RATES OF ASYMPTOTIC REGULARITY OF THE TIKHONOV–MANN ITERATION FOR FAMILIES OF MAPPINGS

HORAŢIU CHEVAL

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In this paper, we generalize the strongly convergent Krasnoselskii–Mann-type iteration for families of nonexpansive mappings defined recently by Boţ and Meier in Hilbert spaces to the abstract setting of W-hyperbolic spaces and we compute effective rates of asymptotic regularity for our generalization. This also extends recent results by Leuştean and the author on the Tikhonov–Mann iteration from single mappings to families of mappings.

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1. INTRODUCTION

In [4], Boţ and Meier propose a strongly convergent Krasnoselskii–Mann-type iteration for finding a common fixed point of a family $(T_n : H \to H)$ of nonexpansive self-mappings of a Hilbert space. They define the sequence (x_n) by

$$x_{n+1} = (1 - \lambda_n)\beta_n x_n + \lambda_n T_n(\beta_n x_n), \tag{1}$$

where $x_0 \in H$ is an arbitrary starting point and (λ_n), (β_n) are sequences in $[0, 1]$. Theorem 3.1 of [4] states that, under some conditions on $(\lambda_n), (\beta_n)$ and (T_n),

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0, \tag{2}$$

and, furthermore, that if (T_n) satisfies an additional asymptotic condition, then (x_n) converges strongly to a common fixed point of (T_n). As a part of the proof of [4, Theorem 3.1], it is also established that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3}$$

Properties (3) and (2) are called the asymptotic, respectively the (T_n)-asymptotic regularity of (x_n), and are important notions in optimization and

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nonlinear analysis, the first one going back to Browder and Petryshyn [7] being later extended by Borwein, Reich and Shafrir [5]. Furthermore, they serve as key steps in many convergence proofs, including the one of [4, Theorem 3.1].

The case when (T_n) is constant in iteration (1) was studied and proven strongly convergent by Yao, Zhou and Liou [26] and recently by Boţ, Csetnek and Meier [3]. Also, the single operator case was generalized by Leuştean and the author [9] to W-hyperbolic spaces, where quadratic rates of asymptotic regularity were obtained. Even better, linear, rates were provided by Kohlenbach, Leuştean and the author [8] in the same setting. A further generalization in the single mapping case was introduced in [11] under the name of the alternating Halpern–Mann iteration, proven there to be strongly convergent in CAT(0) spaces, and later also studied in W-hyperbolic spaces in [20].

In this paper, we generalize iteration (1) from Hilbert spaces to the much more abstract setting of W-hyperbolic space and prove that (3) and (2) also hold for our generalization. Furthermore, our proofs are quantitative, providing explicit *rates of asymptotic regularity* for the iteration. Our results can also be viewed as a generalization of those in [9] from single mappings to families of mappings.

The results in this paper are part of the program of *proof mining* [14, 17] developed by Kohlenbach, which seeks to obtain new quantitative results via the proof-theoretical analysis of mathematical proofs.

2. PRELIMINARY NOTIONS

2.1. Quantitative notions

First, let us recall the quantitative notions in terms of which our main results are expressed. Let (a_n) be a sequence in a metric space (X, d) and $a \in X$ be a point. A function $\varphi : \mathbb{N} \to \mathbb{N}$ is a *rate of convergence* for (a_n) to a if

$$\forall k \in \mathbb{N} \forall n \geq \varphi(k)\Big(d(a_n, a) \leq \frac{1}{k+1}\Big).$$

A function $\chi : \mathbb{N} \to \mathbb{N}$ is a *Cauchy modulus* for (a_n) if

$$\forall k \in \mathbb{N} \forall n \geq \chi(k) \forall j \in \mathbb{N}\Big(d(a_n, a_{n+j}) \leq \frac{1}{k+1}\Big).$$

A *rate of asymptotic regularity* for (a_n) is a rate of convergence to 0 for the sequence $(d(x_n, x_{n+1}))$. Given a mapping $T : X \to X$, a rate of T-asymptotic regularity for (x_n) is a rate of convergence to 0 for $(d(x_n, Tx_n))$, and given a family $(T_n : X \to X)$ of self-mappings of X, a rate of (T_n)-asymptotic regularity of (x_n) is a rate of convergence to 0 for the sequence $(d(x_n, T_nx_n))$.

A rate of divergence for a series $\sum_{n=0}^{\infty} b_n$ of nonnegative real numbers is a function $\theta : \mathbb{N} \to \mathbb{N}$ such that

$$\forall k \in \mathbb{N}\left(\sum_{n=0}^{\theta(k)} b_n \geq k\right).$$

2.2. *W*-hyperbolic spaces

Following [9], we say that a *W-space* is a metric space (X, d) endowed with a mapping $W : X \times X \times [0,1] \to X$. The intended interpretation for $W(x, y, \lambda)$ is that of an abstract convex combination of parameter λ between the two points x and y. Hence, instead of W, throughout the paper, we use the notation

$$(1-\lambda)x + \lambda y = W(x, y, \lambda).$$

Definition 2.1. A W-space (X, d, W) is said to be a *W-hyperbolic space* if it satisfies the following axioms, for all $x, y, z, w \in X$ and $\lambda, \theta \in [0,1]$:

(W1) $d(z, (1-\lambda)x + \lambda y) \leq (1-\lambda)d(z, x) + \lambda d(z, y)$,

(W2) $d((1-\lambda)x + \lambda y, (1-\theta)x + \theta y) = |\lambda - \theta|\, d(x, y)$,

(W3) $(1-\lambda)x + \lambda y = \lambda y + (1-\lambda)x$,

(W4) $d((1-\lambda)x + \lambda z, (1-\lambda)y + \lambda w) \leq (1-\lambda)d(x, y) + \lambda d(z, w)$.

