k-1} = \emptyset\},$$

for $0 \le k \le \ell$, where $F^{-1} = \emptyset$. Then, the set $\mathrm{ch}(\mathcal{A})$ of chambers is decomposed into

$$\mathrm{ch}(\mathcal{A}) = \mathrm{ch}^0_{\mathcal{F}}(\mathcal{A}) \sqcup \mathrm{ch}^1_{\mathcal{F}}(\mathcal{A}) \sqcup \cdots \sqcup \mathrm{ch}^\ell_{\mathcal{F}}(\mathcal{A}).$$

It is known that $\# \mathrm{ch}^k_{\mathcal{F}}(\mathcal{A}) = \mathrm{rank}_{\mathbb{Z}} A^k_{\mathbb{Z}}(\mathcal{A})$ [20, Proposition 2.3.2]. Further, $\mathrm{ch}^k_{\mathcal{F}}(\mathcal{A}) = \{C^k_1, \dots, C^k_{b_k}\}$ naturally determines a basis $[C^k_1], \dots, [C^k_{b_k}] \in A^k_{\mathbb{Z}}(\mathcal{A})$ [21, Theorem 3.2], which is called a *chamber basis*.

Let $\omega \in A^1_{\mathbb{Z}}(\mathcal{A})$. We can express the coboundary map of the Aomoto complex using the chamber basis as

$$\omega \wedge [C_i^k] = \sum_j \Gamma_{ij} \cdot [C_j^{k+1}], \tag{13}$$

with $\Gamma_{ij} \in \mathbb{Z}$. Then [21, Theorem 4.1] asserts that there is a decomposition

$$\Gamma_{ij} = N_{ij} \cdot L_{ij}, \tag{14}$$

with $N_{ij}, L_{ij} \in \mathbb{Z}$ such that the map

$$\nabla_\omega : A^k_{\mathcal{Z}}(\mathcal{A}) \longrightarrow A^{k+1}_{\mathcal{Z}}(\mathcal{A}),\ [C_i^k] \longmapsto \sum_j N_{ij} \cdot (q^{L_{ij}/2} - q^{-L_{ij}/2}) \cdot [C_j^{k+1}] \tag{15}$$

defines a cochain complex $(A^\bullet_{\mathcal{Z}}(\mathcal{A}), \nabla_\omega)$ whose cohomology group at $q = q_0$ is isomorphic to $H^k(M(\mathcal{A}), \mathcal{L}_{\rho(\omega,q_0)})$ for any $q_0 \in \mathbb{C}^\times \setminus \{1\}$. Replace the chamber basis by $\eta_i^k = (q^{1/2} - q^{-1/2})^k [C_i^k]$. The map ∇_ω is expressed as

$$\nabla_\omega(\eta_i^k) = \sum_j N_{ij} \cdot [L_{ij}]_q \cdot \eta_j^{k+1}, \tag{16}$$

which can be considered as a q-deformation of the coboundary map of the Aomoto complex (13).

Now, we consider the case $\ell = 2$. The integer N_{ij} is equal to (up to sign) the degree map introduced in [20, Definition 6.3.3]. When $\ell = 2$, the value is known to be $N_{ij} \in \{0, \pm 1\}$ [22, Definition 3.2 (1)]. Thus, we have

$$N_{ij} \cdot [L_{ij}]_q = [N_{ij} \cdot L_{ij}]_q = [\Gamma_{ij}]_q, \tag{17}$$

which is nothing but the canonical q-deformation of the coboundary map of the Aomoto complex with respect to the chamber basis. This completes the proof of Theorem 4.4.

Remark 4.5. By the proof, the Aomoto complex $(A^\bullet_{\mathbb{Z}}(\mathcal{A}), \omega)$ has a q-deformation which has the universality in the sense of Definition 4.3. However, it is not necessarily the canonical q-deformation with respect to certain basis. The reason is that if $\ell \geq 3$, there are examples such that $N_{ij} \leq -2$ (actually, it can take any value in $\{1, 0. -1, -2, -3, \dots\}$). Thus, (17) does not hold in general.

