Dedicated to Gabriela Kohr

ON MAPPINGS WITH GENERALIZED PARAMETRIC REPRESENTATION

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We survey some results involving mappings with generalized parametric representation with respect to a time-dependent linear operator.

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1. INTRODUCTION

Inspired by the work of John Pfaltzgraff [18] and Tadeusz Poreda [20] in extending the Loewner theory to higher dimensions, Gabriela Kohr [15] introduced in 2001 the family $S^0(\mathbb{B}^n)$ of mappings with parametric representation on the Euclidean unit ball as a natural analog of the class S, rather than the family $S(\mathbb{B}^n)$ of univalent normalized mappings on $\mathbb{B}^n$. She proved properties of $S^0(\mathbb{B}^n)$, which are similar to those of S (by using the Loewner differential equations and the Loewner chains, see [15, Theorems 2.3, 2.4, 2.5]; cf. [19]). Also, she pointed out differences between the dimension one case and the higher dimensions case, see [15, Examples 2.9, 2.10]. Moreover, she gave various examples of mappings that have parametric representation and used their parametric representation to deduce corresponding properties, see [15, Theorems 2.6, 2.7]. Furthermore, Gabriela Kohr proved, along with Hidetaka Hamada and Ian Graham [4], the compactness of the Carathéodory family $\mathcal{M}$ (the analog family of normalized holomorphic functions with positive real part) and, along with Ian Graham and Mirela Kohr [9], the compactness of $S^0(\mathbb{B}^n)$. For a detailed presentation of the Loewner theory in one and higher dimensions and the significance of the above contributions, we strongly recommend the excellent book of Ian Graham and Gabriela Kohr [8].

Taking into account the spirallike mappings with respect to a linear operator, Peter Duren, Ian Graham, Hidetaka Hamada, Gabriela Kohr and Mirela

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Kohr [3, 6] introduced and studied the mappings with A-parametric representation on $\mathbb{B}^n$, when A is a linear operator that satisfies a condition involving its spectrum. Next, Ian Graham, Hidetaka Hamada, Gabriela Kohr and Mirela Kohr [7] considered the generalized parametric representation on $\mathbb{B}^n$ with respect to a time-dependent linear operator.

Following [13], in this paper, we survey some results for mappings with generalized parametric representation on $\mathbb{B}^n$ with respect to a time-dependent linear operator. We present certain results obtained in [5, 7, 11, 12, 16, 17, 21]. We discuss both similarities and differences between the time-dependent case and the time-independent case.

2. PRELIMINARIES

Let $\mathbb{C}^n$ be the space of n complex variables $z = (z_1, \ldots, z_n)$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w}_j$ and the Euclidean norm $\|z\| = \sqrt{\langle z, z\rangle}$. The unit ball $\{z \in \mathbb{C}^n : \|z\| < 1\}$ is denoted by $\mathbb{B}^n$. In the case $n = 1$, the unit disc is denoted by $\mathbb{U}$.

Let

$$H(\mathbb{B}^n) = \big\{f : \mathbb{B}^n \to \mathbb{C}^n : f \text{ is holomorphic}\big\}$$
$$S(\mathbb{B}^n) = \big\{f \in H(\mathbb{B}^n) : f \text{ is univalent with } f(0) = 0 \text{ and } Df(0) = I\big\},$$

where D stands for the Fréchet differential and I is the identity operator. We consider the compact-open topology on these families.

