ON THE CALCULATION OF HIVES ASSOCIATED TO SUMS OF SELFADJOINT MATRICES

HARI BERCOVICI and WING SUET LI

Communicated by Dan Timotin

We present a technique that can be used in the study of a conjecture by Danilov and Koshevoy, concerning triples (A, B, C) of $n \times n$ complex selfadjoint matrices such that $C = A + B$. The conjecture proposes an explicit formula, in terms of traces of compressions of A and B, for one associated hive. We also use this technique to show why an earlier attempt to prove the conjecture fails for $n = 4$.

AMS 2020 Subject Classification: 15A18.

Key words: hive, honeycomb, selfadjoint matrix, eigenvalue.

1. INTRODUCTION

Suppose that n is a natural number, and that A, B, C are selfadjoint $n \times n$ matrices such that $A+B = C$. Knutson and Tao [8] showed that the eigenvalues of A, B, and C appear as the increments of a hive between adjacent boundary points. To make this statement precise, we recall the definition of a hive. Let Δ_n be an equilateral triangle with sides of length n, divided into n^2 equilateral triangles with sides of length 1. These smaller triangles are referred to as *unit triangles*, and their vertices as *lattice points*. The lattices points can be labeled by pairs (p, q) of nonnegative integers such that $p + q \leq n$; see Figure 1. More precisely, we use the top of the triangle as the origin of a system of coordinates, the left side as the p-coordinate axis, and the right side as the q-coordinate axis.

A hive is a real-valued function h, defined on the set of lattice points in Δ_n, that satisfies the following requirement: given distinct lattice points X, Y, Z, W such that XYZ and WYZ are unit triangles, we have

$$h(X) + h(W) \leq h(Y) + h(Z).$$

We say that a hive h is *associated* with the triple (A, B, C) of selfadjoint matrices if the eigenvalues of A, B, and C, listed in nonincreasing order, are

$$h(p, 0) - h(p - 1, 0), \quad p = 1, \ldots, n,$$

The research of Wing Suet Li was supported in part by National Science Foundation grant DMS-2344684.

REV. ROUMAINE MATH. PURES APPL. **69** (2024), *3-4*, 375–395
doi: 10.59277/RRMPA.2024.375.395

$$h(n-q,q)-h(n-q+1,q-1), \quad q=1,\dots,n, \text{ and}$$
$$h(0,q)-h(0,q-1), \quad q=1,\dots,n,$$

respectively. We only consider hives that satisfy $h(0,0)=0$. A result of [8] states that there always exists a hive associated with (A,B,C) if $C=A+B$. There are usually many such hives, but there is no simple procedure for producing one. Trying to remedy this situation, Danilov and Koshevoy [5] defined an explicit function $h_{A,B,C}$ on the lattice points in Δ_n that, they conjectured, is a hive associated with (A,B,C). To define this function, we introduce some notation. Given a pair (P,Q) of mutually orthogonal projections on $\mathbb{C}^n$, set

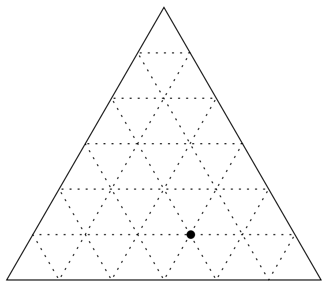


Figure 1 – The point $(2,3)$ in Δ_6

(1) $$t(P,Q)=\mathrm{Tr}\big(AP+(A+B)Q\big)\big(=\mathrm{Tr}\big(A(P+Q)+BQ\big)\big).$$

Then, given a lattice point $X=(p,q)$, we set

(2) $$h_{A,B,C}(X)=\sup t(P,Q),$$

where the supremum is taken over all pairs (P,Q) as above such that P and Q have ranks p and q, respectively.

Danilov and Koshevoy [5] present some evidence in favor of the conjecture showing, for instance, that it is true for $n=3$. The methods used in [5] were applied in [2] in order to propose a proof for arbitrary values of n. However, Lombard [9] provides numerical evidence that this method may not work. The proof proposed in [2] goes as follows. Suppose that X,Y,Z,W are, as before, the vertices of two adjacent unit triangles, and that (P_X,Q_X), (P_W,Q_W) are given pairs of mutually orthogonal projections such that $X=(\mathrm{rank}(P_X),\mathrm{rank}(Q_X))$ and $W=(\mathrm{rank}(P_W),\mathrm{rank}(Q_W))$. Then [2] proposes to construct pairs (P_Y,Q_Y) and (P_Z,Q_Z) of mutually orthogonal projections such that

(3) $$Y=\big(\mathrm{rank}(P_Y),\mathrm{rank}(Q_Y)\big) \text{ and } Z=\big(\mathrm{rank}(P_Z),\mathrm{rank}(Q_Z)\big)$$

as follows. Construct orthonormal bases $\mathcal{B}_X, \mathcal{B}_W$ for P_X, P_W and $\mathcal{B}'_X, \mathcal{B}'_W$ for Q_X, Q_W such that new orthonormal systems $\mathcal{B}_Y$, $\mathcal{B}_Z, \mathcal{B}'_Y, \mathcal{B}'_Z$ of appropriate cardinalities can be formed satisfying

$$\mathcal{B}_X \cup \mathcal{B}_W = \mathcal{B}_Y \cup \mathcal{B}_Z \text{ and } \mathcal{B}'_X \cup \mathcal{B}'_W = \mathcal{B}'_Y \cup \mathcal{B}'_Z, \tag{4}$$

and the elements of $\mathcal{B}_Y$ (respectively, $\mathcal{B}_Z$) are orthogonal to the elements of $\mathcal{B}'_Y$, respectively, $\mathcal{B}'_Z$. (The unions in (4) should be viewed as multisets. In other words, if a vector belongs to both $\mathcal{B}_X$ and $\mathcal{B}_W$, it should also belong to $\mathcal{B}_Y$ and to $\mathcal{B}_Z$.) Then from projections P_Y, Q_Y, P_Z, Q_Z whose ranges are generated by these orthonormal systems. If such bases can be found, we have

$$P_X + P_W = P_Y + P_Z \text{ and } Q_X + Q_W = Q_Y + Q_Z, \tag{5}$$

and this implies that

$$t(P_X, Q_X) + t(P_W, Q_W) = t(P_Y, Q_Y) + t(P_Z, Q_Z). \tag{6}$$

When $t(P_X, Q_X) = h_{A,B,C}(X)$ and $t(P_W, Q_W) = h_{A,B,C}(W)$, we conclude that

$$h_{A,B,C}(X) + h_{A,B,C}(W) \le h_{A,B,C}(Y) + h_{A,B,C}(Z).$$

We show that, for $n \ge 4$, such bases cannot generally be found. In fact, pairs (P_Y, Q_Y) and (P_Z, Q_Z) (mutually orthogonal and of appropriate ranks) satisfying (5) may not exist. At the same time, we present the simplest instances of a construction which produces pairs (P, Q) that achieve the supremum in (2) in some cases in which there is a unique hive (with $h(0,0) = 0$) associated with (A, B, C). The construction shows that $h_{A,B,C}$ is indeed equal to this unique hive h. We illustrate this method for certain matrices of size 4, and it is this illustration that shows why the method of [2] does not work.