5. EXAMPLES AND REMARKS

5.1. Deleted B_3-arrangement

In this section, we exhibit some computations of q-deformations of the Aomoto complex with respect to the chamber basis for the deleted B_3-arrangement.

The deleted B_3-arrangement was studied in detail by Suciu [14, 15] as the first example which has a positive dimensional translated component in the characteristic variety.

Let $\mathcal{A} = \{H_1, \ldots, H_7\}$ be the line arrangement as in Figure 3. Let

$$\omega = e_1 + 2e_2 + 2e_3 + e_4 + 2e_5 + 3e_6 + 4e_7 \in A^1_{\mathbb{Z}}(\mathcal{A}). \tag{18}$$

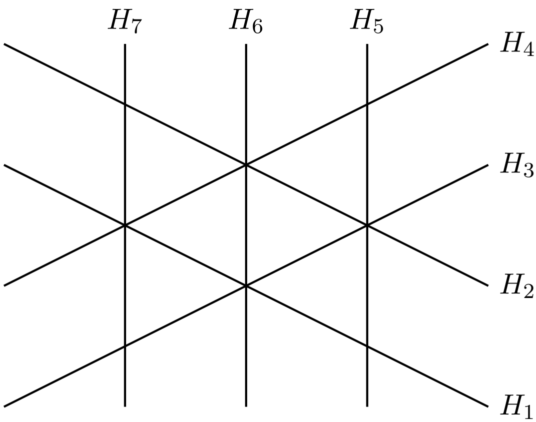

Figure 3 – The deleted B_3-arrangement.

Take the generic flag $F^0 \subset F^1$ as in Figure 4. Then, the set of chambers is decomposed as

$$\begin{aligned}\mathrm{ch}^0(\mathcal{A}) &= \{C_0\}\\ \mathrm{ch}^1(\mathcal{A}) &= \{C_1, C_2, \ldots, C_7\}\\ \mathrm{ch}^2(\mathcal{A}) &= \{D_1, D_2, \ldots, D_{12}\}\end{aligned}$$

We describe the Aomoto complex $(A^\bullet_{\mathbb{Z}}(\mathcal{A}), \omega)$ with respect to the chamber basis $[C_0] = 1 \in A^0_{\mathbb{Z}}(\mathcal{A}), [C_1], \ldots, [C_7] \in A^1_{\mathbb{Z}}(\mathcal{A}), [D_1], \ldots, [D_{12}] \in A^2_{\mathbb{Z}}(\mathcal{A})$. First, we have

$$\omega \cdot 1 = [C_1] + 3[C_2] + 7[C_3] + 10[C_4] + 12[C_5] + 13[C_6] + 15[C_7].$$

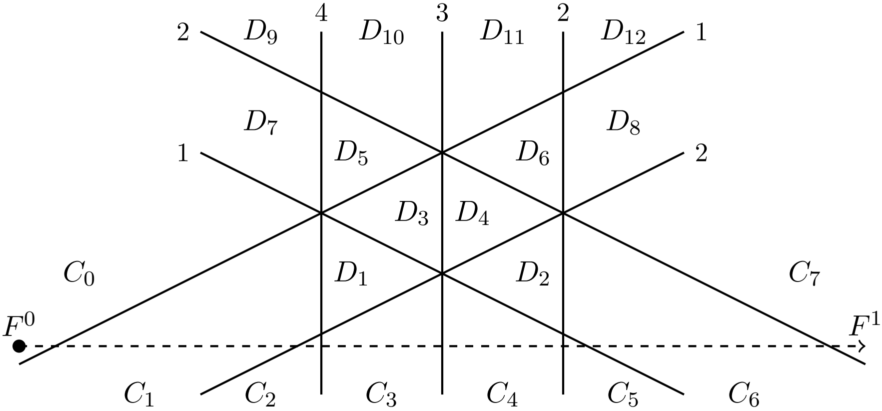

Figure 4 – Generic flag and chambers (the small numbers adjacent to lines are coefficients of ω).

