WEIGHTED HARDY–RELLICH TYPE INEQUALITIES: IMPROVED BEST CONSTANTS AND SYMMETRY BREAKING

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When studying the weighted Hardy–Rellich inequality in L^2 with the full gradient replaced by the radial derivative, the best constant becomes trivially larger or equal than in the first situation. Our contribution is to determine the new sharp constant and to show that for some part of the weights is strictly larger than before. In some cases, we emphasize that the extremals functions of the sharp constant are not radially symmetric.

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1. INTRODUCTION

The celebrated L^2 *Hardy inequality* (e.g., [15, 18]) states that for $N \geq 3$ and $u \in C_c^\infty(\mathbb{R}^N)$ it holds

$$(1) \qquad \int_{\mathbb{R}^N} |\nabla u|^2 dx \geq C_H \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx, \quad C_H(N) := \frac{(N-2)^2}{4},$$

where the constant $C_H(N)$ is sharp.

The *Rellich inequality* (e.g., [19]) asserts that for $N \geq 5$ and $u \in C_c^\infty(\mathbb{R}^N)$, we have

$$(2) \qquad \int_{\mathbb{R}^N} |\Delta u|^2 dx \geq C_R(N) \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^4} dx, \quad C_R(N) := \Big(\frac{N(N-4)}{4}\Big)^2,$$

with the best constant $C_R(N)$. The Hardy and Rellich inequalities are important tools widely used in the analysis of partial differential operators and equations of harmonic and biharmonic-type.

The *Hardy–Rellich inequality* has been studied more recently (see, e.g., [21, 12, 5]). This is, in fact, an improved Hardy inequality (with a larger

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optimal constant) applied to classes of vector fields originated from potential gradients, which arises in fluid mechanics. For $N \geq 3$ and $u \in C_0^\infty(\mathbb{R}^N)$, this leads to

$$\int_{\mathbb{R}^N} |\Delta u|^2 dx \geq C_{HR}(N) \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^2} dx, \tag{3}$$

with the best constant

$$C_{HR}(N) := \begin{cases} \dfrac{N^2}{4}, & N \geq 5 \\ 3, & N = 4 \\ \dfrac{25}{36}, & N = 3. \end{cases}$$

The Hardy–Rellich inequality was firstly extended in [21] to more general singular weights of the form

$$\int_{\mathbb{R}^N} |\Delta u|^2 |x|^m dx \geq C(N,m) \int_{\mathbb{R}^N} |\nabla u|^2 |x|^{m-2} dx, \quad \forall u \in C_c^\infty(\mathbb{R}^N), \tag{4}$$

where the authors proved that for any $N \geq 5$ and any $4 - N < m \leq 0$ the best constant is

$$C(N,m) = \min_{k=0,1,2,\dots} \frac{\left(\frac{(N-4+m)(N-m)}{4} + k(N+k-2)\right)^2}{\left(\frac{N-4+m}{2}\right)^2 + k(N+k-2)}. \tag{5}$$

In particular, according to the computations in [21], if $\frac{N+4-2\sqrt{N^2-N+1}}{3} \leq m \leq 0$ then

$$C(N,m) = \left(\frac{N-m}{2}\right)^2$$

otherwise, if $4 - N < m < \frac{N+4-2\sqrt{N^2-N+1}}{3}$ then

$$C(N,m) < \left(\frac{N-m}{2}\right)^2.$$

To ensure the integrability of the singular term in inequality (4), we need to impose that $|x|^{m-2} \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ which gives us the constraint

$$m > 2 - N \quad (or \;\; m + N - 2 > 0). \tag{6}$$

The weighted inequality (4) was later extended in [12] to all the cases $N \geq 1$ and $m > 2 - N$. Optimal constants of the cases which were not covered in [21] were solved in [12, Theorem 6.1]. Next, we emphasize a brief presentation of these additional cases:

- If $N = 1$ and $m \in \left(1, \frac{7}{3}\right] \cup [3, \infty)$ then $C(1,m) = \left(\frac{1-m}{2}\right)^2$.

- If $N = 1$ and $m \in (\frac{7}{3}; 3)$ then $C(1, m) \leq \big(\frac{1-m}{2}\big)^2$ (there are values m for which the inequality is strict).
- If $N \geq 1$ and $m = 4 - N$ then $C(N, m) = \min\{(N-2)^2, N-1\}$.
- If $N \geq 2$ and $\frac{N+4-2\sqrt{N^2-N+1}}{3} \leq m$ then $C(N, m) = \big(\frac{N-m}{2}\big)^2$.
- If $2 \leq N \leq 3$ and $2 - N < m < \frac{N+4-2\sqrt{N^2-N+1}}{3}$, or cases $N \geq 4$ and $2 - N < m \leq 4 - N$, then $C(N, m) = \frac{\left(\frac{(N-4+m)(N+m)}{4}+N-1\right)^2}{\left(\frac{N-4+m}{2}\right)^2+N-1}$.
- If $N = 3$ and $m \geq \frac{N+4-2\sqrt{N^2-N+1}}{3}$ or $N \geq 4$ and $m > 4-N$ the best constant requires a further subdivision, based on very technical expressions, which could be consulted in [12, Theorem 6.1].