Definition 2.1. Let $L(\mathbb{C}^n)$ denote the space of linear operators from $\mathbb{C}^n$ to $\mathbb{C}^n$ with the standard operator norm. For $A \in L(\mathbb{C}^n)$, let

$$m(A) = \min\{\Re\langle A(z), z\rangle : \|z\| = 1\}.$$

If $A \in L(\mathbb{C}^n)$ with $m(A) \geq 0$, let

$$\mathcal{N}_A(\mathbb{B}^n) = \{h \in H(\mathbb{B}^n) : h(0) = 0, Dh(0) = A \text{ and } \Re\langle h(z), z\rangle \geq 0, z \in \mathbb{B}^n\}.$$

The family $\mathcal{N}_I(\mathbb{B}^n)$ is denoted by $\mathcal{M}(\mathbb{B}^n)$ (see [18]). Graham, Hamada, Kohr [4, Corollary 1.3] proved that $\mathcal{M}(\mathbb{B}^n)$ is a compact family. Moreover, their arguments imply that $\mathcal{N}_A(\mathbb{B}^n)$ is a compact family for every $A \in L(\mathbb{C}^n)$ with $m(A) \geq 0$, see [6, Lemma 1.2]. These families play an important role in geometric function theory in higher dimensions (see [8]).

Taking into account [7] (see also [21]), we consider the following definition.

Definition 2.2. Let $A : [0, \infty) \to L(\mathbb{C}^n)$ be a measurable mapping which is locally integrable on $[0, \infty)$. For every $s \geq 0$, $V(s, \cdot) : [s, \infty) \to L(\mathbb{C}^n)$ is the unique locally absolutely continuous solution of the initial value problem ([2])

$$\text{(1)} \qquad \frac{\partial V}{\partial t}(s, t) = -A(t)V(s, t), \text{ for a.e. } t \in [s, \infty), V(s, s) = I_n.$$

Let $V(t) = V(0,t)$, for all $t \geq 0$.

Remark 2.3 ([2]). Let $A : [0,\infty) \to L(\mathbb{C}^n)$ be a measurable mapping which is locally integrable and let V be given by (1). The following hold:

(i) $V(t)^{-1}$ exists, for every $t \geq 0$, and $\frac{\partial}{\partial t} V(t)^{-1} = V(t)^{-1} A(t)$, for a.e. $t \geq 0$.

(ii) $V(s,t) = V(t)V(s)^{-1}$ for $0 \leq s \leq t < \infty$.

(iii) if $A(t)$ and $\int_s^t A(\tau)d\tau$ commute, for all $t \geq s$, then

$$V(s,t) = e^{-\int_s^t A(\tau)d\tau}, \quad \forall\, t \in [s,\infty).$$

Definition 2.4. A family $\{F_t\}_{t\geq 0} \subset H(\mathbb{B}^n)$ is a subordination chain if:

- $F_s(\mathbb{B}^n) \subseteq F_t(\mathbb{B}^n)$, $0 \leq s \leq t$,
- $F_t(0) = 0$, $t \geq 0$.

Moreover, $\{F_t\}_{t\geq 0}$ is called a Loewner chain, if, in addition,

- F_t is univalent, $t \geq 0$.

Furthermore, $\{F_t\}_{t\geq 0}$ is called a normal Loewner chain, if, in addition,

- $\left\{DF_t(0)^{-1}F_t\right\}_{t\geq 0}$ is a normal family in $S(\mathbb{B}^n)$.

If $\{F_t\}_{t\geq 0}$ is a Loewner chain, then $v_{s,t} = F_t^{-1} \circ F_s$, $0 \leq s \leq t$, forms a family of univalent Schwarz mappings on $\mathbb{B}^n$ (i.e., self-mappings that fix the origin), called the family of transition mappings of $\{F_t\}_{t\geq 0}$.

If $\{F_t\}_{t\geq 0}$ satisfies $DF_t(0) = V(t)^{-1}$, $t \geq 0$, where V is given by (1) associated to a measurable and locally integrable $A : [0,\infty) \to L(\mathbb{C}^n)$, then $\{F_t\}_{t\geq 0}$ is said to be a subordination chain with respect to A. In the case $DF_t(0) = e^t I$, $t \geq 0$, the condition regarding the normality in the above definition was first pointed out in [4, 9] (cf. [18]) to play an important role in Loewner Theory on $\mathbb{B}^n$. The case $DF_t(0) = e^{tA}$, $t \geq 0$, when $A \in L(\mathbb{C}^n)$, was first studied in [6]. The general case of no normalization was first investigated in [7]. A very general study of non-normalized Loewner chains on complex manifolds was considered in [1].