The approach we propose depends on a couple of basic propositions that allow one to construct projections P_X, Q_X associated with a given lattice point X starting with projections associated with a nearby lattice point. (See Propositions 3.1 and 3.2.) This method can be used to prove [5, Conjecture 1] in many other cases, and this will be the subject of a later publication. The applications of Propositions 3.1 and 3.2 are made possible by using the intersection ring of the Grassmannian in order to construct projections to which the technical Lemma 2.1 applies. This lemma is our way of constructing new projections from old ones and we illustrate its application in several simple cases in Section 3. In Section 4, we consider a special class of tripples (A, B, C) of 4×4 matrices to which the results of Section 3 apply. We conclude this section, as well as the paper, with one particular triple in this class, points X, Y, Z, W, and projections P_X, Q_X, P_W, Q_W such that no projections P_Y, Q_Y, P_Z, Q_Z (of appropriate dimensions) satisfying (3) and (5) exist.

2. A LEMMA ABOUT ORTHOGONAL PROJECTIONS

In the following statement, $P^{\perp}$ stands for $I_{\mathcal{H}} - P$, that is, the projection onto the kernel of the orthogonal projection P.

LEMMA 2.1. *Suppose that $\mathcal{H}$ is a Hilbert space and P, L, M are orthogonal projections on $\mathcal{H}$ that satisfy the following conditions:*

(1) $PL\mathcal{H} \subset M\mathcal{H}$, *and*

(2) $P^{\perp}M\mathcal{H} \subset L\mathcal{H}$.

Then:

(i) $P' = P + L - M$ *is also an orthogonal projection.*

(ii) *We have* $P'M\mathcal{H} \subset L\mathcal{H}$ *and* $P'^{\perp}L\mathcal{H} \subset M\mathcal{H}$.

Proof. Denote by W the orthogonal projection onto the closure of $PL\mathcal{H}$, and note that $W \leq P$ and $(P - W)L = 0$. We can rewrite

$$P + L - M = \big((P - W) + L\big) - (M - W).$$

The equality $(P-W)L = 0$ shows that $P-W+L$ is an orthogonal projection. Condition (1) imples that $M-W$ is a projection as well. To conclude the proof of (i), it suffices (and it is also necessary) to prove the inclusion

$$(M - W)\mathcal{H} \subset (P - W + L)\mathcal{H}.$$

Indeed, suppose that $x \in (M - W)\mathcal{H}$, so that $x \in M\mathcal{H}$ and $x \perp W\mathcal{H}$. Observe that $P^{\perp}x \in L\mathcal{H}$ by (2), and

$$\langle Px, PLy\rangle = \langle x, PLy\rangle = \langle x, WPLy\rangle = 0, \quad y \in \mathcal{H}.$$

We also have $P^{\perp}x \in L\mathcal{H}$ by (2), so

$$x = Px + P^{\perp}x = (P - W)Px + L(P^{\perp}x) \in (P - W + L)\mathcal{H},$$

thus proving (i).

To prove the first inclusion in (ii), suppose that $x \in M\mathcal{H}$. Then

$$P'x = Px + Lx - Mx = Px + Lx - x = Lx - P^{\perp}x,$$

and this vector belongs to $L\mathcal{H}$ because $P^{\perp}M\mathcal{H} \subset L\mathcal{H}$. For the second inclusion in (ii), we note that

$$P'^{\perp}x = P^{\perp}x + Mx - Lx = P^{\perp}x + Mx - x = Mx - Px \in M\mathcal{H},$$

provided that $x \in L\mathcal{H}$. $\square$

Conditions (1) and (2) are not necessary for $P-L+M$ to be a projection. The simplest example is obtained by taking $L = M$. More interestingly, note that the identity 2×2 matrix 1_2 can be written in infinitely many ways as $1_2 = P + L = P' + M$ with rank one projections P, L, P', M. In most cases, neither (1) nor (2) is satisfied.

3. PRINCIPAL CONSTRUCTION DEVICE

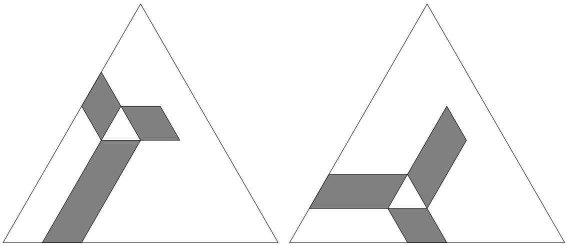
We first establish some notation relating to Grassmannians, flags, and Schubert varieties. Suppose that $n, r \in \mathbb{N}$ and $r \leq n$. We denote by $G(r, n)$ the collection of all r-dimensional vector subspaces of $\mathbb{C}^n$. Let now $\mathcal{E} = \{E_j\}_{j=0}^n$ be a complete flag in $\mathbb{C}^n$, that is, each E_j is a vector subspace of dimension j in $\mathbb{C}^n$, and $E_{j-1} \subset E_j$ for $j = 1, \ldots, n$. Consider also a set $I \subset \{1, \ldots, n\}$ with r elements $i_1 < \cdots < i_r$. One can then define the Schubert variety $\mathfrak{S}(\mathcal{E}, I)$ to consist of those spaces $M \in G(r, n)$ for which the inequalities $\dim(M \cap E_{i_\ell}) \geq \ell$ are satisfied for $\ell = 1, \ldots, r$. (In the notation of [6, I.5],

$$\mathfrak{S}(\mathcal{E}, I) = \overline{W_{a_1, \ldots, a_r}},$$

where $a_\ell = (n-r) - (i_\ell - \ell)$ and $V_\ell = E_\ell$.) For example, $\mathfrak{S}(\mathcal{E}, I) = \{E_r\}$ if $I = \{1, \ldots, r\}$, and $\mathfrak{S}(\mathcal{E}, I) = G(r, n)$ if $I = \{n - i + 1, i = 1, \ldots, r\}$. Every space $M \in \mathfrak{S}(\mathcal{E}, I)$ has an orthonormal basis $\{v_1, \ldots, v_r\}$ such that $v_\ell \in E_{i_\ell}$ for $\ell = 1, \ldots, r$.

We are interested in flags that arise from the eigenvectors of selfadjoint matrices. Thus, suppose that A is such a matrix with eigenvalues $\alpha_1 \geq \cdots \geq \alpha_n$, and $\{e_1, \ldots, e_n\}$ is an orthonormal basis in $\mathbb{C}^n$ such that $Ae_j = \alpha_j e_j$ for $j = 1, \ldots, n$. Then, we can define a flag $\mathcal{E} = \{E_j\}_{j=0}^n$ by letting E_j be the linear span of $\{e_1, \ldots, e_j\}$ for $j = 1, \ldots, n$. Such a flag $\mathcal{E}$ is called an *eigenflag* for A. A matrix A may have several eigenflags, but E_j, $j = 1, \ldots, n-1$, is uniquely determined precisely when $\alpha_j > \alpha_{j+1}$. We also consider the flag $\widetilde{\mathcal{E}} = (\widetilde{E}_j)_{j=0}^n$ defined by $\widetilde{E}_j = E_{n-j+1}^\perp$. It is convenient to use the notation $\widetilde{\alpha}_j = \alpha_{n-j+1}$. Clearly, $\widetilde{\mathcal{E}}_j$ is an eigenflag for $-A$. Given a subspace $M \subset \mathbb{C}^n$, we also use the letter M to denote the corresponding orthogonal projection. In order to avoid confusion, we write $M \vee N$ for the vector space generated by M and N, or for the corresponding projection. Similarly, $M \wedge N$ denotes the (projection onto) the intersection of M and N. The notation $M + N$ is reserved for the sum of the projections M and N; this is not a projection unless $M \perp N$. Similarly, $M - N$ is a projection only when $N \subset M$. Given an arbitrary $I \subset \{1, \ldots, n\}$ and an element $M \in \mathfrak{S}(\mathcal{E}, I)$, we have (see, for instance [7]) the inequality