Subsequent extensions of the weighted Hardy–Rellich type inequalities with reminder terms have been done recently in [1], [11] and [20] by applying factorization methods. Also, recent improvements when adding magnetic fields have been established in [4, 16]. See also very recent results on the Hardy–Rellich inequalities in [2, 17] and in [14] (in the context of solenoidal vector fields) and the references therein.

Overall, for any $N \geq 1$ and $m > 2 - N$ always happens that the best constant in (4) does not pass the threshold

$$C(N, m) \leq \Big(\frac{N-m}{2}\Big)^2. \tag{7}$$

In this paper, we study a weighted Hardy–Rellich type inequality, by replacing the full gradient in (4) with the radial derivative $\partial_r u := \frac{x}{|x|} \cdot \nabla u$ on the right-hand side, namely

$$\int_{\mathbb{R}^N} |\Delta u|^2 |x|^m dx \geq \tilde{C}(N, m) \int_{\mathbb{R}^N} |x \cdot \nabla u|^2 |x|^{m-4} dx, \quad \forall u \in C_c^\infty(\mathbb{R}^N). \tag{8}$$

where $\tilde{C}(N, m)$ denotes the best constant in (8). Since $|\partial_r u| \leq |\nabla u|$ notice that inequality (8) is a relaxation of (4) and, in balance with that, the best constant might be larger or equal, i.e., $C(N, m) \leq \tilde{C}(N, m)$.

To our knowledge, inequality (8) has been partially treated in the literature, especially for test functions $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ as follows. For $m = 0$ the constant $\frac{N^2}{4}$ with $N \geq 2$ was conceived in [9]. It was subsequently extended to the weighted cases $m \neq 0$ (see, e.g., [8], [10]) where the authors obtained (8) with the constant $\frac{(N-m)^2}{4}$. From the quoted papers, it is not clear for which values m the constant $\frac{(N-m)^2}{4}$ could be sharply extended from the space of functions $\{u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})\}$ to $\{u \in C_c^\infty(\mathbb{R}^N)\}$.

Our purpose in this paper is to supply explicitly the best constant $\tilde{C}(N,m)$ of (8) for the full range of parameters $N \geq 1$ and $m > 2-N$, to emphasize situations in which we get an improvement in (8) with respect to (4), i.e., $\tilde{C}(N,m) > C(N,m)$ and to put in evidence the radially symmetry breaking for the optimal approximations of the sharp constants.

2. MAIN RESULT

In order to state the main result, we need to introduce a cut-off function $g \in C^\infty([0,\infty))$, with $0 \leq g \leq 1$, such that

$$g(r) = \begin{cases} 1, & 0 \leq r \leq 1, \\ 0, & r \geq 2. \end{cases} \tag{9}$$

The main result of this paper is the following.

THEOREM 2.1. *Let $N \geq 1$ and $m > 2-N$. Then*

$$\int_{\mathbb{R}^N} |\Delta u|^2 |x|^m dx \geq \tilde{C}(N,m) \int_{\mathbb{R}^N} |x \cdot \nabla u|^2 |x|^{m-4} dx, \quad \forall u \in C_c^\infty(\mathbb{R}^N), \tag{10}$$

where the optimal constant $\tilde{C}(N,m)$ is given as follows:

If $N = 1$ and $m > 1$ or if $N \geq 2$ and

$$m \in [2 - \sqrt{(N-1)^2+1}, 2 + \sqrt{(N-1)^2+1}],$$

then

$$\tilde{C}(N,m) = \Big(\frac{N-m}{2}\Big)^2,$$

which is approximated by the sequence $\{u_\epsilon\}_{\epsilon>0}$ given by

$$u_\epsilon(x) = |x|^{-\frac{N+m-4}{2}+\epsilon} g(|x|). \tag{11}$$

If $N \geq 2$ and $m \in (2-N, 2-\sqrt{(N-1)^2+1})$, then

$$\tilde{C}(N,m) = \frac{\big((m-2)^2 - N^2\big)^2}{4(N+m-4)^2}$$

which is approximated by the sequence $\{u_\epsilon\}_{\epsilon>0}$ given by

$$u_\epsilon(x) = |x|^{-\frac{N+m-4}{2}+\epsilon} g(|x|)\phi_1(x), \tag{12}$$

where ϕ_1 is a spherical harmonic function of degree 1 with $\|\phi_1\|_{L^2(S^{N-1})} = 1$.

If $N \geq 2$ and $m \in (2+\sqrt{(N-1)^2+1}, \infty)$, then

$$\tilde{C}(N,m) = \min_{l \leq k(m)} \tilde{C}(N,m,l) \text{ and}$$

$$\tilde{C}(N,m,l) := \frac{(-N+m-2l)^2\,(2l+m+N-4)^2}{4\,(m+N-4)^2}$$

where $k(m)$ is defined later in (31)–(44). *The constant $\tilde{C}(N,m)$ is approximated by the sequence given by*

$$u_\epsilon(x) = |x|^{-\frac{N+m-4}{2}+\epsilon} g(|x|)\phi_{l_{\min}}(x), \tag{13}$$

where $\phi_{l_{\min}}$ is a spherical harmonic function of degree $l_{\min}$ such that

$$\|\phi_{l_{\min}}\|_{L^2(S^{N-1})} = 1 \text{ and } l_{\min} := \arg \min_{l \le k(m)} \tilde{C}(N,m,l).$$

Remark 2.2. Notice also that in all the situations above

$$\tilde{C}(N,m) \le \Big(\frac{N-m}{2}\Big)^2,$$

but there are cases, see for instance $N = 1$ and $m \in \left(\frac{7}{3}, 3\right)$, when our best constant $\tilde{C}(N,m)$ in (8) improves with respect to the best constant $C(N,m)$ in (4).

