

# POINTED HOPF SUPERALGEBRAS OF DIMENSION UP TO 10

TAIKI SHIBATA, KENICHI SHIMIZU, and RYOTA WAKAO

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By utilizing the technique introduced in our previous work to construct Hopf superalgebras by an inverse procedure of the Radford–Majid bosonization, we classify non-semisimple pointed Hopf superalgebras of dimension up to 10 over an algebraically closed field of characteristic zero.

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## 1. INTRODUCTION

The importance of the study of Hopf algebras in braided monoidal categories was recognized through the investigation of quantum groups and their applications. For example, Nichols algebras have been actively investigated for the classification of pointed Hopf algebras [3]–[7, 10, 13], a class of Hopf algebras including quantized enveloping algebras. The category  $\mathcal{SV}$  of superspaces over a field  $\mathbb{k}$  of characteristic  $\neq 2$  (see Section 2.1) is a simple but significant example of braided monoidal categories. Hopf algebras in  $\mathcal{SV}$  are called *Hopf superalgebras*. The enveloping algebra of a Lie superalgebra and its quantization are important examples of Hopf superalgebras in representation theory, low-dimensional topology, mathematical physics, etc.

The classification problem of finite-dimensional Hopf algebras of a given dimension has been actively studied by many researchers after it was proposed by Kaplansky in 1975; see [9] for a survey. The classification problem of finite-dimensional Hopf superalgebras over an algebraically closed field of characteristic zero could also be fundamental and crucial. Some families of finite-dimensional Hopf superalgebras have been studied. For instance, Andruskiewitsch, Etingof and Gelaki [3] classified a class of triangular Hopf algebras by means of finite supergroups. Andruskiewitsch, Angiono and Yamane [2] developed basic results on finite-dimensional pointed Hopf superalgebras. Regarding the classification problem of finite-dimensional Hopf superalgebras of a given dimension, Aissaoui and Makhlouf [1] classified those of dimension 2,

3 and 4. However, unlike the case of ordinary Hopf algebras, the systematic study of the classification problem seems to be just began.

The goal of this paper is to give a complete list of finite-dimensional pointed Hopf superalgebras of dimension up to 10 over an algebraically closed field of characteristic zero as an intermediate step to classify all Hopf superalgebras of those dimensions.

Our central method has been introduced in [18]. A Hopf superalgebra  $\mathcal{H}$  can be regarded as a Hopf algebra in the braided monoidal category  ${}_{\mathbb{k}\mathbb{Z}_2}^{\mathbb{k}\mathbb{Z}_2}\mathcal{YD}$  of Yetter–Drinfeld modules over  $\mathbb{k}\mathbb{Z}_2$ , and hence, we obtain a Hopf algebra  $\widehat{\mathcal{H}} := \mathcal{H} \# \mathbb{k}\mathbb{Z}_2$  from the Hopf superalgebra  $\mathcal{H}$  by the Radford–Majid bosonization [14, 17]. We note that  $\mathcal{H}$  and  $\widehat{\mathcal{H}}$  share many properties: For example,  $\mathcal{H}$  is semisimple if and only if  $\widehat{\mathcal{H}}$  is.  $\mathcal{H}$  is pointed if and only if  $\widehat{\mathcal{H}}$  is. In [18], for a given Hopf algebra  $A$ , we have discussed how Hopf superalgebras  $\mathcal{H}$  such that  $\mathcal{H}_{\bar{1}} \neq 0$  and  $\widehat{\mathcal{H}} \cong A$  are obtained, where  $\mathcal{H}_{\bar{1}}$  is the odd part of  $\mathcal{H}$ . As a consequence, such Hopf superalgebras are parametrized by *super-data* for  $A$  (see Definition 3.4). This is a pair  $(g, \alpha)$  of group-like elements of  $g \in \mathbf{G}(A)$  and  $\alpha \in \mathbf{G}(A^*)$  such that  $g^2 = 1$ ,  $\alpha^2 = \varepsilon$ ,  $\alpha(g) = -1$ ,  $\alpha \rightharpoonup a \leftarrow \alpha = gag$  for all  $a \in A$  and  $g$  is not central in  $A$ . Furthermore, two Hopf superalgebras arising from two super-data  $(g, \alpha)$  and  $(g', \alpha')$  are isomorphic if and only if there is a Hopf algebra automorphism  $\varphi$  of  $A$  such that  $\varphi(g) = g'$  and  $\alpha' \circ \varphi = \alpha$ . Thus, the classification of Hopf superalgebras  $\mathcal{H}$  such that  $\mathcal{H}_{\bar{1}} \neq 0$  and  $\widehat{\mathcal{H}} \cong A$  are completed by the following strategy: First, determine all super-data for  $A$ . Second, determine Hopf algebra automorphisms of  $A$ . Finally, give a presentation of the resulting Hopf superalgebras.

If a classification of, say, pointed Hopf algebras of dimension  $n$  has been known, then the classification of pointed Hopf superalgebras of dimension  $n/2$  are obtained by the above program. From now on, we work over an algebraically closed field  $\mathbb{k}$  of characteristic zero. Let  $p$  be an odd prime number. By exploiting results on Hopf algebras of dimensions  $2p$  and  $2p^2$  [4, 15, 16], we obtain the following theorem.

**THEOREM** (= Theorems 2.3 and 4.7). *Let  $\mathcal{H}$  be a Hopf superalgebra over  $\mathbb{k}$ .*

1. *If  $\dim(\mathcal{H}) = p$ , then  $\mathcal{H}$  is purely even, that is,  $\mathcal{H}_{\bar{1}} = 0$ .*
2. *If  $\dim(\mathcal{H}) = p^2$  and  $\mathcal{H}$  is non-semisimple pointed,  $\mathcal{H}$  is purely even.*

It is easy to see that the exterior superalgebra  $\bigwedge \mathbb{k}$  is the only (up to isomorphism) Hopf superalgebra of dimension 2 whose odd part is non-trivial. Thanks to the classification of 8-dimensional Hopf algebras by Ştefan [19] and that of 16-dimensional pointed Hopf algebras by Caenepeel, Dăscălescu and

Raianu [10], we classify pointed Hopf superalgebras of dimensions 4 and 8. Hence, we complete the classification of pointed Hopf superalgebras of dimension up to 10.

### Classification lists

To present our classification result uniformly, we introduce the pointed Hopf superalgebra  $\mathcal{A}(\Gamma, \mathcal{D})$  constructed from a finite abelian group  $\Gamma$  and a compatible datum  $\mathcal{D} \in (\Gamma \times \widehat{\Gamma} \times \{0, 1\} \times \{0, 1\})^\theta$  as a kind of super-version of pointed Hopf algebras introduced by Andruskiewitsch and Schneider [5], see Section 4.1 for the precise definition. Here,  $\widehat{\Gamma}$  is the character group of  $\Gamma$  and  $\theta$  is a natural number. Using this notation, we display our classification result below, where 1 denote the trivial character of  $\Gamma$ .

**Dimension  $2p$ .** Let  $p$  denote an odd prime, and let  $\zeta_p \in \mathbb{k}$  denote a fixed primitive  $p$ -th root of unity. Let  $\Gamma$  be the group  $C_p = \langle g \mid g^p = 1 \rangle$  of order  $p$ , and let  $\chi$  be the character defined by  $\chi(g) = \zeta_p$ . A complete list of pairwise non-isomorphic non-semisimple pointed Hopf superalgebras  $\mathcal{H}$  of dimension  $2p$  satisfying  $\mathcal{H}_{\bar{1}} \neq 0$  are given by Table 1 (Theorem 4.8). For explicit relations between  $\mathcal{H}_{2p}^{(i)}$  ( $i \in \{1, 2, 3, 4\}$ ) and  $\mathcal{A}(\Gamma, \mathcal{D})$ , see Section 4.2.

| Hopf superalgebras $\mathcal{H}$<br>with $\mathcal{H}_{\bar{1}} \neq 0$ | $\Gamma$ | $\mathcal{D} = (g_i, \chi_i, \mu_i; \epsilon_i)_{i=1}^\theta$ | The dual $\mathcal{H}^*$ of $\mathcal{H}$   |
|---|----------|---|---|
| $\mathcal{H}_{2p}^{(1)} = \mathbb{k}C_p \otimes \wedge \mathbb{k}$      | $C_p$    | $(1, 1, 0; 1)$  | self-dual                                   |
| $\mathcal{H}_{2p}^{(2)}$  |          | $(g^{(p+1)/2}, 1, 0; 1)$                                      | $\mathcal{H}_{2p}^{(3)}$                    |
| $\mathcal{H}_{2p}^{(3)}$  |          | $(1, \chi, 0; 1)$   | $\mathcal{H}_{2p}^{(2)}$                    |
| $\mathcal{H}_{2p}^{(4)}$  |          | $(g^{(p+1)/2}, 1, 1; 1)$                                      | non-pointed for $p = 3, 5$<br>(Theorem 5.4) |

Table 1 – Non-semisimple pointed Hopf superalgebras of dimension  $2p$

**Dimension 4.** Let  $\Gamma$  be the group  $C_2 = \langle g \mid g^2 = 1 \rangle$  of order two, and let  $\chi$  be the non-trivial character, that is,  $\chi(g) = -1$ . A complete list of pairwise non-isomorphic non-semisimple pointed Hopf superalgebras  $\mathcal{H}$  of dimension 4 satisfying  $\mathcal{H}_{\bar{1}} \neq 0$  are given by Table 2 (Theorem 4.10).

| Hopf superalgebras $\mathcal{H}$<br>with $\mathcal{H}_{\bar{1}} \neq 0$ | $\Gamma$ | $\mathcal{D} = (g_i, \chi_i, \mu_i; \epsilon_i)_{i=1}^{\theta}$ | The dual $\mathcal{H}^*$ of $\mathcal{H}$ |
|---|----------|---|---|
| $\mathcal{H}_4^{(1)} = \bigwedge \mathbb{k}^2$                          | $\{1\}$  | $((1, 1, 0; 1), (1, 1, 0; 1))$                                  | self-dual                                 |
| $\mathcal{H}_4^{(2)} = \mathbb{k}C_2 \otimes \bigwedge \mathbb{k}$      | $C_2$    | $(1, 1, 0; 1)$  | self-dual                                 |
| $\mathcal{H}_4^{(3)}$   |          | $(g, 1, 0; 1)$  | $\mathcal{H}_4^{(4)}$                     |
| $\mathcal{H}_4^{(4)}$   |          | $(1, \chi, 0; 1)$   | $\mathcal{H}_4^{(3)}$                     |

Table 2 – Non-semisimple pointed Hopf superalgebras of dimension 4

**Dimension 8.** Let  $\zeta_4 \in \mathbb{k}$  denote a fixed primitive fourth root of unity. If  $\Gamma$  is  $C_2 = \langle g \mid g^2 = 1 \rangle$  (respectively,  $C_2 \times C_2 = \langle g_1, g_2 \mid g_1^2 = g_2^2 = 1, g_1g_2 = g_2g_1 \rangle$ ,  $C_4 = \langle g \mid g^4 = 1 \rangle$ ), then we take  $\chi$  (respectively,  $\chi_1, \chi_2, \chi$ ) so that  $\chi(g) = -1$  (respectively,  $\chi_i(g_j) = -(-1)^{i+j}$ ,  $\chi(g) = \zeta_4$ ). A complete list of pairwise non-isomorphic non-semisimple pointed Hopf superalgebras  $\mathcal{H}$  of dimension 8 satisfying  $\mathcal{H}_{\bar{1}} \neq 0$  are given by Table 3 (Theorem 4.14).

### Organization of the paper

This paper is organized as follows. In Section 2, we review definitions and properties of Hopf superalgebras. Fundamental results on the bosonization, which are useful for the classification of Hopf superalgebras, are recalled from [18] in Sections 2.2 and 3.1. We say that a Hopf superalgebra  $\mathcal{H}$  is a super-form of a Hopf algebra  $A$  if  $A \cong \widehat{\mathcal{H}}$ . In Section 3, we discuss super-forms of finite-dimensional pointed Hopf algebras with abelian coradical and, relying the classification of such Hopf algebras [6, 7], we show that it is generated by group-like elements and skew-primitive elements. As an example, we determine all super-forms of a Taft algebra.

In Sections 4 and 5, we work over an algebraically closed field of characteristic zero. In Section 4, after introducing the pointed Hopf superalgebra  $\mathcal{A}(\Gamma, \mathcal{D})$ , we classify super-forms of pointed Hopf algebras of dimension up to 20. Non-semisimple pointed Hopf superalgebras of dimensions  $p^2, 2p, 4$  and 8 (where  $p$  is an odd prime number) are classified in Theorems 4.7, 4.8, 4.10 and 4.14. As a result, we obtain the classification of non-semisimple pointed Hopf superalgebra of dimension up to 10, as explained in the above.

In the final Section 5, we determine duals of the Hopf superalgebras appeared in Section 4.

| Hopf superalgebras $\mathcal{H}$<br>with $\mathcal{H}_{\bar{1}} \neq 0$         | $\Gamma$ | $\mathcal{D} = (g_i, \chi_i, \mu_i; \epsilon_i)_{i=1}^{\theta}$ | The dual $\mathcal{H}^*$ of $\mathcal{H}$ |
|---|----------|---|---|
| $\mathcal{H}_8^{(1)} = \bigwedge \mathbb{k}^3$                                  | $\{1\}$  | $((1, 1, 0; 1), (1, 1, 0; 1), (1, 1, 0; 1))$                    | self-dual                                 |
| $\mathcal{H}_8^{(2)} = \mathbb{k}C_2 \otimes \bigwedge \mathbb{k}^2$            | $C_2$    | $((1, 1, 0; 1), (1, 1, 0; 1))$                                  | self-dual                                 |
| $\mathcal{H}_8^{(3)}$   |          | $((1, 1, 0; 1), (g, 1, 0; 1))$                                  | $\mathcal{H}_8^{(4)}$                     |
| $\mathcal{H}_8^{(4)}$   |          | $((1, 1, 0; 1), (1, \chi, 0; 1))$                               | $\mathcal{H}_8^{(3)}$                     |
| $\mathcal{H}_8^{(5)}$   |          | $((g, 1, 0; 1), (g, 1, 0; 1))$                                  | $\mathcal{H}_8^{(6)}$                     |
| $\mathcal{H}_8^{(6)}$   |          | $((1, \chi, 0; 1), (1, \chi, 0; 1))$                            | $\mathcal{H}_8^{(5)}$                     |
| $\mathcal{H}_8^{(7)} = T_4(-1) \otimes \bigwedge \mathbb{k}$                    |          |   | $((g, \chi, 0; 0), (1, 1, 0; 1))$         |
| $\mathcal{H}_8^{(8)} = \mathbb{k}(C_2 \times C_2) \otimes \bigwedge \mathbb{k}$ | $C_2^2$  | $(1, 1, 0; 1)$  | self-dual                                 |
| $\mathcal{H}_8^{(9)}$   |          | $(g_1, 1, 0; 1)$  | $\mathcal{H}_8^{(10)}$                    |
| $\mathcal{H}_8^{(10)}$  |          | $(1, \chi_1, 0; 1)$   | $\mathcal{H}_8^{(9)}$                     |
| $\mathcal{H}_8^{(11)}$  |          | $(g_1, \chi_1, 0; 1)$   | self-dual                                 |
| $\mathcal{H}_8^{(12)} = \mathbb{k}C_4 \otimes \bigwedge \mathbb{k}$             | $C_4$    | $(1, 1, 0; 1)$  | self-dual                                 |
| $\mathcal{H}_8^{(13)}$  |          | $(g, 1, 0; 1)$  | $\mathcal{H}_8^{(15)}$                    |
| $\mathcal{H}_8^{(14)}$  |          | $(g^2, 1, 0; 1)$  | $\mathcal{H}_8^{(16)}$                    |
| $\mathcal{H}_8^{(15)}$  |          | $(1, \chi, 0; 1)$   | $\mathcal{H}_8^{(13)}$                    |
| $\mathcal{H}_8^{(16)}$  |          | $(1, \chi^2, 0; 1)$   | $\mathcal{H}_8^{(14)}$                    |
| $\mathcal{H}_8^{(17)}$  |          | $(g^2, \chi^2, 0; 1)$   | self-dual                                 |
| $\mathcal{H}_8^{(18)}$  |          |   | $(g, 1, 1; 1)$                            |

Table 3 – Non-semisimple pointed Hopf superalgebras of dimension 8

## 2. PRELIMINARIES

In this section, we work over a field  $\mathbb{k}$  of characteristic not equal to 2. The unadorned symbol  $\otimes$  means the tensor product over  $\mathbb{k}$ . We denote by  $\mathbb{Z}_2$  the additive group of integers of modulo 2. The class of  $n \in \mathbb{Z}$  in  $\mathbb{Z}_2$  is written as  $\bar{n}$  or, by abuse of notation, by the same symbol  $n$ .