Next, the coefficient of $\omega \wedge [C_j] = \sum_{i=1}^{12} c_{ij}[D_i]$ is computed as follows ([21, 22]).

(19)

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
D_1	-4	6	-2				
D_2				-1	3	-2	
D_3	-5	7	-3		8	-7	
D_4	-8	10		-3	5	-4	
D_5		8	-4		9	-8	
D_6	-10	12		-5	7		-4
D_7					13	-12	
D_8	-12	14					-2
D_9					15		-12
D_{10}		10	-6		11		-8
D_{11}		13		-6	8		-5
D_{12}		15					-3

By Theorem 4.4, this basis is universal and canonically q-deformable. Hence, we have the following relation (we can also easily check by using Clebsch–

Gordan relation (3)),

$$
\begin{pmatrix}
-[4]_q & [6]_q & -[2]_q & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -[1]_q & [3]_q & -[2]_q & 0 \\
-[5]_q & [7]_q & -[3]_q & 0 & [8]_q & -[7]_q & 0 \\
-[8]_q & [10]_q & 0 & -[3]_q & [5]_q & -[4]_q & 0 \\
0 & [8]_q & -[4]_q & 0 & [9]_q & -[8]_q & 0 \\
-[10]_q & [12]_q & 0 & -[5]_q & [7]_q & 0 & -[4]_q \\
0 & 0 & 0 & 0 & [13]_q & -[12]_q & 0 \\
-[12]_q & [14]_q & 0 & 0 & 0 & 0 & -[2]_q \\
0 & 0 & 0 & 0 & [15]_q & 0 & -[12]_q \\
0 & [10]_q & -[6]_q & 0 & [11]_q & 0 & -[8]_q \\
0 & [13]_q & 0 & -[6]_q & [8]_q & 0 & -[5]_q \\
0 & [15]_q & 0 & 0 & 0 & 0 & -[3]_q
\end{pmatrix}
\begin{pmatrix}
[1]_q \\ [3]_q \\ [7]_q \\ [10]_q \\ [12]_q \\ [13]_q \\ [15]_q
\end{pmatrix} = 0, \tag{20}
$$

and any specialization as $q \in \mathbb{C}^\times \setminus \{1\}$ gives the local system cohomology $H^1(M(\mathcal{A}), \mathcal{L}_{\rho(\omega,q)})$. Here, we consider two cases, $q = -1$ and $q = \exp(\pi\sqrt{-1}/3)$.

5.2. The case $q = -1$

In this case, $q^{1/2} = (-1)^{1/2} = \sqrt{-1}$, and $q^{-1/2} = -\sqrt{-1}$, and $[n]_{-1}$ is as follows.

$$
[n]_{-1} = \begin{cases} 1 & \text{if } n \equiv 1 \mod 4, \\ -1 & \text{if } n \equiv 3 \mod 4 \\ 0 & \text{if } n \equiv 0, 2 \mod 4. \end{cases} \tag{21}
$$

The coefficient matrix (20) is specialized to

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 0 \\
-1 & -1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & -1 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \tag{22}
$$

This matrix has rank 4. Hence $\dim H^1(M(\mathcal{A}), \mathcal{L}_{\rho(\omega,-1)}) = 2$. Indeed, the character $(q_1, \dots, q_7) = (-1, 1, 1, -1, 1, -1, 1)$ is one of the two isolated points at which the twisted cohomology has $\dim H^1(M(\mathcal{A}), \mathcal{L}) = 2$ ([14, Example 10.6]).