Remark 2.3. The approximating sequences u_ϵ in (11), (13), (12) do not belong to the space $C_c^\infty(\mathbb{R}^N)$ but they are in the energy space of the inequality (10), i.e., both terms in (10) are finite for u_ϵ. So, by regularizing u_ϵ near the origin, one can show that the constants remain sharp for functions $u \in C_c^\infty(\mathbb{R}^N)$, see for instance similar arguments in [6, 7].

3. PROOF OF THE MAIN RESULT

We use spherical coordinates instead of cartesian coordinates. This coordinates transformation is given by

$$x \in \mathbb{R}^N \setminus \{0\} \mapsto (r,\sigma) \in (0,\infty) \times S^{N-1}, \quad r = |x|, \quad \sigma = \frac{x}{|x|},$$

where S^{N-1} is the $N-1$-dimensional sphere with respect to the Hausdorff measure in $\mathbb{R}^N$. We use the formula of the Laplacian in spherical coordinates

$$\Delta = \partial^2_{rr} + \frac{N-1}{r}\partial_r + \frac{1}{r^2}\Delta_{S^{N-1}}, \tag{14}$$

where ∂_r and ∂^2_{rr} are first and second order partial derivatives with respect to the radial component r whereas (for fixed r) the Laplace–Beltrami operator with respect to the metric tensor on S^{N-1} and, respectively, the spherical gradient are given by

$$\Delta_{S^{N-1}} u(r\sigma) = \Delta\Big[u\Big(\frac{x}{|x|}\Big)\Big]_{|x=\sigma}, \quad \nabla_{S^{N-1}} u(r\sigma) = \nabla\Big[u\Big(\frac{x}{|x|}\Big)\Big]_{|x=\sigma}.$$

Applying the spherical harmonics decomposition, we can expand $u \in C_c^\infty(\mathbb{R}^N)$ as

$$u(x) = u(r\sigma) = \sum_{k=0}^{\infty} u_k(r)\phi_k(\sigma).$$

The set of functions $\{\phi_k\}_{k\geq 0}$ are spherical harmonics of degree k which consists in an orthogonal basis in $L^2(S^{N-1})$. These functions satisfy the properties

$$(15)\qquad \begin{cases} -\Delta_{S^{N-1}}\phi_k = c_k\phi_k \text{ on } S^{N-1}, \\ -\int_{S^{N-1}} \Delta_{S^{N-1}}\phi_k\phi_l d\sigma = \int_{S^{N-1}} \nabla_{S^{N-1}}\phi_k \cdot \nabla_{S^{N-1}}\phi_l d\sigma, \\ = c_k \int_{S^{N-1}} \phi_k\phi_l d\sigma = c_k\delta_{lk}, \quad k,l\in\mathbb{N}, \end{cases}$$

where $c_k = k(k+N-2)$, $k\geq 0$ are the eigenvalues of the Laplace–Beltrami operator $\Delta_{S^{N-1}}$, where δ_{lk} represents the Kronecker symbol, see, e.g., [13] for more detailed properties of spherical harmonics.

Since in view of (14)

$$\Delta\Big[u_k(|x|)\phi_k\Big(\frac{x}{|x|}\Big)\Big] = \big(\Delta_r u_k(r) - \frac{c_k}{r^2}u_k(r)\big)\phi_k(\sigma),$$

due to (15) similar computations as in [5] lead to

$$(16)\quad \int_{\mathbb{R}^N} |\Delta u|^2|x|^m dx = \sum_{k=0}^{\infty}\int_0^\infty \Big(|\Delta_r u_k|^2 + \frac{c_k^2}{r^4}u_k^2 - \frac{2c_k}{r^2}u_k\Delta_r u_k\Big)r^{N+m-1}dr.$$

Next, we write u_k' and u_k'' to express both first and second derivatives of the Fourier coefficients $\{u_k\}_k$.

We need to compute $\int_0^\infty |\Delta_r u_k| r^{N+m-1}$ and $\int_0^\infty \frac{\Delta_r u_k}{r^2}u_k(r)r^{N+m-1}dr$.

Integration by parts leads to

$$(17)\qquad \begin{aligned} \int_0^\infty |\Delta_r u_k|^2 r^{N+m-1}dr = (N-1)(1-m)\int_0^\infty |u_k'(r)|r^{N+m-3}dr \\ + \int_0^\infty |u_k''(r)|r^{N+m-1}dr, \end{aligned}$$

and

(18)
$$\begin{aligned} \int_0^\infty \frac{\Delta_r u_k}{r^2}u_k(r)r^{N+m-1}dr = \frac{1}{2}\Big(-2\int_0^\infty |u_k'(r)|r^{N+m-3}dr\Big) \\ + (N+m-4)(m-2)\int_0^\infty |u_k(r)|r^{N+m-5}dr. \end{aligned}$$