$$\operatorname{Tr}(AM) \geq \sum_{i \in I} \alpha_i.$$



(a) Proof of (7) for $m = 1$

(b) Proof of (8) for $m = 2$

Figure 2 – Here, $r = 7$, $X = (1, 2)$, $Y = (1, 3)$

This is easily verified by using a basis $\{e_1, \ldots, e_r\}$ in M such that $e_\ell \in E_{i_\ell}$ for $\ell = 1, \ldots, r$. Similarly, if $M \in \mathfrak{G}(\widetilde{\mathcal{E}}, I)$, we have

$$\mathrm{Tr}(AM) \leq \sum_{i \in I} \widetilde{\alpha}_{n-i+1}.$$

Suppose now that B is another selfadjoint $n \times n$ matrix, and $C = A + B$. Denote by $\beta_1 \geq \cdots \geq \beta_n$ (respectively, $\gamma_1 \geq \cdots \geq \gamma_n$) the eigenvalues of B (respectively, C), and let $\mathcal{F}$ and $\mathcal{G}$ be eigenflags for B and C, respectively. Let h be an arbitrary hive associated with (A, B, C). If X and Y are adjacent lattice points in Δ_n, there are several ways to estimate the difference $h(Y) - h(X)$ in terms of the eigenvalues $\alpha_j, \beta_j, \gamma_j$. We list below some of the simplest such inequalities. Suppose that $X = (p, q)$ and $Y = (p, q + 1)$, where $p + q + 1 \leq n$. Then

$$h(Y) - h(X) \leq \alpha_{p+m+1} + \beta_{q-m+1}, \quad m = 0, \ldots, q. \tag{7}$$

This inequality is obtained by applying the hive inequalities to the gray parallelograms in Figure 2(A). Similarly, the inequalities

$$h(Y) - h(X) \geq \alpha_{p+m+q+1} + \beta_{n-p-m}, \quad m = 0, \ldots, n - p - q - 1, \tag{8}$$

are obtained from Figure 2(B). We write similar inequalities for the other two possible positions of the segment XY. If $X = (p, q)$ and $Y = (p + 1, q)$, we have

$$h(Y) - h(X) \leq \gamma_{q+m+1} - \beta_{n-p+m}, \quad m = 0, \ldots, p, \tag{9}$$

and

$$h(Y) - h(X) \geq \gamma_{p+q+m+1} - \beta_{q+m+1}, \quad m = 0, \ldots, n - p - q - 1. \tag{10}$$

Finally, if $X = (p, q)$ and $Y = (p - 1, q + 1)$, we have

$$h(Y) - h(X) \leq \gamma_{q-m+1} - \alpha_{p+q-m}, \quad m = 0, \ldots, q, \tag{11}$$

and

$$h(Y) - h(X) \geq \gamma_{p+q-m} - \alpha_{p-m}, \quad m = 0, \dots, p. \tag{12}$$

We show that the inequalities (7)–(12) are also satisfied when h is replaced by the function $h_{A,B,C}$ of (2). (Of course, there are many more inequalities that $h_{A,B,C}$ would have to satisfy in order to be proven a hive!) The argument is based on the following result, for whose formulation it is to convenient to use, in addition to a pair (P, Q) of mutually orthogonal projections, the complement $R = (P+Q)^\perp$. We use $\mathrm{ran}(S)$ to denote the range of an operator S. We also use the notation

$$\widetilde{I} = \{n + 1 - i : i \in I\}$$

when $I \subset \{1, \dots, n\}$.

PROPOSITION 3.1. *Let A, B, C be selfadjoint $n \times n$ matrices with eigenvalues $(\alpha_j)_{j=1}^n$, $(\beta_j)_{j=1}^n$, $(\gamma_j)_{j=1}^n$ and eigenflags $\mathcal{E}, \mathcal{F}, \mathcal{G}$, respectively. Let P, Q, and R be mutually orthogonal projections such that $P + Q + R = 1_n$, and let I, J, K be subsets of $\{1, \dots, n\}$. Suppose that there exist projections L, M, N with the following properties*:

(1) $L \in \mathfrak{S}(\mathcal{E}, I)$, $M \in \mathfrak{S}(\mathcal{F}, J)$, $N \in \mathfrak{S}(\widetilde{\mathcal{G}}, \widetilde{K})$.

(2) $\mathrm{ran}(PL) \subset \mathrm{ran}(M)$ *and* $\mathrm{ran}(P^\perp M) \subset \mathrm{ran}(L)$.

(3) $\mathrm{ran}(QM) \subset \mathrm{ran}(N)$ *and* $\mathrm{ran}(Q^\perp N) \subset \mathrm{ran}(M)$.

(4) $\mathrm{ran}(RN) \subset \mathrm{ran}(L)$ *and* $\mathrm{ran}(R^\perp L) \subset \mathrm{ran}(N)$.

Then the operators

$$P' = P + L - M,\ Q' = Q + M - N,\ R' = R + N - L$$

are mutually orthogonal projections, $P' + Q' + R' = 1_n$, and

$$t(P', Q') \geq t(P, Q) + \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j - \sum_{k \in K} \gamma_k.$$

Proof. Conditions (2)–(4), and the Lemma 2.1, show that P', Q', and R' are indeed projections. The identity $P' + Q' + R' = 1_n$ is trivially verified, and it also implies that these three projections are mutually orthogonal. Finally, we estimate

$$\begin{aligned} t(P', Q') &= \mathrm{Tr}\big(A(P + L - M) + (A+B)(Q + M - N)\big) \\ &= \mathrm{Tr}\big(AP + (A+B)Q\big) + \mathrm{Tr}(AL) + \mathrm{Tr}(BM) - \mathrm{Tr}\big((A+B)N\big). \end{aligned}$$

The last three traces are evaluated using the observations preceding the proposition:

$$\mathrm{Tr}(AL) \geq \sum_{i \in I} \alpha_i,\ \mathrm{Tr}(BM) \geq \sum_{j \in J} \beta_j,\ \mathrm{Tr}(CN) \leq \sum_{k \in K} \gamma_k,$$

and these inequalities, together with the calculation above, yield the result. □

The following result is obtained by applying Proposition 3.1 to matrices $(-A, -B, -C)$ in place of (A, B, C). We record it here for later use.

PROPOSITION 3.2. *Let A, B, C be selfadjoint $n \times n$ matrices with eigenvalues $(\alpha_j)_{j=1}^n$, $(\beta_j)_{j=1}^n$, $(\gamma_j)_{j=1}^n$ and eigenflags $\mathcal{E}, \mathcal{F}, \mathcal{G}$, respectively. Let P, Q, and R be mutually orthogonal projections such that $P + Q + R = 1_n$, and let I, J, K be subsets of $\{1, \dots, n\}$. Suppose that there exist projections L, M, N with the following properties*:

(1) $L \in \mathfrak{S}(\widetilde{\mathcal{E}}, \widetilde{I})$, $M \in \mathfrak{S}(\widetilde{\mathcal{F}}, \widetilde{J})$, $N \in \mathfrak{S}(\mathcal{G}, K)$.

