SPACES OF k-MODIFIED HARMONIC POLYNOMIALS

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The polynomials u on $\mathbb{R}^d$ ($d \geq 2$) that satisfy the equation $x_d \cdot \Delta u + k \cdot \frac{\partial u}{\partial x_d} = 0$ ($k \in \mathbb{R}$) are called *k-modified harmonic*. In this article, we study the dimension of the space of homogeneous such polynomials of a fixed degree.

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1. INTRODUCTION AND NOTATIONS

In the 1940s and later, Alexander Weinstein studied the partial differential equation

$$x_d \cdot \Delta u(x) + k \cdot \frac{\partial u(x)}{\partial x_d} = 0 \tag{1}$$

for a function u on a domain in $\mathbb{R}^d$, where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$ and $k \in \mathbb{R}$ (see [5]). The u solutions that he obtained were called *generalized axially symmetric potentials*. This term is justified by the following observations.

If a function f on a domain in $\mathbb{R}^{2+n}$ is a solution of the Laplace equation

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_{2+n}^2} = 0 \qquad (n \geq 1)$$

and if f is axially symmetric about the x_1-axis, that is,

$$f(x_1, x_2, \ldots, x_{2+n}) = \phi\big(x_1, \sqrt{x_2^2 + \cdots + x_{2+n}^2}\big)$$

with a suitable function ϕ, then ϕ satisfies the equation

$$y \cdot \left[\frac{\partial^2 \phi(x,y)}{\partial x^2} + \frac{\partial^2 \phi(x,y)}{\partial y^2}\right] + n \cdot \frac{\partial \phi(x,y)}{\partial y} = 0$$

(and vice versa). More generally, if f is defined on a domain in $\mathbb{R}^{d+n}$ and

$$f(x_1, x_2, \ldots, x_{d+n}) = u\big(x_1, x_2, \ldots, x_{d-1}, \sqrt{x_d^2 + x_{d+1}^2 + \cdots + x_{d+n}^2}\big)$$

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with a suitable function u, then f is a solution of the Laplace equation if and only if u satisfies (1) with $k = n$ (see [3], where the reader can also find other results related to our present study).

In this article, following Heinz Leutwiler (see, e. g., [2]), we call the solutions of (1) *k-modified harmonic functions*. This term is justified by the fact that for $d \geq 3$ the operator

$$x_d^{2k/(2-d)}\left(\Delta + \frac{k}{x_d} \cdot \frac{\partial}{\partial x_d}\right)$$

turns out to be the Laplace–Beltrami operator for $\mathbb{R}^{d-1} \times (0, \infty)$ with the line-element

$$\mathrm{d}l^2 = x_d^{2k/(d-2)}\left(\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \cdots + \mathrm{d}x_d^2\right) .$$

Let $\mathcal{H}_n^{(k)}(\mathbb{R}^d)$ be the real vector space of all k-modified harmonic functions that are homogeneous polynomials of degree n on $\mathbb{R}^d$. In what follows, we assume that $d \geq 2$. We denote by $\mathbb{N}$ the set of strictly positive integers and write $\mathbb{N}_0$ for $\mathbb{N} \cup \{0\}$. Let $\mathcal{P}_n(\mathbb{R}^d)$ be the real vector space of all homogeneous polynomials of degree n on $\mathbb{R}^d$. The monomials

$$x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \ldots \cdot x_d^{\alpha_d}$$

for $\alpha_1, \alpha_2, \ldots, \alpha_d \in \mathbb{N}_0$, $\alpha_1 + \alpha_2 + \cdots + \alpha_d = n$ form a basis of $\mathcal{P}_n(\mathbb{R}^d)$. Its elements may be counted as n-combinations with repetitions of $x_1, \ldots, x_d$, so

$$\dim \mathcal{P}_n(\mathbb{R}^d) = \binom{d+n-1}{n} . \tag{2}$$

We now fix n and consider the (well-defined) linear map

$$W_n^{(k)} : \mathcal{P}_n(\mathbb{R}^d) \longrightarrow \mathcal{P}_{n-1}(\mathbb{R}^d) , \quad W_n^{(k)}(u)(x) := x_d \Delta u(x) + k \cdot \frac{\partial u(x)}{\partial x_d} .$$