Takahashi [24] already studied W-spaces satisfying (W1), while full W-hyperbolic spaces were introduced by Kohlenbach in [13]. Examples of W-hyperbolic spaces include all normed spaces, as well as structures from geodesic geometry, such as Busemann spaces [22] and CAT(0) spaces [1, 6].

Throughout this paper, unless otherwise mentioned, (X, d, W) is a W-hyperbolic space.

PROPOSITION 2.2. *The following hold, for all* $x, y, z, w \in X$ *and for all* $\lambda, \theta \in [0,1]$.

(i) $d(x, (1-\lambda)x + \lambda y) = \lambda d(x, y)$ *and* $d(y, (1-\lambda)x + \lambda y) = (1-\lambda)d(x, y)$;

(ii) $d((1-\lambda)x + \lambda z, (1-\theta)y + \theta w) \leq (1-\lambda)d(x, y) + \lambda d(z, w) + |\lambda - \theta|\, d(y, w)$;

(iii) $d((1-\lambda)x + \lambda z, (1-\theta)x + \theta w) \leq \lambda d(z, w) + |\lambda - \theta|\, d(x, w)$.

Proof. See [9, Lemma 2.1]. □

3. MAIN RESULTS

Let $(T_n : X \to X)$ be a sequence of nonexpansive operators (specifically, $d(T_nx, T_ny) \leq d(x, y)$ for $x, y \in X$), (λ_n) and (β_n) be sequences in $[0, 1]$, and $x_0, u \in X$ be two arbitrary points. We define the *Tikhonov–Mann iteration* associated to the family (T_n) with parameters $(\lambda_n), (\beta_n)$, anchor point u and starting point x_0 by

$$x_{n+1} = (1 - \lambda_n)u_n + \lambda_n T_n u_n, \quad \text{where} \tag{4}$$

$$u_n = (1 - \beta_n)u + \beta_n x_n. \tag{5}$$

If X is a Hilbert space, with the choice $u = 0$, we recover the iteration from [4] and if (T_n) is a constant sequence, we get the Tikhonov–Mann iteration from [9].

We consider the following quantitative conditions on the parameters of the iteration.

(C 1_q) $\prod_{n=0}^{\infty} \beta_{n+1} = 0$ with rate of convergence σ,

(C 2_q) $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n|$ is convergent with Cauchy modulus χ_β,

(C 3_q) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n|$ is convergent with Cauchy modulus χ_λ,

(C 4_q) $\lim_{n\to\infty} \beta_n = 1$ with rate of convergence η,

(C 5_q) $\Lambda \in \mathbb{N}^*$ and $N_\Lambda \in \mathbb{N}$ are such that $\lambda_n \geq \frac{1}{\Lambda}$ for all $n \geq N_\Lambda$,

(C 6_q) $\sum_{n=0}^{\infty} d(T_{n+1}u_n, T_n u_n)$ is convergent with Cauchy modulus χ_T.

These are quantitative analogues of the conditions from [4, Theorem 2.1], with the caveat that the condition that $\sum_{n=0}^{\infty}(1 - \beta_n) = \infty$ used in that paper is replaced with the equivalent, when $\beta_n > 0$, condition that $\prod_{n=0}^{\infty} \beta_{n+1} = 0$. The reason we choose this reformulation is that it allows us to get better rates of asymptotic regularity, as observed first by Kohlenbach [15], who used it to obtain polynomial rates of asymptotic regularity for the Halpern iteration for the first time.

For a mapping $T : X \to X$, let us denote by $\mathrm{Fix}(T) = \{x \in X \mid Tx = x\}$ its set of fixed points. In the rest of this paper, let $F = \bigcap_{n\in\mathbb{N}} \mathrm{Fix}(T_n)$ be the set of common fixed points of the family (T_n), and assume it to be nonempty. The following lemma provides some useful upper bounds on the sequences involved.

LEMMA 3.1. *Let $p \in F$ be a common fixed point and*

$$M = \lceil \max\{d(x_0, p), d(u, p)\} \rceil. \tag{6}$$

Then, for all $n \in \mathbb{N}$, the following bounds hold:

(i) $d(x_n, p) \leq M$ *and* $d(x_n, u) \leq 2M$*;*

(ii) $d(u_n, p) \leq M$ *and* $d(u_n, T_n u_n) \leq 2M$*.*

Proof. All the inequalities are easily proved by adapting the proofs of [9, Lemma 3.1], replacing T with T_n. □

The following lemma establishes the main recursive inequality on (x_n) that allows us to obtain its asymptotic regularity.