(2) $\operatorname{ran}(P^\perp L) \subset \operatorname{ran}(M)$ *and* $\operatorname{ran}(PM) \subset \operatorname{ran}(L)$.

(3) $\operatorname{ran}(Q^\perp M) \subset \operatorname{ran}(N)$ *and* $\operatorname{ran}(QN) \subset \operatorname{ran}(M)$.

(4) $\operatorname{ran}(R^\perp N) \subset \operatorname{ran}(L)$ *and* $\operatorname{ran}(RL) \subset \operatorname{ran}(N)$.

Then the operators

$$P' = P - L + M, \ Q' = Q - M + N, \ R' = R - N + L$$

are mutually orthogonal projections, $P' + Q' + R' = 1_n$, and

$$t(P', Q') \geq t(P, Q) - \sum_{i \in I} \alpha_i - \sum_{j \in J} \beta_j + \sum_{k \in K} \gamma_k.$$

Remark 3.3. Some of the hypotheses of Proposition 3.2 are redundant. For instance, the second inclusion in condition (2) follows from the first inclusions in (3) and (4):

$$\operatorname{ran}(PM) = \operatorname{ran}(R^\perp Q^\perp M) \subset \operatorname{ran}(R^\perp N) \subset \operatorname{ran}(L).$$

Similarly, the first halves of conditions (2)–(4) in Proposition 3.1 are redundant. However, we find that the search for the projections L, M, N is often facilitated by writing out a redundant set of conditions.

As a first illustration of the way these two propositions above are applied, we prove the analog of (7) with $h_{A,B,C}$ in place of h.

Example 3.4. Suppose that $X = (p, q)$ and $Y = (p, q + 1)$ are in Δ_n, let P, Q be arbitrary mutually orthogonal projections of ranks $p, q+1$, respectively, and set $R = (P + Q)^\perp$. We may assume that $t(P, Q) = h_{A,B,C}(Y)$. Now, fix $m \in \{0, \dots, q\}$. Proposition 3.2 shows that the inequality

$$h_{A,B,C}(X) \geq h_{A,B,C}(Y) - \alpha_{p+m+1} - \beta_{q-m+1}$$

can be proved if we can find projections L, M, N (of ranks $1, 1, 0$, respectively) satisfying the hypotheses for $I = \{p + m + 1\}$, $J = \{q - m + 1\}$, and $K = \varnothing$. Indeed, the projections P', Q' have then ranks p, q, respectively, and

$$t(P', Q') \geq h_{A,B,C}(Y) - \alpha_{p+m+1} - \beta_{q-m+1}.$$

We organize the requirements on L, M, N in Table 1. The second column indicates vectors that should belong to each space according to condition (1), and the third column indicates subspaces that must be included to satisfy conditions (2)–(4) of the proposition and their consequences.

L	$\widetilde{e}_{n-p-m}$	$R^{\perp}N \supset PM$
M	$\widetilde{f}_{n-q+m}$	$P^{\perp}L \supset QN$
N	0	$Q^{\perp}M \supset RL$

Table 1 – Requirements for Example 3.4

For instance, L must have dimension 1 and be generated by a nonzero vector in $\widetilde{E}_{n-p-m}$, and it must contain the range of $R^{\perp}N$ (which is $\{0\}$), and $Q^{\perp}\widetilde{f}_{n-q+m}$ must belong to N, so $Q^{\perp}\widetilde{f}_{n-q+m} = 0$, that is, $\widetilde{f}_{n-q+m}$ is in the range of Q. Also, $P^{\perp}\widetilde{e}_{n-p-m}$ must belong to $M \subset \widetilde{F}_{n-q-m}$. We show that appropriate vectors $\widetilde{f}_{n-q+m}$ and $\widetilde{e}_{n-p-m}$ can be chosen. Note first that $\widetilde{f}_{n-q+m}$ must be chosen from the space

$$S_1 = \widetilde{F}_{n-q+m} \wedge Q$$

that has dimension at least $m + 1$. For the choice of $\widetilde{e}_{n-p-m}$, we distinguish two cases. If $\widetilde{E}_{n-p-m} \wedge P \neq 0$, then we choose $\widetilde{e}_{n-p-m}$ to be an arbitrary nonzero vector in this intersection, and we choose $\widetilde{f}_{n-q+m}$ to be an arbitrary nonzero vector from S_1. If $\widetilde{E}_{n-p-m} \wedge P = 0$ (which is the generic case), the space $S_2 = \widetilde{E}_{n-p-m} \vee P$ has dimension $(n-p-m)+p = n-m$, and therefore, $S_1 \wedge S_2 \neq 0$. In this case, we choose a nonzero vector $f_{n-q+m} \in S_1 \wedge S_2$, and we write $\widetilde{f}_{n-q+m} = \widetilde{e}_{n-p-m} + x$, with $\widetilde{e}_{n-p-m} \in \widetilde{E}_{n-p-m}$. and $x \in P$. In both cases, one verifies with no difficulty that the spaces L generated by $\widetilde{e}_{n-p-m}$, M generated by f_{n-q+m}, and $N = 0$ satisfy the requirements of Proposition 3.2.

Example 3.5. We prove next the inequality

$$h_{A,B,C}(Y) \geq h_{A,B,C}(X) + \alpha_{p+q+m+1} + \beta_{n-p-m},$$

analogous to (8), when $X = (p, q)$, $Y = (p, q+1)$, and $m \in \{0, \ldots, n-p-q-1\}$. This time, we choose P, Q of ranks p, q such that $t(P, Q) = h(X)$, and we look for spaces L, M, N satisfying the hypotheses of Proposition 3.1 for sets

$I = \{p+m+q+1\}$, $J = \{n-p-m\}$, and $K = \varnothing$. These hypotheses are summarized in Table 2.

L	$e_{p+q+m+1}$	$P^{\perp}M \supset RN$
M	f_{n-p-m}	$Q^{\perp}N \supset PL$
N	0	$R^{\perp}L \supset QM$

Table 2 – Requirements for Example 3.5

Since R has codimension $p+q$, the space $S_1 = E_{p+q+m+1} \cap R$ has dimension at least $m+1$. As in the preceding example, we distinguish two cases. If $P \cap F_{n-p-m} \neq 0$, we choose a nonzero vector f_{n-p-m} in that space and choose an arbitrary $e_{p+q+m+1} \in S_1 \backslash \{0\}$. In the contrary case, the space $S_2 = P \vee F_{n-p-m}$ has dimension $n-m$ and hence, $S_1 \wedge S_2$ contains a nonzero vector $e_{p+q+m+1} = x + f_{n-p-m}$, where $x \in P$. One then verifies that the spaces $L = \mathbb{C}e_{p+q+m+1}, M = \mathbb{C}f_{n-p-m}, N = 0$ satisfy all the requirements.

Remark 3.6. It is possible that $\alpha_{p+q+m+1} + \beta_{n-p-m} = \alpha_{p+m+1} + \beta_{q-m+1}$. This happens when $\alpha_{p+m+1} = \alpha_{p+m+2} \cdots = \alpha_{p+q+m+1}$ and $\beta_{q-m+1} = \beta_{q-m+2} = \cdots = \beta_{n-p-m}$. In this case, the exact value of $h(Y) - h(X)$ is known for every hive h associated with (A, B, C), and the two examples above show that $h_{A,B,C}(Y) - h_{A,B,C}(X)$ is also equal to this value. In particular, if the equality $h_{A,B,C}(X) = h(X)$ holds, it follows that $h_{A,B,C}(Y) = h(Y)$ as well.