Definition 2.5 ([11]). $\widetilde{\mathcal{A}}$ is the family of mappings $A : [0,\infty) \to L(\mathbb{C}^n)$ that are measurable and satisfy:

- $m(A(\tau)) \geq 0$, for a.e. $\tau \geq 0$;
- $\operatorname{ess\,sup}_{s\geq 0}\|A(s)\| < \infty$;

- $\sup_{s\geq 0}\int_s^\infty \|V(s,t)^{-1}\|e^{-2\int_s^t m(A(\tau))d\tau}dt < \infty$, where V is given by (1).

The above set of conditions for a time-dependent linear operator A preserves various important results from the Loewner Theory for time-independent linear operators. We show this in the next sections. Before we move on, let us mention some special cases. If A is constant and equal to I, then $A \in \widetilde{\mathcal{A}}$. If A is constant and equal to a linear operator $\mathbf{A}$, then Graham, Hamada, Kohr, Kohr [6] proved that the third condition for $A \in \widetilde{\mathcal{A}}$ is equivalent with the elegant condition $k_+(\mathbf{A}) < 2m(\mathbf{A})$, where

$$k_+(\mathbf{A}) = \max\{\Re\lambda : \lambda \text{ is an eigenvalue of } \mathbf{A}\}.$$

3. THE LOEWNER DIFFERENTIAL EQUATIONS

Definition 3.1 ([7, Definition 1.5], [11]). Let $A : [0,\infty) \to L(\mathbb{C}^n)$ be a measurable mapping which is locally integrable. So, $h : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ is a Herglotz vector field with respect to A, if:

- $h(z,\cdot)$ is measurable and locally integrable on $[0,\infty)$, $z \in \mathbb{B}^n$,
- $h(\cdot,t) \in \mathcal{N}_{A(t)}(\mathbb{B}^n)$, for a.e. $t \geq 0$.

Next, $v(z,T,t;h)$ is the solution of the Loewner differential equation related to h:

$$\frac{dv}{dt} = -h(v,t), \text{ a.e. } t \in [T,\infty), \tag{2}$$

such that $v(z,T,T;h) = z$, $z \in \mathbb{B}^n, T \geq 0$. By [7] and [21], (2) always has a solution $v(z,T,\cdot;h)$ on $[T,\infty)$, for every $z \in \mathbb{B}^n$; moreover, $\{v(\cdot,s,t;h)\}_{0\leq s\leq t}$ is a family of Schwartz mappings on $\mathbb{B}^n$.

Taking into account [5, 7] (see also [1, 21]) we consider, in this section, the implications of the conditions for $A \in \widetilde{\mathcal{A}}$ in characterizing certain solutions of the generalized Loewner differential equation associated to a Herglotz vector field h with respect to A:

$$\frac{\partial F_t(z)}{\partial t} = DF_t(z)h(z,t), \quad \text{a.e. } t \geq 0, \quad z \in \mathbb{B}^n. \tag{3}$$

In the following, in view of [5], we say that $\{F_t\}_{t\geq 0} \subset H(\mathbb{B}^n)$ is a standard solution of (3), if $t \mapsto F_t(z)$ is locally absolutely continuous, locally uniformly with respect to $z \in \mathbb{B}^n$, $F_t(0) = 0$, $t \geq 0$, and, of course, satisfies (3). Note that, in view of (1), any standard solution $\{F_t\}_{t\geq 0}$ of (3) satisfies $DF_t(0) = V(t)^{-1}$, $t \geq 0$, if $DF_0(0) = I$.