Then (16) becomes

(19)
$$\begin{aligned} \int_{\mathbb{R}^N} |\Delta u|^2|x|^m dx = \sum_{k=0}^{\infty}\Big(\int_0^\infty |u_k''|^2 r^{N+m-1}dr \\ + (2c_k + (N-1)(1-m))\int_0^\infty |u_k'|^2 r^{N+m-3}dr \\ + (c_k^2 - c_k(m-2)(N+m-4))\int_0^\infty |u_k|^2 r^{N+m-5}dr\Big). \end{aligned}$$

and more easily, since $\partial_r u = \frac{x}{|x|} \cdot \nabla u$, we get

$$\int_{\mathbb{R}^N} |x \cdot \nabla u|^2 |x|^{m-4} dx = \sum_{k=0}^{\infty} \int_0^{\infty} |u_k'|^2 r^{N+m-3} dr. \tag{20}$$

3.1. The case $N = 1$ and $m > 1$

The inequality (10) reduces to:

$$\int_{\mathbb{R}} |u''|^2 r^m dr \geq \tilde{C}(1,m) \int_{\mathbb{R}} |u'|^2 r^{m-2} dr, \quad \forall u \in C_c^{\infty}(\mathbb{R}). \tag{21}$$

The relation (21) comes from

$$\begin{aligned} \int_{\mathbb{R}} (u')^2 r^{m-2} dr &= \int_{\mathbb{R}} (u')^2 \Big(\frac{r^{m-1}}{m-1}\Big)' dr \\ &= -\frac{2}{m-1} \int_{\mathbb{R}} u' u'' r^{m-1} dr \\ &= -\frac{2}{m-1} \int_{\mathbb{R}} u' r^{\frac{m}{2}} u'' r^{\frac{m-2}{2}} dr \\ &\leq -\frac{2}{m-1} \Big(\int_{\mathbb{R}} (u'')^2 r^m dr \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}} (u')^2 r^{m-2} dr \Big)^{\frac{1}{2}}. \end{aligned}$$

Integration by parts and Cauchy–Schwarz inequality lead that (21) holds with the constant

$$\tilde{C}(1,m) \geq \Big(\frac{m-1}{2}\Big)^2. \tag{22}$$

3.2. The case $N \geq 2$

In this case, it remains to compare the right-hand sides in (19), (20). We apply the well-known 1-d weighted Hardy inequalities (see, e.g., [21], [3, Proposition 2.4])

$$\int_0^{\infty} |u_k''|^2 r^{N+m-1} dr \geq \Big(\frac{N+m-2}{2}\Big)^2 \int_0^{\infty} |u_k'|^2 r^{N+m-3} dr, \tag{23}$$

$$\int_0^{\infty} |u_k'|^2 r^{N+m-3} dr \geq \Big(\frac{N+m-4}{2}\Big)^2 \int_0^{\infty} |u_k|^2 r^{N+m-5} dr. \tag{24}$$

Case A: $(2-m)(N+m-4)\geq 0$. We distinguish two sub-cases as follows.

A1) $2-m\geq 0$ and $N+m-4\geq 0$. These imply $4-N\leq m\leq 2$.

A2) $2-m\leq 0$ and $N+m-4\leq 0$. Since $N\geq 2$ it follows that $m=N=2$.

Gluing both situations, we can summarize that Case A is equivalent to

$$4-N\leq m\leq 2. \tag{25}$$

Then, we obtain from (19), (20) and (23) that

$$\begin{aligned}\int_{\mathbb{R}^N}|\Delta u|^2|x|^m dx &\geq \sum_{k=0}^{\infty}\Bigg(\Big(\frac{N+m-2}{2}\Big)^2\int_0^\infty |u_k'|^2 r^{N+m-3}dr\\ &\qquad +(2c_k+(N-1)(1-m))\int_0^\infty |u_k'|^2 r^{N+m-3}dr\Bigg)\\ &\geq \Bigg(\Big(\frac{N+m-2}{2}\Big)^2+(N-1)(1-m)\Bigg)\\ &\qquad\times\sum_{k=0}^{\infty}\int_0^\infty |u_k'|^2 r^{N+m-3}dr\\ &=\Big(\frac{N-m}{2}\Big)^2\sum_{k=0}^{\infty}\int_0^\infty |u_k'|^2 r^{N+m-3}dr\\ &=\Big(\frac{N-m}{2}\Big)^2\int_{\mathbb{R}^N}|x\cdot\nabla u|^2|x|^{m-4}dx.\end{aligned} \tag{26}$$

Hence

$$\tilde{C}(N,m)\geq\Big(\frac{N-m}{2}\Big)^2. \tag{27}$$

Case B: $(2-m)(N+m-4)<0$.