Example 3.7. We continue with the analog of (9):

$$h_{A,B,C}(X) \geq h_{A,B,C}(Y) + \beta_{n-p+m} - \gamma_{q+m+1},$$

where $X = (p, q)$, $Y = (p+1, q)$, and $m \in \{0, \ldots, p\}$. We start with projections P, Q of ranks $p+1, q$ and apply Proposition 3.1 with $I = \varnothing$, $J = \{n-p+m\}$, and $K = \{q+m+1\}$. The relevant requirements on the spaces L, M, N are summarized in Table 3.

L	0	$P^{\perp}M \supset RN$
M	f_{n-p+m}	$Q^{\perp}N \supset PL$
N	$\widetilde{g}_{n-q-m}$	$R^{\perp}L \supset QM$

Table 3 – Requirements for Example 3.7

The space $S_1 = F_{n-p+m} \wedge P$ has dimension at least $m+1$. Further, if $Q \wedge \widetilde{G}_{n-q-m} \neq 0$, we choose an arbitrary nonzero vector g_{n-q+m} in that intersection and an arbitrary $f_{n-p+m} \in S_1 \backslash \{0\}$. Otherwise, the space $S_2 =$

$Q^\perp \widetilde{G}_{n-q-m}$ has dimension $n-q-m$. Since both S_1 and S_2 are subspaces of $Q^\perp$ (which has dimension $n-q$), we have $S_1 \wedge S_2 \neq 0$. Then, we choose $f_{n-p+m} = x + \widetilde{g}_{n-q-m} \neq 0$ in this intersection. In both cases, the spaces L, M, N determined by these vectors satisfy the conditions summarized in the table.

Example 3.8. The analog of (10) is:

$$h_{A,B,C}(Y) \geq h_{A,B,C}(X) + \gamma_{p+q+m+1} - \beta_{q+m+1},$$

where $X = (p,q)$, $Y = (p+1,q)$, and $m \in \{0, \ldots, n-p-q-1\}$. Thus, we start with projections P, Q of ranks p, q, and look for spaces L, M, N satisfying the conditions summarized in Table 4.

L	0	$R^\perp N \supset PM$
M	$\widetilde{f}_{n-q-m}$	$P^\perp L \supset QN$
N	$g_{p+q+m+1}$	$Q^\perp M \supset RL$

Table 4 – Requirements for Example 3.8

To find the relevant vectors, we note that the condition $R^\perp N \subset L = 0$ implies that $g_{p+q+m+1}$ belongs to the space $S = G_{p+q+m+1} \wedge R$ of dimension at least $m+1$. Then, we note that $(Q \vee S) \wedge \widetilde{F}_{n-q-m}$ has dimension at least

$$(q+m+1) + (n-q-m) - n = 1,$$

so we can choose a nonzero vector

$$x + g_{p+q+1} = \widetilde{f}_{n-q-m}$$

in this space, with $x \in Q$. One can then construct the spaces L, M, N. In case $Q \wedge \widetilde{F}_{n-q-m} \neq 0$, the vector $g_{p+q+1} \in S$ can be chosen arbitrarily.

Example 3.9. The analog of (11) is:

$$h_{A,B,C}(X) \geq h_{A,B,C}(Y) - \gamma_{q-m+1} + \alpha_{p+q-m},$$

where $X = (p,q)$, $Y = (p+1, q-1)$, and $m \in \{0, \ldots, q\}$. Thus, we start with projections P, Q of ranks $p-1, q+1$, and look for spaces L, M, N subject to the conditions summarized in Table 5.

L	e_{p+q-m}	$P^\perp M \supset RN$
M	0	$Q^\perp N \supset PL$
N	$\widetilde{g}_{n-q+m}$	$R^\perp L \supset QM$

Table 5 – Requirements for Example 3.9

We have $Q^{\perp}N = 0$, so N is contained in the space $S = Q \cap \widetilde{G}_{n-q+m}$ of dimension at least $m+1$. Then $(R \vee S) \wedge E_{p+q-m}$ has rank at least

$$(n-p-q)+(m+1)+(p+q-m)=1.$$

Choose $e_{p+q-m} = x + \widetilde{g}_{n-q+m}$, $x \in R$, in this space and construct the spaces L, M, N from these vectors. In the nongeneric case, $E_{p+q-m} \wedge R \neq 0$, and then $\widetilde{g}_{n-q+m}$ can be chosen arbitrarily in S.

Example 3.10. Finally, the analog of (12) is:

$$h_{A,B,C}(Y) \geq h_{A,B,C}(X) - \alpha_{p-m} + \gamma_{p+q-m},$$

where $X = (p,q)$,$Y = (p+1, q-1)$, and $m \in 0, \ldots, p$. We start with projections P, Q of ranks p, q, and look for spaces L, M, N subject to the conditions summarized in Table 6.

L	$\widetilde{e}_{n-p+m+1}$	$R^{\perp}N \supset PM$
M	0	$P^{\perp}L \supset QN$
N	g_{p+q-m}	$Q^{\perp}M \supset RL$

Table 6 – Requirements for Example 3.10

The vector $\widetilde{e}_{n-p+m+1}$ must be chosen from the space $S = \widetilde{E}_{n-p+m+1} \wedge P$ of dimension $\geq m+1$. We note, as in the preceding example, that $(R \vee S) \wedge G_{p+q-m}$ has rank at least 1, so we can choose $g_{p+q-m} = x + \widetilde{e}_{n-p+m+1}$, $x \in R$, in this space. There is again a nongeneric case in which $\widetilde{E}_{n-p+m+1}$ intersects R in a nonzero space. The remaining details are easily verified.

The attentive reader noticed that the six examples above are rather similar, though the reasoning behind solving for the spaces L, M, N may be somewhat different in each case. The arguments show that, generically, these spaces are uniquely determined.

4. **SOME RIGID HIVES OF SIZE 4**

We undertake now a detailed study of certain triples $(A, B, C = A + B)$ of selfadjoint 4×4 matrices. Namely, we assume that the eigenvalues of A and B satisfy the equalities $\alpha_1 = \alpha_2$, $\alpha_3 = \alpha_4$, and $\beta_1 = \beta_2$. By adding appropriate multiples of the identity matrix to A, B, and C, we can and do assume that A and B are nonnegative and noninvertible, that is, $\alpha_4 = \beta_4 = 0$. For each triple (A, B, C) of this type, there is a unique associated hive h, normalized by

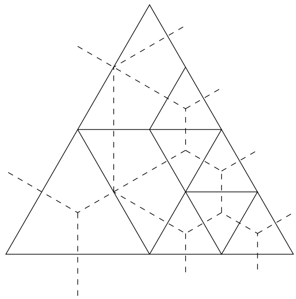


Figure 3 – The $2, 2|2, 1, 1|1, 1, 1, 1$ locking pattern and its dual honeycomb

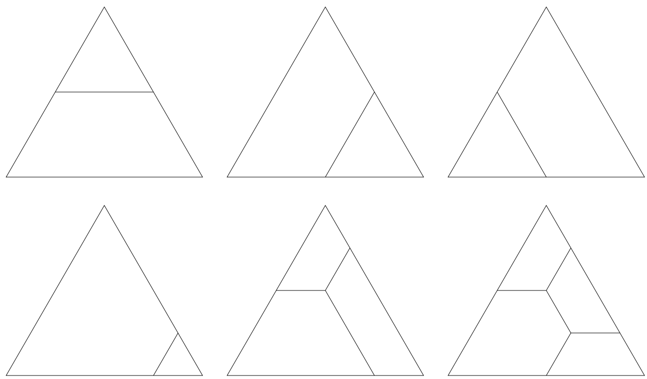


Figure 4 – Flat regions for the extreme hives

$h(0, 0) = 0$, which is one reason for choosing these particular restrictions on the eigenvalues of A and B. This set of conditions is called a *locking pattern* in [4]. We illustrate in Figure 3 the "flat" pieces associated with this locking pattern, that is, areas in which any hive respecting this pattern must be an affine function. We also draw (with dotted lines) the dual honeycomb which is extreme (derived from a tree).