5.3. **The case** $q = \exp(\pi\sqrt{-1}/3)$

Let $q = \zeta := \exp(\pi\sqrt{-1}/3)$ be a primitive 6-th root of 1. In this case, $q^{1/2} = \exp(\pi\sqrt{-1}/6)$, $q^{-1/2} = \exp(-\pi\sqrt{-1}/6)$, and we have

$$[n]_\zeta = \begin{cases} 0 & \text{if } n \equiv 0, 6 \mod 12, \\ 1 & \text{if } n \equiv 1, 5 \mod 12, \\ \sqrt{3} & \text{if } n \equiv 2, 4 \mod 12, \\ 2 & \text{if } n \equiv 3 \mod 12, \\ -1 & \text{if } n \equiv 7, 11 \mod 12, \\ -\sqrt{3} & \text{if } n \equiv 8, 10 \mod 12, \\ -2 & \text{if } n \equiv 9 \mod 12. \end{cases} \tag{23}$$

The coefficient matrix (20) is specialized to

$$\begin{pmatrix} -\sqrt{3} & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -\sqrt{3} & 0 \\ -1 & -1 & -2 & 0 & -\sqrt{3} & 1 & 0 \\ \sqrt{3} & -\sqrt{3} & 0 & -2 & 1 & -\sqrt{3} & 0 \\ 0 & -\sqrt{3} & -\sqrt{3} & 0 & -2 & -\sqrt{3} & 0 \\ \sqrt{3} & 0 & 0 & -1 & -1 & 0 & -\sqrt{3} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 & 0 & 0 & -\sqrt{3} \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & -\sqrt{3} & 0 & 0 & -1 & 0 & \sqrt{3} \\ 0 & 1 & 0 & 0 & -\sqrt{3} & 0 & -1 \\ 0 & 2 & 0 & 0 & 0 & 0 & -2 \end{pmatrix} \tag{24}$$

This matrix has rank 5. Hence $\dim H^1(M(\mathcal{A}), \mathcal{L}_{\rho(\omega,\zeta)}) = 1$. Indeed, the character $(q_1, \dots, q_7) = (\zeta, \zeta^2, \zeta^2, \zeta, \zeta^2, \zeta^3, \zeta^4)$ (where $\zeta = \exp(\pi\sqrt{-1}/3)$) is a point in the translated component

$$\Omega = \{(t, -t^{-1}, -t^{-1}, t, t^2, -1, t^{-2}) \mid t \in \mathbb{C}^\times\}$$

described in [14, Example 10.6].

5.4. **Concluding remarks**

Once a universal canonically q-deformable basis of the Aomoto complex is obtained, the local system cohomology group $H^k(M(\mathcal{A}), \mathcal{L})$ is computed combinatorially (for rank one local systems expressed as $\mathcal{L} = \mathcal{L}_{\rho(\omega,q_0)}$). However, at this moment, the utilization of real structures (chambers) is necessary to select such a nice basis. Naturally, the following problems emerge.

Problem 5.1. 1. Can one characterize universal canonically q-deformable bases of the Aomoto complex combinatorially for complexfied real line arrangements?

2. Do such bases exist for higher-dimensional complex hyperplane arrangements?

A universal canonically q-deformable basis of the Aomoto complex requires highly non-trivial relations among q-integers such as (20). However, as we saw in Section 2, all the relations among q-integers are derived from the Clebsch–Gordan relations (Proposition 2.2) which has an origin in the irreducible decomposition of the tensor product of highest weight representations of $U_q(\mathfrak{sl}_2)$ (Remark 2.3).

Problem 5.2. Can we categorify the q-deformation of the Aomoto complex in a similar manner that (4) categorifies (3)?

Another interesting topic related to q-deformation is the notion of *cyclic sieving phenomena* (CSP). Many enumerative problems are naturally equipped with a cyclic group action. The number of objects which are invariant under a cyclic subgroup is sometimes computed from the q-deformation of the original enumerative formula as the special value at a root of 1. See [12] for more details on CSP. Universal canonically q-deformable bases of the Aomoto complex enable us to compute the monodromy eigenspaces of the cohomology of a cyclic covering space of $M(\mathcal{A})$ by specializing the q-deformation at a root of 1. There are notable parallel structures between CSP and the framework of this paper. Exploring the relationship between CSP and the topology of cyclic covering spaces of arrangements complements promises to be an interesting research direction.

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