The group algebra of  $\mathbb{Z}_2$  is denoted by  $\mathbb{k}\mathbb{Z}_2$ . When we consider  $\mathbb{k}\mathbb{Z}_2$ , we identify  $\mathbb{Z}_2$  with the multiplicative group  $\{\mathbf{e}, \boldsymbol{\sigma}\}$  of order two, where  $\mathbf{e}$  is the identity element and  $\boldsymbol{\sigma}^2 = \mathbf{e}$ , for notational convenience.

Given a vector space  $X$ , we denote by  $X^*$  the dual space of  $X$ . Let  $C$  be a coalgebra with comultiplication  $\Delta$ . We use the Heyneman–Sweedler notation

$\Delta(c) = c_{(1)} \otimes c_{(2)}$  to express the comultiplication of  $c \in C$ . The dual space  $C^*$  is an algebra with respect to the multiplication given by  $fg(c) = f(c_{(1)})g(c_{(2)})$  for  $f, g \in C^*$  and  $c \in C$ . We note that  $C$  is a  $C^*$ -bimodule by the actions given by

$$f \rightharpoonup c := c_{(1)}f(c_{(2)}), \quad c \leftharpoonup f := f(c_{(1)})c_{(2)} \quad (f \in C^*, c \in C).$$

### 2.1. Hopf superalgebras

We denote by  $\mathcal{SV}$  the category of *superspaces*. Namely, an object of this category is a vector space  $V = V_0 \oplus V_1$  graded by the group  $\mathbb{Z}_2$  and a morphism is a linear map respecting the  $\mathbb{Z}_2$ -grading. For a homogeneous element  $0 \neq v \in V_0 \cup V_1$ , we denote its degree by  $|v|$ . We say that  $v \in V$  is an *even* (respectively, *odd*) element if  $|v| = 0$  respectively,  $|v| = 1$ . For simplicity, when we write  $|v|$ ,  $v$  is always supposed to be homogeneous. We say that  $V \in \mathcal{SV}$  is *purely even* if  $V_1 = 0$ .

The dual space  $V^*$  of  $V \in \mathcal{SV}$  is a superspace by letting  $(V^*)_\epsilon := (V_\epsilon)^*$  ( $\epsilon \in \{0, 1\}$ ). There is a natural tensor product in the category  $\mathcal{SV}$ . The category  $\mathcal{SV}$  is, in fact, a symmetric tensor category with respect to the natural isomorphism

$$V \otimes W \longrightarrow W \otimes V; \quad v \otimes w \longmapsto (-1)^{|v||w|} w \otimes v \quad (V, W \in \mathcal{SV}),$$

called the *supersymmetry*.

A *superalgebra* (respectively, *supercoalgebra*, *Hopf superalgebra*) is an algebra (respectively, coalgebra, Hopf algebra) in the symmetric tensor category  $\mathcal{SV}$ . A left supermodule over a superalgebra  $\mathcal{A}$  is a superspace  $V$  equipped with a morphism  $\mathcal{A} \otimes V \rightarrow V$  in  $\mathcal{SV}$  satisfying the associativity and the unit axioms. The definition of a right  $\mathcal{A}$ -supermodule should be clear. Supercomodules over a supercoalgebra are defined analogously.

*Definition 2.1.* Let  $\mathcal{H}$  be a Hopf superalgebra.

1.  $\mathcal{H}$  is said to be *(co)semisimple* if the category of left  $\mathcal{H}$ -super(co)modules is semisimple.
2.  $\mathcal{H}$  is said to be *pointed* if any simple right  $\mathcal{H}$ -supercomodule is one-dimensional.
3.  $\mathcal{H}$  is said to have the *Chevalley property* if the tensor product of any two simple right  $\mathcal{H}$ -supercomodule is semisimple.

The above definition of the Chevalley property is a super-analogue of the Chevalley property of ordinary Hopf algebras considered in [9] and [12].

We warn that this term has been used in a different meaning in literature including [3].

We are especially interested in finite-dimensional non-semisimple pointed Hopf superalgebras. In later, we show that a finite-dimensional semisimple pointed Hopf superalgebra is purely even and is isomorphic to a group algebra of a finite group (see Proposition 2.5).

Let  $\mathcal{H}$  be a Hopf superalgebra with comultiplication  $\Delta_{\mathcal{H}}$  and counit  $\varepsilon_{\mathcal{H}}$ . As in the non-super situation, the set

$$\mathbf{G}(\mathcal{H}) := \{g \in \mathcal{H}_0 \mid \varepsilon_{\mathcal{H}}(g) = 1, \Delta_{\mathcal{H}}(g) = g \otimes g\}$$

becomes a group under the multiplication of  $\mathcal{H}$ . An element of  $\mathbf{G}(\mathcal{H})$  is called a *group-like* element of  $\mathcal{H}$ . For a fixed  $g \in \mathbf{G}(\mathcal{H})$ , an element  $z \in \mathcal{H}$  is said to be  *$g$ -skew primitive* if it satisfies  $\Delta_{\mathcal{H}}(z) = g \otimes z + z \otimes 1$ . A 1-skew primitive element is simply called a *primitive element* of  $\mathcal{H}$ .

If  $\mathcal{H}$  is a finite-dimensional Hopf superalgebra, then one can make  $\mathcal{H}^*$  into a Hopf superalgebra, called the *dual* Hopf superalgebra of  $\mathcal{H}$ , in a similar way as the ungraded context. Let  $\mathcal{H}$  and  $\mathcal{A}$  be Hopf superalgebras, and let  $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{A} \rightarrow \mathbb{k}$  be a bilinear map satisfying  $\langle \mathcal{H}_{\bar{\varepsilon}}, \mathcal{A}_{\bar{\nu}} \rangle = 0$  for  $\varepsilon, \nu \in \{0, 1\}$  with  $\varepsilon \neq \nu$ . The map  $\langle \cdot, \cdot \rangle$  is called a *Hopf pairing* if it satisfies

$$\begin{aligned} \langle xy, a \rangle &= \langle x, a_{(1)} \rangle \langle y, a_{(2)} \rangle, & \langle x, ab \rangle &= \langle x_{(1)}, a \rangle \langle x_{(2)}, b \rangle, \\ \langle x, 1_{\mathcal{A}} \rangle &= \varepsilon_{\mathcal{H}}(x), & \langle 1_{\mathcal{H}}, a \rangle &= \varepsilon_{\mathcal{A}}(a) \end{aligned}$$

for  $x, y \in \mathcal{H}$  and  $a, b \in \mathcal{A}$ . If  $\langle \cdot, \cdot \rangle$  is a Hopf pairing and  $\mathcal{A}$  is finite-dimensional, then one sees that the map  $\mathcal{H} \rightarrow \mathcal{A}^*$ ;  $x \mapsto (a \mapsto \langle x, a \rangle)$  is a Hopf superalgebra map. Note that there exists another definition of a Hopf pairing taking into account the supersymmetry. However, one can show that if the base field  $\mathbb{k}$  is algebraically closed then these definitions coincide, see [18, Remark 3.10].

*Example 2.2.* Let  $V$  be a vector space. The exterior superalgebra  $\bigwedge V$  over  $V$  has a natural structure of a pointed Hopf superalgebra so that each  $z \in V$  is odd primitive. The canonical pairing  $\langle \cdot, \cdot \rangle : V \times V^* \rightarrow \mathbb{k}$  uniquely extends to a non-degenerate Hopf pairing  $\langle \cdot, \cdot \rangle : \bigwedge V \times \bigwedge V^* \rightarrow \mathbb{k}$  given by

$$\langle v_1 \wedge \cdots \wedge v_n, f_1 \wedge \cdots \wedge f_m \rangle = \delta_{n,m} \det(\langle v_i, f_j \rangle)_{i,j} \quad (n, m \in \mathbb{N}),$$

where  $\delta_{n,m}$  is the Kronecker symbol. In particular,  $\bigwedge V$  is self-dual if  $V$  is finite-dimensional.

## 2.2. Bosonization of Hopf superalgebras

Let  $H$  be a Hopf algebra, in general. By Radford [17] and Majid [14], up to isomorphism, there is a one-to-one correspondence between Hopf algebras

equipped with a split epimorphism onto  $H$  and Hopf algebras in the category  ${}^H_H\mathcal{YD}$  of Yetter–Drinfeld modules over  $H$ . For a Hopf algebra  $\mathcal{B}$  in  ${}^H_H\mathcal{YD}$ , we denote the corresponding Hopf algebra by  $\mathcal{B}\#H$ , called the *bosonization* of  $\mathcal{B}$  by  $H$ .

By definition, an object of  ${}^{\mathbb{k}\mathbb{Z}_2}_{\mathbb{k}\mathbb{Z}_2}\mathcal{YD}$  is an object  $V \in \mathcal{SV}$  equipped with a left action of  $\mathbb{k}\mathbb{Z}_2$  such that  $\sigma.V_\epsilon \subset V_\epsilon$  for  $\epsilon \in \{0, 1\}$ . A superspace  $V$  becomes an object of  ${}^{\mathbb{k}\mathbb{Z}_2}_{\mathbb{k}\mathbb{Z}_2}\mathcal{YD}$  by defining the left action of  $\mathbb{k}\mathbb{Z}_2$  by  $\sigma^i.v := (-1)^{i|v|}v$  for  $i \in \{0, 1\}$  and  $v \in V$ . In this way, we can regard  $\mathcal{SV} \subset {}^{\mathbb{k}\mathbb{Z}_2}_{\mathbb{k}\mathbb{Z}_2}\mathcal{YD}$  as braided monoidal categories.

Let  $\mathcal{H}$  be a Hopf superalgebra with comultiplication  $\Delta_{\mathcal{H}}$ , counit  $\varepsilon_{\mathcal{H}}$  and antipode  $S_{\mathcal{H}}$ . Since  $\mathcal{SV} \subset {}^{\mathbb{k}\mathbb{Z}_2}_{\mathbb{k}\mathbb{Z}_2}\mathcal{YD}$ , we can consider the bosonization

$$\widehat{\mathcal{H}} := \mathcal{H}\#\mathbb{k}\mathbb{Z}_2$$

of  $\mathcal{H}$  by  $\mathbb{k}\mathbb{Z}_2$ . More precisely, as a vector space  $\widehat{\mathcal{H}}$  is just  $H \otimes \mathbb{k}\mathbb{Z}_2$  and  $\widehat{\mathcal{H}}$  becomes an ordinary Hopf algebra whose structure is described as follows:

- (multiplication)  $(h \otimes \sigma^i)(h' \otimes \sigma^j) = (-1)^{i|h'|}hh' \otimes \sigma^{i+j}$ .
- (unit)  $1_{\widehat{\mathcal{H}}} = 1_{\mathcal{H}} \otimes e$ .
- (comultiplication)  $\Delta_{\widehat{\mathcal{H}}}(h \otimes \sigma^i) = h_{(1)} \otimes \sigma^{i+|h_{(2)}|} \otimes h_{(2)} \otimes \sigma^i$ .
- (counit)  $\varepsilon_{\widehat{\mathcal{H}}}(h \otimes \sigma^i) = \varepsilon_{\mathcal{H}}(h)$ .
- (antipode)  $S_{\widehat{\mathcal{H}}}(h \otimes \sigma^i) = (-1)^{i+|h|}S_{\mathcal{H}}(h) \otimes \sigma^{i+|h|}$ .

Here,  $h, h' \in \mathcal{H}$  and  $i, j \in \{0, 1\}$ . One easily sees  $\mathbf{G}(\widehat{\mathcal{H}}) \cong \mathbf{G}(\mathcal{H}) \times \mathbb{Z}_2$  as groups. If  $\mathcal{H}_{\bar{1}} \neq 0$ , then  $\widehat{\mathcal{H}}$  is neither commutative nor cocommutative. As an application of this observation, we obtain.

**THEOREM 2.3** ([18]). *All Hopf superalgebras of odd prime dimensions are purely even.*

*Proof.* Suppose that there is a Hopf superalgebra  $\mathcal{H}$  of dimension  $p$  for some odd prime number  $p$  such that  $\mathcal{H}_{\bar{1}} \neq 0$ . By the above observation,  $\widehat{\mathcal{H}}$  is a Hopf algebra of dimension  $2p$  that is neither commutative nor cocommutative. However, according to the classification of Hopf algebras of dimension  $2p$  due to Masuoka [15] and Ng [16], there is no such Hopf algebra. Thus, we have a contradiction.  $\square$

Since the category of left  $\mathcal{H}$ -super(co)modules and the category of left  $\widehat{\mathcal{H}}$ -(co)modules are equivalent, we get the following.

**PROPOSITION 2.4.** *The following holds:*



1.  $\mathcal{H}$  is (co)semisimple if and only if  $\widehat{\mathcal{H}}$  is (co)semisimple.
2.  $\mathcal{H}$  is pointed if and only if  $\widehat{\mathcal{H}}$  is pointed.
3.  $\mathcal{H}$  has the Chevalley property if and only if  $\widehat{\mathcal{H}}$  has the Chevalley property.

As another application of the bosonization technique, we prove the following.

**PROPOSITION 2.5.** *A finite-dimensional semisimple pointed Hopf superalgebra is purely even and isomorphic to the group algebra of a finite group.*

*Proof.* Let  $\mathcal{H}$  be such a Hopf superalgebra. Then the bosonization  $\widehat{\mathcal{H}}$  is a semisimple Hopf algebra. The Larson–Radford theorem implies that  $\widehat{\mathcal{H}}$  is cosemisimple as a coalgebra. This implies that  $\mathcal{H}$  is also cosemisimple as a supercoalgebra. By the assumption that  $\mathcal{H}$  is pointed,  $\mathcal{H}$  is spanned by group-like elements. Hence,  $\mathcal{H}$  is a group algebra.  $\square$

Suppose that  $\mathcal{H}$  is finite-dimensional. For the bosonization  $\widehat{\mathcal{H}}^*$  of  $\mathcal{H}^*$ , one easily sees that the bilinear map

$$\widehat{\mathcal{H}}^* \times \widehat{\mathcal{H}} \longrightarrow \mathbb{k}; \quad (f \otimes \sigma^i, h \otimes \sigma^j) \longmapsto (-1)^{ij} f(h)$$

is a non-degenerate Hopf pairing. As a consequence, we get the following.

**PROPOSITION 2.6.** *As a Hopf algebra, the bosonization of  $\mathcal{H}^*$  is isomorphic to the dual of  $\widehat{\mathcal{H}}$ .*

### 2.3. Coinvariant subalgebras

We give a summary of the classification method of finite-dimensional Hopf superalgebras proposed in [18]. Roughly speaking, this is done by enumerating all Hopf superalgebras  $\mathcal{H}$  such that  $\widehat{\mathcal{H}} \cong A$  for each finite-dimensional Hopf algebra  $A$ . As an intermediate step for accomplishing this, we first mention Hopf algebras  $\mathcal{H}$  in  ${}_{\mathbb{k}\mathbb{Z}_2}^{\mathbb{k}\mathbb{Z}_2}\mathcal{YD}$  such that  $\widehat{\mathcal{H}} \cong A$ .