For its kernel, it obviously holds that $\ker W_n^{(k)} = \mathcal{H}_n^{(k)}(\mathbb{R}^d)$. So, if $W_n^{(k)}$ is surjective, then

$$\dim \mathcal{H}_n^{(k)}(\mathbb{R}^d) = \dim \mathcal{P}_n(\mathbb{R}^d) - \dim \mathcal{P}_{n-1}(\mathbb{R}^d)$$

$$= \binom{d+n-1}{n} - \binom{d+n-2}{n-1} = \binom{d+n-2}{n} .$$

In this article, we investigate the issue of surjectivity of $W_n^{(k)}$ (Section 2) and find out the dimension of $\mathcal{H}_n^{(k)}(\mathbb{R}^d)$ in all but a finite number of cases (referring to the parameter k). In Section 4, we also present explicit bases in almost all those cases.

2. THE OPERATOR $W_n^{(k)}$

Let $u \in \mathcal{P}_n(\mathbb{R}^d)$, which we write in ascending powers of x_d:

$$u(x_1, \ldots, x_d) = \sum_{j=0}^{n} x_d^j p_j(x_1, \ldots, x_{d-1}),$$

where $p_j \in \mathcal{P}_{n-j}(\mathbb{R}^{d-1})$. Now, we apply the linear map $W_n^{(k)}$ and write the result in the same order:

$$\begin{aligned}
W_n^{(k)}(u) = x_d \Delta u + k \cdot \frac{\partial u}{\partial x_d} &= x_d \tilde{\Delta} u + x_d \cdot \frac{\partial^2 u}{\partial x_d^2} + k \cdot \frac{\partial u}{\partial x_d} \\
&= \sum_{j=0}^{n} x_d^{j+1} \tilde{\Delta} p_j + x_d \sum_{j=0}^{n} j(j-1) x_d^{j-2} p_j + k \sum_{j=0}^{n} j x_d^{j-1} p_j \\
&= \sum_{j=0}^{n-1} x_d^{j+1} \tilde{\Delta} p_j + \sum_{j=1}^{n} j(j-1+k) x_d^{j-1} p_j \\
&= \sum_{l=1}^{n} x_d^{l} \tilde{\Delta} p_{l-1} + \sum_{l=0}^{n-1} (l+1)(l+k) x_d^{l} p_{l+1} \\
&= k p_1 + \underbrace{x_d^n \tilde{\Delta} p_{n-1}}_{=0} + \sum_{l=1}^{n-1} x_d^l \cdot \left[\tilde{\Delta} p_{l-1} + (l+1)(l+k) p_{l+1} \right].
\end{aligned}$$

($\tilde{\Delta}$ denotes the Laplacian in the first $d-1$ coordinates.)

To check if $W_n^{(k)}$ is surjective, we consider several special cases.

Case I. $W_n^{(k)}(u)(x) = q(x_1, \ldots, x_{d-1}) \in \mathcal{P}_{n-1}(\mathbb{R}^{d-1})$, $q \neq 0$.

Firstly, this equation requires $p_1(x_1, \ldots, x_{d-1}) = \frac{q(x_1, \ldots, x_{d-1})}{k}$, which is only possible if $k \neq 0$. Secondly, it is purposive to take $p_m = 0$ for even m, while p_1 determines the rest of the coefficient polynomials step by step:

$$p_3 = \frac{-\tilde{\Delta} p_1}{3(2+k)}, \quad p_5 = \frac{-\tilde{\Delta} p_3}{5(4+k)}, \ldots, \; p_n = \frac{-\tilde{\Delta} p_{n-2}}{n(n-1+k)} \text{ if } n \in 2\mathbb{N}_0 + 1,$$

$$p_{n-1} = \frac{-\tilde{\Delta} p_{n-3}}{(n-1)(n-2+k)} \quad \text{if } n \in 2\mathbb{N}.$$

Obviously, this is only possible if $k \notin \{-2, -4, \ldots, -2\lfloor \frac{n-1}{2} \rfloor\}$.