The extremal structure of the set Γ of hives (such that $h(0, 0) = 0$) associated with these triples (A, B, C) can be explained using the results of [3]. Namely, Γ has six extreme rays, and every element of Γ can be represented uniquely as a sum of extreme hives, one from each extreme ray. The extreme hives arise from the six restrictive flatness patterns represented in Figure 4.

Rather than list the values of these six kinds of extreme hives, we list in Table 7 the eigenvalues of the triples (A, B, C) that produce such hives. In this table, a, b, c, d, e, f represent nonnegative real numbers.

α	β	γ
$a, a, 0, 0$	$0, 0, 0, 0$	$a, a, 0, 0$
$0, 0, 0, 0$	$b, b, 0, 0$	$b, b, 0, 0$
$c, c, 0, 0$	$c, c, 0, 0$	c, c, c, c
$0, 0, 0, 0$	$d, d, d, 0$	$d, d, d, 0$
$e.e, 0, 0$	$e, e, e, 0$	$2e, e, e, e$
$f, f, 0, 0$	$f, f, 0, 0$	$2f, f, f, 0$

Table 7 – Eigenvalues for the extreme hives

Thus, the eigenvalues of a general triple (A, B, C) that satisfies the above requirements are described below.

$$
\begin{aligned}
\alpha_1 = \alpha_2 &= a + c + e + f \\
\alpha_3 = \alpha_4 &= 0 \\
\beta_1 = \beta_2 &= b + c + d + e + f \\
\beta_3 &= d + e \\
\beta_4 &= 0 \\
\gamma_1 &= a + b + c + d + 2e + 2f \\
\gamma_2 &= a + b + c + d + e + f \\
\gamma_3 &= c + d + e + f \\
\gamma_4 &= c + e
\end{aligned} \tag{13}
$$

Conversely, the constants a, b, c, d, e, f can be calculated from the eigenvalues of A, B, and C as follows:

$$
\begin{aligned}
a &= \gamma_2 - \beta_1 \\
b &= \beta_1 - \gamma_3 \\
c &= \alpha_1 + \beta_1 - \gamma_1 \\
d &= \alpha_1 + \beta_1 + \beta_3 - \gamma_1 - \gamma_4 \\
e &= \alpha_1 + \beta_1 + \beta_3 - \gamma_2 - \gamma_3 \\
f &= \gamma_1 + \gamma_3 - \alpha_1 - \beta_1 - \beta_3.
\end{aligned} \tag{14}
$$

It can be seen from Figure 3 that A and B have no common reducing spaces (equivalently, (A, B, C) is *irreducible*) precisely when all constants a, b, c, d, e, f are different from zero. (This can also be deduced from the main result of [1] because the corresponding hive has six *attachment points.*) We summarize some useful properties of these irreducible triples.

Lemma 4.1. *Suppose that* (A,B,C) *is an irreducible triple of* 4×4 *selfadjoint matrices such that* $C=A+B$, $\alpha_1=\alpha_2$, $\alpha_3=\alpha_4=0$, $\beta_1=\beta_2$, *and* $\beta_3=0$. *Then*:

(1) $\alpha_2>\alpha_3$,

(2) $\beta_2>\beta_3>\beta_4$,

(3) $\gamma_1>\gamma_2>\gamma_3>\gamma_4>0$,

(4) $E_2\wedge F_2=E_2^\perp\wedge F_2=0$,

(5) $G_1^\perp\wedge E_2$, $G_1^\perp\wedge E_2^\perp$, $G_1^\perp\wedge F_2$, *and* $G_1^\perp\wedge F_2^\perp$ *have rank one, and*

(6) $(E_2\vee G_1)(G_1^\perp\wedge F_2)\neq 0$.

Proof. The inequalities (1), (2), and (3) are seen immediately by inspecting (13).

If E_2 and F_2 have a common nonzero vector x, then x is a common eigenvector for A and B and hence, it generates a reducing space. The second equality in (4) follows for the same reason.

Since $G_1^\perp$ has rank three and E_2 has rank two, $G_1^\perp\wedge E_2$ has rank one or two. The second possibility amounts to $G_1\le E_2^\perp$, in which case G_1 is an eigenspace for both A and C, hence a reducing subspace. The projection $G_1^\perp\wedge E_2^\perp$ has rank one for the same reason. Suppose now that $G_1\le F_2^\perp$ and let g_1 be a unit vector in G_1. Then

$$\gamma_1=\langle Cg_1,g_1\rangle=\langle Ag_1,g_1\rangle+\langle Bg_1,g_1\rangle\le\alpha_1+\beta_3,$$

or, equivalently according to (13),

$$a+b+c+d+2e+2f\le(a+c+e+f)+(d+e),$$

which is not true because, for instance, $b>0$. To conclude the proof of (5), suppose that $G_1\subset F_2$. Then G_1 is an eigenspace for B and C, hence a reducing space.

To prove (6), suppose now $(E_2\vee G_1)(G_1^\perp\wedge F_2)=0$ and x is a unit vector in the range of $G_1^\perp\wedge F_2$. Then $E_2x=0$, so x is in both $E_2^\perp$ and F_2, contrary to the already proved assertion (4). □

We show that [5, Conjecture 1] is correct for the triples under consideration. We also determine, for every point $X=(p,q)\in\Delta_4$, pairwise orthogonal projections P_X,Q_X of ranks p,q such that $t(P_X,Q_X)=h_{A,B,C}(X)$. Most of these pairs of projections are uniquely determined in the irreducible case. We list the values of h and the corresponding projections in Table 8. The

X	$h(X)$	P_X	Q_X
$(0,0)$	0	0	0
$(1,0)$	α_1	$*\ E$	0
$(2,0)$	$\alpha_1+\alpha_2=2\alpha_1$	E_2	0
$(3,0)$	$2\alpha_1+\alpha_3$	$*\ E_3$	0
$(4,0)$	$2\alpha_1+2\alpha_3=\mathrm{Tr}(A)$	E_4	0
$(3,1)$	$\mathrm{Tr}(A)+\beta_1$	$*\ F^\perp$	$*\ F$
$(2,2)$	$\mathrm{Tr}(A)+2\beta_1$	$F_2^\perp$	F_2
$(1,3)$	$\mathrm{Tr}(A)+2\beta_1+\beta_3$	$F_3^\perp$	F_3
$(0,4)$	$\mathrm{Tr}(A)+\mathrm{Tr}(B)=\mathrm{Tr}(C)$	0	G_4
$(0,3)$	$\gamma_1+\gamma_2+\gamma_3$	0	G_3
$(0,2)$	$\gamma_1+\gamma_2$	0	G_2
$(0,1)$	γ_1	0	G_1
$(1,1)$	$\alpha_1+\gamma_1$	$E_2\wedge G_1^\perp$	G_1
$(2,1)$	$2\alpha_1+\alpha_3+\beta_1$	$*\ (E_2\vee F)-F$	$*\ F$
$(1,2)$	$\alpha_1+\alpha_3+\beta_1+\gamma_1$	$(E_2\wedge G_1^\perp)$ $+[(E_2^\perp\vee G_1)(F_2\wedge G_1^\perp)]$ $-(F_2\wedge G_1^\perp)$	$G_1+(F_2\wedge G_1^\perp)$

Table 8 – Values of the unique hive

nonunique entries are marked with an asterisk. In this table, E (respectively, F) denotes an arbitrary projection of rank one such that $E\le E_2$ (respectively, $F\le F_2$), E_3 denotes any projection of rank three such that $E_2\le E_3$, and $[(E_2^\perp\vee G_1)(F_2\wedge G_1^\perp)]$ denotes the rank one projection onto the range of $(E_2^\perp\vee G_1)(F_2\wedge G_1^\perp)$.

