A SHORT SURVEY ON FROBENIUS COMPLEXITY AND GRADINGS WITH RATIONAL TWIST FOR RINGS OF PRIME CHARACTERISTIC

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We present a quick introduction to Frobenius complexity and gradings with rational twist for commutative rings of prime characteristic, providing a place to start for those interested in exploring these topics.

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1. INTRODUCTION

The concept of Frobenius complexity of a local ring has grown out of the interest in understanding the generation ring of Frobenius operators over the injective hull of a local ring of prime characteristic. In a seminar paper in 2001, Lyubeznik and Smith have asked whether finite generation occurs all the time, see [11], and this was answered in 2009 by Katzman who provided a counterexample to the finite generation question in [9]. Subsequently, several mathematicians have investigated this ring leading to a definition of Frobenius complexity due to Enescu and Yao in [5]. This definition can be put in a larger context, that of rational twist for graded rings, see [7]. We discuss these concepts below by listing many of the contributions that currently exist on this topic hoping to facilitate further research. We hope that the reader is able to use our paper as a road map for exploring the presented subject. Due to space limitations, some results did not make it in this survey and we invite the reader to check the cited references.

Katzman's example is a Stanley–Reisner ring and Boix, Álvarez Montaner, and Zarzuela have systematically studied this class of rings in [2]. More detailed work for the Stanley–Reisner case was done more recently by Boix and Zarzuela in [4], and Ilioaea in [8]. In 2014, Katzman, Schwede, Singh, and Zhang [10] have provided a new perspective by introducing the twisted construction, which led to the definition of Frobenius complexity by Enescu

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and Yao in 2016 [5]. Examples of concrete classes of rings have been studied by Enescu and Yao, Page, and Miyazaki. In terms of generating functions, Boix, in his thesis [3], and Álvarez Montaner, in [1], have associated various generating functions to Cartier algebras or rings of Frobenius operators. In particular, Álvarez Montaner has defined the generating function associated to the complexity sequence for a skew algebra and studied it for the ring of Frobenius operators on the injective hull of the residue field of a local ring for the examples in the literature where the complexity sequence was understood. Enescu and Yao have taken a slightly different route, by associating a generating function to the complexity sequence after applying the twisted construction, leading to a concept called graded rational twist. This concept provides a clear avenue outside of the world of Frobenius operators on the injective hull of the residue field of a local ring. It leads to an investigation of the interaction between the grading and the characteristic of the ring, which we think it is interesting in its own right. The survey provides a description of some of the existing results in the literature, without any new results.

2. COMPLEXITY OF SKEW ALGEBRAS

Let us review the definition of the complexity of a skew algebra. A detailed introduction, with proofs, can be found in Section 2 from [5] where the concept in treated in more generality. Here, we restrict ourselves to skew algebras, which cover the case of commutative algebra as well as the twisted construction which is noncommutative, a concept to be defined later in the paper.

Definition 2.1. Let R be a commutative ring. Let $A = \oplus_{e \geq 0} A_e$ be a $\mathbb{N}$-graded ring, not necessarily commutative such that $R = A_0$. We say that A is a (left) *R-skew algebra* if $aR \subseteq Ra$ for all homogeneous elements $a \in A$. A right R-skew algebra can be defined in similar fashion. In this paper, our R-skew algebras are left R-skew algebras and therefore, we simply refer to them as R-skew algebras.

Clearly, if A itself is commutative then it is automatically an R-skew algebra. Other examples, not commutative, are given later.

Definition 2.2. 1. Let $G_e(A) = G_e$ be the subring of A generated by the elements of degree less or equal to e. We set $G_{-1} = A_0$.

2. We use $k_e = k_e(A)$ to denote the minimal number of homogeneous generators of G_e as a subring of A over A_0. (So $k_{-1} = k_0 = 0$.) We say that A is *degree-wise finitely generated* if $k_e < \infty$ for all $e \geq 0$.

3. For a degree-wise finitely generated ring A, we say that a set X of homogeneous elements of A *minimally generates* A if for all e, we have that $X_{\leq e} = \{a \in X : \deg(a) \leq e\}$ is a minimal set of generators for G_e with $k_e = \|X_{\leq e}\|$ for every $e \geq 0$. (Here, $\|\cdot\|$ denotes cardinality in the sequel.) Also, let $X_e = \{a \in X : \deg(a) = e\}$.