Case II. $W_n^{(k)}(u)(x) = x_d^{n-1}$.

a. Let n be even.

Here, it is purposive to take $p_m = 0$ for odd m, whereas for the rest, for example $p_n = 0$, $p_{n-2} = \frac{x_1^2}{2}$, and

$$p_{n-2m}(x) = \frac{(n-2)\cdots(n-2m+2)\cdot(k+n-3)\cdots(k+n-2m+1)}{(-1)^{m-1}(2m)!} x_1^{2m}$$

for $2 \le m \le \frac{n}{2}$.

b. Let n be odd.

Since

$$W_n^{(k)}(ax_d^n) = x_d \cdot an(n-1)x_d^{n-2} + k \cdot anx_d^{n-1} = an(n-1+k)x_d^{n-1},$$

we just have to set $a = \frac{1}{n(n-1+k)}$ if $k \ne 1-n$. If $k = 1-n$, it is purposive to take $p_m = 0$ for even m, while $p_1 = 0$ is obviously necessary (unless $n = 1$, in which case, we have $k = 0$, violating the restriction in Case I). But then, step by step, we have to take $p_3 = 0, \ldots, p_{n-2} = 0$, and then the coefficient of x_d^{n-1} in $W_n^{(k)}(u)$ vanishes too. Therefore, this case is only possible if $k \ne 1-n$.

Case III. $W_n^{(k)}(u)(x) = x_d^l q(x_1, \ldots, x_{d-1}) \ne 0$, $q \in \mathcal{P}_{n-1-l}(\mathbb{R}^{d-1})$, for $1 \le l \le n-2$.

a. Let l be odd.

Here, it is purposive to take $p_m = 0$ for odd m as well as for all $m \ge l+1$, p_{l-1} such that $\tilde{\Delta} p_{l-1} = q$, and inductively, p_{l-2j-1} such that

$$\tilde{\Delta} p_{l-2j-1} = -(l-2j+1)(l-2j+k)p_{l-2j+1}\,, \quad 1 \le j \le \frac{l-1}{2}\,.$$

This is possible, because the so-called Poisson equation with a polynomial right side has a polynomial solution (see [1]).

b. Let l be even.

Here, it is purposive to take $p_m = 0$ for even m as well as for all $m \le l-1$,

$$p_{l+1} = \frac{q}{(l+1)(l+k)}\,,$$

and the rest of p_m with odd m appropriately. Obviously, this is only possible if $k \ne -m$ for any even m, $l \le m \le n-1$.

The investigation of these three special cases has proven the following.

THEOREM 2.1. *The operator* $W_n^{(k)} : \mathcal{P}_n(\mathbb{R}^d) \longrightarrow \mathcal{P}_{n-1}(\mathbb{R}^d)$,

$$W_n^{(k)}(u)(x) = x_d \cdot \Delta u(x) + k \cdot \frac{\partial u(x)}{\partial x_d},$$

is surjective if and only if $-k \notin 2\mathbb{N}_0 \cap [0, n-1]$. *In this case,*

$$\dim \mathcal{H}_n^{(k)}(\mathbb{R}^d) = \binom{d+n-2}{n}.$$

3. THE EXCEPTIONAL VALUE $k = 0$

If $k = 0$, Case III in the previous section remains possible, likewise Case II unless $n = 1$ (then it coincides with Case I), but Case I does not. So, the image of $W_n^{(0)}$ is the space of those homogeneous polynomials in $\mathcal{P}_{n-1}(\mathbb{R}^d)$ that are multiples of x_d. This space is clearly isomorphic to $\mathcal{P}_{n-2}(\mathbb{R}^d)$, so

$$\dim \mathcal{H}_n^{(0)}(\mathbb{R}^d) = \binom{d+n-1}{n} - \binom{d+n-3}{n-2} = \binom{d+n-2}{n} + \binom{d+n-3}{n-1}.$$

4. A BASIS OF $\mathcal{H}_n^{(k)}(\mathbb{R}^d)$

In this section, we determine a basis of the space $\mathcal{H}_n^{(k)}(\mathbb{R}^d)$, as defined in the introduction, for all but a finite number of values of k.