Remark 3.2. By [21, Proposition 1.3.6] (see also [8], Chapter 8), every Loewner chain $\{F_t\}_{t\geq 0}$ with $t\mapsto DF_t(0)$ of local bounded variation is a solution of (3) with respect a certain Herglotz vector field. Moreover, we have that $\frac{\partial DF_t(0)}{\partial t} = DF_t(0)A(t)$, for a.e. $t\geq 0$ (compare this with Remark 2.3 *i*)), for some $A:[0,\infty)\to L(\mathbb{C}^n)$ measurable with $m(A(t))\geq 0$, $t\geq 0$.

The next theorem provides a connection between (2) and (3), through normal Loewner chains. It was basically proved in [7] (see also [21, Sections 1.5] for the same result under more general conditions; cf. [17]). The case of a time-independent linear operator was established in [6] (see also [8, Chapter 8]).

THEOREM 3.3. *Let $A\in\widetilde{\mathcal{A}}$ and h be a Herglotz vector field with respect to A. Then the following limit exists for every $T\geq 0$, locally uniformly with respect to z,*

$$\lim_{t\to\infty} V(T,t)^{-1}v(z,T,t;h) = F_T(z)$$

and $\{F_t\}_{t\geq 0}$ is a normal Loewner chain and a standard solution of (3) *associated to h.*

Conversely, if $\{F_t\}_{t\geq 0}$ is a standard solution of (3) *associated to h and $\{V(t)^{-1}F_t\}_{t\geq 0}$ is normal, then, for every $T\geq 0$ and $z\in\mathbb{B}^n$,*

$$F_T(z) = \lim_{t\to\infty} V(T,t)^{-1}v(z,T,t;h).$$

The normal Loewner chain $\{F_t\}_{t\geq 0}$ that satisfies (3), given by Theorem 3.3, is called the canonical solution of (3) (see [5]). In view of the proof of Theorem 3.3, $\{v(\cdot,s,t;h)\}_{0\leq s\leq t}$ is the family of transition mappings of $\{F_t\}_{t\geq 0}$ (see [5, 7, 21]). Hence, the following corollary holds (see [17]).

COROLLARY 3.4. *Let $A\in\widetilde{\mathcal{A}}$. If $\{F_t\}_{t\geq 0}$ is a normal Loewner chain with respect to A and $\{v_{s,t}\}_{0\leq s\leq t}$ is its family of transition mappings, then $F_T(z) = \lim_{t\to\infty} V(T,t)^{-1}v_{T,t}(z)$, for every $T\geq 0$.*

Remark 3.5. Muir provided recently an interesting example (see [17, Example 6.11]) of two normal Loewner chains $\{F_t\}_{t\geq 0}$ and $\{G_t\}_{t\geq 0}$, with respect to a time-dependent linear operator $A\notin\widetilde{\mathcal{A}}$, that share the same family $\{v_{s,t}\}_{0\leq s\leq t}$ of transition mappings, however they are distinct.

Remark 3.6 ([11]). If $A\in\widetilde{\mathcal{A}}$ and $\{F_t\}_{t\geq 0}$ is a normal Loewner chain with respect to A, then $\cup_{t\geq 0}F_t(\mathbb{B}^n)=\mathbb{C}^n$.

The following theorem is an extension of [5, Theorem 1.1]. One way to prove it is to use the results in [1] and [21, Section 1.5] (see also [17]). This characterizes the standard solutions of (3). For time-independent linear operators, see [3, 10].

THEOREM 3.7. *Let $A \in \widetilde{\mathcal{A}}$ and h be a Herglotz vector field with respect to A. Let $\{F_t\}_{t\geq 0}$ be the canonical solution of* (3) *associated to h and let $\{G_t\}_{t\geq 0} \subset H(\mathbb{B}^n)$. Then $\{G_t\}_{t\geq 0}$ is a standard solution of* (3) *if and only if there exists $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ holomorphic with $\Phi(0) = 0$ such that $G_t = \Phi \circ F_t$, for all $t \geq 0$.*

Note that $\{G_t\}_{t\geq 0}$ in the above theorem is a subordination chain and, moreover, it is a Loewner chain if and only if Φ is biholomorphic (see [5, 21]).