Proposition 4.2. *Suppose that (A,B,C) is an irreducible triple of 4×4 selfadjoint matrices such that $A+B=C$, $\alpha_1=\alpha_2$, $\alpha_3=\alpha_4=0$, $\beta_1=\beta_2$, and $\beta_4=0$, and let h be the unique associated hive. Then $h_{A,B,C}(X)=h(X)$ for every lattice point X in Δ_4. Moreover, all the pairs (P,Q) of mutually orthogonal projections satisfying $X=(\mathrm{rank}(P),\mathrm{rank}(Q))$ and $h_{A,B,C}(X)=t(P,Q)$ are described in Table 8.*

Proof. The values of the hive are deduced immediately from the various inequalities required for every two adjacent small triangles. The fact that $h_{A,B,C}$ equals this hive h at each boundary lattice point is immediate; in fact, this is true for every triple $(A,B,C=A+B)$ in every dimension. Similarly, the spaces P,Q associated to each boundary point are derived immediately from arbitrary eigenflags for A,B, and C. All of these spaces are uniquely determined except for E,E_3, and F. We proceed with the three remaining lattice points, starting with $(1,1)$.

Suppose that P and Q are arbitrary mutually orthogonal projections of rank one. Then $\mathrm{Tr}(AP) \le \alpha_1$ and $\mathrm{Tr}((A+B)Q) = \mathrm{Tr}(CQ) \le \gamma_1$. Adding these two inequalities, we obtain $t(P,Q) \le \alpha_1 + \gamma_1$ and so, $h_{A,B,C}((1,1)) \le h((1,1))$. On the other hand, $E_2 \wedge G_1^\perp$ has rank at least one. Choose a rank one projection $P \le E_2 \wedge G_1^\perp$ and set $Q = G_1$. Then P and Q are mutually orthogonal and $t(P,Q) = \alpha_1 + \gamma_1$. This proves that $h_{A,B,C}((1,1)) = h((1,1))$ and provides explicit spaces that realize the supremum in the definition of $h_{A,B,C}(1,1)$. To show that these are the only spaces with this property, consider arbitrary P and Q such that $t(P,Q) = \alpha_1 + \gamma_1$. Since $\mathrm{Tr}(AP) \le \alpha_1$ and $\mathrm{Tr}(CQ) \le \gamma_1$, we must have equality in both cases. The uniqueness of E_2 and G_1 implies $P \le E_2$ and $Q = G_1$. Moreover, since $P \perp Q$, we have $P \le E_2 \wedge G_1^\perp$. Thus, P is uniquely determined because $E_2 \wedge G_1^\perp$ has rank one by Lemma 4.1(5).

Next, consider $(2,1)$. Suppose that P and Q are arbitrary mutually orthogonal projections of ranks 2 and 1. We have

$$t(P,Q) = \mathrm{Tr}\big(A(P+Q)\big) + \mathrm{Tr}(BQ) \le 2\alpha_1 + \alpha_3 + \beta_1$$

because $\alpha_1, \alpha_1, \alpha_3$ are the top three eigenvalues of A and β_1 is the top eigenvalue of B. Thus, $h_{A,B,C}((2,1)) \le h((2,1))$. On the other hand, suppose that F is an arbitrary rank one projection such that $F \le F_2$. Lemma 4.1(4) shows that $E_2 \vee F$ has rank three, and thus, $P = (E_2 \vee F) - F$ has rank two. If we set $Q = F$, we see immediately that $t(P,Q) = h(P,Q)$. Conversely, suppose that P and Q satisfy the equality $t(P,Q) = h(P,Q)$. Then, we have $\mathrm{Tr}(A(P+Q)) = 2\alpha_1 + \alpha_3$ and $\mathrm{Tr}(BQ) = \beta_1$. It follows that $E_2 \le P + Q$ and Q is generated by a unit vector $f \in F_2$. Thus, $P + Q = E_2 \vee Q$, so $P = (E_2 \vee Q) - Q$, and this completes the description of all maximizing pairs (P,Q).

Finally, consider $(1,2)$. Inequalities (7) and (8), with $m = 1$ in both cases, show that

$$\alpha_4 + \beta_2 \le h\big((1,2)\big) - h\big((1,1)\big) \le \alpha_3 + \beta_1.$$

Under the current assumptions, both inequalities are equalities, and Remark 3.6 shows that we also have

$$h_{A,B,C}\big((1,2)\big) - h_{A,B,C}\big((1,1)\big) = \alpha_3 + \beta_1.$$

Since $h_{A,B,C}((1,1)) = h((1,1))$, it follows that $h_{A,B,C}((1,2)) = h((1,2))$ as well.

Suppose now that P and Q are mutually orthogonal projections of ranks one and two such that $t(P,Q) = h_{A,B,C}((1,2))$; that such projections exist was seen in Examples 3.4 and 3.5. From 3.4, there exist spaces L, M, N satisfying the conditions in that example, such that the projections $P' = P - L + M$, $Q' = Q - M$ satisfy $t(P',Q') \ge h_{A,B,C}((1,1))$. As shown earlier in this proof, such projections P', Q' are uniquely determined, namely, $P' = E_2 \wedge G_1^\perp$ and $Q' = G_1$. Therefore, we have

$$\begin{aligned} P &= P' + L - M, \quad Q = G_1 + M, \\ &= (E_2 \wedge G_1^\perp) + L - M, \\ Q &= Q' + M = G_1 + M, \end{aligned}$$

and Lemma 2.1(ii) shows that $P'L \subset M$ and $P'^\perp M \subset L$. The space M must contain a vector from F_2 and be orthogonal to G_1, so $M \leq F_2 \wedge G_1^\perp$. Therefore, $M = F_2 \wedge G_1^\perp$ by Lemma 4.1(5). Since $P'^\perp = E_2^\perp \vee G_1$, part (6) of the same lemma shows that If $P'^\perp M \neq 0$, and thus, L is the projection onto the range of $P'^\perp M$. This concludes the proof of the proposition. □

Remark 4.3. In the algebraic operation defining $P_{(1,2)}$, no two terms cancel each other. In particular, the space $P_{(1,1)}$ is not contained in $P_{(1,2)}$, but $P_{(1,2)}$ is not orthogonal to $P_{(1,1)}$. This indicates that the space $P_{(1,2)}$ is not usually generated by a vector from $P_{(1,1)}$ or $P_{(2,2)}$.