PROPOSITION 2.3 ([5], Proposition 2.3). *Let R be a commutative ring. Let A be a degree-wise finitely generated R-skew algebra and X a set of homogeneous elements of A. Then*

1. *The minimal number of generators of $\frac{A_e}{(G_{e-1})_e}$ as an R-bimodule is a set $k_e - k_{e-1}$ for all $e \geq 0$.*

2. *If X generates A as a ring over R, then $\|X_e\| \geq k_e - k_{e-1}$ for all $e \geq 0$.*

Let $f(n)$ and $g(n)$ be real-valued functions defined on the set of natural numbers. We say that $f(n) = O(g(n))$ if there exists $M > 0$ and a nonnegative integer n_0 such that $|f(n)| \leq M \cdot |g(n)|$ for all $n \geq n_0$.

Definition 2.4 ([5], Definition 2.5). Let R be a commutative ring. Let A be a degree-wise finitely generated R-skew algebra. The sequence $\{k_e\}_e$ is called the *growth* sequence for A. The *complexity sequence* is given by the set $\{c_e(A) = k_e - k_{e-1}\}_{e \geq 0}$. The *complexity* of A is

$$\inf\{n \in \mathbb{R}_{>0} : c_e(A) = O(n^e)\}$$

and it is denoted by $\mathrm{cx}(A)$. Therefore, $\mathrm{cx}(A) = \infty$ if there is no $n > 0$ such that $c_e(A) = O(n^e)$.

PROPOSITION 2.5 ([5], Corollary 2.10). *Let A be a degree-wise finitely generated R-skew algebra. Then $c_e(A)$ equals the minimal number of generators of $\frac{A_e}{(G_{e-1})_e}$ as a left R-module for all e.*

3. THE TWISTED CONSTRUCTION AND THE RING OF FROBENIUS OPERATORS ON THE INJECTIVE HULL

3.1. The twisted construction

Let R be a commutative ring of prime characteristic $p > 0$. Let $\mathscr{R}$ be an $\mathbb{N}$-graded commutative ring with $\mathscr{R}_0 = R$.

Definition 3.1 ([10]). Define the twisted construction on $\mathscr{R}$ by

$$T(\mathscr{R}) := \oplus_{e \geq 0} \mathscr{R}_{p^e - 1}.$$

This is an $\mathbb{N}$-graded ring by defining the multiplication by

$$a * b = ab^{p^e}$$

for all $a \in \mathscr{R}_{p^e-1}$, $b \in \mathscr{R}_{p^{e'}-1}$ and then extending it via linearity. The degree e piece of $T(\mathscr{R})$ is $T_e(\mathscr{R}) = \mathscr{R}_{p^e-1}$.

We have obtained a noncommutative ring. Note that, since $R = \mathscr{R}_0$ and $a * r = ar^{p^e} \subset Ra$ for all $r \in R = \mathscr{R}_0, a \in \mathscr{R}_e$, $T(\mathscr{R})$ is naturally an R-skew algebra.

3.2. **Frobenius operators**

Let R be a Noetherian ring of prime characteristic p and R-module M. By an eth Frobenius action, we mean a map $\phi : M \to M$. This is an R-additive map such that $\phi(rm) = r^q\phi(m)$, for all $r \in R, m \in M$, where $q = p^e$. Let $\mathscr{F}^e(M)$ be the collection of all eth Frobenius operators on M.

Definition 3.2. We define *the algebra of Frobenius operators* on M by

$$\mathscr{F}(M) = \oplus_{e \geq 0}\mathscr{F}^e(M),$$

with the multiplication on $\mathscr{F}(M)$ determined by composition of functions; that is, if $\phi \in \mathscr{F}^e(M), \psi \in \mathscr{F}^{e'}(M)$ then $\phi\psi := \phi \circ \psi \in \mathscr{F}^{e+e'}(M)$. Hence, in general, $\phi\psi \neq \psi\phi$.

Note that there exists a natural ring homomorphism $R \to \mathrm{End}_R(M) = \mathscr{F}^0(M)$ given by $r \to \theta_r$, where $\theta_r(m) = rm$, for all $m \in M$. This way, we obtain an R-graded module, by defining the multiplication by r to equal the composition with θ_r. Specifically, $r\phi := \theta_r \circ \phi$ and $\phi r := \phi \circ \theta_r$ for all $r \in R$ and $\phi \in \mathscr{F}(M)$.

If $\phi \in \mathscr{F}^e(M)$, then $\phi r = r^p\phi$, because for all $m \in M$, $(\phi r)(m) = \phi(rm) = r^p\phi(m) = (r^p\phi)(m)$. This shows that, when $R = \mathrm{End}_R(M) = \mathscr{F}^0(M)$, $\mathscr{F}(M)$ is a skew R-algebra.

We are interested in the case of $(R, \mathfrak{m}, k)$ a local complete ring and $M = E_R(k)$ the injective hull of the residue field of R. By Matlis duality, $R = \mathrm{End}_R(E_R(k))$.