*Definition 2.7.* Let  $A$  be a finite-dimensional Hopf algebra with counit  $\varepsilon_A$ . A pair  $(g, \alpha) \in \mathbf{G}(A) \times \mathbf{G}(A^*)$  is called an *admissible datum* for  $A$  if it satisfies

$$g^2 = 1, \quad \alpha^2 = \varepsilon_A \quad \text{and} \quad \alpha(g) = -1.$$

The set of all admissible data for  $A$  is denoted by  $\text{AD}(A)$ . For  $(g, \alpha), (g', \alpha') \in \text{AD}(A)$ , we write  $(g, \alpha) \sim (g', \alpha')$  if there exists  $\varphi \in \text{Aut}_{\text{Hopf}}(A)$  such that  $\varphi(g) = g'$  and  $\alpha = \alpha' \circ \varphi$ , where  $\text{Aut}_{\text{Hopf}}(A)$  is the group of all Hopf algebra automorphisms on  $A$ . It is obvious that the relation  $\sim$  becomes an equivalence relation on  $\text{AD}(A)$ .

For  $(g, \alpha) \in \text{AD}(A)$ , one easily sees that the map

$$(2.1) \quad \pi_{(g,\alpha)} : A \longrightarrow \mathbb{k}\mathbb{Z}_2; \quad a \longmapsto \frac{\varepsilon_A(a)}{2}(\mathbf{e} + \boldsymbol{\sigma}) + \frac{\alpha(a)}{2}(\mathbf{e} - \boldsymbol{\sigma})$$

is a split epimorphism with section  $\mathbb{k}\mathbb{Z}_2 \rightarrow A$ ;  $\boldsymbol{\sigma}^i \mapsto g^i$ . Moreover, any split epimorphism  $A \rightarrow \mathbb{k}\mathbb{Z}_2$  arises from an element of  $\text{AD}(A)$  in this way [18]. For the split epimorphism  $\pi := \pi_{(g,\alpha)}$ , the algebra

$$A^{\text{coinv}(g,\alpha)} := \{a \in A \mid a_{(1)} \otimes \pi(a_{(2)}) = a \otimes \mathbf{e}\}$$

of  $\pi$ -coinvariants of  $A$  is given by the following formula:

$$(2.2) \quad A^{\text{coinv}(g,\alpha)} = \{b \in A \mid b = \alpha \rightharpoonup b\}.$$

By the inverse procedure of the bosonization [14, 17], we see that  $A^{\text{coinv}(g,\alpha)}$  becomes a Hopf algebra in  $\frac{\mathbb{k}\mathbb{Z}_2}{\mathbb{k}\mathbb{Z}_2}\mathcal{YD}$  and get the following result.

**PROPOSITION 2.8** ([18]). *Let  $\mathcal{X}$  be the class of all Hopf algebras in  $\frac{\mathbb{k}\mathbb{Z}_2}{\mathbb{k}\mathbb{Z}_2}\mathcal{YD}$  whose bosonization is isomorphic to  $A$  as a Hopf algebra. The assignment  $(g, \alpha) \mapsto A^{\text{coinv}(g,\alpha)}$  gives a bijection from  $\text{AD}(A)/\sim$  to  $\mathcal{X}/\cong$ .*

### 3. SUPER-DATA AND SUPER-FORMS OF HOPF ALGEBRAS

In this section, we also work over a field  $\mathbb{k}$  of characteristic not equal to 2.

#### 3.1. Super-data and super-forms

We summarize results obtained in [18]. Let  $A$  be a finite-dimensional Hopf algebra.

*Definition 3.1.* We say that a Hopf superalgebra  $\mathcal{H}$  is a *super-form* of  $A$  if  $\widehat{\mathcal{H}}$  is isomorphic to  $A$ . If  $\mathcal{H}$  is purely even, then  $\mathcal{H}$  is called a *trivial super-form* of  $A$ .

By the properties of the bosonization, we obtain the following.

**PROPOSITION 3.2.** *Let  $\mathcal{H}$  be a super-form of  $A$ .*

1. *The groups  $\mathbf{G}(A)$  and  $\mathbf{G}(A^*)$  are decomposed into direct products with  $\mathbb{Z}_2$ .*
2. *If the super-form  $\mathcal{H}$  is non-trivial, then  $A$  is neither commutative nor cocommutative.*

3. If  $A$  is pointed (respectively, is (co)semisimple, has the Chevalley property), then  $\mathcal{H}$  is pointed (respectively, is (co)semisimple, has the Chevalley property).

By Proposition 2.8, the isomorphism classes of super-forms of  $A$  are in bijection with the equivalence classes of  $(g, \alpha) \in \text{AD}(A)$  such that  $A^{\text{coinv}(g, \alpha)} \in \frac{\mathbb{k}\mathbb{Z}_2}{\mathbb{k}\mathbb{Z}_2} \mathcal{YD}$  belongs to  $\mathcal{SV}$ . We have found the following easy criteria for  $A^{\text{coinv}(g, \alpha)}$  to be a Hopf superalgebra.

**THEOREM 3.3** ([18]).  $\mathcal{H} := A^{\text{coinv}(g, \alpha)}$  is a super-form of  $A$  if and only if

$$\alpha \rightharpoonup a \leftarrow \alpha = gag \quad \text{for all } a \in A.$$

If the above equivalent condition is satisfied, then the  $\mathbb{Z}_2$ -grading of  $\mathcal{H}$  is given by

$$\mathcal{H}_\epsilon = \{b \in A \mid gbg = (-1)^\epsilon b\}$$

for each  $\epsilon \in \{0, 1\}$ . In particular, the super-form  $\mathcal{H}$  of  $A$  is non-trivial if and only if  $g \notin Z(A)$ , where  $Z(A)$  is the center of the algebra  $A$ .

Taking the above into account, we introduce.

*Definition 3.4.* An admissible datum  $(g, \alpha)$  for  $A$  is called a *super-datum* for  $A$  if  $g \notin Z(A)$  and  $\alpha \rightharpoonup a \leftarrow \alpha = gag$  for all  $a \in A$ . The set of all super-datum for  $A$  is denoted by  $\text{SD}(A)$ .

By the above definition and Proposition 2.8, the map  $(g, \alpha) \mapsto A^{\text{coinv}(g, \alpha)}$  gives a one-to-one correspondence between  $\text{SD}(A)/\sim$  and the isomorphism classes of Hopf superalgebras  $\mathcal{H}$  such that  $\mathcal{H}_1 \neq 0$  and  $\widehat{\mathcal{H}} \cong A$ .

### 3.2. Super-forms of pointed Hopf algebras with abelian coradical

Let  $A$  be a finite-dimensional pointed Hopf algebra such that the group  $\mathbf{G}(A)$  is abelian. We note that such Hopf algebras were studied extensively and the classification of them was finally completed by Angiono and García in [6] based on numerous pioneering works on Nichols algebras and arithmetic root systems; see [7] for a survey. According to the classification results, we see that there exists a finite number of group-like elements  $c_1, \dots, c_\theta \in \mathbf{G}(A)$  and the same number of  $c_i$ -skew primitive elements  $x_i \in A$  ( $i \in \{1, \dots, \theta\}$ ) such that  $A$  is generated by  $\mathbf{G}(A) \cup \{x_i\}_{i=1}^\theta$  as an algebra. Moreover, for each  $i \in \{1, \dots, \theta\}$ , there exists a non-trivial group homomorphism  $\chi_i : \mathbf{G}(A) \rightarrow \mathbb{k}^\times$  such that

$$\gamma x_i \gamma^{-1} = \chi_i(\gamma) x_i$$

for all  $\gamma \in \mathbf{G}(A)$ . We give the following technical remark.

**LEMMA 3.5.** For all  $\alpha \in \mathbf{G}(A^*)$  and  $i \in \{1, \dots, \theta\}$ , we have  $\alpha(x_i) = 0$ .

*Proof.* Let  $i \in \{1, \dots, \theta\}$ . Since  $\chi_i$  is non-trivial, there exists  $\gamma \in \mathbf{G}(A)$  such that  $\chi_i(\gamma) \neq 1$ . We have  $\alpha(x_i) = \alpha(\gamma x_i \gamma^{-1}) = \chi_i(\gamma)\alpha(x_i)$ , and hence  $\alpha(x_i)$  must be zero. Thus, the proof is done.  $\square$

By Lemma 3.5, the set of all super-data  $\text{SD}(A)$  for  $A$  is explicitly given by

$$(3.1) \quad \text{SD}(A) = \left\{ (g, \alpha) \in \text{AD}(A) \mid \begin{array}{l} g \text{ does not belong to } Z(A) \text{ and} \\ g x_i g = \alpha(c_i) x_i \text{ for all } i = 1, \dots, \theta \end{array} \right\}.$$

For  $\alpha \in \mathbf{G}(A^*)$ , we put

$$\mathbf{G}(A)^\alpha := \{ \gamma \in \mathbf{G}(A) \mid \alpha(\gamma) = 1 \}$$

for simplicity. Note that if  $\alpha^2 = \varepsilon_A$ , then  $\alpha(\gamma) = \pm 1$  for all  $\gamma \in \mathbf{G}(A)$ .

**THEOREM 3.6.** *Let  $(g, \alpha) \in \text{SD}(A)$ . As an algebra, the coinvariant subalgebra  $\mathcal{H} := A^{\text{coinv}(g, \alpha)}$  of  $A$  is generated by  $\mathbf{G}(A)^\alpha \cup \{x_i\}_{i=1}^\theta$ . The supercoalgebra structure of  $\mathcal{H}$  is described as follows.*

1.  $\mathbf{G}(\mathcal{H}) = \mathbf{G}(A)^\alpha$ .
2. For each  $i \in \{1, \dots, \theta\}$ ,  $x_i$  is even  $c_i$ -skew primitive in  $\mathcal{H}$  if  $\alpha(c_i) = 1$ .
3. For each  $i \in \{1, \dots, \theta\}$ ,  $x_i$  is odd  $c_i g$ -skew primitive in  $\mathcal{H}$  if  $\alpha(c_i) = -1$ .

*Proof.* Let  $\mathcal{H}'$  be the subalgebra of  $A$  generated by  $\mathbf{G}(A)^\alpha \cup \{x_i\}_{i=1}^\theta$ . We have  $\alpha \rightarrow \gamma = \alpha(\gamma)\gamma$  for all  $\gamma \in \mathbf{G}(A)$  and  $\alpha \rightarrow x_i = x_i$  for all  $i \in \{1, \dots, \theta\}$  by Lemma 3.5. Thus  $\mathcal{H}' \subset \mathcal{H}$ . Since  $\alpha(g) = -1$ , we have  $\mathbf{G}(A) = \mathbf{G}(A)^\alpha \sqcup g\mathbf{G}(A)^\alpha$  and, from this, we deduce  $A = \mathcal{H}' + g\mathcal{H}'$ . By considering the dimension, we conclude  $\mathcal{H} = \mathcal{H}'$ .  $\square$

## 4. CLASSIFICATION OF POINTED HOPF SUPERALGEBRAS

In this section, the base field  $\mathbb{k}$  is supposed to be an algebraically closed field of characteristic zero. The aim of this section is to classify non-semisimple pointed Hopf superalgebras of dimension up to 10.

### 4.1. The Hopf superalgebra $\mathcal{A}(\Gamma, \mathcal{D})$

To present our classification result uniformly, we introduce a family of pointed Hopf superalgebras  $\mathcal{A}(\Gamma, \mathcal{D})$  as in the same fashion as Andruskiewitsch and Schneider [5].

Let  $\Gamma$  be a finite abelian group with unit 1, and let  $\widehat{\Gamma}$  be the group of group homomorphisms from  $\Gamma$  to  $\mathbb{k}^\times$ . The trivial character is denoted by  $1 \in \widehat{\Gamma}$ . Let  $\theta$  be a natural number. We consider a datum

$$\mathcal{D} := (g_i, \chi_i, \mu_i; \epsilon_i)_{i=1}^\theta \in (\Gamma \times \widehat{\Gamma} \times \{0, 1\} \times \{0, 1\})^\theta$$

satisfying the following three conditions.

1. For each  $i \in \{1, \dots, \theta\}$ ,  $N_i \in 2\mathbb{Z}$  if  $\epsilon_i = 1$ .
2. For each  $i \in \{1, \dots, \theta\}$ ,  $\mu_i = 0$  if  $\chi_i^{N_i} \neq 1$ .
3. For each  $i, j \in \{1, \dots, \theta\}$  with  $i \neq j$ ,  $\chi_i(g_j)\chi_j(g_i) = 1$ .

Here, we put  $N_i := \text{ord}((-1)^{\epsilon_i}\chi_i(g_i))$  for each  $i \in \{1, \dots, \theta\}$ .

We let  $\mathcal{A}(\Gamma, \mathcal{D})$  denote the superalgebra generated by  $u_g$  with  $|u_g| = 0$  ( $g \in \Gamma$ ) and  $z_i$  with  $|z_i| = \epsilon_i$  ( $i \in \{1, \dots, \theta\}$ ) subject to

$$\begin{aligned} u_1 &= 1, & u_g u_h &= u_{gh}, & u_g z_i &= \chi_i(g) z_i u_g, \\ z_i^{N_i} &= \mu_i (1 - u_{g_i}^{N_i}), & z_i z_j &= (-1)^{\epsilon_i \epsilon_j} \chi_j(g_i) z_j z_i, \end{aligned}$$

where  $g, h \in \Gamma$  and  $i, j \in \{1, \dots, \theta\}$  with  $i \neq j$ .

**THEOREM 4.1.** *The superalgebra  $\mathcal{A}(\Gamma, \mathcal{D})$  has a unique structure of a Hopf superalgebra whose comultiplication  $\Delta$  is given by*

$$\Delta(u_g) = u_g \otimes u_g \quad (g \in \Gamma), \quad \Delta(z_i) = u_{g_i} \otimes z_i + z_i \otimes 1 \quad (i = 1, \dots, \theta).$$

*The Hopf superalgebra  $\mathcal{A}(\Gamma, \mathcal{D})$  is pointed with  $\mathbf{G}(\mathcal{A}(\Gamma, \mathcal{D})) \cong \Gamma$ .*

In the following, we identify  $g \in \Gamma$  with  $u_g \in \mathcal{A}(\Gamma, \mathcal{D})$  and regard  $\Gamma \subset \mathcal{A}(\Gamma, \mathcal{D})$ .

*Remark 4.2.* For each  $i \in \{1, \dots, \theta\}$ , we let  $\mathbb{k}_{g_i}^{\chi_i}[\epsilon_i]$  be a one-dimensional left Yetter–Drinfeld supermodule over  $\mathbb{k}\Gamma$  with basis  $z_i$  such that

$$(\text{parity}) |z_i| = \epsilon_i, \quad (\text{action}) g \cdot z_i = \chi_i(g) z_i \quad (g \in \Gamma), \quad (\text{coaction}) z_i \mapsto g_i \otimes z_i.$$

Then  $V := \mathbb{k}_{g_1}^{\chi_1}[\epsilon_1] \oplus \dots \oplus \mathbb{k}_{g_\theta}^{\chi_\theta}[\epsilon_\theta]$  becomes a left Yetter–Drinfeld supermodule over  $\mathbb{k}\Gamma$ . The associated braiding is

$$V \otimes V \rightarrow V \otimes V; \quad z_i \otimes z_j \mapsto (-1)^{\epsilon_i \epsilon_j} \chi_j(g_i) z_j \otimes z_i.$$

By [2, Section 1.7], we obtain the *Nichols superalgebra*  $\mathcal{B}(V)$  of  $V$  and we may consider the bosonization  $\mathcal{B}(V)\#\mathbb{k}\Gamma$  of  $\mathcal{B}(V)$  by  $\mathbb{k}\Gamma$ . One easily sees that the graded Hopf superalgebra of  $\mathcal{A}(\Gamma, \mathcal{D})$  associated to the radical filtration is isomorphic to the Hopf superalgebra  $\mathcal{B}(V)\#\mathbb{k}\Gamma$ .

*Example 4.3.* The exterior superalgebra (Example 2.2) of the vector space of dimension  $\theta$  is isomorphic to  $\mathcal{A}(\Gamma, \mathcal{D})$  with  $\Gamma = \{1\}$  and  $\mathcal{D} = (1, 1, 0; 1)_{i=1}^\theta$ .