Corollary 4.4. *Under the hypotheses of Proposition* 4.2*, set* $X = (1,1)$*,* $Y = (2,1)$*,* $Z = (1,2)$*,* $W = (2,2)$*,* $P_X = G_1^\perp \wedge E_2$*,* $Q_X = G_1$*,* $P_W = F_2^\perp$*,* $Q_W = F_2$*, and* $F_1 = F_2 - G_1^\perp \wedge F_2$*. Let* (P_Y, Q_Y) *and* (P_Z, Q_Z) *be pairs of mutually orthogonal projections such that sets* $Y = (\mathrm{rank}(P_Y), \mathrm{rank}(Q_Y))$*,* $Z = (\mathrm{rank}(P_Z), \mathrm{rank}(Q_Z))$*, and*

$$t(P_Y, Q_Y) + t(P_Z, Q_Z) \geq t(P_X, Q_X) + t(P_W, Q_W). \tag{15}$$

Then $t(P_Y, Q_Y) = h(Y)$*,* $t(P_Z, Q_Z) = h(Z)$*, and therefore there exists a projection* $F \leq F_2$ *of rank one such that*

$$\begin{aligned} &P_Y = E_2 \wedge F - F, \quad Q_Y = F, \\ &P_Z = (G_1^\perp \wedge E_2) + \left[(G_1 \vee E_2^\perp)(G_1^\perp \wedge F_2)\right] - (G_1^\perp \wedge F_2), \text{ and} \\ &Q_Z = G_1 + (G_1^\perp \wedge F_2). \end{aligned}$$

1. *The equality*

   $$Q_Y + Q_Z = Q_X + Q_W. \tag{16}$$

   holds precisely when $F = F_1$ *is the projection onto the range of* $F_2 G_1$*.*

2. *If* (16) *holds, then*

   $$P_Y + P_Z = P_X + P_W. \tag{17}$$

   if and only if

   $$\left[(G_1 \vee E_2^\perp)(G_1^\perp \wedge F_2)\right] = E_2^\perp \wedge F_1^\perp. \tag{18}$$

   When this last equality holds, we have $(G_1^\perp \wedge E_2) \vee (G_1^\perp \wedge F_2) \geq E_2^\perp \wedge F_1^\perp$*.*

Proof. Suppose that (15) is satisfied. Since we have $t(P_Y, Q_Y) \le h(Y)$, $t(P_Z, Q_Z) \le h(Z)$, $t(P_X, Q_X) = h(X)$, $t(P_W, Q_W) = h(W)$, and $h(Y)+h(Z) = h(X) + h(W)$, we conclude that $t(P_Y, Q_Y) = h(Y)$ and $t(P_Z, Q_Z) = h(Z)$. The formulas for P_X, Q_X, P_Y, and Q_Y follow from Proposition 4.2. To verify assertion 1, we rewrite (16):

$$(G_1 + G_1^\perp \wedge F_2) + F = G_1 + F_2$$

and thereby deduce that $F = F_1$. Suppose now that condition (16) is satisfied, so $F = F_1$. Then (17) is equivalent to

$$(P_Y + Q_Y) + (P_Z + Q_Z) = (P_X + Q_X) + (P_W + Q_W).$$

Thus,

$$\begin{aligned}(E_2 \vee F_1) + \big((G_1^\perp \wedge E_2) + \big[(G_1 \vee E_2^\perp)(G_1^\perp \wedge F_2)\big] + G_1\big) \\ = (G_1^\perp \wedge E_2 + G_1) + 1_4,\end{aligned}$$

and, after cancellations, we obtain

$$E_2 \vee F_1 + \big[(G_1 \vee E_2^\perp)(G_1^\perp \wedge F_2)\big] = 1_4.$$

This is equivalent to assertion (18) because

$$1_4 - E_2 \vee F_1 = (E_2 \vee F_1)^\perp = E_2^\perp \wedge F_1^\perp.$$

To verify the last assertion, let ξ be a nonzero vector in $G_1^\perp \wedge F_2$. We have $\xi - (G_1^\perp \wedge E_2)\xi \neq 0$ for otherwise ξ would be a common nonzero vector of E_2 and F_2. If (18) holds, this vector generates $E_2^\perp \wedge F_1^\perp$. □

It is easy to see that condition (18) does not hold for generic flags $\mathcal{E}, \mathcal{F}$, and $\mathcal{G}$. The eigenflags of A, B, and C are perhaps not generic, but it is fairly easy to find examples for which (18) does not hold. We present one such example next.

Example 4.5. The matrices

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 5 & -1 & 1 \\ 1 & -1 & 4 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

satisfy $C = A + B$, A has eigenvalues $4, 4, 0, 0$, and C has eigenvalues

$$\begin{aligned}\gamma_1 &= 5.818114\ldots \\ \gamma_2 &= 5.081282\ldots \\ \gamma_3 &= 1.799919\ldots \\ \gamma_4 &= 0.300684\ldots\end{aligned}$$

The formulas (14) show that the constants a, b, c, d, e, f are nonzero, so the triple (A, B, C) is irreducible. To see whether (18) is verified in this case, we observe that the ranges of E_2, F_2, and G_1 are generated by

$$\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\0\end{bmatrix}\right\}, \quad \left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right\}, \text{ and } \begin{bmatrix}\xi_1 = 1.523818\ldots\\ \xi_2 = 4.23607\ldots\\ \xi_3 = -0.94177\ldots\\ \xi_4 = 1\end{bmatrix},$$

respectively. The numerical calculations were done with the help of *Mathematica*, and exact values can be obtained as well, for instance

$$\gamma_1 = \frac{1}{4}\Big(13 + \sqrt{5} + \sqrt{6(13 - \sqrt{5})}\Big),$$
$$\xi_1 = \frac{\gamma_1}{\gamma_1 - 2},$$

with similar formulas for ξ_2 and ξ_3. It is easy to verify that F_1, $G_1^\perp \wedge F_2$, $G_1^\perp \wedge E_2$, and $E_2^\perp \wedge F_1^\perp$ are generated by the vectors

$$\begin{bmatrix}\xi_1\\ \xi_2\\ 0\\ 0\end{bmatrix}, \begin{bmatrix}-\xi_2\\ \xi_1\\ 0\\ 0\end{bmatrix}, \begin{bmatrix}\xi_2 - \xi_3\\ -\xi_1 - 2\xi_3 - \xi_4\\ \xi_1 + 2\xi_2 + \xi_4\\ \xi_2 - \xi_3\end{bmatrix}, \text{ and } \begin{bmatrix}-\xi_2\\ \xi_1\\ \xi_1\\ \xi_2 - 2\xi_1\end{bmatrix},$$

respectively. By Corollary 4.4(2), the equation (18) would require that the last three of these vectors be linearly dependent, and this is not true. For instance, the determinant

$$\begin{vmatrix}-\xi_2 & \xi_2 - \xi_3 & -\xi_2\\ \xi_1 & -\xi_1 - 2\xi_3 - \xi_4 & \xi_1\\ 0 & \xi_1 + 2\xi_2 + \xi_4 & \xi_1\end{vmatrix} = \xi_1[\xi_2(2\xi_3 + \xi_4) + \xi_1\xi_3]$$

is easily seen to be strictly negative.

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Hari Bercovici
Mathematics Department,
Indiana University,
Bloomington, IN 47405, USA
`bercovic@iu.edu`

Wing Suet Li
School of Mathematics,
Georgia Institute of Technology,
Atlanta, GA 30332, USA
`li@math.gatech.edu`