The ring of Frobenius operators on the injective hull $E_R(k)$ of the residue field of a local ring of positive characteristic has been studied by many researchers in commutative algebra. The twisted construction appears naturally in this context as shown by the theorem stated below.

Let $(R, \mathfrak{m}, k)$ be a normal complete local ring. Let ω denote the canonical module of R and ω^{-1} denote the inverse of the canonical module of R in its divisor class group. Let $\omega^{(-n)}$ denote the nth symbolic power of ω^{-1} and consider the symbolic Rees algebra $\mathscr{R}(\omega^{-1}) = \oplus_{n\geq 0}\omega^{(-n)}$ which is called the anticanonical cover of R.

The following result is very important for what follows.

THEOREM 3.3 ([10]). *Let $(R, \mathfrak{m}, k)$ be a normal, complete local ring of positive characteristic. The,n we have an isomorphism of graded rings*

$$\mathscr{F}(E) \simeq T\big(\mathscr{R}(\omega^{-1})\big).$$

3.3. **Frobenius complexity of a local ring**

Let us now define the Frobenius complexity of a local ring of prime characteristic, due to Enescu and Yao, see [5].

Definition 3.4 ([5], Definition 2.13). Let $(R, \mathfrak{m}, k)$ be a local ring of prime characteristic p. Denote by E the injective hull of the residue field of R. Denote $k_e(R) := k_e(\mathscr{F}(E))$, for all e, and call these numbers the *Frobenius growth sequence* of R. Then $c_e = c_e(R) := k_e(R) - k_{e-1}(R)$ defines the *Frobenius complexity sequence* of R.

The *complexity* of $\mathscr{F}(E)$ is

$$\inf\{n \in \mathbb{R}_{>0} : c_e = O(n^e)\}$$

and it is denoted by $\mathrm{cx}(\mathscr{F}(E))$. Therefore, if there is no $n > 0$ such that $c_e(\mathscr{F}(E)) = O(n^e)$, then $\mathrm{cx}(\mathscr{F}(E)) = \infty$.

We define the *Frobenius complexity* of the ring R by

$$\mathrm{cx}_F(R) = \log_p\big(\mathrm{cx}(\mathscr{F}(E))\big),$$

if $\mathrm{cx}(\mathscr{F}(E))$ is nonzero and finite. If the Frobenius growth sequence of the ring R is eventually constant (i.e., $\mathrm{cx}(\mathscr{F}(E)) = 0$), then the Frobenius complexity of R is set to be $-\infty$. If $\mathrm{cx}(\mathscr{F}(E)) = \infty$, the Frobenius complexity if R is set to be ∞.

We list the basic results on Frobenius complexity of local rings.

THEOREM 3.5 ([5, Corollary 2.12, Theorems 4.7 and 4.9]). *Let $(R, \mathfrak{m}, k)$ be a local ring.*

1. *If R is 0-dimensional then $\mathrm{cx}_F(R) = -\infty$.*

2. *If R is normal, complete and has dimension at most two, then the complexity $\mathrm{cx}_F(R) \leq 0$.*

3. *If R is normal, complete and such that the anticanonical cover is finitely generated over R, then $\mathrm{cx}_F(R) < \infty$.*

There is a more precise result that bounds the Frobenius complexity of a local ring when the anticanonical cover is finitely generated, due to Page, see Proposition 2.6 in [14]. We need the following definition first which is also due to Page.

Definition 3.6. Let R be a local normal ring that admits a canonical module ω. Let $\mathscr{R}(\omega^{-1}) = \oplus_{n\geq 0}\omega^{(-n)}$ be its anticanonical cover and assume that $\mathscr{R}(\omega^{-1})$ is finitely generated as an R-algebra. The *anticanonical spread* of R is the Krull dimension of $\mathscr{R}(\omega^{-1}) \otimes k$ and is denoted by $sp_R(\omega^{-1})$.

PROPOSITION 3.7 ([14]). *If $(R, \mathfrak{m}, k)$ is a local normal ring of prime characteristic such that $\mathscr{R}(\omega^{-1})$ if finitely generated, then*

$$\mathrm{cx}_F(R) \leq sp_R(\omega^{-1}) - 1.$$

Page has asked whether for Hibi rings equality holds (when we let $p \to \infty$) and this was answered in the affirmative by Miyazaki in Theorem 8.5 in [12].

Katzman, Schwede, Singh, and Zhang and Enescu and Yao have proved the following theorem that describes the $\mathbb{Q}$-Gorenstein case.