B1) $2-m>0$ and $N+m-4<0$. These combined with (6) are equivalent to

$$2-N<m<4-N. \tag{28}$$

B2) $m-2<0$ and $N+m-4>0$. These are equivalent to

$$m>2. \tag{29}$$

We first look at the spherical part (which contains the terms multiplied with c_k) in the relation (19). Taking into account (24) and the fact that $c_k\geq 0$

for any $k \geq 0$, we successively have

(30)
$$\begin{aligned} 2c_k \int_0^\infty |u_k'|^2 r^{N+m-3} dr + (c_k^2 - c_k(m-2)(N+m-4)) \int_0^\infty |u_k|^2 r^{N+m-5} dr \\ \geq c_k \Big(2\Big(\frac{N+m-4}{2}\Big)^2 + c_k - (m-2)(N+m-4)\Big) \int_0^\infty |u_k|^2 r^{N+m-5} dr. \end{aligned}$$

In view of this, let us denote

$$I_k(m,N) := 2\Big(\frac{N+m-4}{2}\Big)^2 + c_k - (m-2)(N+m-4). \tag{31}$$

In view of identity (19), Hardy inequalities (23), (24) and (30), (31), we get

$$\begin{aligned} \int_{\mathbb{R}^N} |\Delta u|^2 |x|^m dx \geq \Big(\frac{N-m}{2}\Big)^2 \sum_{k=0}^\infty \int_0^\infty |u_k'|^2 r^{N+m-3} dr \\ + \sum_{k=0}^\infty c_k I_k(m,N) \int_0^\infty |u_k|^2 r^{N+m-5} dr. \end{aligned} \tag{32}$$

Next, we want to investigate when $I_k(m,N) \geq 0$. We start with estimating

$$\begin{aligned} I_1(m,N) &= 2\Big(\frac{N+m-4}{2}\Big)^2 + N - 1 - (m-2)(N+m-4) \\ &= -\frac{1}{2}m^2 + 2m + \frac{1}{2}N^2 - N - 1. \end{aligned}$$

Furthermore, the equation $I_1(m,N) = 0$ in the unknown m has the discriminant $\Delta = (N-1)^2 + 1 \geq 0$ and the roots

$$m_{1,2} = 2 \pm \sqrt{(N-1)^2 + 1}$$

which imply

$$I_1(m,N) \geq 0, \qquad \text{iff } m \in [2 - \sqrt{(N-1)^2+1},\ 2 + \sqrt{(N-1)^2+1}].$$

Therefore, we conclude that in the case B1, we have

$$I_1(m,N) \geq 0, \qquad \text{iff } m \in [2 - \sqrt{(N-1)^2+1},\ 4 - N)$$

whereas in the case B2, we have

$$I_1(m,N) \geq 0, \qquad \text{iff } m \in (2, 2 + \sqrt{(N-1)^2+1}].$$

Since $\{I_k\}$ is an increasing sequence with respect to k we get that

$$I_k(m,N) \geq 0, \quad \forall k \geq 1, \tag{33}$$

for any m satisfying

$$m \in [2 - \sqrt{(N-1)^2+1},\ 4 - N) \cup (2, 2 + \sqrt{(N-1)^2+1}]. \tag{34}$$

This, together with (19) and (32), yield

$$\tilde{C}(N,m) \geq \left(\frac{N-m}{2}\right)^2, \quad \forall m \text{ as in (34).} \tag{35}$$

It remains to analyze the complementary "bad cases" of (34) for which $I_1(m,N) < 0$:

$$m \in \underbrace{(2-N, 2-\sqrt{(N-1)^2+1})}_{\text{remaining cases of B1}} \cup \underbrace{(2+\sqrt{(N-1)^2+1}, \infty)}_{\text{remaining cases of B2}}. \tag{36}$$

If $k=2$, we obtain

$$\begin{aligned} I_2(m,N) &= 2\left(\frac{N+m-4}{2}\right)^2 + 2N - (m-2)(N+m-4) \\ &= -\frac{m^2}{2} + 2m + \frac{N^2}{2}. \end{aligned}$$

Therefore,

$$I_2(m,N) \geq 0, \qquad \text{iff } m \in [2-\sqrt{N^2+4},\ 2+\sqrt{N^2+4}]. \tag{37}$$

This leads to

$$I_k(m,N) \geq 0, \quad \forall k \geq 2, \quad m \in (2-N, 2-\sqrt{(N-1)^2+1}), \tag{38}$$

which cover the remaining cases of B1.

Inequality (10) **in the remaining cases of B1:**

$$m \in (2-N, 2-\sqrt{(N-1)^2+1}).$$

The right-hand side in (19) can be bounded from below in terms of a parameter $\varepsilon > 0$ (which are well stated later) as follows, in view of (32) and (24):

$$\begin{aligned} \int_{\mathbb{R}^N} |\Delta u|^2 |x|^m dx \geq \sum_{k=0,k\neq 1}^{\infty} &\Bigg(\Big(\frac{N-m}{2}\Big)^2 \int_0^\infty |u_k'|^2 r^{N+m-3} dr \\ &+ c_k I_k(m,N) \int_0^\infty |u_k|^2 r^{N+m-5} dr \Bigg) \\ &+ \Big(\frac{N-m}{2}\Big)^2 \int_0^\infty |u_1'|^2 r^{N+m-3} dr \\ &+ c_1 I_1(m,N) \int_0^\infty |u_1|^2 r^{N+m-5} dr \\ \geq \sum_{k=0,k\neq 1}^{\infty} &\Big(\frac{N-m}{2}\Big)^2 \int_0^\infty |u_k'|^2 r^{N+m-3} dr \end{aligned} \tag{39}$$

$$
\begin{aligned}
&+\sum_{k=0,k\neq 1}^{\infty} c_k I_k(m,N)\int_0^\infty |u_k|^2 r^{N+m-5}dr\\
&+\Big(\Big(\frac{N-m}{2}\Big)^2-\varepsilon\Big)\int_0^\infty |u_1'|^2 r^{N+m-3}dr\\
&+\Big((2c_1+\varepsilon)\Big(\frac{N+m-4}{2}\Big)^2\\
&\quad+c_1^2-c_1(m-2)(m+N-4)\Big)\int_0^\infty |u_1|^2 r^{N+m-5}dr.
\end{aligned}
$$