LEMMA 3.2. *Let* $p \in F$ *and* M *be defined by* (6)*. For all* $n \in \mathbb{N}$*, the following hold:*

$$d(u_{n+1}, u_n) \leq \beta_{n+1} d(x_{n+1}, x_n) + 2M\,|\beta_{n+1} - \beta_n|\,; \tag{7}$$

$$\begin{aligned} d(x_{n+2}, x_{n+1}) \leq\ & \beta_{n+1} d(x_{n+1}, x_n) + d(T_{n+1}u_n, T_n u_n) \\ & + 2M(|\lambda_{n+1} - \lambda_n| + |\beta_{n+1} - \beta_n|). \end{aligned} \tag{8}$$

Proof. The proofs follow those of [9, Proposition 3.2.(6), (7)]. For (7), we have that, for all $n \in \mathbb{N}$,

$$\begin{aligned} d(u_{n+1}, u_n) &\leq \beta_{n+1} d(x_{n+1}, x_n) + |\beta_{n+1} - \beta_n|\, d(u, x_n) \quad \text{by Proposition 2.2.(iii)} \\ &\leq \beta_{n+1} d(x_{n+1}, x_n) + 2M\,|\beta_{n+1} - \beta_n| \quad \text{from Lemma 3.1.(i).} \end{aligned}$$

For (8), let $n \in \mathbb{N}$. Then

$$\begin{aligned} d(x_{n+2}, x_{n+1}) &\leq (1 - \lambda_{n+1}) d(u_{n+1}, u_n) + \lambda_n d(T_{n+1}u_{n+1}, T_n u_n) \\ &\qquad + |\lambda_{n+1} - \lambda_n|\, d(u_n, T_n u_n) \quad \text{by Lemma 2.2.(ii)} \\ &\leq (1 - \lambda_{n+1}) d(u_{n+1}, u_n) + \lambda_n d(T_{n+1}u_{n+1}, T_n u_n) \\ &\qquad + 2M\,|\lambda_{n+1} - \lambda_n| \quad \text{by Lemma 3.1.(ii)} \\ &\leq (1 - \lambda_{n+1}) d(u_{n+1}, u_n) + 2M\,|\lambda_{n+1} - \lambda_n| \\ &\qquad + \lambda_n (d(T_{n+1}u_{n+1}, T_{n+1}u_n) + d(T_{n+1}u_n, T_n u_n)) \\ &\leq (1 - \lambda_{n+1}) d(u_{n+1}, u_n) + 2M\,|\lambda_{n+1} - \lambda_n| \\ &\qquad + \lambda_n (d(u_{n+1}, u_n) + d(T_{n+1}u_n, T_n u_n)) \\ &\qquad \text{since } T_{n+1} \text{ is nonexpansive} \\ &= d(u_{n+1}, u_n) + \lambda_n d(T_{n+1}u_n, T_n u_n) + 2M\,|\lambda_{n+1} - \lambda_n| \\ &\leq \beta_{n+1} d(x_{n+1}, x_n) + \lambda_n d(T_{n+1}u_n, T_n u_n) \\ &\qquad + 2M(|\lambda_{n+1} - \lambda_n| + |\beta_{n+1} - \beta_n|) \quad \text{from (7)} \\ &\leq \beta_{n+1} d(x_{n+1}, x_n) + d(T_{n+1}u_n, T_n u_n) \\ &\qquad + 2M(|\lambda_{n+1} - \lambda_n| + |\beta_{n+1} - \beta_n|) \\ &\qquad \text{because } 0 \leq \lambda_n \leq 1. \end{aligned}$$

□

The next inequalities are used to derive the (T_n)-asymptotic regularity of (x_n), and their proofs follow [9, Proposition 3.2].

LEMMA 3.3. *For all $n \in \mathbb{N}$, the following hold:*

(i) $d(u_n, T_n x_n) \leq (1-\beta_n) d(u, T_n x_n) + \beta_n d(x_n, T_n x_n)$;

(ii) $d(x_n, T_n x_n) \leq d(x_n, x_{n+1}) + (1-\beta_n) d(u, x_n) + (1-\lambda_n) d(x_n, T_n x_n)$;

(iii) $\lambda_n d(x_n, T_n x_n) \leq d(x_n, x_{n+1}) + 2M(1-\beta_n)$.

Proof.

(i) For all $n \in \mathbb{N}$,

$$\begin{aligned} d(u_n, T_n x_n) &\leq d(u_n, (1-\beta_n)u + \beta_n T_n x_n) + d((1-\beta_n)u + \beta_n T_n x_n, T_n x_n) \\ &\leq d(u_n, (1-\beta_n)u + \beta_n T_n x_n) + (1-\beta_n) d(u, T_n x_n) \\ &\qquad \text{by Proposition 2.2.(i)} \\ &\leq \beta_n d(x_n, T_n x_n) + (1-\beta_n) d(u, T_n x_n) \quad \text{by (W4).} \end{aligned}$$