*Example 4.4* (Super-forms of the Taft algebra). We fix a natural number  $n \geq 2$  and a primitive  $n$ -th root of unity  $\omega \in \mathbb{k}$ . We write down all non-trivial super-forms of the *Taft algebra*

$$T_{n^2}(\omega) := \mathbb{k}\langle c, x \mid c^n = 1, x^n = 0, cx = \omega xc \rangle,$$

where  $c$  is group-like and  $x$  is  $c$ -skew primitive.  $T_{n^2}(\omega)$  is a non-semisimple pointed Hopf algebra of dimension  $n^2$  such that  $\mathbf{G}(T_{n^2}(\omega)) \cong \mathbb{Z}_n$ . It is easy to see that

$$\text{AD}(T_{n^2}(\omega)) = \text{SD}(T_{n^2}(\omega)) = \begin{cases} \{(c^{n/2}, \alpha)\} & \text{if } n \text{ is even and } n/2 \text{ is odd,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Here,  $\alpha : T_{n^2}(\omega) \rightarrow \mathbb{k}$  is defined as  $\alpha(c) = -1$  and  $\alpha(x) = 0$ . In the following, we suppose that  $n$  is even and  $n/2$  is odd. Let  $\mathcal{H}$  be the super-form of  $T_{n^2}(\omega)$  associated to the super-datum  $(c^{n/2}, \alpha)$ . Then there is an isomorphism

$$\mathcal{T}_n(\omega^2) := \mathcal{A}(C_{n/2}, (g^{(n+2)/4}, \chi, 0; 1)) \cong \mathcal{H}; \quad g \mapsto c^2, \quad z \mapsto x$$

of Hopf superalgebras, where  $\chi$  is the character of  $C_{n/2} = \langle g \mid g^{n/2} = 1 \rangle$  defined by  $\chi(g) = \omega^2$ .

By the above discussion, we conclude that  $\mathcal{T}_n(\omega^2)$  is a unique super-form of  $T_{n^2}(\omega)$  up to isomorphisms. By Proposition 2.6 and the self-duality of  $T_{n^2}(\omega)$ , the Hopf superalgebra  $\mathcal{T}_n(\omega^2)$  is self-dual. As a side note,  $\mathcal{T}_2(1)$  is isomorphic to the exterior superalgebra  $\bigwedge \mathbb{k}$ .

### 4.2. Dimensions $2p$ and $p^2$

In this section, we discuss super-forms of the Hopf algebras  $\mathcal{A}(\omega, i, \mu)$  introduced by Andruskiewitsch and Natale [4, Appendix]. As an application, we classify Hopf superalgebras of dimensions  $2p$  and  $p^2$ , where  $p$  is an odd prime number.

Let  $\ell$  and  $q$  be two distinct prime numbers, let  $j \in \{1, \ell r \mid r = 1, \dots, q-1\}$ , and let  $\omega$  be a root of unity such that  $\text{ord}(\omega^j) = q$ . Let  $\mu \in \{0, 1\}$  also be a parameter, which is allowed to be non-zero only if  $j = 1$ . Then the algebra

$$\mathcal{A}(\omega, j, \mu) := \mathbb{k}\langle c, x \mid c^{\ell q} = 1, x^q = \mu(1 - c^q), cx = \omega xc \rangle$$

is a pointed Hopf algebra of dimension  $\ell q^2$  such that

$$\Delta(c) = c \otimes c, \quad \Delta(x) = c^j \otimes x + x \otimes 1.$$

Below, we determine the set of admissible data and the set of super-data for this Hopf algebra. As a preparation, we prove.

LEMMA 4.5. *There exists an algebra map  $A := \mathcal{A}(\omega, j, \mu) \rightarrow \mathbb{k}$  of order two if and only if  $\mu = 0$  or  $q = 2$ . If this equivalent condition is satisfied, then such an algebra map  $\alpha : A \rightarrow \mathbb{k}$  is determined by  $\alpha(c) = -1$  and  $\alpha(x) = 0$ .*

*Proof.* Suppose that  $\alpha : A \rightarrow \mathbb{k}$  is an algebra map of order two. Then by Lemma 3.5, we have  $\alpha(x) = 0$ . We also have  $\alpha(c) = +1$  or  $\alpha(c) = -1$ . If the former holds, then we have  $\alpha = \varepsilon$ , a contradiction. Hence  $\alpha(c) = -1$ . The relation  $x^q = \mu(1 - c^q)$  implies

$$0 = \alpha(x)^q = \mu(1 - \alpha(c)^q) = \mu(1 - (-1)^q),$$

and therefore  $\mu = 0$  or  $q = 2$ . Conversely, the algebra map  $\alpha : A \rightarrow \mathbb{k}$  determined by  $\alpha(c) = -1$  and  $\alpha(x) = 0$  is of order two. The proof is done.  $\square$

In the following, we let  $\alpha$  denote the algebra map given in Lemma 4.5. Since  $\ell$  and  $q$  are distinct prime numbers and  $\mathbf{G}(\mathcal{A}(\omega, j, \mu)) \cong \mathbb{Z}_{\ell q} \cong \mathbb{Z}_{\ell} \times \mathbb{Z}_q$ , we have the following:

1. If  $\ell = 2$ , then  $\mu = 0$  and  $\text{AD}(\mathcal{A}(\omega, j, \mu)) = \{(c^q, \alpha)\}$ .
2. If  $q = 2$ , then  $j \in \{1, \ell\}$  and  $\text{AD}(\mathcal{A}(\omega, j, \mu)) = \{(c^\ell, \alpha)\}$ .
3. If both  $\ell$  and  $q$  are odd,  $\text{AD}(\mathcal{A}(\omega, j, \mu)) = \emptyset$ .

In particular, we get  $\text{AD}(\mathcal{A}(\omega, j, \mu)) \neq \emptyset$  if and only if  $\ell = 2$  or  $q = 2$ .

PROPOSITION 4.6. *The Hopf algebra  $A := \mathcal{A}(\omega, j, \mu)$  admits a superform if and only if  $q = 2$ . If this equivalent condition is satisfied, then we have  $\text{SD}(A) = \{(c^\ell, \alpha)\}$ .*

*Proof.* First, suppose that  $\ell = 2$ . Then  $A^{\text{coinv}(c^q, \alpha)}$  is generated by  $\{c^2, x\}$  as an algebra. If  $j = 1$ , then by definition  $\text{ord}(\omega) = q$ , and hence, we get

$$\alpha(c^j)x = (-1)^jx = -x \neq x = \omega^q x = c^q x c^q.$$

In this case,  $\text{SD}(A) = \emptyset$ . If  $j = 2r$  for some  $r \in \{1, \dots, q-1\}$ , then by definition 2 divides  $\text{ord}(\omega)$  and  $q$  is odd. Thus, in this case,  $\omega^q \neq 1 = (-1)^j = \alpha(c^j)$ , and hence, we get  $\text{SD}(A) = \emptyset$ .

Next, suppose that  $q = 2$ . In this case, we note that  $j \in \{1, \ell\}$  and  $\ell$  is odd. Then  $A^{\text{coinv}(c^\ell, \alpha)}$  is generated by  $\{c^2, x\}$  as an algebra. Since  $\omega^\ell = -1$ , we get

$$c^\ell x c^\ell = \omega^\ell x = -x = (-1)^j x = \alpha(c^j)x$$

for each  $j \in \{1, \ell\}$ . Thus, in this case we have  $\text{SD}(A) = \{(c^\ell, \alpha)\}$ .  $\square$

THEOREM 4.7. *Let  $p$  be an odd prime number. Any non-semisimple pointed Hopf superalgebra of dimension  $p^2$  is purely even.*

*Proof.* Let  $\mathcal{H}$  be a non-semisimple pointed Hopf superalgebra of dimension  $p^2$ . Since  $\widehat{\mathcal{H}}$  is a non-semisimple pointed Hopf algebra of dimension  $2p^2$ , it is isomorphic to one of the following Hopf algebras (see [4, Lemma A.1]).

$$\mathcal{A}(\tau, 1, 0), \mathcal{A}(\tau, 1, 1), \mathcal{A}(\omega, 2r, 0) \text{ or } \mathcal{A}(\tau, 2, 0),$$

where  $r \in \{1, \dots, p - 1\}$ ,  $\tau \in \mathbb{k}$  is a primitive  $p$ -th root of unity and  $\omega \in \mathbb{k}$  is a fixed primitive  $2p$ -th root of unity. Since none of them admits a super-form by Proposition 4.6, the claim follows.  $\square$

**THEOREM 4.8.** *Let  $p$  be an odd prime number, and let  $\mathcal{H}$  be a Hopf superalgebra of dimension  $2p$  satisfying  $\mathcal{H}_1 \neq 0$ . If  $\mathcal{H}$  is non-semisimple and pointed, then  $\mathcal{H}$  is isomorphic to one of the Hopf superalgebras given in Table 1. Moreover, the Hopf superalgebras in Table 1 are pairwise non-isomorphic.*

*Proof.* Since  $\widehat{\mathcal{H}}$  is a non-semisimple pointed Hopf algebra of dimension  $4p$ , it is isomorphic to one of the following Hopf algebras ([4, Lemma A.1]).

$$\mathcal{A}(-1, 1, 0), \mathcal{A}(-1, 1, 1), \mathcal{A}(\omega, p, 0) \text{ or } \mathcal{A}(-1, p, 0),$$

where  $\omega \in \mathbb{k}$  is a fixed primitive  $2p$ -th root of unity. In the following, we describe the structure of the corresponding Hopf superalgebra explicitly. To do this, we let  $\Gamma$  be the group  $C_p = \langle g \mid g^p = 1 \rangle$  of order  $p$ , and let  $\chi$  be the character defined by  $\chi(g) := \omega^2$ .

First, by Theorem 3.6, the Hopf superalgebra  $\mathcal{H}_{2p}^{(1)} := \mathcal{A}(-1, p, 0)^{\text{coinv}(c^p, \alpha)}$  is generated by  $c^2$  and  $x$ . Also, we see that  $x$  is odd primitive in  $\mathcal{H}_{2p}^{(1)}$ . Thus,  $g \mapsto c^2, z \mapsto x$  gives a Hopf superalgebra isomorphism  $\mathcal{A}(\Gamma, (1, 1, 0; 1)) \cong \mathcal{H}_{2p}^{(1)}$ . One sees that the Hopf superalgebra structure of  $\mathcal{H}_{2p}^{(2)} := \mathcal{A}(\omega, p, 0)^{\text{coinv}(c^p, \alpha)}$  is given by the same formula as  $\mathcal{H}_{2p}^{(1)}$  above. Thus,  $g \mapsto c^2, z \mapsto x$  gives a Hopf superalgebra isomorphism  $\mathcal{A}(\Gamma, (1, \chi, 0; 1)) \cong \mathcal{H}_{2p}^{(2)}$ .

Next, by Theorem 3.6, the Hopf superalgebra  $\mathcal{H}_{2p}^{(3)} := \mathcal{A}(-1, 1, 0)^{\text{coinv}(c^p, \alpha)}$  is generated by  $c^2$  and  $x$ . Also, we see that  $x$  is odd  $c^{p+1}$ -skew primitive in  $\mathcal{H}_{2p}^{(3)}$ . Therefore,  $g \mapsto c^2, z \mapsto x$  gives a Hopf superalgebra isomorphism  $\mathcal{A}(\Gamma, (g^{(p+1)/2}, 1, 0; 1)) \cong \mathcal{H}_{2p}^{(3)}$ . One sees that the Hopf superalgebra structure of  $\mathcal{H}_{2p}^{(4)} := \mathcal{A}(-1, 1, 1)^{\text{coinv}(c^p, \alpha)}$  is given by the same formula as  $\mathcal{H}_{2p}^{(3)}$  above. Thus, the assignment  $g \mapsto c^2, z \mapsto x$  gives a Hopf superalgebra isomorphism  $\mathcal{A}(\Gamma, (g^{(p+1)/2}, 1, 1; 1)) \cong \mathcal{H}_{2p}^{(4)}$ .  $\square$

*Remark 4.9.* By definition, we have  $\mathcal{A}(\Gamma, (1, \chi, 0; 1)) = \langle g, z \mid g^p = 1, z^2 = 0, gz = \omega^2 zg \rangle$ , where  $g$  is group-like and  $z$  is odd primitive. Suppose that  $\zeta_p \in \mathbb{k}$



is a primitive  $p$ -th root of unity. Then one easily sees that

$$\mathcal{A}(\Gamma, (1, \chi, 0; 1)) \cong \mathbb{k}\langle g, z \mid g^p = 1, z^2 = 0, gz = \zeta_p z g \rangle,$$

where  $g$  is group-like and  $z$  is odd primitive.

It is easy to see that the exterior superalgebra  $\bigwedge \mathbb{k}$  is the only (up to isomorphism) Hopf superalgebra of dimension 2 whose odd part is non-zero. By this observation and Theorems 2.3, 4.7 and 4.8, to complete the classification of non-semisimple pointed Hopf superalgebras  $\mathcal{H}$  (with  $\mathcal{H}_{\bar{1}} \neq 0$ ) of dimension up to 10, it remains to address the cases of  $\dim(\mathcal{H}) = 4$  and  $\dim(\mathcal{H}) = 8$ .

### 4.3. Dimension 4

In this section, we show the following.

**THEOREM 4.10.** *Let  $\mathcal{H}$  be a Hopf superalgebra of dimension 4 satisfying  $\mathcal{H}_{\bar{1}} \neq 0$ . If  $\mathcal{H}$  is non-semisimple and pointed, then  $\mathcal{H}$  is isomorphic to one of the Hopf superalgebras given in Table 2. Moreover, the Hopf superalgebras in Table 2 are pairwise non-isomorphic.*

While this has indeed been proven in [18], below we give an alternative proof of Theorem 4.10 by demonstrating a more conceptual approach using our Theorem 3.6.

Classification of non-semisimple pointed Hopf algebras of dimension 8 ( $= 2 \times 4$ ) has been done by Ştefan [19]. According to the result, such a Hopf algebra is isomorphic to one of the following one.

- $A_{C_2} := \mathbb{k}\langle c, x_1, x_2 \mid c^2 = 1, cx_i = -x_i c, x_i x_j = -x_j x_i \ (i, j \in \{1, 2\}) \rangle$ , where  $c$  is group-like and  $x_1, x_2$  are  $c$ -skew primitive.
- $A_{C_2 \times C_2} := \mathbb{k}\langle c, d, x \mid c^2 = d^2 = 1, cd = dc, cx = -xc, dx = -xd, x^2 = 0 \rangle$ , where  $c, d$  are group-like and  $x$  is  $c$ -skew primitive.
- $A'_{C_4} := \mathbb{k}\langle c, x \mid c^4 = 1, cx = -xc, x^2 = 0 \rangle$ , where  $c$  is group-like and  $x$  is  $c$ -skew primitive.
- $A''_{C_4} := \mathbb{k}\langle c, x \mid c^4 = 1, cx = \zeta_4 xc, x^2 = 0 \rangle$ , where  $c$  is group-like and  $x$  is  $c$ -skew primitive.
- $\mathbb{k}\langle x_1, x_2 \mid x_1^4 = 1, x_2 x_1 = \zeta_4 x_1 x_2, x_2^2 = 0 \rangle$ , where  $\Delta(x_1) = x_1 \otimes x_1 - 2x_1 x_2 \otimes x_1^3 x_2$ ,  $\varepsilon(x_1) = 1$ ,  $S(x_1) = x_1^3$ ,  $\Delta(x_2) = x_2 \otimes x_1^2 - 1 \otimes x_2$ ,  $\varepsilon(x_2) = 0$  and  $S(x_2) = -x_2 x_1^2$ .

By Proposition 3.2, we see that the Hopf algebras  $A_{C_2}$  and  $A_{C_2 \times C_2}$  are the only ones which might have super-forms. Note that  $\mathbf{G}(A_{C_2}) \cong \mathbb{Z}_2$  and  $\mathbf{G}(A_{C_2 \times C_2}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$  are both abelian.

### 4.3.1 The $A_{C_2}$ case

First, we treat  $A^{(1)} := A_{C_2}$ . By definition, the group-like element  $c \in A^{(1)}$  is of order two. One sees that  $\alpha \in \mathbf{G}((A^{(1)})^*)$  defined by  $\alpha(c) = -1, \alpha(x_1) = \alpha(x_2) = 0$  is the only algebra map of order two. Thus, we have  $\text{AD}(A^{(1)}) = \{(c, \alpha)\}$ . Moreover, we see that  $\text{SD}(A^{(1)}) = \{(c, \alpha)\}$ . By Theorem 3.6, the coinvariant subalgebra  $\mathcal{H}_4^{(1)} := (A^{(1)})^{\text{coinv}(c, \alpha)}$  is generated by  $x_1$  and  $x_2$ . Also,  $x_1$  and  $x_2$  are odd primitive. Thus, we get the following result.