The coefficient of the integral term $\int_0^\infty |u_1|^2 r^{N+m-5}dr$ in (39) becomes

$$
\begin{aligned}
a_1(N,m,\varepsilon) := (2(N-1)+\varepsilon)\Big(\frac{N+m-4}{2}\Big)^2&\\
+(N-1)^2-(N-1)(m-2)(m+N-4).&
\end{aligned}\tag{40}
$$

We choose $\varepsilon_1>0$ such that $a_1(N,m,\varepsilon_1)=0$ and we get

$$
\varepsilon_1=\frac{2(N-1)(m^2-N^2-4m+2N+2)}{(m+N-4)^2}.\tag{41}
$$

Indeed, $\varepsilon_1>0$ because the inequality $m^2-N^2-4m+2N+2>0$ holds if and only if

$$
m\in(-\infty,2-\sqrt{(N-1)^2+1})\cup(2+\sqrt{(N-1)^2+1},\infty),
$$

set which contains the remaining cases $m\in(2-N,2-\sqrt{(N-1)^2+1})$ of the case B1. From (41), we obtain that

$$
\Big(\frac{N-m}{2}\Big)^2-\varepsilon_1=\frac{\big((m-2)^2-N^2\big)^2}{4(N+m-4)^2}>0.
$$

Coming back to (39), we have

$$
\int_{\mathbb{R}^N}|\Delta u|^2|x|^m dx\geq\frac{\big((m-2)^2-N^2\big)^2}{4(N+m-4)^2}\sum_{k=0}^{\infty}\int_0^\infty |u_k'|^2 r^{N+m-3}dr,\tag{42}
$$

and therefore,

$$
\tilde{C}(N,m)\geq\frac{\big((m-2)^2-N^2\big)^2}{4(N+m-4)^2}.\tag{43}
$$

Inequality (10) **in the remaining cases of B2:**

$$
m\in(2+\sqrt{(N-1)^2+1},\infty).
$$

We want to apply the same idea as in the previous case. We define the number

$$
k(m):=\min\{k\in\mathbb{N}\mid I_k(m,N)\geq 0\}.\tag{44}
$$

Then

$$I_1(m,N) < I_2(m,N) < \cdots < I_{k(m)-1} < 0 \tag{45}$$
$$I_k(m,N) \geq I_{k(m)}(m,N) \geq 0, \text{ for all } k \geq k(m).$$

The inequality (32) is rewritten as:

$$\begin{aligned}
\int_{\mathbb{R}^N} |\Delta u|^2 |x|^m dx &\geq \Big(\frac{N-m}{2}\Big)^2 \Big(\sum_{k\geq k(m)} \int_0^\infty |u_k'|^2 r^{N+m-3} dr \\
&\quad + \int_0^\infty |u_0'|^2 r^{N+m-3} dr \Big) + \sum_{k=1}^{k(m)-1} c_k I_k(m,N) \int_0^\infty |u_k|^2 r^{N+m-5} dr \\
&= \Big(\frac{N-m}{2}\Big)^2 \Big(\sum_{k\geq k(m)} \int_0^\infty |u_k'|^2 r^{N+m-3} dr + \int_0^\infty |u_0'|^2 r^{N+m-3} dr \Big) \\
&\quad + \sum_{k=1}^{k(m)-1} \Big(\Big(\Big(\frac{N-m}{2} \Big)^2 - \varepsilon_k \Big) \int_0^\infty |u_k'|^2 r^{N+m-3} dr \\
&\quad + c_k I_k(m,N) \int_0^\infty |u_k|^2 r^{N+m-5} dr + \varepsilon_k \int_0^\infty |u_k'|^2 r^{N+m-3} dr \Big) \\
&\geq \Big(\frac{N-m}{2}\Big)^2 \Big(\sum_{k\geq k(m)} \int_0^\infty |u_k'|^2 r^{N+m-3} dr + \int_0^\infty |u_0'|^2 r^{N+m-3} dr \Big) \\
&\quad + \sum_{k=1}^{k(m)-1} \Big(\Big(\Big(\frac{N-m}{2} \Big)^2 - \varepsilon_k \Big) \int_0^\infty |u_k'|^2 r^{N+m-3} dr + \\
&\quad + \Big(c_k I_k(m,N) + \varepsilon_k \Big(\frac{N+m-4}{2} \Big)^2 \Big) \int_0^\infty |u_k|^2 r^{N+m-5} dr \Big).
\end{aligned} \tag{46}$$

We denote the coefficient of the zero order term above by

$$a_k(N,m,\varepsilon_k) := c_k I_k(m,N) + \varepsilon_k \Big(\frac{N+m-4}{2} \Big)^2. \tag{47}$$