If u is a k-modified harmonic function, then the functions $\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_{d-1}}$ are obviously k-modified harmonic too. Furthermore, if u is k-modified harmonic, then so is its *k-modified Kelvin transform*,

$$K[u](x_1, \ldots, x_d) := r^{2-k-d} \cdot u\left(\frac{x_1}{r^2}, \ldots, \frac{x_d}{r^2}\right),$$

where $r := \sqrt{x_1^2 + \cdots + x_d^2}$. This can be verified by an elementary, but lengthy computation.

Now, since $u(x_1, \ldots, x_d) := r^{2-k-d}$ is k-modified harmonic (being the k-modified Kelvin transform of 1), so are its partial derivatives

$$u_{\alpha_1 \cdots \alpha_{d-1}} := \frac{\partial^n u}{\partial x_1^{\alpha_1} \cdots \partial x_{d-1}^{\alpha_{d-1}}}$$

for $\alpha_1, \ldots, \alpha_{d-1} \in \mathbb{N}_0$, $\alpha_1 + \cdots + \alpha_{d-1} = n$, as well as their k-modified Kelvin transforms

$$\begin{aligned} v_{\alpha_1 \cdots \alpha_{d-1}}(x_1, \ldots, x_d) &:= K[u_{\alpha_1 \cdots \alpha_{d-1}}](x_1, \ldots, x_d) \\ &= r^{2-k-d} u_{\alpha_1 \cdots \alpha_{d-1}}\left(\frac{x_1}{r^2}, \ldots, \frac{x_d}{r^2}\right) = r^{2n+d+k-2} u_{\alpha_1 \cdots \alpha_{d-1}}(x_1, \ldots, x_d), \end{aligned} \tag{3}$$

since $u_{\alpha_1\cdots\alpha_{d-1}}$ is homogeneous of degree $2-k-d-n$ (in fact, r^{2-k-d} is homogeneous of degree $2-k-d$, and every partial derivative reduces the degree of homogeneity by 1). Inductively, it follows that $u_{\alpha_1\cdots\alpha_{d-1}}$ has the form $r^{2-k-d-2n}\cdot P$ with a polynomial P, so the functions $v_{\alpha_1\cdots\alpha_{d-1}}$ are polynomials, in fact homogeneous polynomials of degree n: $v_{\alpha_1\cdots\alpha_{d-1}} \in \mathcal{H}_n^{(k)}(\mathbb{R}^d)$.

The number of the tuples $(\alpha_1,\ldots,\alpha_{d-1}) \in \mathbb{N}_0$ such that $\alpha_1+\cdots+\alpha_{d-1}=n$ is $\binom{d+n-2}{n}$ (cf. (2)). Considering the last equation in the introduction, the question arises whether the $\binom{d+n-2}{n}$ polynomials $v_{\alpha_1\cdots\alpha_{d-1}} \in \mathcal{H}_n^{(k)}(\mathbb{R}^d)$ form a basis of $\mathcal{H}_n^{(k)}(\mathbb{R}^d)$. According to Theorem 2.1, a necessary condition for this is $-k \notin 2\mathbb{N}_0 \cap [0, n-1]$.

Before we take a closer look at the polynomials $v_{\alpha_1\cdots\alpha_{d-1}}$, we give a list of them for $\alpha_1+\cdots+\alpha_{d-1} \le 3$, from which it already can be seen that by no means they form a basis in every case.

$v_{0\ldots0} = 1$

$v_{0\ldots010\ldots0} = (2-k-d)x_i$ (the index 1 is at the i-th position)

$v_{0\ldots020\ldots0} = (2-k-d)[r^2-(k+d)x_i^2]$ (the index 2 is at the i-th position)

$v_{0\ldots010\ldots010\ldots0} = (k+d-2)(k+d)x_ix_j$ (the indices 1 are at the positions i and j)

$v_{0\ldots030\ldots0} = (k+d-2)(k+d)x_i[3r^2-(k+d+2)x_i^2]$ (the index 3 is at the i-th position)

$v_{0\ldots020\ldots010\ldots0} = (k+d-2)(k+d)x_j[r^2-(k+d+2)x_i^2]$ (2 is at the i-th, 1 at the j-th position)

$v_{0\ldots010\ldots010\ldots010\ldots0} = (2-k-d)(k+d)(k+d+2)x_ix_jx_l$ (the indices 1 are at the positions i, j, l).