(ii) For all $n \in \mathbb{N}$,

$$\begin{aligned} d(x_n, T_n x_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T_n x_n) \\ &\leq d(x_n, x_{n+1}) + (1-\lambda_n) d(u_n, T_n x_n) + \lambda_n d(T_n u_n, T_n x_n) \\ &\qquad \text{by (W1)} \\ &\leq d(x_n, x_{n+1}) + (1-\lambda_n) d(u_n, T_n x_n) + \lambda_n d(u_n, x_n) \\ &\qquad \text{by the nonexpansiveness of } T_n \\ &\leq d(x_n, x_{n+1}) + (1-\lambda_n)(1-\beta_n) d(u, T_n x_n) \\ &\qquad + (1-\lambda_n)\beta_n d(x_n, T_n x_n) + \lambda_n d(u_n, x_n) \\ &= d(x_n, x_{n+1}) + (1-\lambda_n)(1-\beta_n) d(u, T_n x_n) \\ &\qquad + (1-\lambda_n)\beta_n d(x_n, T_n x_n) + \lambda_n (1-\beta_n) d(u, x_n) \\ &\qquad \text{by (W4)} \\ &\leq d(x_n, x_{n+1}) + (1-\lambda_n)(1-\beta_n)(d(u, x_n) + d(x_n, T_n x_n)) \\ &\qquad + (1-\lambda_n)\beta_n d(x_n, T_n x_n) + \lambda_n (1-\beta_n) d(u, x_n) \\ &= d(x_n, x_{n+1}) + (1-\beta_n) d(u, x_n) + (1-\lambda_n) d(x_n, T_n x_n). \end{aligned}$$

(iii) Starting from (ii)

$$d(x_n, T_n x_n) \leq d(x_n, x_{n+1}) + (1-\beta_n) d(u, x_n) + (1-\lambda_n) d(x_n, T_n x_n),$$

move the last term to the left-hand side to get that

$$\lambda_n d(x_n, T_n x_n) \leq d(x_n, x_{n+1}) + (1-\beta_n) d(u, x_n).$$

Apply Lemma 3.1.(i) to get the conclusion. □

Finally, before the main theorems of the paper, let us give sufficient conditions for a family (T_n) to satisfy $(\mathrm{C}\,6_q)$.

PROPOSITION 3.4. *Let (γ_n) be a sequence of positive reals satisfying the following conditions:*

$(\mathrm{C}\,7_q)$ $\sum\limits_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n|$ *is convergent with Cauchy modulus* χ_γ*;*

$(\mathrm{C}\,8_q)$ $\Gamma \in \mathbb{N}^*$ *and* $N_\Gamma \in \mathbb{N}$ *are such that* $\gamma_n \geq \frac{1}{\Gamma}$ *for all* $n \geq N_\Gamma$*.*

Suppose the family of operators $(T_n : X \to X)$ satisfies the following condition with respect to (γ_n): for all $x \in X$ and $m, n \in \mathbb{N}$,

$$d(T_m x, T_n x) \leq \frac{|\gamma_m - \gamma_n|}{\gamma_n} d(T_n x, x). \tag{9}$$

Then, (T_n) satisfies condition $(\mathrm{C}\,6_q)$ with χ_T given by

$$\chi_T(k) = \max\left\{N_\Gamma, \chi_\gamma(2M\Gamma(k+1) - 1)\right\}.$$

Proof. Let $k, j \in \mathbb{N}$ and $n \geq \chi_T(k)$. Then,

$$\begin{aligned}
\sum_{i=n+1}^{n+j} d(T_{i+1}u_i, T_i u_i) &\overset{(9)}{\leq} \sum_{i=n+1}^{n+j} \frac{|\gamma_n - \gamma_{i+1}|}{\gamma_i} d(u_i, T_i u_i) \\
&\leq \Gamma \sum_{i=n+1}^{n+j} |\gamma_n - \gamma_{i+1}| d(u_i, T_i u_i) \quad \text{by } (\mathrm{C}\,8_q) \\
&\leq 2M\Gamma \sum_{i=n+1}^{n+j} |\gamma_n - \gamma_{i+1}| \quad \text{by Lemma 3.1.(ii)} \\
&\leq 2M\Gamma \frac{1}{2M\Gamma(k+1)} \quad \text{by } (\mathrm{C}\,7_q) \\
&= \frac{1}{k+1}.
\end{aligned}$$

□

Conditions $(\mathrm{C}\,7_q)$ and $(\mathrm{C}\,8_q)$ are quantitative versions of those imposed in [4, Theorem 3.1] and are used there to derive (9). Reference [18] provides a large class of mappings satisfying Condition (9), introduced under the name of Condition $(C1)$ in that paper: it is shown there that if X is a CAT(0) space and the family (T_n) is *jointly* (P_2) *with respect to* (γ_n), then (T_n) satisfies (9).

3.1. General rates of asymptotic regularity

THEOREM 3.5. *Let $p \in F$ be a common fixed point of (T_n) and M be defined by* (6). *Furthermore, define*

(10)

$$\chi(k) = \max\{\chi_T(2(k+1)-1), \chi_\lambda(8M(k+1)-1), \chi_\beta(8M(k+1)-1)\}.$$

Suppose that conditions ($\mathrm{C}\,1_q$), ($\mathrm{C}\,2_q$), ($\mathrm{C}\,3_q$) *and* ($\mathrm{C}\,6_q$) *are satisfied and that* $\psi_0 : \mathbb{N} \to \mathbb{N}^*$ *is such that*

$$\frac{1}{\psi_0(k)} \le \prod_{n=0}^{\chi(3k+2)} \beta_{n+1}.$$

Then (x_n) *is asymptotically regular with rate*

$$\Sigma(k) = \max\{\sigma(6M(k+1)\psi_0(k)-1), \chi(3k+2)+1\}+1.$$

Proof. We apply Proposition 5.2.(ii) of [9], which is a particular case of quantitative versions of a well-known Lemma by Xu [25] proved in [16, 21], with

$$\begin{aligned} s_n &= d(x_n, x_{n+1}), \\ a_n &= 1-\beta_n, \\ c_n &= d(T_{n+1}u_n, T_n u_n) + 2M(|\lambda_{n+1}-\lambda_n| + |\beta_{n+1}-\beta_n|), \\ L &= 2M. \end{aligned}$$

We proceed to show that the conditions of that proposition are fulfilled.