We choose ε_k such that $a_k(N,m,\varepsilon_k) = 0$ and we obtain

$$\varepsilon_k = \frac{-4 c_k I_k(m,N)}{(N+m-4)^2} > 0. \tag{48}$$

For consistency, we want to make sure that $\big(\frac{N-m}{2}\big)^2 - \varepsilon_k$ is positive for every k, m and N. Indeed,

$$\begin{aligned}
\Big(\frac{N-m}{2}\Big)^2 - \varepsilon_k &= \Big(\frac{N-m}{2}\Big)^2 + \frac{4 c_k I_k(m,N)}{(N+m-4)^2} \\
&= \frac{(-N+m-2k)^2 \, (2k+m+N-4)^2}{4\,(m+N-4)^2} > 0.
\end{aligned} \tag{49}$$

As a consequence of (46), we get

$$\int_{\mathbb{R}^N} |\Delta u|^2 |x|^m dx \geq \min_{l \leq k(m)} \left(\left(\frac{N-m}{2} \right)^2 - \varepsilon_l \right) \sum_{k=0}^{\infty} \int_0^{\infty} |u_k'|^2 r^{N+m-3} dr, \tag{50}$$

where ε_l is as in (48). So, in view of (49), we have

$$\tilde{C}(N,m) \geq \min_{l \leq k(m)} \tilde{C}(N,m,l),$$

where

$$\tilde{C}(N,m,l) := \frac{(-N+m-2l)^2 \, (2l+m+N-4)^2}{4\,(m+N-4)^2}.$$

3.3. **Optimality**

In this section, we aim to prove that all the lower bound constants obtained in (22), (27), (35), (43) and (49) are the sharp constants. For that, it suffices to prove the existence of approximating sequences in (10) for the quoted constants.

Step I. The cases with radially symmetric approximations:

- $N = 1$ and $m > 1$
- $N \geq 2$ and m as in the Case A (condition (25))
- $N \geq 2$ and m as in the "good" cases B (condition (34))

The above cases can be treated similarly because the same sequence with radial symmetry can be built to approach the constants (22), (27) and (35).

To prove that, let us consider the radially symmetric sequence

$$u_\epsilon(x) = |x|^{-\frac{N+m-4}{2}+\epsilon} g(|x|) = r^{-\frac{N+m-4}{2}+\epsilon} g(r) =: U_\epsilon(r)$$

with g given in (9). Replacing u with u_ϵ in (19) and arguing as in [5], since the spherical part is missing, we obtain

$$\int_{\mathbb{R}^N} |\Delta u_\epsilon|^2 |x|^m dx = \int_0^{\infty} r^{N-1+m} |U_\epsilon''(r)|^2 dr + (N-1)(1-m) \int_0^{\infty} r^{N+m-3} |U_\epsilon'(r)|^2 dr$$

and

$$\int_{\mathbb{R}^N} |x \cdot \nabla u_\epsilon|^2 |x|^{m-4} dx = \int_0^{\infty} |U_\epsilon'(r)|^2 r^{N+m-3} dr.$$

From the definition of U_ϵ, we have

(51)

$$\int_0^\infty r^{N+m-3}|U'(r)|^2dr$$
$$= \int_0^\infty r^{N+m-3}\Big(\Big(-\frac{N+m-4}{2}+\epsilon\Big)^2 r^{-(N+m-2)+2\epsilon}g^2(r)$$
$$+ r^{-(N+m-4)+2\epsilon}g'(r)^2\Big)dr$$
$$+ \int_0^\infty r^{N+m-3}\Big(2\Big(-\Big(\frac{N+m-4}{2}\Big)+2\epsilon\Big)r^{-(N+m-3)+2\epsilon}g'(r)g(r)\Big)dr$$
$$= \frac{1}{2\epsilon}\Big(-\Big(\frac{N-4+m}{2}\Big)+\epsilon\Big)^2+\mathcal{O}(1).$$

Also, since

$$U_\epsilon''(r) = \Big(-\Big(\frac{N-4+m}{2}\Big)+\epsilon\Big)\Big(-\Big(\frac{N-2+m}{2}\Big)+\epsilon\Big)r^{-\frac{N+m}{2}+\epsilon}g(r)+\mathcal{O}(1),$$

we obtain

$$\int_0^\infty r^{N+m-1}|U_\epsilon''(r)|^2dr$$

(52)
$$= \frac{1}{2\epsilon}\Big(-\Big(\frac{N+m-4}{2}\Big)+\epsilon\Big)^2\Big(-\Big(\frac{N+m-2}{2}\Big)+\epsilon\Big)^2+\mathcal{O}(1).$$

Due to (51) and (52), we obtain

$$\frac{\int_{\mathbb{R}^N}|\Delta u_\epsilon|^2|x|^m dx}{\int_{\mathbb{R}^N}|x\cdot\nabla u_\epsilon|^2|x|^{m-4}dx}$$
$$= \frac{\frac{1}{2\epsilon}\left(-\frac{N+m-4}{2}+\epsilon\right)^2\left(-\frac{N+m-2}{2}+\epsilon\right)^2}{\frac{1}{2\epsilon}\left(-\frac{N+m-4}{2}+\epsilon\right)^2+\mathcal{O}(1)}$$
$$+ \frac{\frac{1}{2\epsilon}(N-1)(1-m)\left(-\frac{N+m-4}{2}+\epsilon\right)^2+\mathcal{O}(1)}{\frac{1}{2\epsilon}\left(-\frac{N+m-4}{2}+\epsilon\right)^2+\mathcal{O}(1)}$$
$$= \frac{\left(-\frac{N+m-4}{2}+\epsilon\right)^2\left(\left(-\frac{N+m-2}{2}+\epsilon\right)^2+(N-1)(1-m)\right)+\mathcal{O}(\epsilon)}{\left(-\left(\frac{N+m-4}{2}\right)+\epsilon\right)^2+\mathcal{O}(\epsilon)}$$
$$\searrow \Big(\frac{N+m-2}{2}\Big)^2+(N-1)(1-m) = \frac{N^2-2Nm+m^2}{4} = \Big(\frac{N-m}{2}\Big)^2,$$

as $\epsilon \searrow 0$.