4. GENERALIZED SPIRALLIKE MAPPINGS

In this section, we consider some particular normal Loewner chains, given by spirallike mappings. Also, we take a look at Loewner chains of a certain order in relationship to spiral-shaped mappings.

Definition 4.1 ([1, 16]). Let $A : [0, \infty) \to L(\mathbb{C}^n)$ be a measurable and locally integrable mapping such that $m(A(t)) \geq 0$, for a.e. $t \geq 0$. A mapping $f \in H(\mathbb{B}^n)$ is said to be generalized spiral-shaped with respect to A, if f is univalent and $V(s,t)f(z) \in f(\mathbb{B}^n)$, for all $z \in \mathbb{B}^n$ and $0 \leq s \leq t < \infty$.

If f is a normalized (i.e., $f \in S(\mathbb{B}^n)$) generalized spiral-shaped mapping, then f is said to be a generalized spirallike mapping, see [7]. Moreover, if, in addition, A is constant, then we have the usual definition of an A-spirallike mapping ($e^{-tA}f(z) \in f(\mathbb{B}^n)$, for all $z \in \mathbb{B}^n, t \geq 0$), which can be characterized analytically using $\mathcal{N}_A(\mathbb{B}^n)$ (see [8, Theorem 6.4.10]). On the other hand, if f is generalized spiral-shaped with respect to a constant A, then f is called spiral-shaped, see [1].

The next proposition from [11] (see also [7]) shows a characterization of generalized spirallike mappings in terms, on one hand, of normal Loewner chains and, on the other hand, spirallike mappings. A detailed proof of it can be found in [14].

PROPOSITION 4.2. *Let $A \in \widetilde{\mathcal{A}}$ and let $f \in S(\mathbb{B}^n)$. Then the following statements are equivalent:*

(i) *f is a generalized spirallike mapping with respect to A.*

(ii) *f is $A(t)$-spirallike, for a.e. $t \geq 0$.*

(iii) *$F : \mathbb{B}^n \times [0, \infty) \to \mathbb{C}^n$ given by $F(z,t) = V(t)^{-1}f(z)$, $z \in \mathbb{B}^n$, $t \geq 0$, is a normal Loewner chain with respect to A.*

Recently, Muir [16] obtained a similar characterization for generalized spiral-shaped mappings, refining the above result. To present it, we need the following definition from [1] (see also [16]): $\{F_t\}_{t\geq 0}$ is called a Loewner chain of order $p \in [1,\infty]$, if $F_s(\mathbb{B}^n) \subseteq F_t(\mathbb{B}^n)$, $0 \leq s \leq t$, and there $t \mapsto F_t(z)$ is locally L^p-continuous on $[0,\infty)$, locally uniformly with respect to $z \in \mathbb{B}^n$. For various properties of these Loewner chains, see [1].

Remark 4.3. In view of [17, Theorem 3.3] and [21, Proposition 1.3.4], if $A : [0,\infty) \to L(\mathbb{C}^n)$ is a measurable mapping such that $\|A\|$ is locally L^p, for some $p \in [1,\infty]$, then every Loewner chain with respect to A is a Loewner chain of order p.

Proposition 4.4. *Let $A : [0,\infty) \to L(\mathbb{C}^n)$ be a measurable mapping such that $\|A\|$ is locally L^p, for some $p \in [1,\infty]$, and $m(A(t)) > 0$, for a.e. $t \geq 0$. Let $f \in H(\mathbb{B}^n)$ be univalent. Then the following statements are equivalent:*

(i) *f is a generalized spiral-shaped mapping with respect to A.*

(ii) *f is spiral-shaped with respect to $A(t)$, for a.e. $t \geq 0$.*

(iii) *$F : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ given by $F(z,t) = V(t)^{-1}f(z)$, $z \in \mathbb{B}^n$, $t \geq 0$, is a Loewner chain of order p.*

5. GENERALIZED PARAMETRIC REPRESENTATION

A natural extension of the generalized spirallike mappings is given by the mappings with generalized parametric representation, which we consider in this section.