THEOREM 3.8 ([10, Proposition 4.1] and [5, Theorem 4.5]). *If $(R, \mathfrak{m}, k)$ is normal and $\mathbb{Q}$-Gorenstein, then the order of its canonical module in the divisor class group is relatively prime to p if and only if $\mathrm{cx}_F(R) = -\infty$.*

4. RATIONAL TWIST

Let R be a commutative ring of prime characteristic p and $\mathscr{R}$ be an $\mathbb{N}$-graded R-skew left algebra. Recall that $T(\mathscr{R})$ is naturally a skew R-algebra. In this section, we review some recent notions defined by Enescu and Yao, related to the complexity sequence of the twisted construction. Some of these ideas go back to Boix, in his thesis [3], and Álvarez Montaner, in [1], who have looked at the same concepts for the case of rings of Frobenius operators.

Definition 4.1 ([7], Definition 2.1). Let R be a commutative ring of prime characteristic p and $\mathscr{R}$ be an $\mathbb{N}$-graded R-skew left algebra. Let $T = T(\mathscr{R})$ be its twisted construction.

1. Let $\{c_e\}_{e\geq 0}$ be the complexity sequence for $T = T(\mathscr{R})$. The *twisted generating function* of $\mathscr{R}$ is

$$\mathcal{C}_{\mathscr{R}}(z) := \sum_{e=0}^{\infty} c_e z^e \in \mathbb{Q}[\![z]\!],$$

2. We say that $\mathscr{R}$ has a grading with *rational twist*, or simply that the graded ring $\mathscr{R}$ has *rational twist*, if the twisted generating function is a rational function in z. More precisely, $\mathcal{C}(z) = \mathcal{C}_{\mathscr{R}}(z) = \frac{P(z)}{Q(z)}$ with $P(z)$, $Q(z) \in \mathbb{Q}[z]$. (One can assume that $P(z)$, $Q(z) \in \mathbb{Q}[z]$ do not have common roots in $\mathbb{C}$.)

We remind the reader a standard result on generating functions and recurrence relations, in a form due to Stanley, see [15], p. 464, Theorem 4.1.1. We use this result in our definitions later.

THEOREM 4.2. *Let $\alpha_1, \dots, \alpha_d$ be complex numbers, $d \geq 1$ and $\alpha_d \neq 0$. Let $f : \mathbb{N} \to \mathbb{C}$ be a function. The following assertions are equivalent:*

1. *The generating function of the sequence f satisfies*
$$\sum_{n \geq 0} f(n)x^n = \frac{P(x)}{Q(x)},$$
where $Q(x) = 1 + \alpha_1 x + \cdots + \alpha_d x^d$ and $P(x)$ is a polynomial of degree less than d.

2. *For all $n \geq 0$,*
$$f(n+d) + \alpha_1 f(n+d-1) + \cdots + \alpha_d f(n) = 0.$$

3. *For all $n \geq 0$,*
$$f(n) = \sum_{i=1}^{k} P_i(n)\gamma_i^n,$$
where $1 + \alpha_1 x + \cdots + \alpha_d x^d = \prod_{i=1}^{k}(1 - \gamma_i x)^{d_i}$, the γ_i's are distinct and nonzero and $P_i(n)$ is a polynomial of degree less than d_i.

Definition 4.3 ([7], Definition 3.4). We say that $\mathscr{R}$ has rational twist with *dominant eigenvalue* if $\mathcal{C}(z)$ is a rational function of the form $\mathcal{C}(z) = \frac{P(z)}{Q(z)}$, where $P(z) \in \mathbb{Q}[z]$ and $Q(z) \in \mathbb{Q}[z]$ do not have common roots in $\mathbb{C}$, such that either $Q(z)$ is constant or $Q(z)$ has a unique simple root $1/\gamma$ of minimal absolute value.

In the case where $Q(z)$ has a unique simple root $1/\gamma$ of minimal absolute value, Theorem 4.2 gives, for $e \gg 0$,
$$c_e = \rho\gamma^e + \text{lower order terms } o(\gamma^e).$$
We call the number ρ is the *twisted complexity multiplicity* of $\mathscr{R}$, or simply *t-multiplicity*, and we call γ the *dominant eigenvalue* of $\mathscr{R}$.

The reader should note that in Miyazaki's terminology $\log_p(\gamma)$ is called the T-complexity of $\mathscr{R}$, see [12], Definition 7.2. Specifically, the *T-complexity* of $\mathscr{R}$ is $Tcx(\mathscr{R}) = \log_p(\mathrm{cx}(T(\mathscr{R}))$.