At this point, we introduce a notation, which facilitates the further study of the polynomials $v_{\alpha_1\cdots\alpha_{d-1}}$. We set $X_1 := x_1^2, \ldots, X_d := x_d^2$ and relate every function $f(X_1,\ldots,X_d)$ to the function

$$g(x_1,\ldots,x_d) := f(x_1^2,\ldots,x_d^2) = f(X_1,\ldots,X_d)\big|_{X_1=x_1^2,\ldots,X_d=x_d^2}.$$

Here and in the sequel, we assume that $x_1,\ldots,x_d \ge 0$. This correspondence leads to relations between the partial derivatives of f and g:

$$\begin{aligned}\frac{\partial g}{\partial x_i}(x_1,\ldots,x_d) &= \frac{\partial f}{\partial X_i}(X_1,\ldots,X_d)\Big|_{X_1=x_1^2,\ldots,X_d=x_d^2}\cdot 2x_i \\ &= \Big[\frac{\partial f}{\partial X_i}(X_1,\ldots,X_d)\cdot 2\sqrt{X_i}\Big]\Big|_{X_1=x_1^2,\ldots,X_d=x_d^2}\end{aligned}$$

for $1 \le i \le d$, which we express in the shorter form

$$\frac{\partial g}{\partial x_i} = 2\sqrt{X_i}\cdot\frac{\partial f}{\partial X_i}.$$

Under this convention, it further holds for $i, j \in \{1, \dots, d\}$:

$$\frac{\partial^2 g}{\partial x_i^2} = 2 \cdot \frac{\partial f}{\partial X_i} + 4X_i \cdot \frac{\partial^2 f}{\partial X_i^2}, \qquad \frac{\partial^2 g}{\partial x_i \partial x_j} = 4\sqrt{X_i X_j} \cdot \frac{\partial^2 f}{\partial X_i \partial X_j},$$

$$\frac{\partial^3 g}{\partial x_i^3} = 12\sqrt{X_i} \cdot \frac{\partial^2 f}{\partial X_i^2} + 8X_i\sqrt{X_i} \cdot \frac{\partial^3 f}{\partial X_i^3},$$

$$\frac{\partial^4 g}{\partial x_i^4} = 12 \cdot \frac{\partial^2 f}{\partial X_i^2} + 48X_i \cdot \frac{\partial^3 f}{\partial X_i^3} + 16X_i^2 \cdot \frac{\partial^4 f}{\partial X_i^4},$$

$$\frac{\partial^5 g}{\partial x_i^5} = 120X_i^{\frac{1}{2}} \cdot \frac{\partial^3 f}{\partial X_i^3} + 160X_i^{\frac{3}{2}} \cdot \frac{\partial^4 f}{\partial X_i^4} + 32X_i^{\frac{5}{2}} \cdot \frac{\partial^5 f}{\partial X_i^5}, \quad \text{etc.}$$

The next three lemmas are crucial.

Lemma 4.1. 1. *For $\alpha \in 2\mathbb{N}_0$ and $i \in \{1, \dots, d\}$ it holds:*

$$\frac{\partial^\alpha g}{\partial x_i^\alpha} = \sum_{j=0}^{\frac{\alpha}{2}} c_{i,\alpha,j} X_i^j \cdot \frac{\partial^{\frac{\alpha}{2}+j} f}{\partial X_i^{\frac{\alpha}{2}+j}}$$

with certain $c_{i,\alpha,j} \in \mathbb{N}$ (not null).