PROPOSITION 4.11. *The exterior superalgebra  $\bigwedge \mathbb{k}^2$  is the only Hopf superalgebra whose bosonization is isomorphic to  $A^{(1)} (= A_{C_2})$ .*

### 4.3.2 The $A_{C_2 \times C_2}$ case

Next, we treat  $A^{(2)} := A_{C_2 \times C_2}$ . Both  $\mathbf{G}(A^{(2)})$  and  $\mathbf{G}((A^{(2)})^*)$  are isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$  and are given by

$$\mathbf{G}(A^{(2)}) = \{1, c, d, cd\} \quad \text{and} \quad \mathbf{G}((A^{(2)})^*) = \{\varepsilon, \alpha_1, \alpha_2, \alpha_3 := \alpha_1\alpha_2\},$$

where  $\alpha_1$  and  $\alpha_2$  are algebra maps  $A^{(2)} \rightarrow \mathbb{k}$  determined by  $\alpha_1(c) = -1, \alpha_1(d) = 1, \alpha_2(c) = 1, \alpha_2(d) = -1$  and  $\alpha_1(x) = \alpha_2(x) = 0$ . Thus, the set of all admissible data for  $A^{(2)}$  is

$$\text{AD}(A^{(2)}) = \{(c, \alpha_1), (cd, \alpha_1), (d, \alpha_2), (cd, \alpha_2), (c, \alpha_3), (d, \alpha_3)\}.$$

By (3.1), we get  $\text{SD}(A^{(2)}) = \{(c, \alpha_1), (c, \alpha_3), (d, \alpha_3)\}$ . The following is easy to see.

LEMMA 4.12. *We have  $\text{Aut}_{\text{Hopf}}(A^{(2)}) = \{\varphi_u\}_{u \in \mathbb{k}^\times}$ , where  $\varphi_u$  is determined by  $\varphi_u|_{\mathbf{G}(A^{(2)})} = \text{id}$  and  $\varphi_u(x) = ux$ .*

By Lemma 4.12, a complete set of representatives of  $\text{SD}(A^{(2)})/\sim$  is given by  $\{(c, \alpha_1), (c, \alpha_3), (d, \alpha_3)\}$ . In the following, we determine structure of each coinvariant subalgebras of  $A^{(2)}$  using Theorem 3.6. To do this, we let  $\Gamma$  be the group  $C_2 = \langle g \mid g^2 = 1 \rangle$  of order two, and let  $\chi$  be the non-trivial character of  $C_2$ .

- Set  $\mathcal{H}_4^{(2)} := (A^{(2)})^{\text{coinv}(c, \alpha_1)}$ . As an algebra,  $\mathcal{H}_4^{(2)}$  is generated by  $\{d, x\}$  and  $x$  is odd primitive. Thus, the assignment  $g \mapsto d, z \mapsto x$  gives a Hopf superalgebra isomorphism  $\mathcal{A}(\Gamma, (1, 1, 0; 1)) \cong \mathcal{H}_4^{(2)}$ .
- Set  $\mathcal{H}_4^{(3)} := (A^{(2)})^{\text{coinv}(c, \alpha_3)}$ . As an algebra,  $\mathcal{H}_4^{(3)}$  is generated by  $\{cd, x\}$  and  $x$  is odd primitive. Thus, the assignment  $g \mapsto cd, z \mapsto x$  gives a Hopf superalgebra isomorphism  $\mathcal{A}(\Gamma, (1, \chi, 0; 1)) \cong \mathcal{H}_4^{(3)}$ .

- Set  $\mathcal{H}_4^{(4)} := (A^{(2)})^{\text{coinv}(d, \alpha_3)}$ . As an algebra,  $\mathcal{H}_4^{(4)}$  is generated by  $\{cd, x\}$  and  $x$  is odd  $cd$ -skew primitive. Thus, the assignment  $g \mapsto cd, z \mapsto x$  gives a Hopf superalgebra isomorphism  $\mathcal{A}(\Gamma, (g, 1, 0; 1)) \cong \mathcal{H}_4^{(4)}$ .

One sees  $\mathcal{H}_4^{(2)} \cong \mathbb{k}C_2 \otimes \bigwedge \mathbb{k}$ .

Therefore, we get the following result.

**PROPOSITION 4.13.** *The Hopf superalgebras  $\mathcal{A}(\Gamma, \mathcal{D})$  with  $\mathcal{D} \in \{(1, 1, 0; 1), (1, \chi, 0; 1), (g, 1, 0; 1)\}$  are the only ones whose bosonization is isomorphic to  $A^{(2)} (= AC_2 \times C_2)$ . Moreover, these are pairwise non-isomorphic.*

By Propositions 4.11 and 4.13, the proof of Theorem 4.10 is done.

### 4.4. Dimension 8

In this section, we show the following.

**THEOREM 4.14.** *Let  $\mathcal{H}$  be a Hopf superalgebra of dimension 8 satisfying  $\mathcal{H}_{\bar{1}} \neq 0$ . If  $\mathcal{H}$  is non-semisimple and pointed, then  $\mathcal{H}$  is isomorphic to one of the Hopf superalgebras given in Table 3. Moreover, the Hopf superalgebras in Table 3 are pairwise non-isomorphic.*

Classification of non-semisimple pointed Hopf algebras of dimension 16 ( $= 2 \times 8$ ) has been done by Caenepeel, Dăscălescu and Raianu [10]. According to their result, there are 29 such Hopf algebras. By Proposition 3.2, we see that among those 29 Hopf algebras, the ones listed in Table 4 are candidates for those having super-forms. In the following, we classify Hopf superalgebras  $\mathcal{H}$  such that  $\mathcal{H}_{\bar{1}} \neq 0$  and  $\widehat{\mathcal{H}} \cong A^{(j)}$  for each  $j = 1, \dots, 14$ .

#### 4.4.1 The $A^{(1)}$ case

The Hopf algebra  $A^{(1)}$  is given by

$$A^{(1)} = \mathbb{k}\langle c, x_1, x_2, x_3 \mid c^2 = 1, x_i x_j = -x_j x_i, c x_i = -x_i c \ (i, j \in \{1, 2, 3\}) \rangle,$$

where  $c$  is group-like,  $x_1, x_2$  and  $x_3$  are  $c$ -skew primitive.

One easily sees that  $\text{AD}(A^{(1)}) = \text{SD}(A^{(1)}) = \{(c, \alpha)\}$ , where  $\alpha(c) = -1$  and  $\alpha(x_i) = 0$  ( $i \in \{1, 2, 3\}$ ). By Theorem 3.6, the coinvariant subalgebra  $\mathcal{H}_8^{(1)} := (A^{(1)})^{\text{coinv}(c, \alpha)}$  is generated by  $\{x_1, x_2, x_3\}$ . Also,  $x_1, x_2$  and  $x_3$  are odd primitive. The above argument shows the following result.

**PROPOSITION 4.15.** *The exterior superalgebra  $\bigwedge \mathbb{k}^3$  is the only Hopf superalgebra whose bosonization is isomorphic to  $A^{(1)}$ .*

| Hopf algebras | Notation in [10]  |
|---------------|---|
| $A^{(1)}$     | $H(C_2, (2, 2, 2), (c^*, c^*, c^*), (c, c, c), (0, 0, 0))$  |
| $A^{(2)}$     | $H(C_2 \times C_2, (2, 2), (c^*, c^*), (c, c), (0, 0))$   |
| $A^{(3)}$     | $H(C_2 \times C_2, (2, 2), (c^*, c^*d^*), (c, c), (0, 0))$  |
| $A^{(4)}$     | $H(C_2 \times C_2, (2, 2), (c^*, d^*), (c, d), (0, 0))$   |
| $A^{(5)}$     | $H(C_2 \times C_2, (2, 2), (c^*d^*, c^*d^*), (c, d), (0, 0))$   |
| $A^{(6)}$     | $H(C_2 \times C_2, (2, 2), (c^*d^*, c^*d^*), (c, d), (0, 0), (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}))$ |
| $A^{(7)}$     | $H(C_2 \times C_2 \times C_2, 2, c^*, c, 0)$  |
| $A^{(8)}$     | $H(C_4 \times C_2, (c^*)^2, c, 0)$  |
| $A^{(9)}$     | $H(C_4 \times C_2, d^*, cd, 0)$   |
| $A^{(10)}$    | $H(C_4 \times C_2, c^*, c^2, 0)$  |
| $A^{(11)}$    | $H(C_4 \times C_2, d^*, d, 0)$  |
| $A^{(12)}$    | $H(C_4 \times C_2, c^*, c^2d, 0)$   |
| $A^{(13)}$    | $H(C_4 \times C_2, (c^*)^2, c, 1)$  |
| $A^{(14)}$    | $H(C_4 \times C_2, d^*, cd, 1)$   |

Table 4 – Hopf algebras that may have super-forms

### 4.4.2 The $A^{(2)}$ case

The Hopf algebra  $A^{(2)}$  is given by

$$A^{(2)} = \mathbb{k} \left\langle c, d, x_1, x_2 \mid \begin{array}{l} c^2 = d^2 = 1, cd = dc, x_i x_j = -x_j x_i, \\ cx_i = -x_i c, dx_i = x_i d \ (i, j \in \{1, 2\}) \end{array} \right\rangle,$$

where  $c$  and  $d$  are group-like,  $x_1$  and  $x_2$  are  $c$ -skew primitive.

First, note that both  $\mathbf{G}(A^{(2)})$  and  $\mathbf{G}((A^{(2)})^*)$  are isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$  and are given by

$$\mathbf{G}(A^{(2)}) = \{1, c, d, cd\} \quad \text{and} \quad \mathbf{G}((A^{(2)})^*) = \{\varepsilon, \alpha_1, \alpha_2, \alpha_3 := \alpha_1 \alpha_2\},$$

where  $\alpha_1$  and  $\alpha_2$  are algebra maps  $A^{(2)} \rightarrow \mathbb{k}$  determined by  $\alpha_1(c) = -1, \alpha_1(d) = 1, \alpha_2(c) = 1, \alpha_2(d) = -1$  and  $\alpha_1(x_i) = \alpha_2(x_i) = 0$  for  $i \in \{1, 2\}$ . Thus, we get

$$\text{AD}(A^{(2)}) = \{(c, \alpha_1), (cd, \alpha_1), (d, \alpha_2), (cd, \alpha_2), (c, \alpha_3), (d, \alpha_3)\}.$$

By (3.1), we have  $\text{SD}(A^{(2)}) = \{(c, \alpha_1), (cd, \alpha_1), (c, \alpha_3)\}$ . Let  $\text{GL}_2(\mathbb{k})$  denote the general linear group of degree two over  $\mathbb{k}$ . The following is easy to see.

LEMMA 4.16. *We have  $\text{Aut}_{\text{Hopf}}(A^{(2)}) = \{\varphi_P\}_{P \in \text{GL}_2(\mathbb{k})}$ , where  $\varphi_P$  is determined by  $\varphi_P|_{\mathbf{G}(A^{(2)})} = \text{id}$  and  $\varphi_P(x_i) = p_{i,1}x_1 + p_{i,2}x_2$  for  $i \in \{1, 2\}$  with  $p_{i,j}$  the  $(i, j)$ -entry of the matrix  $P$ .*

Hence, we get  $\#(\text{SD}(A^{(2)})) = \#(\text{SD}(A^{(2)})/\sim)$ . Let  $\Gamma$  be the group  $C_2 = \langle g \mid g^2 = 1 \rangle$  of order two, and let  $\chi$  be the non-trivial character.

**PROPOSITION 4.17.** *The Hopf superalgebras  $\mathcal{A}(\Gamma, \mathcal{D})$  with  $\mathcal{D} \in \{((1, 1, 0; 1), (1, 1, 0; 1)), ((g, 1, 0; 1), (g, 1, 0; 1)), ((1, \chi, 0; 1), (1, \chi, 0; 1))\}$  are the only ones whose bosonization is isomorphic to  $A^{(2)}$ . Moreover, these are pairwise non-isomorphic.*

*Proof.* In the following, determine structure of each coinvariant subalgebras of  $A^{(2)}$  using Theorem 3.6 one by one.

- The Hopf superalgebra  $\mathcal{H}_8^{(2)} := (A^{(2)})^{\text{coinv}(c, \alpha_1)}$  is generated by  $d, x_1, x_2$ . As an element in  $\mathcal{H}_8^{(2)}$ ,  $x_i$  is odd primitive ( $i \in \{1, 2\}$ ). Thus,  $g \mapsto d, z_i \mapsto x_i$  gives a Hopf superalgebra isomorphism  $\mathcal{A}(\Gamma, ((1, 1, 0; 1), (1, 1, 0; 1))) \cong \mathcal{H}_8^{(2)}$ .
- The Hopf superalgebra  $\mathcal{H}_8^{(5)} := (A^{(2)})^{\text{coinv}(cd, \alpha_1)}$  is generated by  $d, x_1, x_2$ . As an element in  $\mathcal{H}_8^{(5)}$ ,  $x_i$  is odd  $d$ -skew primitive ( $i \in \{1, 2\}$ ). Thus, the assignment  $g \mapsto d, z_i \mapsto x_i$  gives a Hopf superalgebra isomorphism  $\mathcal{A}(\Gamma, ((g, 1, 0; 1), (g, 1, 0; 1))) \cong \mathcal{H}_8^{(5)}$ .
- The case  $\mathcal{H}_8^{(6)} := (A^{(2)})^{\text{coinv}(c, \alpha_3)}$ . As an element in  $\mathcal{H}_8^{(6)}$ ,  $x_i$  is odd primitive ( $i \in \{1, 2\}$ ). Thus, the assignment  $g \mapsto cd, z_i \mapsto x_i$  gives a Hopf superalgebra isomorphism  $\mathcal{A}(\Gamma, ((1, \chi, 0; 1), (1, \chi, 0; 1))) \cong \mathcal{H}_8^{(6)}$ .

This completes the proof.  $\square$

One easily sees  $\mathcal{H}_8^{(2)} \cong \mathbb{k}C_2 \otimes \bigwedge \mathbb{k}^2$ .

### 4.4.3 The $A^{(3)}$ case

The Hopf algebra  $A^{(3)}$  is given by

$$A^{(3)} = \mathbb{k} \left\langle c, d, x_1, x_2 \mid \begin{array}{l} c^2 = d^2 = 1, cd = dc, x_i x_j = -x_j x_i, \\ cx_i = -x_i c, dx_i = -(-1)^i x_i d \ (i, j \in \{1, 2\}) \end{array} \right\rangle,$$

where  $c$  and  $d$  are group-like,  $x_1$  and  $x_2$  are  $c$ -skew primitive.

First, note that both  $\mathbf{G}(A^{(3)})$  and  $\mathbf{G}((A^{(3)})^*)$  are isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$  and are given by

$$\mathbf{G}(A^{(3)}) = \{1, c, d, cd\} \quad \text{and} \quad \mathbf{G}((A^{(3)})^*) = \{\varepsilon, \alpha_1, \alpha_2, \alpha_3 := \alpha_1 \alpha_2\},$$

where  $\alpha_1$  and  $\alpha_2$  are algebra maps  $A^{(3)} \rightarrow \mathbb{k}$  determined by  $\alpha_1(c) = -1$ ,  $\alpha_1(d) = 1$ ,  $\alpha_2(c) = 1$ ,  $\alpha_2(d) = -1$  and  $\alpha_1(x_i) = \alpha_2(x_i) = 0$  for  $i \in \{1, 2\}$ . Thus, we get

$$\text{AD}(A^{(3)}) = \{(c, \alpha_1), (cd, \alpha_1), (d, \alpha_2), (cd, \alpha_2), (c, \alpha_3), (d, \alpha_3)\}.$$

By (3.1), we have  $\text{SD}(A^{(3)}) = \{(c, \alpha_1), (c, \alpha_3)\}$ . The following is easy to see.