Claim. χ, as defined by (10), is a Cauchy modulus for $\sum_{n=0}^{\infty} c_n$.

Proof of claim. For brevity, denote $\widehat{c}_n = \sum_{i=0}^{n} c_i$, $\widehat{t}_n = \sum_{i=0}^{n} d(T_{i+1}u_i, T_i u_i)$, $\widehat{\lambda}_n = \sum_{i=0}^{n} |\lambda_{n+1}-\lambda_n|$ and $\widehat{\beta}_n = \sum_{i=0}^{n} |\beta_{n+1}-\beta_n|$, so that

$$\widehat{c}_n = \widehat{t}_n + 2M(\widehat{\lambda}_n + \widehat{\beta}_n).$$

Let $k \in \mathbb{N}$ and $n \ge \chi(k)$ and $j \in \mathbb{N}$. Given the definition of χ and the fact that χ_T, χ_λ and χ_β are Cauchy moduli for $(\widehat{t}_n)$, $(\widehat{\lambda}_n)$ and $(\widehat{\beta}_n)$ respectively, we get that

$$\begin{aligned} \widehat{c}_{n+j} - \widehat{c}_n &= \widehat{t}_{n+j} - \widehat{t}_n + 2M(\widehat{\lambda}_{n+j} - \widehat{\lambda}_n + \widehat{\beta}_{n+j} - \widehat{\beta}_n) \\ &\le \frac{1}{2(k+1)} + 2M\Big(\frac{1}{8M(k+1)} + \frac{1}{8M(k+1)}\Big) \\ &= \frac{1}{k+1}. \end{aligned}$$

□

This claim, together with Lemma 3.2.(8), shows that we are in the conditions to apply [9, Proposition 5.2.(ii)] and obtain our result.

3.2. **General rates of (T_n)-asymptotic regularity**

The following lemma shows that in the presence of conditions $(\mathrm{C}\,4_q)$, $(\mathrm{C}\,5_q)$, the asymptotic regularity of (x_n) also implies its (T_n)-asymptotic regularity, with an explicit translation of rates.

LEMMA 3.6. *Suppose φ is a rate of asymptotic regularity for (x_n) and assume that conditions $(\mathrm{C}\,4_q)$, $(\mathrm{C}\,5_q)$ hold. Then, $\widetilde{\varphi}$ defined by*

$$\widetilde{\varphi}(k) = \max\{N_\Lambda, \varphi(2\Lambda(k+1)-1), \eta(4M\Lambda(k+1)-1)\}$$

is a rate of (T_n)-asymptotic regularity for (x_n).

Proof. Let $k \in \mathbb{N}$ and $n \geq \widetilde{\varphi}(k)$. We need to show that $d(x_n, T_n x_n) \leq \frac{1}{k+1}$.

As $n \geq N_\Lambda$, Condition $(\mathrm{C}\,5_q)$ shows that $\lambda_n > 0$ and that

$$\frac{1}{\lambda_n} \leq \Lambda. \tag{11}$$

Since $n \geq \varphi(2\Lambda(k+1)-1)$, the fact that φ is a rate of asymptotic regularity yields

$$d(x_n, x_{n+1}) \leq \frac{1}{2\Lambda(k+1)}. \tag{12}$$

Finally, because $n \geq \eta(4M\Lambda(k+1)-1)$ and η is a rate of convergence for $\lim_{n\to\infty}(1-\beta_n) = 0$, we get

$$1-\beta_n \leq \frac{1}{4M\Lambda(k+1)}. \tag{13}$$

From Lemma 3.2.(iii), we know that

$$d(x_n, T_n x_n) \leq \frac{1}{\lambda_n} d(x_n, x_{n+1}) + \frac{1}{\lambda_n} 2M(1-\beta_n),$$

which, together with (11), (12) and (13) yields

$$d(x_n, T_n x_n) \leq \Lambda \frac{1}{2\Lambda(k+1)} + \Lambda 2M \frac{1}{4M\Lambda(k+1)} = \frac{1}{k+1}$$

thus, proving the claim. □

THEOREM 3.7. *Suppose conditions $(\mathrm{C}\,1_q)$, $(\mathrm{C}\,2_q)$, $(\mathrm{C}\,3_q)$, $(\mathrm{C}\,4_q)$, $(\mathrm{C}\,5_q)$ and $(\mathrm{C}\,6_q)$ hold. Let Σ be defined as in Theorem 3.5. Then (x_n) is (T_n)-asymptotically regular with rate*

$$\widetilde{\Sigma}(k) = \max\{N_\Lambda, \Sigma(2\Lambda(k+1)-1), \eta(4M\Lambda(k+1)-1)\}. \tag{14}$$

Proof. Apply Lemma 3.6 together with Theorem 3.5. □

Other than the parameters (λ_n), (β_n) of the iteration, the obtained rates of $((T_n)$-)asymptotic regularity depend weakly on the space X and on the mappings (T_n), only via M and χ_T. The rates are less uniform compared to the rates obtained in the single mapping case in [9], where only M is present. Indeed, if (T_n) is constant, then χ_T can simply be taken as $k \mapsto 0$. However, Proposition 3.4 shows that for a large class of mappings, the dependence on χ_T can be reduced to one only on the real parameters (γ_n).