Step II. The "bad" cases of B and non-radially symmetric optimal approximations:

- $N \geq 2$ and m as in the "bad" cases of B1 (i.e., $m \in (2-N, 2-\sqrt{(N-1)^2+1})$)

- $N \geq 2$ and m as in the "bad" cases of B2
(i.e., $m \in (2 + \sqrt{(N-1)^2+1}, \infty)$)

In the first "bad" case $N \geq 2$ and $m \in (2-N, 2-\sqrt{(N-1)^2+1})$, we consider the sequence

$$u_\epsilon(x) = |x|^{-\frac{N+m-4}{2}+\epsilon} g(|x|)\phi_1\Big(\frac{x}{|x|}\Big) = r^{-\frac{N+m-4}{2}+\epsilon} g(r)\phi_1(\sigma) =: U_\epsilon(r)\phi_1(\sigma)$$

with g as in (9).

If we replace once more u with the above u_ϵ in (19), we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} |\Delta u_\epsilon|^2 |x|^m dx &= \int_0^\infty r^{N-1+m} |U_\epsilon''(r)|^2 dr \\ &\quad + (2c_1 + (N-1)(1-m)) \int_0^\infty r^{N+m-3} |U_\epsilon'(r)|^2 dr \\ &\quad + \big(c_1^2 - c_1(m-2)(N+m-4)\big) \int_0^\infty r^{N+m-5} |U_\epsilon(r)|^2 dr\end{aligned}$$

and

$$\int_{\mathbb{R}^N} |x \cdot \nabla u_\epsilon|^2 |x|^{m-4} dx = \int_0^\infty |U_\epsilon'(r)|^2 r^{N+m-3} dr.$$

Due to (51) and (52) and the fact that

$$\int_0^\infty r^{N+m-5} |U_\epsilon(r)|^2 dr = \frac{1}{2\epsilon} + \mathcal{O}(1), \tag{53}$$

we have

$$\begin{aligned}\int_{\mathbb{R}^N} |\Delta u_\epsilon|^2 |x|^m dx = \frac{1}{2\epsilon}\Bigg(&\Big(-\frac{N+m-4}{2}+\epsilon\Big)^2\Big(-\frac{N+m-2}{2}+\epsilon\Big)^2 \\ &+ (2c_1 + (N-1)(1-m))\Big(-\frac{N+m-4}{2}+\epsilon\Big)^2 \\ &+ c_1^2 - c_1(m-2)(N+m-4)\Bigg) + \mathcal{O}(1)\end{aligned} \tag{54}$$

and

$$\int_{\mathbb{R}^N} |x \cdot \nabla u_\epsilon|^2 |x|^{m-4} dx = \frac{1}{2\epsilon}\Big(-\frac{N+m-4}{2}+\epsilon\Big)^2 + \mathcal{O}(1). \tag{55}$$

Due to (54) and (55), we successively obtain

$$\begin{aligned}\frac{\int_{\mathbb{R}^N} |\Delta u_\epsilon|^2 |x|^m dx}{\int_{\mathbb{R}^N} |x \cdot \nabla u_\epsilon|^2 |x|^{m-4} dx} =\searrow \Big(-\frac{N+m-2}{2}\Big)^2 + 2c_1 + (N-1)(1-m) \\ + \frac{4\left(c_1^2 - c_1(m-2)(N+m-4)\right)}{(N+m-4)^2}\end{aligned}$$

$$= \frac{\left((m-2)^2 - N^2\right)^2}{4\left(N+m-4\right)^2}$$

as $\epsilon \searrow 0$.

For the second "bad" case $N \geq 2$ and $m \in (2 + \sqrt{(N-1)^2+1}, \infty)$, we consider the sequence

$$u_\epsilon(x) = |x|^{-\frac{N+m-4}{2}+\epsilon} g(|x|)\phi_{l_{\min}}(x), \tag{56}$$

where $\phi_{l_{\min}}$ is a spherical harmonic function of degree $l_{\min}$ such that we have $\|\phi_{l_{\min}}\|_{L^2(S^{N-1})} = 1$ and $l_{\min} := \arg\min_{l \leq k(m)} \tilde{C}(N,m,l)$, where

$$\tilde{C}(N,m,l) := \frac{\left(-N+m-2l\right)^2 \left(2l+m+N-4\right)^2}{4\left(m+N-4\right)^2}$$

and $k(m)$ was defined in (31)–(44).

Similarly (but more technically) as above, one can show that

$$\frac{\int_{\mathbb{R}^N} |\Delta u_\epsilon|^2 |x|^m dx}{\int_{\mathbb{R}^N} |x \cdot \nabla u_\epsilon|^2 |x|^{m-4} dx} \searrow \tilde{C}(N,m,l)$$

as $\epsilon \searrow 0$. The details are left to the reader. The optimality is showed and the proof of the main result is complete now.

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