2. *For $\alpha \in 2\mathbb{N}_0 + 1$ and $i \in \{1, \dots, d\}$ it holds:*

$$\frac{\partial^\alpha g}{\partial x_i^\alpha} = \sum_{j=0}^{\frac{\alpha-1}{2}} c_{i,\alpha,j+\frac{1}{2}} X_i^{j+\frac{1}{2}} \cdot \frac{\partial^{\frac{\alpha+1}{2}+j} f}{\partial X_i^{\frac{\alpha+1}{2}+j}}$$

with certain $c_{i,\alpha,j+\frac{1}{2}} \in \mathbb{N}$ (not null).

For a proof, the reader is referred to [4].

Lemma 4.2. *The functions $X_1^{i_1} \cdot \ldots \cdot X_{d-1}^{i_{d-1}}$, where $i_1, \dots, i_{d-1}$ run through $\frac{1}{2}\mathbb{N} \cup \{0\}$, are linearly independent.*

Proof. After the substitution $X_1 = x_1^2, \dots, X_{d-1} = x_{d-1}^2$, these functions become the monomials $x_1^{2i_1} \cdot \ldots \cdot x_{d-1}^{2i_{d-1}}$, which are linearly independent. □

Lemma 4.3. *For a function of the form $f(X_1, \dots, X_d) = \frac{1}{(X_1+\dots+X_d)^\tau}$ it holds:*

$$\frac{\partial^l f}{\partial X_1^l} = \dots = \frac{\partial^l f}{\partial X_d^l} = \frac{(-1)^l \cdot (\tau)_l}{(X_1 + \dots + X_d)^{\tau+l}},$$

where $(\tau)_l := \tau(\tau+1)\cdots(\tau+l-1)$ is the Pochhammer symbol.

Proof. The claim follows easily by induction. □

For the function in the last lemma, it follows that

$$\frac{\partial^{\alpha_1+\cdots+\alpha_{d-1}} f(X_1,\ldots,X_d)}{\partial X_1^{\alpha_1}\cdots\partial X_{d-1}^{\alpha_{d-1}}} = \frac{(-1)^{\alpha_1+\cdots+\alpha_{d-1}}(\tau)_{\alpha_1+\cdots+\alpha_{d-1}}}{(X_1+\cdots+X_d)^{\tau+\alpha_1+\cdots+\alpha_{d-1}}}. \tag{4}$$

To resume the study of the polynomials $v_{\alpha_1\cdots\alpha_{d-1}}$ we set $\tau = \frac{k+d}{2}-1$. Since $f(X_1,\ldots,X_d)|_{X_1=x_1^2,\ldots,X_d=x_d^2} = r^{-2\tau} = r^{2-k-d}$ for $r=\sqrt{x_1^2+\cdots+x_d^2}$, the question is whether the functions

$$\frac{\partial^n}{\partial x_1^{\alpha_1}\cdots\partial x_{d-1}^{\alpha_{d-1}}}\left[f(X_1,\ldots,X_d)|_{X_1=x_1^2,\ldots,X_d=x_d^2}\right] = \frac{v_{\alpha_1\cdots\alpha_{d-1}}(x_1,\ldots,x_d)}{r^{2n+d+k-2}}$$

for $\alpha_1,\ldots,\alpha_{d-1}\in\mathbb{N}_0$, $\alpha_1+\cdots+\alpha_{d-1}=n$ (see (3)), are linearly independent.

The following reasoning is similar with that in the last part of [4], there formulated for the case $k=d-2$.

By reductio ad absurdum, we assume that there exists a linear combination

$$\sum_{\substack{\alpha_1+\cdots+\alpha_{d-1}=n\\ \alpha_1,\ldots,\alpha_{d-1}\geq 0}} C_{\alpha_1,\ldots,\alpha_{d-1}}\cdot\frac{\partial^n}{\partial x_1^{\alpha_1}\cdots\partial x_{d-1}^{\alpha_{d-1}}}\left[f(X_1,\ldots,X_d)|_{X_1=x_1^2,\ldots,X_d=x_d^2}\right]=0, \tag{5}$$

where not all $C_{\alpha_1,\ldots,\alpha_{d-1}}$ vanish.