LEMMA 4.18. *We have  $\text{Aut}_{\text{Hopf}}(A^{(3)}) = \langle \tau, \varphi_{a_1, a_2} \mid a_1, a_2 \in \mathbb{k}^\times \rangle$ , where*

$$\begin{aligned} \tau(c) &= d, & \tau(d) &= cd, & \tau(x_1) &= x_2, & \tau(x_2) &= x_1, \\ \varphi_{a_1, a_2}(c) &= c, & \varphi_{a_1, a_2}(d) &= d, & \varphi_{a_1, a_2}(x_i) &= a_i x_i & (i \in \{1, 2\}). \end{aligned}$$

So, we get  $\text{SD}(A^{(3)})/\sim = \{[(c, \alpha_1)]\}$ . Let  $\Gamma$  be the group  $C_2 = \langle g \mid g^2 = 1 \rangle$  of order two, and let  $\chi$  be the non-trivial character.

PROPOSITION 4.19. *The Hopf superalgebra  $\mathcal{A}(\Gamma, ((1, 1, 0; 1), (1, \chi, 0; 1)))$  is the only one whose bosonization is isomorphic to  $A^{(3)}$ .*

*Proof.* The coinvariant subalgebra  $\mathcal{H}_8^{(4)} := (A^{(3)})^{\text{coinv}(c, \alpha_1)}$  is generated by  $d, x_1, x_2$ . By Theorem 3.6, as elements in  $\mathcal{H}_8^{(4)}$ , one sees that  $x_1, x_2$  are odd primitive. Thus, the assignment  $g \mapsto d, z_i \mapsto x_i$  gives a Hopf superalgebra isomorphism  $\mathcal{A}(\Gamma, ((1, 1, 0; 1), (1, \chi, 0; 1))) \cong \mathcal{H}_8^{(4)}$ .  $\square$

#### 4.4.4 The $A^{(4)}$ case

The Hopf algebra  $A^{(4)}$  is given by

$$A^{(4)} = \mathbb{k} \left\langle c, d, x_1, x_2 \mid \begin{array}{l} c^2 = d^2 = 1, cd = dc, x_i x_j = x_j x_i, x_i^2 = 0, \\ cx_i = (-1)^i x_i c, dx_i = -(-1)^i x_i d \ (i, j \in \{1, 2\}) \end{array} \right\rangle,$$

where  $c$  and  $d$  are group-like,  $x_1$  (respectively,  $x_2$ ) is  $c$ -skew (respectively,  $d$ -skew) primitive.

First, note that both  $\mathbf{G}(A^{(4)})$  and  $\mathbf{G}((A^{(4)})^*)$  are isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$  and are given by

$$\mathbf{G}(A^{(4)}) = \{1, c, d, cd\} \quad \text{and} \quad \mathbf{G}((A^{(4)})^*) = \{\varepsilon, \alpha_1, \alpha_2, \alpha_3 := \alpha_1 \alpha_2\},$$

where  $\alpha_1$  and  $\alpha_2$  are algebra maps  $A^{(4)} \rightarrow \mathbb{k}$  determined by  $\alpha_1(c) = -1$ ,  $\alpha_1(d) = 1$ ,  $\alpha_2(c) = 1$ ,  $\alpha_2(d) = -1$  and  $\alpha_1(x_i) = \alpha_2(x_i) = 0$  for  $i \in \{1, 2\}$ . Thus, we get

$$\text{AD}(A^{(4)}) = \{(c, \alpha_1), (cd, \alpha_1), (d, \alpha_2), (cd, \alpha_2), (c, \alpha_3), (d, \alpha_3)\}.$$

By (3.1), we have  $\text{SD}(A^{(4)}) = \{(c, \alpha_1), (c, \alpha_3)\}$ . The following is easy to see.

LEMMA 4.20. *We have  $\text{Aut}_{\text{Hopf}}(A^{(4)}) = \langle \tau, \varphi_{a_1, a_2} \mid a_1, a_2 \in \mathbb{k}^\times \rangle$ , where*

$$\begin{aligned} \tau(c) &= d, & \tau(d) &= c, & \tau(x_1) &= x_2, & \tau(x_2) &= x_1, \\ \varphi_{a_1, a_2}(c) &= c, & \varphi_{a_1, a_2}(d) &= d, & \varphi_{a_1, a_2}(x_i) &= a_i x_i & (i \in \{1, 2\}). \end{aligned}$$

Thus, we get  $(c, \alpha_1) \sim (d, \alpha_2)$  and  $\text{SD}(A^{(4)})/\sim = \{[(c, \alpha_1)]\}$ . Let  $\Gamma$  be the group  $C_2 = \langle g \mid g^2 = 1 \rangle$  of order two, and let  $\chi$  be the non-trivial character.

PROPOSITION 4.21. *The Hopf superalgebra  $\mathcal{A}(\Gamma, ((g, \chi, 0; 0), (1, 1, 0; 1)))$  is the only one whose bosonization is isomorphic to  $A^{(4)}$ .*

*Proof.* The coinvariant subalgebra  $\mathcal{H}_8^{(7)} := (A^{(4)})^{\text{coinv}(c, \alpha_1)}$  is generated by  $d, x_1, x_2$ . By Theorem 3.6, as elements in  $\mathcal{H}_8^{(7)}$ , one sees that  $x_1$  is odd primitive and  $x_2$  is even  $d$ -skew primitive. Thus, the assignment  $g \mapsto d, z_1 \mapsto x_2, z_2 \mapsto x_1$  gives a Hopf superalgebra isomorphism  $\mathcal{A}(\Gamma, ((g, \chi, 0; 0), (1, 1, 0; 1))) \cong \mathcal{H}_8^{(7)}$ .

One sees  $\mathcal{H}_8^{(7)} \cong T_4(-1) \otimes \bigwedge \mathbb{k}$ .  $\square$

### 4.4.5 The $A^{(5)}$ case

The Hopf algebra  $A^{(5)}$  is given by

$$A^{(5)} = \mathbb{k} \left\langle c, d, x_1, x_2 \mid \begin{array}{l} c^2 = d^2 = 1, cd = dc, x_i x_j = -x_j x_i, \\ cx_i = -x_i c, dx_i = -x_i d \ (i, j \in \{1, 2\}) \end{array} \right\rangle,$$

where  $c$  and  $d$  are group-like,  $x_1$  (respectively,  $x_2$ ) is  $c$ -skew (respectively,  $d$ -skew) primitive.

First, note that both  $\mathbf{G}(A^{(5)})$  and  $\mathbf{G}((A^{(5)})^*)$  are isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$  and are given by

$$\mathbf{G}(A^{(5)}) = \{1, c, d, cd\} \quad \text{and} \quad \mathbf{G}((A^{(5)})^*) = \{\varepsilon, \alpha_1, \alpha_2, \alpha_3 := \alpha_1 \alpha_2\},$$

where  $\alpha_1$  and  $\alpha_2$  are algebra maps  $A^{(5)} \rightarrow \mathbb{k}$  determined by  $\alpha_1(c) = -1, \alpha_1(d) = 1, \alpha_2(c) = 1, \alpha_2(d) = -1$  and  $\alpha_1(x_i) = \alpha_2(x_i) = 0$  for  $i \in \{1, 2\}$ . Thus, we get

$$\text{AD}(A^{(5)}) = \{(c, \alpha_1), (cd, \alpha_1), (d, \alpha_2), (cd, \alpha_2), (c, \alpha_3), (d, \alpha_3)\}.$$

By (3.1), we have  $\text{SD}(A^{(5)}) = \{(c, \alpha_1), (c, \alpha_3)\}$ . The following is easy to see.

LEMMA 4.22. *We have  $\text{Aut}_{\text{Hopf}}(A^{(5)}) = \langle \tau, \varphi_{a_1, a_2} \mid a_1, a_2 \in \mathbb{k}^\times \rangle$ , where*

$$\begin{aligned} \tau(c) &= d, & \tau(d) &= c, & \tau(x_1) &= x_2, & \tau(x_2) &= x_1, \\ \varphi_{a_1, a_2}(c) &= c, & \varphi_{a_1, a_2}(d) &= d, & \varphi_{a_1, a_2}(x_i) &= a_i x_i & (i \in \{1, 2\}). \end{aligned}$$

Thus, we get  $(c, \alpha_3) \sim (d, \alpha_3)$ . Hence, we get  $\text{SD}(A^{(5)})/\sim = \{[(c, \alpha_3)]\}$ . Let  $\Gamma$  be the group  $C_2 = \langle g \mid g^2 = 1 \rangle$  of order two.

PROPOSITION 4.23. *The Hopf superalgebra  $\mathcal{A}(\Gamma, ((1, 1, 0; 1), (g, 1, 0; 1)))$  is the only one whose bosonization is isomorphic to  $A^{(5)}$ .*

*Proof.* The coinvariant subalgebra  $\mathcal{H}_8^{(3)} := (A^{(5)})^{\text{coinv}(c, \alpha_3)}$  is generated by  $cd, x_1, x_2$ . By Theorem 3.6, as elements in  $\mathcal{H}_8^{(3)}$ , one sees that  $x_1$  is odd primitive and  $x_2$  is odd  $cd$ -skew primitive. Thus, the assignment  $g \mapsto cd, z_i \mapsto x_i$  gives a Hopf superalgebra isomorphism  $\mathcal{A}(\Gamma, ((1, 1, 0; 1), (g, 1, 0; 1))) \cong \mathcal{H}_8^{(3)}$ .

□

### 4.4.6 The $A^{(6)}$ case

The Hopf algebra  $A^{(6)}$  is given by

$$A^{(6)} = \mathbb{k} \left\langle c, d, x_1, x_2 \mid \begin{array}{l} c^2 = d^2 = 1, cd = dc, x_2x_1 = -x_1x_2 + c - 1, x_i^2 = 0, \\ cx_i = -x_ic, dx_i = -x_id \ (i \in \{1, 2\}) \end{array} \right\rangle,$$

where  $c$  and  $d$  are group-like,  $x_1$  (respectively,  $x_2$ ) is  $c$ -skew (respectively,  $d$ -skew) primitive.

First, note that  $\mathbf{G}(A^{(6)}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$  and  $\mathbf{G}((A^{(6)})^*) \cong \mathbb{Z}_2$  and are given by

$$\mathbf{G}(A^{(6)}) = \{1, c, d, cd\} \quad \text{and} \quad \mathbf{G}((A^{(6)})^*) = \{\varepsilon, \alpha\},$$

where  $\alpha$  is an algebra map  $A^{(6)} \rightarrow \mathbb{k}$  determined by the set  $\alpha(c) = 1, \alpha(d) = -1, \alpha(x_i) = 0 \ (i \in \{1, 2\})$ . Thus, we get

$$\text{AD}(A^{(6)}) = \{(d, \alpha), (cd, \alpha)\}.$$

By (3.1), one easily sees that  $\text{SD}(A^{(6)}) = \emptyset$ . Thus, we have the following result.

PROPOSITION 4.24. *There is no Hopf superalgebra whose bosonization is isomorphic to  $A^{(6)}$ .*

### 4.4.7 The $A^{(7)}$ case

The Hopf algebra  $A^{(7)}$  is given by

$$A^{(7)} = \mathbb{k} \left\langle c, d, e, x \mid \begin{array}{l} c^2 = d^2 = e^2 = 1, cd = dc, ce = ec, de = ed, x^2 = 0, \\ cx = -xc, dx = xd, ex = xe \end{array} \right\rangle,$$

where  $c, d$  and  $e$  are group-like,  $x$  is  $c$ -skew primitive.

First, note that both  $\mathbf{G}(A^{(7)})$  and  $\mathbf{G}((A^{(7)})^*)$  are isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$  and are given by

$$\mathbf{G}(A^{(7)}) = \langle c, d, e \rangle \quad \text{and} \quad \mathbf{G}((A^{(7)})^*) = \langle \alpha_1, \alpha_2, \alpha_3 \rangle,$$

where  $\alpha_1, \alpha_2$  and  $\alpha_3$  are algebra maps  $A^{(7)} \rightarrow \mathbb{k}$  determined by  $\alpha_1(c) = -1, \alpha_1(d) = 1, \alpha_1(e) = 1, \alpha_2(c) = 1, \alpha_2(d) = -1, \alpha_2(e) = 1, \alpha_3(c) = 1, \alpha_3(d) = 1,$



$\alpha_3(e) = -1$  and  $\alpha_1(x) = \alpha_2(x) = \alpha_3(x) = 0$ , respectively. Set  $\alpha_4 := \alpha_1\alpha_2$ ,  $\alpha_5 := \alpha_1\alpha_3$ ,  $\alpha_6 := \alpha_2\alpha_3$  and  $\alpha_7 := \alpha_1\alpha_2\alpha_3$ . By (3.1), one sees that the set  $\text{SD}(A^{(7)})$  is given as follows.

$$\{(c, \alpha_1), (c, \alpha_4), (c, \alpha_5), (c, \alpha_7), (cd, \alpha_1), (cd, \alpha_5), \\ (ce, \alpha_1), (ce, \alpha_4), (cde, \alpha_1), (cde, \alpha_7)\}.$$

The following is easy to see.

LEMMA 4.25. *We have  $\text{Aut}_{\text{Hopf}}(A^{(7)}) = \langle \sigma, \tau, \varphi_u \mid u \in \mathbb{k}^\times \rangle$ , where*

$$\begin{aligned} \sigma(c) = c, \quad \sigma(d) = de, \quad \sigma(e) = e, \quad \sigma(x) = x, \\ \tau(c) = c, \quad \tau(d) = e, \quad \tau(e) = d, \quad \tau(x) = x, \\ \varphi_u|_{\mathbf{G}(A^{(7)})} = \text{id}, \quad \varphi_u(x) = ux. \end{aligned}$$

We get the following.

LEMMA 4.26. *The set  $\{(c, \alpha_1), (c, \alpha_4), (cd, \alpha_1), (cd, \alpha_5)\}$  is a complete set of representatives of  $\text{SD}(A^{(7)})/\sim$ .*

Let  $\Gamma$  be the group  $C_2 \times C_2 = \langle g_1, g_2 \mid g_1^2 = g_2^2 = 1, g_1g_2 = g_2g_1 \rangle$ , and let  $\chi_i$  be the character defined by  $\chi_i(g_j) = -(-1)^{i+j}$  ( $i, j \in \{1, 2\}$ ).

PROPOSITION 4.27. *The Hopf superalgebras  $\mathcal{A}(\Gamma, \mathcal{D})$  with  $\mathcal{D} \in \{(1, 1, 0; 1), (g_1, 1, 0; 1), (1, \chi_1, 0; 1), (g_1, \chi_1, 0; 1)\}$  are the only ones whose bosonization is isomorphic to  $A^{(7)}$ . Moreover, these are pairwise non-isomorphic.*

*Proof.* In the following, determine structure of each coinvariant subalgebras of  $A^{(7)}$  using Theorem 3.6 one by one.

- The Hopf superalgebra  $\mathcal{H}_8^{(8)} := (A^{(7)})^{\text{coinv}(c, \alpha_1)}$  is generated by  $d, e, x$ . As an element in  $\mathcal{H}_8^{(8)}$ ,  $x$  is odd primitive. Thus, the assignment  $g_1 \mapsto d, g_2 \mapsto e, z \mapsto x$  gives a Hopf superalgebra isomorphism  $\mathcal{A}(\Gamma, (1, 1, 0; 1)) \cong \mathcal{H}_8^{(8)}$ .
- The Hopf superalgebra  $\mathcal{H}_8^{(10)} := (A^{(7)})^{\text{coinv}(c, \alpha_4)}$  is generated by  $cd, e, x$ . As an element in  $\mathcal{H}_8^{(10)}$ ,  $x$  is odd primitive. Thus,  $g_1 \mapsto cd, g_2 \mapsto e, z \mapsto x$  gives a Hopf superalgebra isomorphism  $\mathcal{A}(\Gamma, (1, \chi_1, 0; 1)) \cong \mathcal{H}_8^{(10)}$ .
- The Hopf superalgebra  $\mathcal{H}_8^{(9)} := (A^{(7)})^{\text{coinv}(cd, \alpha_1)}$  is generated by  $d, e, x$ . As an element in  $\mathcal{H}_8^{(9)}$ ,  $x$  is odd  $d$ -skew primitive. Thus, the assignment  $g_1 \mapsto d, g_2 \mapsto e, z \mapsto x$  gives a Hopf superalgebra isomorphism  $\mathcal{A}(\Gamma, (g, 1, 0; 1)) \cong \mathcal{H}_8^{(9)}$ .