Definition 5.1 ([11]). For $T \geq 0$ and $A : [0,\infty) \to L(\mathbb{C}^n)$ measurable and locally integrable, let

$$\widetilde{S}_A^T(\mathbb{B}^n) = \big\{f \in S(\mathbb{B}^n) : \exists\, h \text{ Herglotz vector field with respect to A such that } f = \lim_{t\to\infty} V(T,t)^{-1}v(\cdot,T,t;h)\big\},$$

where $V(s,t)$ is the unique solution on $[s,\infty)$ of the initial value problem (1). The mappings in $\widetilde{S}_A^T(\mathbb{B}^n)$ are said to have generalized parametric representation.

The results presented in Section 3 imply the following theorem. For a detailed proof, see [11, Theorem 3.3] (cf. [7, 21]).

Theorem 5.2. *Let $A \in \widetilde{\mathcal{A}}$. Then, for every $T \geq 0$,*

$$\widetilde{S}_A^T(\mathbb{B}^n) = \big\{f \in S(\mathbb{B}^n) : \exists\, \{F_t\}_{t\geq 0} \text{ normal Loewner chain such that } DF_t(0) = V(t)^{-1}, t \geq 0, \text{ and } f = V(T)F_T\big\}.$$

If A is constant and equal to I, then the above theorem was proved in [4, 9]. In this case, we have the well-known family $S^0(\mathbb{B}^n)$, introduced by Kohr [15]. If A is constant and equal to a linear operator $\mathbf{A}$, then we have the family $S^0_{\mathbf{A}}(\mathbb{B}^n)$, introduced by Graham, Hamada, Kohr, Kohr [6], who proved the above theorem with the condition $k_+(\mathbf{A}) < 2m(\mathbf{A})$. We see in the next section that the choice of T is irrelevant in the case of time-independent linear operators.

Corollary 5.3. *Any mapping that is generalized spirallike with respect to an $A \in \widetilde{\mathcal{A}}$ has generalized parametric representation.*

Corollary 5.4. *Let $A \in \widetilde{\mathcal{A}}$. If $\{F_t\}_{t\geq 0}$ is a normal Loewner chain with respect to A, then $V(t)F_t \in \widetilde{S}^t_A(\mathbb{B}^n)$, for every $t \geq 0$.*

Remark 5.5. Recently, Muir (see [17, Example 6.11]) gave an example of a normal Loewner chain $\{F_t\}_{t\geq 0}$ with respect to a time-dependent linear operator $A \notin \widetilde{\mathcal{A}}$ for which the above corollary fails to hold.

Remark 5.6. The set of conditions for $A \in \widetilde{\mathcal{A}}$ imply that $\widetilde{S}^T_A(\mathbb{B}^n)$ is compact, which is another similarity between $\widetilde{S}^T_A(\mathbb{B}^n)$ and $S^0(\mathbb{B}^n)$. This was proved in [6] for A time-independent, and in [11] for A time-dependent.

Remark 5.7. The set of conditions for $A \in \widetilde{\mathcal{A}}$ provide various extremal properties and convergence results for $\widetilde{S}^T_A(\mathbb{B}^n)$, see [13].

6. GENERALIZED PARAMETRIC REPRESENTATION INDEPENDENT OF TIME

In view of the examples given in [11], there exist time-dependent linear operators $A \in \widetilde{\mathcal{A}}$ such that $\widetilde{S}^s_A(\mathbb{B}^n) \neq \widetilde{S}^t_A(\mathbb{B}^n)$, for some $t > s \geq 0$. In this section, we discuss the case $\widetilde{S}^s_A(\mathbb{B}^n) = \widetilde{S}^t_A(\mathbb{B}^n)$, for all $t > s \geq 0$. In fact, we focus on the special situation $\widetilde{S}^t_A(\mathbb{B}^n) = S^0_{\mathbf{A}}(\mathbb{B}^n)$, for all $t \geq 0$, when A is a time-dependent operator and $\mathbf{A}$ is a time-independent operator.