In the following example, we compute explicit rates for a concrete choice of (β_n), (λ_n) and (γ_n). The rates obtained for this example are the same as those obtained for the single mapping case in [9, Corollary 4.3], which demonstrates how under the hypotheses of Proposition 3.4, the additional dependence on the family (T_n) is eliminated.

Example 3.8. Let $\lambda_n = \lambda \in (0,1)$, $\beta_n = 1 - \frac{1}{n+1}$ and $\gamma_n = 1 + \frac{1}{n+1}$ and consider the iteration (x_n) given by (4) with these parameters and let (T_n) be a family of nonexpansive mappings satisfying (9) with respect to (γ_n). Then

(i) (x_n) is asymptotically regular with rate

$$k \mapsto 144M^2(k+1)^2 - 6M(k+1);$$

(ii) (x_n) is (T_n)-asymptotically regular with rate

$$k \mapsto 576M^2 \left\lceil \frac{1}{\lambda} \right\rceil^2 (k+1)^2 - 12M \left\lceil \frac{1}{\lambda} \right\rceil (k+1).$$

Proof. (i) We apply Theorem 3.5. Let us first show that its assumptions are fulfilled. Because

$$\prod_{i=0}^{n} \beta_{n+1} = \frac{1}{n+2}, \quad \sum_{i=0}^{n} |\beta_{i+1} - \beta_i| = 1 - \frac{1}{n+2} \text{ and } \sum_{i=0}^{n} |\lambda_{i+1} - \lambda_i| = 0,$$

we get that $(\mathrm{C}\,1_q)$, $(\mathrm{C}\,2_q)$ and $(\mathrm{C}\,3_q)$ are satisfied, respectively, with

$$\sigma(k) = k, \quad \chi_\beta(k) = k, \quad \chi_\lambda(k) = 0.$$

$(\mathrm{C}\,4_q)$ holds with

$$\eta(k) = k.$$

$(\mathrm{C}\,5_q)$ is satisfied with,

$$N_\Lambda = 0, \quad \Lambda = \left\lceil \frac{1}{\lambda} \right\rceil.$$

Furthermore, as

$$\sum_{i=0}^{n} |\gamma_{i+1} - \gamma_i| = 1 - \frac{1}{n+2},$$

(γ_n) fulfills (C 7_q) and (C 8_q) with

$$\Gamma = 1, \quad N_\Gamma = 0 \quad \text{and} \quad \chi_\gamma(k) = k.$$

Thus, by Proposition 3.4, it follows that (T_n) satisfies (C 6_q) with

$$\chi_T(k) = \max\{N_\Gamma, 2M\Gamma(k+1) - 1\} = 2M(k+1) - 1.$$

It follows that χ defined by (10) is equal in this case to

$$\begin{aligned}\chi(k) &= \max\{\chi_T(2(k+1)-1), \chi_\lambda(8M(k+1)-1), \chi_\beta(8M(k+1)-1)\}\\ &= \max\{4M((k+1)) - 1, 0, 8M(k+1) - 1\}\\ &= 8M(k+1) - 1.\end{aligned}$$

Finally, ψ_0 can be taken as $\psi_0(k) = \chi(3k+2) = 24M(k+1) - 1$. Thus,

$$\begin{aligned}\Sigma(k) &= \max\{\sigma(6M(k+1)\psi_0(k) - 1), \chi(3k+2) + 1\} + 1\\ &= 144M^2(k+1)^2 - 6M(k+1).\end{aligned}$$

(ii) Applying Theorem 3.7 with the rate Σ from (i), we get that

$$\begin{aligned}\widetilde{\Sigma}(k) &= \max\{N_\Lambda, \Sigma(2\Lambda(k+1) - 1), \eta(4M\Lambda(k+1) - 1)\}\\ &= 576M^2 \left\lceil \frac{1}{\lambda} \right\rceil^2 (k+1)^2 - 12M \left\lceil \frac{1}{\lambda} \right\rceil (k+1).\end{aligned}$$ □

3.3. **Linear rates of $((T_n)$-)asymptotic regularity**

In this section, we compute linear rates of asymptotic regularity for iteration (1), by applying a lemma on real numbers introduced by Sabach and Shtern [23], that was originally used there to obtain the linear asymptotic regularity of a viscosity-type Halpern iteration, and was recently employed to the same end for a variety of iterations [8, 10, 19, 20].

The following is a particular case of [23, Lemma 3], as reformulated in [20].

LEMMA 3.9. *Let $L > 0$, and define, for all $n \in \mathbb{N}$, $a_n = \frac{2}{n+2}$. Suppose (s_n) is a sequence of nonnegative reals such that $s_0 \le L$ and that*

(15) $$s_{n+1} \le (1 - a_{n+1})s_n + (a_n - a_{n+1})L$$

for all $n \in \mathbb{N}$. Then,

$$s_n \le \frac{2L}{n+2}$$

for all $n \in \mathbb{N}$.