- The Hopf superalgebra  $\mathcal{H}_8^{(11)} := (A^{(7)})^{\text{coinv}(cd, \alpha_5)}$  is generated by  $d, ce, x$ . As an element in  $\mathcal{H}_8^{(11)}$ ,  $x$  is odd  $d$ -skew primitive. Thus, the assignment  $g_1 \mapsto d, g_2 \mapsto ce, z \mapsto x$  gives a Hopf superalgebra isomorphism  $\mathcal{A}(\Gamma, (g_1, \chi_1, 0; 1)) \cong \mathcal{H}_8^{(11)}$ .

This completes the proof.  $\square$

### 4.4.8 The $A^{(8)}$ case

Let  $\zeta_4 \in \mathbb{k}$  be a primitive fourth root of unity. The Hopf algebra  $A^{(8)}$  is given by

$$A^{(8)} = \mathbb{k}\langle c, d, x \mid c^4 = d^2 = 1, cd = dc, x^2 = 0, cx = -xc, dx = xd \rangle,$$

where  $c$  and  $d$  are group-like,  $x$  is  $c$ -skew primitive.

First, note that both  $\mathbf{G}(A^{(8)})$  and  $\mathbf{G}((A^{(8)})^*)$  are isomorphic to  $\mathbb{Z}_4 \times \mathbb{Z}_2$  and are given by

$$\mathbf{G}(A^{(8)}) = \langle c, d \rangle \quad \text{and} \quad \mathbf{G}((A^{(8)})^*) = \langle \alpha, \beta \rangle,$$

where  $\alpha$  and  $\beta$  are algebra maps  $A^{(8)} \rightarrow \mathbb{k}$  determined by  $\alpha(c) = \zeta_4, \alpha(d) = 1, \alpha(x) = 0$  and  $\beta(c) = 1, \beta(d) = -1, \beta(x) = 0$ , respectively. Set  $\alpha_1 := \alpha^2, \alpha_2 := \beta, \alpha_3 := \alpha^2\beta$ . We get

$$\text{AD}(A^{(8)}) = \{(d, \alpha_2), (c^2d, \alpha_2), (d, \alpha_3), (c^2d, \alpha_3)\}.$$

By (3.1), one easily sees that  $\text{SD}(A^{(8)}) = \emptyset$ . Thus, we have the following result.

**PROPOSITION 4.28.** *There is no Hopf superalgebra whose bosonization is isomorphic to  $A^{(8)}$ .*

### 4.4.9 The $A^{(9)}$ case

Let  $\zeta_4 \in \mathbb{k}$  be a primitive fourth root of unity. The Hopf algebra  $A^{(9)}$  is given by

$$A^{(9)} = \mathbb{k}\langle c, d, x \mid c^4 = d^2 = 1, cd = dc, x^2 = 0, cx = xc, dx = -xd \rangle,$$

where  $c$  and  $d$  are group-like,  $x$  is  $cd$ -skew primitive.

First, note that both  $\mathbf{G}(A^{(9)})$  and  $\mathbf{G}((A^{(9)})^*)$  are isomorphic to  $\mathbb{Z}_4 \times \mathbb{Z}_2$  and are given by

$$\mathbf{G}(A^{(9)}) = \langle c, d \rangle \quad \text{and} \quad \mathbf{G}((A^{(9)})^*) = \langle \alpha, \beta \rangle,$$

where  $\alpha$  and  $\beta$  are algebra maps  $A^{(9)} \rightarrow \mathbb{k}$  determined by  $\alpha(c) = \zeta_4, \alpha(d) = 1, \alpha(x) = 0$  and  $\beta(c) = 1, \beta(d) = -1, \beta(x) = 0$ , respectively. Set  $\alpha_1 := \alpha^2, \alpha_2 := \beta, \alpha_3 := \alpha^2\beta$ . We get

$$\text{AD}(A^{(9)}) = \{(d, \alpha_2), (c^2d, \alpha_2), (d, \alpha_3), (c^2d, \alpha_3)\}.$$

By (3.1), we have  $SD(A^{(9)}) = \{(d, \alpha_2), (c^2d, \alpha_2)\}$ . The following is easy to see.

LEMMA 4.29. *We have  $\text{Aut}_{\text{Hopf}}(A^{(9)}) = \langle \sigma, \varphi_u \mid u \in \mathbb{k}^\times \rangle$ , where  $\sigma(c) = c, \sigma(d) = c^2d, \sigma(x) = x, \varphi_u|_{\mathbf{G}(A^{(9)})} = \text{id}$  and  $\varphi_u(x) = ux$ .*

Hence, we get  $SD(A^{(9)})/\sim = \{[(d, \alpha_2)]\}$ . Let  $\Gamma$  be the group  $C_4 = \langle g \mid g^4 = 1 \rangle$  of order four.

PROPOSITION 4.30. *The Hopf superalgebra  $\mathcal{A}(\Gamma, (g, 1, 0, 1))$  is the only one whose bosonization is isomorphic to  $A^{(9)}$ .*

*Proof.* The Hopf superalgebra  $\mathcal{H}_8^{(13)} := (A^{(9)})^{\text{coinv}(d, \alpha_2)}$  is generated by  $c, x$ . As an element in  $\mathcal{H}_8^{(13)}$ ,  $x$  is odd  $c$ -skew primitive. Thus, the assignment  $g \mapsto c, z \mapsto x$  gives a Hopf superalgebra isomorphism  $\mathcal{A}(\Gamma, (g, 1, 0; 1)) \cong \mathcal{H}_8^{(13)}$ .  $\square$

### 4.4.10 The $A^{(10)}$ case

Let  $\zeta_4 \in \mathbb{k}$  be a primitive fourth root of unity. The Hopf algebra  $A^{(10)}$  is given by

$$A^{(10)} = \mathbb{k}\langle c, d, x \mid c^4 = d^2 = 1, cd = dc, x^2 = 0, cx = \zeta_4xc, dx = -xd \rangle,$$

where  $c$  and  $d$  are group-like,  $x$  is  $c^2$ -skew primitive.

First, note that both  $\mathbf{G}(A^{(10)})$  and  $\mathbf{G}((A^{(10)})^*)$  are isomorphic to  $\mathbb{Z}_4 \times \mathbb{Z}_2$  and are given by

$$\mathbf{G}(A^{(10)}) = \langle c, d \rangle \quad \text{and} \quad \mathbf{G}((A^{(10)})^*) = \langle \alpha, \beta \rangle,$$

where  $\alpha$  and  $\beta$  are algebra maps  $A^{(10)} \rightarrow \mathbb{k}$  determined by  $\alpha(c) = \zeta_4, \alpha(d) = 1, \alpha(x) = 0$  and  $\beta(c) = 1, \beta(d) = -1, \beta(x) = 0$ , respectively. Set  $\alpha_1 := \alpha^2, \alpha_2 := \beta, \alpha_3 := \alpha^2\beta$ . We get

$$\text{AD}(A^{(10)}) = \{(d, \alpha_2), (c^2d, \alpha_2), (d, \alpha_3), (c^2d, \alpha_3)\}.$$

Since  $c^2d \in Z(A^{(10)})$ , we have  $SD(A^{(10)}) = \emptyset$  by (3.1). Thus, we get the following result.

PROPOSITION 4.31. *There is no Hopf superalgebra whose bosonization is isomorphic to  $A^{(10)}$ .*

### 4.4.11 The $A^{(11)}$ case

Let  $\zeta_4 \in \mathbb{k}$  be a primitive fourth root of unity. The Hopf algebra  $A^{(11)}$  is given by

$$A^{(11)} = \mathbb{k}\langle c, d, x \mid c^4 = d^2 = 1, cd = dc, x^2 = 0, cx = xc, dx = -xd \rangle,$$

where  $c$  and  $d$  are group-like,  $x$  is  $d$ -skew primitive.

First, note that both  $\mathbf{G}(A^{(11)})$  and  $\mathbf{G}((A^{(11)})^*)$  are isomorphic to  $\mathbb{Z}_4 \times \mathbb{Z}_2$  and are given by

$$\mathbf{G}(A^{(11)}) = \langle c, d \rangle \quad \text{and} \quad \mathbf{G}((A^{(11)})^*) = \langle \alpha, \beta \rangle,$$

where  $\alpha$  and  $\beta$  are algebra maps  $A^{(11)} \rightarrow \mathbb{k}$  determined by  $\alpha(c) = \zeta_4, \alpha(d) = 1, \alpha(x) = 0$  and  $\beta(c) = 1, \beta(d) = -1, \beta(x) = 0$ , respectively. Set  $\alpha_1 := \alpha^2, \alpha_2 := \beta, \alpha_3 := \alpha^2\beta$ . We get

$$\text{AD}(A^{(11)}) = \{(d, \alpha_2), (c^2d, \alpha_2), (d, \alpha_3), (c^2d, \alpha_3)\}.$$

By (3.1), we see that  $\text{AD}(A^{(11)})$  coincides with  $\text{SD}(A^{(11)})$ . The following is easy to see.

LEMMA 4.32. *We have  $\text{Aut}_{\text{Hopf}}(A^{(11)}) = \langle \tau, \varphi_u \mid u \in \mathbb{k}^\times \rangle$ , where  $\tau(c) = c^3, \tau(d) = d, \tau(x) = x, \varphi_u|_{\mathbf{G}(A^{(11)})} = \text{id}$  and  $\varphi_u(x) = ux$ .*

Hence, we get  $\#(\text{SD}(A^{(11)})) = \#(\text{SD}(A^{(11)})/\sim)$ . Let  $\Gamma$  be the group  $C_4 = \langle g \mid g^4 = 1 \rangle$  of order four, and let  $\chi$  be the character defined by  $\chi(g) = \zeta_4$ , where  $\zeta_4 \in \mathbb{k}$  is a primitive fourth rot of unity.

PROPOSITION 4.33. *The Hopf superalgebras  $\mathcal{A}(\Gamma, \mathcal{D})$  with  $\mathcal{D} \in \{(1, 1, 0; 1), (g^2, 1, 0; 1), (1, \chi^2, 0; 1), (g^2, \chi^2, 0; 1)\}$  are the only ones whose bosonization is isomorphic to  $A^{(11)}$ . Moreover, these are pairwise non-isomorphic.*

*Proof.* In the following, determine structure of each coinvariant subalgebras of  $A^{(11)}$  using Theorem 3.6 one by one.

- The Hopf superalgebra  $\mathcal{H}_8^{(12)} := (A^{(11)})^{\text{coinv}(d, \alpha_2)}$  is generated by  $c, x$ . As an element in  $\mathcal{H}_8^{(12)}$ ,  $x$  is odd primitive. Thus, the assignment  $g \mapsto c, z \mapsto x$  gives a Hopf superalgebra isomorphism  $\mathcal{A}(\Gamma, (1, 1, 0; 1)) \cong \mathcal{H}_8^{(12)}$ . Note that  $\mathcal{H}_8^{(12)} \cong \mathbb{k}C_4 \otimes \bigwedge \mathbb{k}$ .
- The Hopf superalgebra  $\mathcal{H}_8^{(14)} := (A^{(11)})^{\text{coinv}(c^2d, \alpha_2)}$  is generated by  $c, x$ . As an element in  $\mathcal{H}_8^{(14)}$ ,  $x$  is odd  $c^2$ -skew primitive. Thus, the assignment  $g \mapsto c, z \mapsto x$  gives a Hopf superalgebra isomorphism  $\mathcal{A}(\Gamma, (g^2, 1, 0; 1)) \cong \mathcal{H}_8^{(14)}$ .
- The Hopf superalgebra  $\mathcal{H}_8^{(16)} := (A^{(11)})^{\text{coinv}(d, \alpha_3)}$  is generated by  $cd, x$ . As an element in  $\mathcal{H}_8^{(16)}$ ,  $x$  is odd primitive. Thus, the assignment  $g \mapsto cd, z \mapsto x$  gives a Hopf superalgebra isomorphism  $\mathcal{A}(\Gamma, (1, \chi^2, 0; 1)) \cong \mathcal{H}_8^{(16)}$ .

- The Hopf superalgebra  $\mathcal{H}_8^{(17)} := (A^{(11)})^{\text{coinv}(c^2d, \alpha_2)}$  is generated by  $cd, x$ . As an element in  $\mathcal{H}_8^{(17)}$ ,  $x$  is odd  $c^2$ -skew primitive. Thus, the assignment  $g \mapsto cd, z \mapsto x$  gives a Hopf superalgebra isomorphism of type  $\mathcal{A}(\Gamma, (g^2, \chi^2, 0; 1)) \cong \mathcal{H}_8^{(17)}$ .

This completes the proof.  $\square$

### 4.4.12 The $A^{(12)}$ case

Let  $\zeta_4 \in \mathbb{k}$  be a primitive fourth root of unity. The Hopf algebra  $A^{(12)}$  is given by

$$A^{(12)} = \mathbb{k}\langle c, d, x \mid c^4 = d^2 = 1, cd = dc, x^2 = 0, cx = \zeta_4xc, dx = xd \rangle,$$

where  $c$  and  $d$  are group-like,  $x$  is  $c^2d$ -skew primitive.

First, note that both  $\mathbf{G}(A^{(12)})$  and  $\mathbf{G}((A^{(12)})^*)$  are isomorphic to  $\mathbb{Z}_4 \times \mathbb{Z}_2$  and are given by

$$\mathbf{G}(A^{(12)}) = \langle c, d \rangle \quad \text{and} \quad \mathbf{G}((A^{(12)})^*) = \langle \alpha, \beta \rangle,$$

where  $\alpha$  and  $\beta$  are algebra maps  $A^{(12)} \rightarrow \mathbb{k}$  determined by  $\alpha(c) = \zeta_4, \alpha(d) = 1, \alpha(x) = 0$  and  $\beta(c) = 1, \beta(d) = -1, \beta(x) = 0$ , respectively. Set  $\alpha_1 := \alpha^2, \alpha_2 := \beta, \alpha_3 := \alpha^2\beta$ . We get

$$\text{AD}(A^{(12)}) = \{(d, \alpha_2), (c^2d, \alpha_2), (d, \alpha_3), (c^2d, \alpha_3)\}.$$

By (3.1), we have  $\text{SD}(A^{(12)}) = \{(c^2d, \alpha_2), (c^2d, \alpha_3)\}$ . The following is easy to see.

**LEMMA 4.34.** *We have  $\text{Aut}_{\text{Hopf}}(A^{(12)}) = \langle \tau, \varphi_u \mid u \in \mathbb{k}^\times \rangle$ , where  $\tau(c) = cd, \tau(d) = d, \tau(x) = x, \varphi_u|_{\mathbf{G}(A^{(12)})} = \text{id}$  and  $\varphi_u(x) = ux$ .*

Hence, we get  $\text{SD}(A^{(12)})/\sim = \{(c^2d, \alpha_2)\}$ . Let  $\Gamma$  be the group  $C_4 = \langle g \mid g^4 = 1 \rangle$  of order four, and let  $\chi$  be the character defined by  $\chi(g) = \zeta_4$ .

**PROPOSITION 4.35.** *The Hopf superalgebra  $\mathcal{A}(\Gamma, (1, \chi, 0; 1))$  is the only ones whose bosonization is isomorphic to  $A^{(12)}$ .*

*Proof.* The Hopf superalgebra  $\mathcal{H}_8^{(15)} := (A^{(12)})^{\text{coinv}(c^2d, \alpha_2)}$  is generated by  $c, x$ . As an element in  $\mathcal{H}_8^{(15)}$ ,  $x$  is odd primitive. Thus, the assignment  $g \mapsto c, z \mapsto x$  gives a Hopf superalgebra isomorphism  $\mathcal{A}(\Gamma, (1, \chi, 0; 1)) \cong \mathcal{H}_8^{(15)}$ .  $\square$

### 4.4.13 The $A^{(13)}$ case

The Hopf algebra  $A^{(13)}$  is given by

$$A^{(13)} = \mathbb{k}\langle c, d, x \mid c^4 = d^2 = 1, cd = dc, x^2 = c^2 - 1, cx = -xc, dx = xd \rangle,$$

where  $c$  and  $d$  are group-like,  $x$  is  $c$ -skew primitive.