Proposition 6.1 ([17, Theorem 4.1]). *Let $a : [0, \infty) \to (0, \infty)$ be a measurable and locally integrable function such that $\int_0^\infty a(t)dt = \infty$. Also, let $\mathbf{A} \in L(\mathbb{C}^n)$ be such that $m(\mathbf{A}) > 0$ and let $A : [0, \infty) \to L(\mathbb{C}^n)$ be given by $A(t) = a(t)\mathbf{A}$, $t \geq 0$. Then $\widetilde{S}^T_A(\mathbb{B}^n) = S^0_{\mathbf{A}}(\mathbb{B}^n)$, for all $T \geq 0$.*

The above result significantly improves [11, Proposition 3.7].

Proposition 6.2 ([12, Proposition 4.3]). *Let $a : [0, \infty) \to [\alpha, \beta]$, where $0 < \alpha < \beta < \infty$, be a measurable function. Also, let $A : [0, \infty) \to L(\mathbb{C}^n)$ be such that $A(t) + A(t)^* = a(t)I$, $t \geq 0$. Then $A \in \widetilde{\mathcal{A}}$ and $\widetilde{S}^T_A(\mathbb{B}^n) = S^0(\mathbb{B}^n)$, for all $T \geq 0$.*

PROPOSITION 6.3 ([17, Theorem 4.4]). *Let $a : [0, \infty) \to \mathbb{C}$ be a measurable and locally integrable function such that $\operatorname{Re} a(t) > 0$, for a.e. $t \geq 0$, and $\int_0^\infty \operatorname{Re} a(t)dt = \infty$. Also, let $\mathbf{A} \in L(\mathbb{C}^n)$ be Hermitian positive definite and let $A : [0, \infty) \to L(\mathbb{C}^n)$ be given by $A(t) = a(t)\mathbf{A}$, $t \geq 0$. Then $\widetilde{S}_A^T(\mathbb{B}^n) = S^0(\mathbb{B}^n)$, for all $T \geq 0$.*

Even though Propositions 6.2 and 6.3 have the same conclusion, they are quite different in view of [12, Example 4.2]. On the other hand, Proposition 6.3 implies the following result of Muir, which improves [11, Corollary 3.8].

COROLLARY 6.4 ([17, Corollary 4.6]). *Let $a : [0, \infty) \to \mathbb{C}$ be a measurable and locally integrable function such that $\operatorname{Re} a(t) > 0$, for a.e. $t \geq 0$, and $\int_0^\infty \operatorname{Re} a(t)dt = \infty$. Then $\widetilde{S}_a^T(\mathbb{U}) = S^0(\mathbb{U}) = S$, for all $T \geq 0$.*

According to [17, Corollary 4.11], if, in the above corollary, we consider the opposite condition: $\int_0^\infty \operatorname{Re} a(t)dt < \infty$, then we have that $\widetilde{S}_a^s(\mathbb{U}) \neq \widetilde{S}_a^t(\mathbb{U})$, for all $t > s \geq 0$.

We finish with some questions. Some partial answers have been presented above (cf. [13]).

Question 6.5. Under which necessary conditions for $A \in \widetilde{\mathcal{A}}$ do we have $\widetilde{S}_A^T(\mathbb{B}^n) = \widetilde{S}_A^0(\mathbb{B}^n)$, for all $T \geq 0$?

Question 6.6. Let $A \in \widetilde{\mathcal{A}}$ and $T \geq 0$. Does there exist $\mathbf{A} \in L(\mathbb{C}^n)$ such that $k_+(\mathbf{A}) < 2m(\mathbf{A})$ and $\widetilde{S}_A^T(\mathbb{B}^n) = S_{\mathbf{A}}^0(\mathbb{B}^n)$?

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