Theorem 3.10. *Let* $\beta_n = 1 - \frac{2}{n+2}$, $\lambda_n = \lambda \in (0,1)$ *and* $\gamma_n = \frac{n+3}{n+2}$ *and suppose the family* (T_n) *satisfies* (9) *with respect to* (γ_n). *Then, the iteration* (1) *with parameters* (β_n) *and* (λ_n) *satisfies, for all* $n, m \in \mathbb{N}$,

$$d(x_n, x_{n+1}) \leq \frac{6M}{n+2};$$
$$d(x_n, T_n x_n) \leq \frac{10M}{\lambda(n+2)};$$
$$d(x_n, T_m x_n) \leq \frac{20M}{\lambda(n+2)}.$$

Thus, the mappings $k \mapsto 6M(k+1) - 2$, $k \mapsto 10M\left\lceil\frac{1}{\lambda}\right\rceil(k+1) - 2$ *and* $k \mapsto 20M\left\lceil\frac{1}{\lambda}\right\rceil(k+1) - 2$ *are rates of asymptotic,* (T_n)*-asymptotic and* T_m*-asymptotic regularity for* (x_n)*, respectively.*

Proof. For the first inequality, we apply Lemma 3.9 with

$$s_n = d(x_n, x_{n+1}),$$
$$a_n = 1 - \beta_n,$$
$$L = 3M.$$

Note first that

$$|\gamma_{n+1} - \gamma_n| = \frac{1}{(n+2)(n+3)},$$
$$\frac{|\gamma_{n+1} - \gamma_n|}{\gamma_n} = \frac{n+2}{n+3} \cdot \frac{1}{(n+2)(n+3)} = \frac{1}{(n+3)^2}$$

and that

$$\beta_{n+1} - \beta_n = \frac{2}{(n+2)(n+3)}.$$

We now show that the main condition (15) is satisfied. By Lemma 3.2 and (9), for all $n \in \mathbb{N}$,

$$\begin{aligned}
s_{n+1} &\leq \beta_{n+1}s_n + 2M(\beta_{n+1} - \beta_n) + d(T_{n+1}u_n, T_n u_n) \\
&\leq \beta_{n+1}s_n + 2M(\beta_{n+1} - \beta_n) + \frac{|\gamma_{n+1} - \gamma_n|}{\gamma_n} d(T_n u_n, u_n) \\
&\leq \beta_{n+1}s_n + 2M(\beta_{n+1} - \beta_n) + 2M\frac{|\gamma_{n+1} - \gamma_n|}{\gamma_n} \\
&= \beta_{n+1}s_n + (\beta_{n+1} - \beta_n)\Big(2M + 2M\frac{|\gamma_{n+1} - \gamma_n|}{\gamma_n(\beta_{n+1} - \beta_n)}\Big) \\
&= \beta_{n+1}s_n + (\beta_{n+1} - \beta_n)\Big(2M + \frac{2M}{(n+3)^2} \cdot \frac{(n+2)(n+3)}{2}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \beta_{n+1}s_n + (\beta_{n+1} - \beta_n)\Big(2M + M\frac{n+2}{n+3}\Big) \\
&\le \beta_{n+1}s_n + (\beta_{n+1} - \beta_n)L,
\end{aligned}
$$

thus, proving the first claim. For the second claim, using the previous and Lemma 3.3.(iii), we have

$$
\begin{aligned}
d(x_n, T_n x_n) &\le \frac{1}{\lambda} d(x_n, x_{n+1}) + \frac{4M}{\lambda(n+2)} \\
&\le \frac{6M}{\lambda(n+2)} + \frac{4M}{\lambda(n+2)} = \frac{10M}{\lambda(n+2)}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
d(x_n, T_m x_n) &\le d(x_n, T_n x_n) + \frac{|\gamma_n - \gamma_m|}{\gamma_m} d(x_n, T_n x_n) \\
&= d(x_n, T_n x_n) + \frac{|n-m|}{(m+2)(n+3)} d(x_n, T_n x_n) \\
&\le 2d(x_n, T_n x_n) \qquad \square
\end{aligned}
$$

and the claim follows from the previous result.

3.4. **Relation to the modified Halpern iteration**

In this section, we show that the relation between the modified Halpern, originally introduced in [12], and the Tikhonov–Mann iterations studied in [8] for the single mapping case extends to families of mappings: the $((T_n)$-)asymptotic regularity of one iteration implies that of the other, with an explicit translation of rates.