First, note that  $\mathbf{G}(A^{(13)}) \cong \mathbb{Z}_4 \times \mathbb{Z}_2$  and  $\mathbf{G}((A^{(13)})^*) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$  and are given by

$$\mathbf{G}(A^{(13)}) = \langle c, d \rangle \quad \text{and} \quad \mathbf{G}((A^{(13)})^*) = \{\varepsilon, \alpha_1, \alpha_2, \alpha_3 := \alpha_1\alpha_2\},$$

where  $\alpha_1$  and  $\alpha_2$  are algebra maps  $A^{(13)} \rightarrow \mathbb{k}$  determined by the equalities  $\alpha_1(c) = -1, \alpha_1(d) = 1, \alpha_1(x) = 0$  and  $\alpha_2(c) = 1, \alpha_2(d) = -1, \alpha_2(x) = 0$ , respectively. Thus, we get

$$\text{AD}(A^{(13)}) = \{(d, \alpha_2), (c^2d, \alpha_2), (d, \alpha_3), (c^2d, \alpha_3)\}.$$

By (3.1), we have  $\text{SD}(A^{(13)}) = \emptyset$ . Thus, we have the following result.

**PROPOSITION 4.36.** *There is no Hopf superalgebra whose bosonization is isomorphic to  $A^{(13)}$ .*

### 4.4.14 The $A^{(14)}$ case

The Hopf algebra  $A^{(14)}$  is given by

$$A^{(14)} = \mathbb{k}\langle c, d, x \mid c^4 = d^2 = 1, cd = dc, x^2 = c^2 - 1, cx = xc, dx = -xd \rangle,$$

where  $c$  and  $d$  are group-like,  $x$  is  $cd$ -skew primitive.

First, note that  $\mathbf{G}(A^{(14)}) \cong \mathbb{Z}_4 \times \mathbb{Z}_2$  and  $\mathbf{G}((A^{(14)})^*) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$  and are given by

$$\mathbf{G}(A^{(14)}) = \langle c, d \rangle \quad \text{and} \quad \mathbf{G}((A^{(14)})^*) = \{\varepsilon, \alpha_1, \alpha_2, \alpha_3 := \alpha_1\alpha_2\},$$

where  $\alpha_1$  and  $\alpha_2$  are algebra maps  $A^{(14)} \rightarrow \mathbb{k}$  determined by the equalities  $\alpha_1(c) = -1, \alpha_1(d) = 1, \alpha_1(x) = 0$  and  $\alpha_2(c) = 1, \alpha_2(d) = -1, \alpha_2(x) = 0$ , respectively. Thus, we get

$$\text{AD}(A^{(14)}) = \{(d, \alpha_2), (c^2d, \alpha_2), (d, \alpha_3), (c^2d, \alpha_3)\}.$$

By (3.1), we have  $\text{SD}(A^{(14)}) = \{(d, \alpha_2), (c^2d, \alpha_2)\}$ . The following is easy to see.

**LEMMA 4.37.** *We have  $\text{Aut}_{\text{Hopf}}(A^{(14)}) = \langle \tau, \varphi_u \mid u \in \mathbb{k}^\times \rangle$ , where  $\tau(c) = c^3, \tau(d) = c^2d, \tau(x) = x, \varphi_u|_{\mathbf{G}(A^{(14)})} = \text{id}$  and  $\varphi_u(x) = ux$ .*

Hence, we get  $\text{SD}(A^{(14)})/\sim = \{[(d, \alpha_2)]\}$ . Let  $\Gamma$  be the group  $G_4 = \langle g \mid g^4 = 1 \rangle$  of order four.

PROPOSITION 4.38. *The Hopf superalgebra  $\mathcal{A}(\Gamma, (g, 1, 1; 1))$  is the only ones whose bosonization is isomorphic to  $A^{(14)}$ .*

*Proof.* The Hopf superalgebra  $\mathcal{H}_8^{(18)} := (A^{(14)})^{\text{coinv}(d, \alpha_2)}$  is generated by  $c, x$ . As an element in  $\mathcal{H}_8^{(18)}$ ,  $x$  is odd  $c$ -skew primitive. Thus, the assignment  $g \mapsto c, z \mapsto x$  gives a Hopf superalgebra isomorphism  $\mathcal{A}(\Gamma, (g, 1, 1; 1)) \cong \mathcal{H}_8^{(18)}$ . The proof of Theorem 4.14 is done.  $\square$

### 5. DUALS OF POINTED HOPF SUPERALGEBRAS

In this section, we also work over an algebraically closed field  $\mathbb{k}$  of characteristic zero. We have classified pointed Hopf superalgebras of dimension up to 10. The aim of this section is to determine duals of these Hopf superalgebras.

#### 5.1. Duals of $\mathcal{H}_{2p}^{(i)}$

Let  $p$  be an odd prime number, and let  $\zeta_p \in \mathbb{k}$  be a fixed primitive  $p$ -th root of unity. We identify duals of pointed Hopf superalgebras  $\mathcal{H}_{2p}^{(i)}$  ( $i \in \{1, 2, 3, 4\}$ ) introduced in Section 4.2.

THEOREM 5.1. *There are isomorphisms*

$$(\mathcal{H}_{2p}^{(1)})^* \cong \mathcal{H}_{2p}^{(1)} \quad \text{and} \quad (\mathcal{H}_{2p}^{(2)})^* \cong \mathcal{H}_{2p}^{(3)}$$

*of Hopf superalgebras. Moreover, the dual of  $\mathcal{H}_{2p}^{(4)}$  is non-pointed.*

*Proof.* By our construction, there are Hopf algebra isomorphisms

$$\widehat{\mathcal{H}_{2p}^{(1)}} \cong \mathcal{A}(-1, p, 0), \quad \widehat{\mathcal{H}_{2p}^{(2)}} \cong \mathcal{A}(-1, 1, 0), \quad \widehat{\mathcal{H}_{2p}^{(3)}} \cong \mathcal{A}(-\zeta_p, p, 0).$$

By Proposition 2.6,  $\mathcal{A}(-1, p, 0)^*$  is isomorphic to the bosonization of  $(\mathcal{H}_{2p}^{(1)})^*$ . Since  $\mathcal{A}(-1, p, 0)$  is isomorphic to  $T_4(-1) \otimes \mathbb{k}\mathbb{Z}_p$ , we see that  $\mathcal{A}(-1, p, 0)^*$  is isomorphic to  $\mathcal{A}(-1, p, 0)$ . Since  $\mathcal{H}_{2p}^{(1)}$  is a unique non-trivial super-form of  $\mathcal{A}(-1, p, 0)$  up to isomorphisms, we conclude that  $(\mathcal{H}_{2p}^{(1)})^*$  is isomorphic to  $\mathcal{H}_{2p}^{(1)}$ .

The second isomorphism in the statement of this theorem is obtained in a similar manner: By Remark (b) mentioned after [4, Lemma A.1], we see that  $\mathcal{A}(-1, 1, 0)$  and  $\mathcal{A}(-\zeta_p, p, 0)$  are dual to each other. Since  $\mathcal{H}_{2p}^{(2)}$  and  $\mathcal{H}_{2p}^{(3)}$  are a unique non-trivial super-form of  $\mathcal{A}(-1, 1, 0)$  and  $\mathcal{A}(-\zeta_p, p, 0)$ , respectively, we conclude that  $\mathcal{H}_{2p}^{(2)}$  and  $\mathcal{H}_{2p}^{(3)}$ .

Finally, we mention the dual of  $\mathcal{H}_{2p}^{(4)}$ . By Remark (c) mentioned after [4, Lemma A.1], the dual of  $\mathcal{A}(-1, 1, 0)$  is not pointed. Hence, by Proposition 2.4,  $(\mathcal{H}_{2p}^{(4)})^*$  is not pointed.  $\square$

### 5.2. Duals of $\mathcal{H}_4^{(i)}$

We identify duals of pointed Hopf superalgebras  $\mathcal{H}_4^{(i)}$  ( $i \in \{1, 2, 3, 4\}$ ) introduced in Section 4.3.

**THEOREM 5.2.** *There are isomorphisms of Hopf superalgebras*

$$(\mathcal{H}_4^{(1)})^* \cong \mathcal{H}_4^{(1)}, \quad (\mathcal{H}_4^{(2)})^* \cong \mathcal{H}_4^{(2)} \quad \text{and} \quad (\mathcal{H}_4^{(3)})^* \cong \mathcal{H}_4^{(4)}.$$

*Proof.* The first isomorphism follows from Example 2.2. For  $(i, j) = (2, 2), (3, 4)$ , one can check that there is a non-degenerate Hopf pairing  $\mathcal{H}_4^{(i)} \times \mathcal{H}_4^{(j)} \rightarrow \mathbb{k}$  given by  $\langle g, g \rangle = -1, \langle z, z \rangle = 1$ , and  $\langle g, z \rangle = \langle z, g \rangle = 0$ .  $\square$

### 5.3. Duals of $\mathcal{H}_8^{(i)}$

We identify duals of pointed Hopf superalgebras  $\mathcal{H}_8^{(i)}$  ( $i \in \{1, 2, \dots, 18\}$ ) introduced in Section 4.4.

**THEOREM 5.3.** *There is an isomorphism  $(\mathcal{H}_8^{(i)})^* \cong \mathcal{H}_8^{(j)}$  of Hopf superalgebras for each pair  $(i, j) = (1, 1), (2, 2), (3, 4), (5, 6), (7, 7), (8, 8), (9, 10), (11, 11), (12, 12), (13, 15), (14, 16), (17, 17)$ . Moreover, the dual of  $\mathcal{H}_8^{(18)}$  is non-pointed.*

*Proof.* For  $(i, j) = (3, 4), (7, 7), (13, 15)$ , the proof goes along the same line as that of Theorem 5.1. More precisely, for these pairs  $(i, j)$ , the bosonization of  $\mathcal{H}_8^{(i)}$  and that of  $\mathcal{H}_8^{(j)}$  are dual to each other by Proposition 2.6 and [8]. According to our classification result,  $\mathcal{H}_8^{(\ell)}$  ( $\ell \in \{i, j\}$ ) is a unique non-trivial super-form of its bosonization. Thus, by Proposition 2.6, we conclude that  $\mathcal{H}_8^{(i)}$  and that of  $\mathcal{H}_8^{(j)}$  are dual to each other.

The above argument cannot be applied to other cases. By direct computation, there is the following non-degenerate Hopf pairing  $\langle \cdot, \cdot \rangle : \mathcal{H}_8^{(i)} \times \mathcal{H}_8^{(j)} \rightarrow \mathbb{k}$ .

- For  $(i, j) = (2, 2), (5, 6)$ , define

$$\langle g, g \rangle = -1, \quad \langle z_s, z_t \rangle = \delta_{s,t} \quad \text{and} \quad \langle g, z_s \rangle = \langle z_s, g \rangle = 0 \quad (s, t \in \{1, 2\}).$$



- For  $(i, j) = (8, 8), (9, 10), (11, 11)$ , define

$$\langle g_s, g_t \rangle = (-1)^{\delta_{s,t}}, \quad \langle z, z \rangle = 1 \quad \text{and} \quad \langle g_s, z \rangle = \langle z, g_s \rangle = 0 \quad (s, t \in \{1, 2\}).$$

- For  $(i, j) = (12, 12), (14, 16), (17, 17)$ , define

$$\langle g, g \rangle = \zeta_4, \quad \langle z, z \rangle = 1 \quad \text{and} \quad \langle g, z \rangle = \langle z, g \rangle = 0,$$

where  $\zeta_4$  is a primitive fourth root of unity.

The Hopf superalgebra  $\mathcal{H}_8^{(18)}$  is the only one whose bosonization is isomorphic to  $A^{(14)}$ . By [8], we know that the dual of  $A^{(14)}$  is not pointed. Thus, the claim follows.  $\square$

### 5.4. Concluding remarks

In this paper, we have classified non-semisimple pointed Hopf superalgebras of dimensions 4, 8,  $2p$  and obtained Tables 1, 2 and 3. The dual of some of them are not pointed. By the classification result of low-dimensional Hopf algebras, we prove the following.

**THEOREM 5.4.** *Suppose that  $\mathcal{H}$  is non-semisimple non-pointed Hopf superalgebra such that  $\mathcal{H}_{\bar{1}} \neq 0$ .*

1. *If  $\dim(\mathcal{H}) = 6$ , then  $\mathcal{H}$  is isomorphic to  $(\mathcal{H}_6^{(4)})^*$ .*
2. *If  $\dim(\mathcal{H}) = 8$  and  $\mathcal{H}$  does not have the Chevalley property, then  $\mathcal{H}$  is isomorphic to  $(\mathcal{H}_8^{(18)})^*$ .*
3. *If  $\dim(\mathcal{H}) = 10$ , then  $\mathcal{H}$  is isomorphic to  $(\mathcal{H}_{10}^{(4)})^*$ .*

*Proof.* We consider the case where  $\dim(\mathcal{H})$  is either 6 or 10. According to Cheng and Ng [11, Theorem II], a non-semisimple Hopf algebra of dimension  $4p$  with  $p \in \{3, 5, 7, 11\}$  is pointed or dual-pointed (meaning that the dual is pointed). By Proposition 2.4 and the assumption that  $\mathcal{H}$  is non-semisimple non-pointed,  $\mathcal{H}^*$  must be pointed. Thus,  $\mathcal{H}^*$  is isomorphic to the Hopf superalgebra  $\mathcal{H}_{2p}^{(4)}$ , where  $p = \dim(\mathcal{H})/2$ . Thus  $\mathcal{H} \cong (\mathcal{H}_{2p}^{(4)})^*$ .

To complete the proof, we assume that  $\dim(\mathcal{H}) = 8$  and  $\mathcal{H}$  does not have the Chevalley property. According to García and Vay [13, Theorem 1.3], a non-semisimple non-pointed Hopf algebra without the Chevalley property is isomorphic to the dual of  $A^{(14)}$  of Table 4. Thus,  $\widehat{\mathcal{H}}$  is isomorphic to  $(A^{(14)})^*$ . Since  $(\mathcal{H}_8^{(18)})^*$  is a unique super-form of  $(A^{(14)})^*$ , we conclude that  $\mathcal{H}$  is isomorphic to  $(\mathcal{H}_8^{(18)})^*$ . The proof is done.  $\square$

According to the classification result of Hopf algebras of low-dimensions (see the survey [9]), non-semisimple Hopf algebras of dimension  $\leq 20$ , except dimension 16, are pointed or dual-pointed. This implies that non-semisimple Hopf superalgebras of dimension  $\leq 10$ , except dimension 8, are pointed or dual-pointed.

Given a primitive fourth root  $\zeta_4$  of unity, we denote by  $H_{16}(\zeta_4)$  the Hopf algebra of dimension 16 with the Chevalley property introduced in [12]. By [13], every Hopf algebra of dimension 16 that is neither pointed nor dual-pointed has the Chevalley property and isomorphic to either of  $H_{16}(\zeta_4)$  or  $H_{16}(-\zeta_4)$ . In [18], we have determined all super-forms of  $H_{16}(\pm\zeta_4)$  and obtained 8 new Hopf superalgebras of dimension 8. Thus, combining [18] with Theorems 2.3 and 5.4, we have completed the classification of non-semisimple Hopf superalgebras of dimension  $\leq 10$ . The remaining case, semisimple Hopf superalgebras of dimension  $\leq 10$ , are discussed in our forthcoming paper.

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*Department of Applied Mathematics,  
Okayama University of Science,  
1-1 Ridai-cho Kita-ku Okayama-shi, Okayama  
700-0005, Japan  
shibata@ous.ac.jp*

*Department of Mathematical Sciences,  
Shibaura Institute of Technology,  
307 Fukasaku, Minuma-ku, Saitama-shi, Saitama  
337-8570, Japan  
kshimizu@shibaura-it.ac.jp*

*Department of Applied Mathematics,  
Okayama University of Science,  
1-1 Ridai-cho Kita-ku Okayama-shi, Okayama  
700-0005, Japan  
r23nda8mr@ous.jp*