Corollary 4.4. *Let $(R, \mathfrak{m}, k)$ be a normal, complete local ring of prime characteristic p. If the graded ring $\mathscr{R}(\omega^{-1})$ has rational twist with dominant eigenvalue γ, then*

$$\mathrm{cx}_F(R) = \log_p(\gamma) = Tcx(\mathscr{R}(\omega^{-1}).$$

Theorem 4.5 ([5, 7]). *Let K be a field of characteristic p and $m \geq 3$ be an integer. Consider the determinantal ring of 2×2 minors in a matrix of indeterminates of size $m \times (m-1)$ over K, and denote by S_m the completion of the ring at its maximal homogenous ideal. Let γ be the dominant eigenvalue of $K[x_1, \ldots, x_m]$ considered with standard grading. Then*

1. $\mathrm{cx}_F(S_m) = \log_p(\gamma)$,
2. $p^{m-2} < \gamma < p^{m-1}$,
3. $\lim_{p\to\infty}(\mathrm{cx}_F(S_m)) = m - 1$,
4. $\lim_{p\to\infty} \frac{\gamma}{p^{m-1}} = 1 - \frac{1}{(m-1)!}$.

In our previous papers [5, 6], we examined the dominant eigenvalue of the polynomial ring with standard grading and the Veronese ring of a polynomial ring with standard grading.

Theorem 4.6 ([6, Corollary 3.13, Subsection 3.2]). *Let R be a Noetherian ring, $r \geq 1$ and $m \geq r + 2$ be integers, and $\mathscr{R} = V_r(R[x_1, \ldots, x_m])$ be the r-th Veronese ring of $R[x_1, \ldots, x_m]$ with standard grading. Then $\mathscr{R}$ has rational twist with dominant eigenvalue for p large enough. If γ_p denotes the dominant eigenvalue of $\mathscr{R}$ in characteristic p, then $\lim_{p\to\infty} \log_p(\gamma_p) = m - 1$.*

We have obtained more general results for affine semigroups rings in [7]. Specifically, let R be a commutative ring of prime characteristic p, and let m, $d_1, \ldots, d_m$ be positive integers. Let A be a finitely generated semigroup of $(\mathbb{N}^m, +)$, with the assumption $(0, \ldots, 0) \in A$. Consider the polynomial ring $R[x_1, \ldots, x_m]$, with general grading $\deg(x_i) = d_i$ for all $i = 1, \ldots, m$. Let $R[A]$ denote the semigroup ring of A over R, so $R[A]$ is a graded subring of $R[x_1, \ldots, x_m]$.

Let $\alpha_1, \ldots, \alpha_h$ be a minimal generating set for A, with total degrees $f_1, \ldots, f_h$. That is, $|\alpha_i| = f_i$, where we find that for $\alpha = (a_1, \ldots, a_m) \in \mathbb{N}^m$, $|\alpha| := |\alpha|_d := a_1 d_1 + \cdots + a_m d_m$. For every $i \in \mathbb{N}$, denote $A_i = \{\alpha \in A : |\alpha| = i\}$.

Definition 4.7 ([7], Definition 2.4). Let $A \subseteq \mathbb{N}^m$, for some $m \in \mathbb{N}$ be an affine semigroup.

1. We say that A is *closed under differences* or simply *CD* if:
$$\alpha, \beta \in A \text{ and } \alpha - \beta \in \mathbb{N}^m \implies \alpha - \beta \in A.$$

2. We say that A is *closed under left twisted differences* or simply *CLTD* if:
$$\alpha' \in A_{p^{e'}-1}, \alpha'' \in \mathbb{N}^m \text{ and } \alpha' + p^{e'}\alpha'' \in A_{p^{e'+e''}-1} \implies \alpha'' \in A$$
for all $e' > 0$, $e'' > 0$.

3. We say that A is *closed under right twisted differences* or simply *CRTD* if:
$$\alpha' \in \mathbb{N}^m, \alpha'' \in A_{p^{e''}-1} \text{ and } \alpha' + p^{e'}\alpha'' \in A_{p^{e'+e''}-1} \implies \alpha' \in A$$
for all $e' > 0$, $e'' > 0$.

Recall that the twisted construction for our graded ring $\mathscr{R} = R[A]$ is $T(\mathscr{R}) = T(R[A]) = \oplus_e T_e(R[A])$, where $T_e(R[A]) = (R[A])_{p^e-1}$ for every integer $e \geq 0$. When we mention the degree of a monomial, we agree that it refers to its (total) degree in $\mathscr{R} = R[A]$. Thus, a monomial in T_e is a monomial of (total) degree $p^e - 1$. In particular, $T_0 = R$.

4.1. **Concatenation**

We give a concrete interpretation of the operation T in terms of writing integers in p-basis.

Let $\alpha = (\alpha_i : i = 1, \ldots, m), \beta = (\beta_i : i = 1, \ldots, m)$. Note that in base p
$$p^e - 1 = \overline{p-1p-1\cdots p-1}_e,$$
where the index indicates how many digits we use in base p.