Let $\widehat{\alpha_1}$ be the highest value of α_1 such that $C_{\alpha_1,\ldots,\alpha_{d-1}}\neq 0$ for certain $\alpha_2,\ldots,\alpha_{d-1}$. Let $\widehat{\alpha_2}$ be the highest value of α_2 such that $C_{\widehat{\alpha_1},\alpha_2,\alpha_3,\ldots,\alpha_{d-1}}\neq 0$ for certain $\alpha_3,\ldots,\alpha_{d-1}$. Continuing inductively, let eventually $\widehat{\alpha_{d-2}}$ be the highest value of α_{d-2} for which $C_{\widehat{\alpha_1},\widehat{\alpha_2},\ldots,\widehat{\alpha_{d-3}},\alpha_{d-2},\alpha_{d-1}}\neq 0$ for a certain α_{d-1}. Obviously, there is only one such value of α_{d-1}, namely

$$\widehat{\alpha_{d-1}} := n-\widehat{\alpha_1}-\widehat{\alpha_2}-\cdots-\widehat{\alpha_{d-2}}.$$[1]

According to Lemma 4.1, the term of the highest order monomial $X_1^{j_1}\cdots X_{d-1}^{j_{d-1}}$ in $\frac{\partial^n}{\partial x_1^{\alpha_1}\cdots\partial x_{d-1}^{\alpha_{d-1}}}[f(X_1,\ldots,X_d)|_{X_1=x_1^2,\ldots,X_d=x_d^2}]$ is

$$c_{1,\alpha_1,\frac{\alpha_1}{2}}\cdot\ldots\cdot c_{d-1,\alpha_{d-1},\frac{\alpha_{d-1}}{2}}\cdot X_1^{\frac{\alpha_1}{2}}\cdots X_{d-1}^{\frac{\alpha_{d-1}}{2}}\cdot\frac{\partial^{\alpha_1+\cdots+\alpha_{d-1}}f(X_1,\ldots,X_d)}{\partial X_1^{\alpha_1}\cdots\partial X_{d-1}^{\alpha_{d-1}}}.$$

Therefore, after setting $X_d = 1-X_1-\cdots-X_{d-1}$, the product $X_1^{\frac{\widehat{\alpha_1}}{2}}\cdots X_{d-1}^{\frac{\widehat{\alpha_{d-1}}}{2}}$ appears only once in (5), and its coefficient is, according to (4),

$$C_{\widehat{\alpha_1},\ldots,\widehat{\alpha_{d-1}}}\cdot c_{1,\widehat{\alpha_1},\frac{\widehat{\alpha_1}}{2}}\cdot\ldots\cdot c_{d-1,\widehat{\alpha_{d-1}},\frac{\widehat{\alpha_{d-1}}}{2}}\cdot(-1)^n(\tau)_n,$$

[1]In fact, $C_{\widehat{\alpha_1},\ldots,\widehat{\alpha_{d-1}}}$ is the last non-null coefficient with respect to the following lexicographic ordering of $(d-1)$-tuples (considered in ascending order): $(\alpha_1,\ldots,\alpha_{d-1}) < (\alpha_1',\ldots,\alpha_{d-1}')$ if and only if there exists $l\in\{1,\ldots,d-1\}$ such that $\alpha_i=\alpha_i'$ for $i<l$, and $\alpha_l<\alpha_l'$.

which does not vanish unless $-\tau \in \mathbb{N}_0 \cap [0, n-1]$, a fact that would contradict Lemma 4.2.

At this point, we have established the following fact.

THEOREM 4.4. *If* $-k \notin d-2+2(\mathbb{N}_0 \cap [0, n-1])$, *the polynomials* $v_{\alpha_1 \cdots \alpha_{d-1}}$, $\alpha_1 + \cdots + \alpha_{d-1} = n$ *are linearly independent. If additionally* $-k \notin 2\mathbb{N}_0 \cap [0, n-1]$, *they form a basis of* $\mathcal{H}_n^{(k)}(\mathbb{R}^d)$.

The treatment in the case of the exceptional values of k seems to be a difficult issue.

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