Write $\alpha_i = \overline{u_{1,i}u_{2,i}\cdots u_{e',i}}$ and $\beta_i = \overline{v_{1,i}v_{2,i}\cdots v_{e'',i}}$ in base p, that is, all integers $u_{j,i}, v_{j,i}$ are between 0 and $p-1$. Note that, in base p,
$$\alpha_i + p^{e'}\beta_i = \overline{v_{1,i}v_{2,i}\cdots v_{e'',i}u_{1,i}u_{2,i}\cdots u_{e',i}}.$$
In conclusion, if $\alpha \in T_{e'}, \beta \in T_{e''}$, then the semigroup operation $*$ becomes, when written in base p,
$$\alpha * \beta = (\overline{v_{1,i}v_{2,i}\cdots v_{e'',i}u_{1,i}u_{2,i}\cdots u_{e',i}} : i = 1, \ldots, m).$$

The reader should note that the condition $\alpha \in T_{e'}$ is translated as the sum of all $\alpha_i, i = 1, \ldots, m$ equals to $p^{e'} - 1 = \overline{p-1p-1\cdots p-1}_{e'}$ when written in base p. Similarly, for $\beta \in T_{e''}$.

Theorem 4.8 ([7], Theorem 2.10). *Let $R[A]$ and $T = T(R[A])$ be as above.*

1. *For every nonnegative integer $e \geq 0$, let t_e denote the rank of $T_e(R[A])$ as an R-module. Then the generating function $\mathcal{T}(z) = \sum_{e\geq 0} t_e z^e$ is a rational function in z over $\mathbb{Q}$.*

2. *Let c_e denote the complexity sequence of $T(R[A])$. If A satisfies CLTD or CRTD, then the twisted generating function of $R[A]$, $\mathcal{C}(z) = \sum_{e\geq 0} c_e z^e$, is a rational function in z over $\mathbb{Q}$. In conclusion, the graded ring $\mathscr{R} = R[A]$ has rational twist.*

Example 4.9. 1. $\mathbb{N}^m$ is CD, CRTD, and CLTD.

2. Let $V_r = \{\alpha \in \mathbb{N}^m : r \mid |\alpha|\}$, the r-Veronese sub-semigroup of $\mathbb{N}^m$. Then V_r satisfies CD, CRTD, and CLTD.

3. In general, if A is CD, then A is CRTD.

As an immediate consequence, we have the following result, per [7].

Corollary 4.10. *Let R be a commutative ring of prime characteristic p.*

1. *Let $\mathscr{R} = R[x_1, \ldots, x_m]$ graded with $\deg(x_i) = d_i$, $i = 1, \ldots, m$. Then the graded ring $\mathscr{R}$ has rational twist.*

2. *Let $\mathscr{R} = V_r(R[x_1, \ldots, x_m])$, where $r \geq 1$, be the rth Veronese subring of the graded polynomial ring $R[x_1, \ldots, x_m]$ with $\deg(x_i) = d_i$, $i = 1, \ldots, m$. Then the graded ring $\mathscr{R}$ has rational twist.*

4.2. **Applications**

Definition 4.11. Let K be a field and $m > n \geq 2$. Let $S_{m,n}$ denote the completion of $K[x_1, \ldots, x_m] \sharp K[y_1, \ldots, y_n]$ with respect to the ideal generated by all homogeneous elements of positive degree. Let $\mathscr{R}_{m,n}$ be the anticanonical cover of $S_{m,n}$.

For any positive integers p and m (with p prime), let us further denote by $M_{p,m}(i)$ (or simply $M(i)$ if p and m are understood) the rank of $(R[x_1, \ldots, x_m]/(x_1^p, \ldots, x_m^p))_i$ over R, for all $i \in \mathbb{Z}$. Note that $M_{p,m} = 0$ exactly when $i > m(p-1)$ or $i < 0$. In fact, we have the following identity that can be used to define $M_{p,m}(i)$ in the terms of the Poincaré series (which is a polynomial):

$$\sum_{i=-\infty}^{\infty} M_{p,m}(i)t^i = \sum_{i=0}^{m(p-1)} M_{p,m}(i)t^i = \left(\frac{1-t^p}{1-t}\right)^m = \left(1 + \cdots + t^{p-1}\right)^m.$$

Following [6], we have the following definition.

Definition 4.12. In what follows, we call

$$U(p, r, m) = U := \left[u_{ij}\right]_{(m-r-1)\times(m-r-1)} \quad \text{with} \quad u_{ij} := M_{p,m}(r(p-1)+ip-j)$$

as the *determining matrix* for p, r, m.

THEOREM 4.13 ([6], Theorem 3.4). *Consider* $T = T(V_r(R[x_1, \ldots, x_m]))$ *as above with* $m = r + 2$. *Then*

$$c_e(T) = \binom{rp}{m-1}\binom{p+m-2}{m-1}^{e-2}\binom{p+m-3}{m-1}$$

for all $e \geq 2$ *and* $\mathrm{cx}(T) = \binom{p+m-2}{m-1}$.

THEOREM 4.14 ([6], Corollary 3.13). *Let* $\mathscr{R} = R[x_1, \ldots, x_m]$. *If* $p \gg 0$, *then*

$$(m-r-1)\binom{m-1+p-(m-r-1)}{m-1} \leq \mathrm{cx}(T(V_r(\mathscr{R}))$$
$$\leq (m-r-1)\binom{m-1+\lceil\frac{m(p-1)}{2}\rceil}{m-1}$$

and therefore, $\lim_{p\to\infty} \log_p \mathrm{cx}(T(V_r(\mathscr{R})) = m-1$.

THEOREM 4.15 ([6], Theorem 4.1). *Let* K *be a field of characteristic* p, $S_{m,n}$ *and* $\mathscr{R}_{m,n}$ *be as in Definition* 4.11 *with* $m > n \geq 2$. *Let* $E_{m,n}$ *denote the injective hull of the residue field of* $S_{m,n}$.

1. *The ring of Frobenius operators of* $S_{m,n}$ *(i.e.,* $\mathscr{F}(E_{m,n})$*) is never finitely generated over* $\mathscr{F}_0(E_{m,n})$.
2. *When* $n = 2$, *we have* $\mathrm{cx}_F(S_{m,2}) = \log_p \binom{p+m-2}{m-1}$.
3. *We have* $\lim_{p\to\infty} \mathrm{cx}_F(S_{m,n}) = m-1$.
4. *For* $p \gg 0$ *or whenever the determining matrix* $U = U(p, m, m-n)$ *has all positive entries, we have* $\mathrm{cx}_F(S_{m,n}) = \log_p(\lambda)$, *in which* λ *is the Perron root for* U.

The study of the generation of the Frobenius ring of operators on the injective hull of the residue field for a completion of a Stanley–Reisner originates with Katzman. Later, Álvarez Montaner, Boix and Zarzuela have investigated this problem in detail in [2] and later in [4]. Álvarez Montaner, Boix and Zarzuela fully described, for this class of rings, when the Frobenius ring of operators $\mathscr{F}(E_R)$ is finitely generated, see Theorem 3.5 in [2]. Based on their work, Ilioaea has computed the complexity sequence.

THEOREM 4.16 ([8]). *Let K be a field of prime characteristic p and ring $R = K[[X_1, \dots, X_n]]/I$, where I is a square free monomial ideal. Let $\{c_e\}$ be the complexity sequence for the ring of Frobenius operators on the injective hull of R. Then $\{c_e\}_{e \geq 0}$ has the form $0, \mu + 1, \mu, \mu, \mu, \dots$, for a certain positive integer μ.*

Another class of rings which was investigated in detail by Page and, independently, Miyazaki in terms of Frobenius complexity is that of Hibi rings. The reader is referred to [14] and [12], for their work.

4.3. **Cases with dominant eigenvalue for the polynomial ring**

Let $\mathscr{R} = R[x_1, \dots, x_m]$ with $\deg(x_1) = d_1, \dots, \deg(x_m) = d_m$, which means that we take $A = \mathbb{N}^m$. As before, let $D = \operatorname{lcm}(d_1, \dots, d_m)$. We assume that $p \equiv 1 \bmod D$.

For every $n \in \mathbb{N}$, let $h(n) := \operatorname{rank}_R(\mathscr{R}_n)$. There exist $h_i(x) \in \mathbb{Q}[x]$, $i = 0, \dots, D-1$, such that

$$h(n) = \operatorname{rank}_R(\mathscr{R}_n) = h_i(n) \quad \text{if} \quad n \equiv i \mod D.$$

for all $n \geq 0$. Let $\eta_0(x) = h_0(x-1)$.

The following are the assumptions we make in this subsection (see Section 3 in [7]).

Assumption 4.17. This is essential in our next theorem:

1. $m = 2$ and $D > 1$, or $m \geq 3$.

2. $p \equiv 1 \mod D$.

3. The positive integers $d_1, \dots, d_m$ are pairwise relatively prime.

4. The coefficients of $\eta_0(x) = h_0(x-1) = \sum_{j=0}^{m-1} a_{0,j} x^j$ are nonnegative.

THEOREM 4.18 ([7], Theorem 3.16). *Recall that $\mathscr{R} = R[x_1, \dots, x_m]$ with grading given by $\deg(x_i) = d_i$, for $i = 1, \dots, m$. Under the conditions stated in Assumption 4.17 the following assertions hold.*

1. *The complexity sequence is given by*

$$c_e = \rho_1 \gamma_1^e + \cdots + \rho_l \gamma_l^e,$$

with $p^{j_h} < \gamma_{l-h} < p^{j_{h+1}}$, for all $h = 0, \dots, l-1$, for all $e \geq 1$.

2. *$\mathscr{R}$ has rational twist of dominant eigenvalue.*

3. *The dominant eigenvalue is* γ_1 *with* $\gamma_1 < p^{m-1}$ *and* $\lim_{p\to\infty} \frac{\gamma_1}{p^{m-1}} = 1 - \frac{1}{(m-1)D!}$. *In general, for* $p \gg 0$, *we have that* $p^{m-2} < \gamma_1$. *Additionally, if* $a_{0,m-2} \neq 0$, *then* $p^{m-2} < \gamma_1$, *for all* p. *Here, all considerations regarding* p *are for values of* p *such that* $p \equiv 1 \mod D$.

Example 4.19. We discuss now an example to illustrate some of the features of the theory presented here. Let r be a fixed positive integer. Let p a prime number. we have that

$$A_r = \langle (1,1), (r, r-1) \rangle$$

an affine semigroup in $\mathbb{N}^2$. We denote by T the twisted construction of A_r. The elements of T_e have the form $m(1,1) + n(r, r-1)$ such that $2m + (2r-1)n = p^e - 1$, with m, n nonnegative integers. Recall that the operation $*$ on T correspond to the aforementioned concatenation.

The complexity sequence c_e corresponds the number of vectors from T_e that are not in the union of all $T_{e'} * T_{e''}$ where $e' + e'' = e, 0 < e', e'' < e$. Or in the language of concatenation, the vectors in T_e that cannot be obtained via the concatenation of elements from $T_{e'}, T_{e''}$ where $e' + e'' = e, 0 < e', e'' < e$.

Let now $r = p$ and $e \geq 2$. Let us notice that $(p^{e-1}, p^{e-1} - 1)$ is in T_e because

$$(p^{e-1} - p)(1,1) + (p, p-1) = (p^{e-1}, p^{e-1} - 1).$$

Note that in base p,

$$p^{e-1} = \overline{(p-1)0\cdots 0}_e \text{ and } p^{e-1} = \overline{0(p-1)\cdots(p-1)}_e.$$

So

$$\left(\overline{(p-1)0\cdots 0}_e, \overline{0(p-1)\cdots(p-1)}_e\right) \in T_e.$$

It is now clear that this element cannot be obtained by concatenation because no vector in T can have a zero in the first coordinate. This shows concretely that $c_e \neq 0$ for all $e \geq 2$ in $T = T(A_p)$. Of course, counting all vectors that give c_e is a much harder task.

Now let us consider $r = 2, p = 3$. A similar argument as above shows that $c_e \neq 0$ for $e \geq 2$ in T for A_2 and $p = 3$. Fix $e \geq 2$. The elements of T_e are vectors $(m + 2n, m + n)$ such that $2m + 3n = 3^e - 1$. Note that $(1,1) + (3^{e-1} - 1)(2,1) = (2 \cdot 3^{e-1} - 1, 3^{e-1})$ is also in T_e.

As before, $3^{e-1} = \overline{20\cdots 0}_e$ and $2 \cdot 3^{e-1} - 1 = \overline{02\cdots 2}_e$ it is now clear that $(\overline{02\cdots 2}_e, \overline{20\cdots 0}_e)$ cannot be obtained via concatenation. So $c_e \neq 0$ for all $e \geq 2$.

The example A_r with $r = 2$ was first considered in [13], where the author has also observed that the twisted construction for an affine semigroup in $\mathbb{N}$ is always finitely generated.

Fix again r positive integer and p prime. Note that A_r can be identified to the semigroup generated by two vectors $x = (1, 0)$, $y = (0, 1)$ where $deg(x) = 2, deg(y) = 2r - 1$. Then $lcm(2, 2r - 1) = 4r - 2$. The case $p \equiv 1 \mod (4r - 2)$ was treated in [7] where the authors showed that there exists a constant ρ such that the complexity sequence $\{c_e\}_e$ is given by $c_e = \rho \cdot (\frac{p(4r-2)-p+1}{4r-2}))^e$ for $e \geq 2$. In fact, based on the work in [7], one can compute c_e for all prime p with $gcd(p, 4r - 2) = 1$, a tedious computation. This puts in perspective the discussion above.

We end with a question that we find of interest.

Question 4.20. What affine semigroup rings have graded rational twist?

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