The modified Halpern iteration as defined for W-hyperbolic spaces in [8] can naturally be extended to families of mappings as follows. Define, for all $n \in \mathbb{N}$,

$$
\begin{aligned}
y_{n+1} &= (1 - \beta_{n+1})u + \beta_{n+1} v_n, \quad \text{where} \\
v_n &= (1 - \lambda_n) y_n + \lambda_n T_n y_n.
\end{aligned}
$$

PROPOSITION 3.11. *Assume that* $y_0 = (1 - \beta_0)u + \beta_0 x_0$. *Then, for all* $n \in \mathbb{N}$,

$$
u_n = y_n \quad \textit{and} \quad x_{n+1} = v_n.
$$

Proof. The proof goes by induction on n just like the proof of [8, Proposition 3.2], replacing T with T_n. $\square$

Proposition 3.12. *Assume that* (C 4_q) *holds, that* $y_0 = (1-\beta_0)u+\beta_0 x_0$, *define*

$$\alpha(k) = \eta(2M(k+1)-1),$$

and suppose Σ *is a rate of* $((T_n)$-$)$-*asymptotic regularity for one of the sequences* (x_n) *or* (y_n). *Then, the other sequence is* $((T_n)$-$)$ *asymptotically regular with rate*

$$\Sigma'(k) = \max\left\{\alpha(3k+2), \Sigma(3k+2)\right\}.$$

Proof. Noting that

$$d(x_n, u_n) \overset{\text{Proposition 2.2.(iii)}}{=} (1-\beta_n)d(x_n, u) \overset{\text{Lemma 3.1.(i)}}{\leq} (1-\beta_n)2M,$$

the fact that $\lim_{n\to\infty} d(x_n, u_n) = 0$ with rate of convergence α is proved as in [8, Lemma 4.1]. Using this, the fact that Σ' is a rate of $((T_n)$-$)$asymptotic regularity is proved the same as in [8, Proposition 4.2], replacing T with T_n. □

4. RATES OF ASYMPTOTIC REGULARITY FOR THE TIKHONOV-FORWARD-BACKWARD ALGORITHM WITH VARIABLE STEP-SIZE

Let us first recall some notions from convex optimization and monotone operator theory. Let H be a Hilbert space. Next, for a set-valued operator $A : H \rightrightarrows H$, its graph is $\mathrm{gra}(A) = \{(x, u) \in H \mid u \in Ax\}$ and its set of zeroes is $\mathrm{zer}(A) = \{x \in H \mid 0 \in Ax\}$. Further, given $A : H \rightrightarrows H$, the inverse of A is the operator $A^{-1} : H \rightrightarrows H$, $A^{-1}u = \{x \mid u \in Ax\}$ and the scaling of A by $\gamma \in \mathbb{R}$ is the operator $\gamma A : H \rightrightarrows H$, $(\gamma A)x = \{\gamma u \mid u \in Ax\}$. Given two set-valued operators $A, B : H \rightrightarrows H$, their sum, $A+B : H \rightrightarrows H$, is defined by $(A+B)x = \{u+v \mid u \in Ax, v \in Bx\}$ and their composition, $AB : H \rightrightarrows H$, by $(AB)x = \{v \in H \mid \text{ there exists } u \in Ax \text{ such that } v \in Bu\}$.

We say that $A : H \rightrightarrows H$ is *monotone* if, for all $(x, u), (y, v) \in \mathrm{gra}(A)$, we have that $\langle x-y, u-v\rangle \geq 0$. A is called *maximally monotone* if there exists no other monotone operator $B : H \rightrightarrows H$ such that $\mathrm{gra}(A) \subsetneq \mathrm{gra}(B)$. For a maximally monotone operator A, its *resolvent* of order $\gamma > 0$ is defined by

$$\mathrm{J}_{\gamma A} = (\mathrm{Id} + \gamma A)^{-1}$$

and it is known to be a single-valued, nonexpansive mapping, where the identity mapping is $\mathrm{Id} : H \to H$. A single-valued operator $B : H \to H$ is said to be β-cocoercive for some $\beta > 0$ if, for all $x, y \in H$, $\langle x-y, Bx-By\rangle \geq \beta \|Tx-Ty\|^2$.

The *forward-backward* algorithm is one of the procedures widely employed for finding a point in $\mathrm{zer}(A+B)$ where A is maximally monotone and B is cocoercive. We refer to [2] for a more detailed account of this algorithm and the theory of monotone operators in general.

Based on their strongly convergent Krasnoselskii–Mann iteration for families of mappings, the authors of [4] define a version of the forward-backward algorithm with variable step-size and prove its strong convergence. In this section, we give a generalized version of this iteration based on (4) and compute rates of $((T_n)$-)asymptotic regularity for it.

Let in the following $A : H \rightrightarrows H$ be maximally monotone and $B : H \to H$ be β-cocoercive for some $\beta > 0$. The Tikhonov-forward-backward algorithm with variable step size associated with A and B is defined by

$$\begin{aligned} x_{n+1} &= (1-\lambda_n)u_n + \lambda_n \, \mathrm{J}_{\gamma_n A}(u_n - \gamma_n B u_n), \quad \text{where} \\ u_n &= (1-\beta_n)u + \beta_n x_n, \end{aligned}$$

where $(\beta_n) \subset (0,1]$, $(\gamma_n) \subset (0, 2\beta)$ and $\lambda_n \subset (0, \frac{1}{\alpha_n}]$.

Corollary 4.1. *Assume* $(\mathrm{C}\,1_q)$, $(\mathrm{C}\,2_q)$, $(\mathrm{C}\,3_q)$, $(\mathrm{C}\,7_q)$, $(\mathrm{C}\,8_q)$ *are satisfied. Let* $\chi_T : \mathbb{N} \to \mathbb{N}$ *be defined by:*

$$\chi_T(k) = \max\left\{N_\Gamma, \chi_\gamma(2M\Gamma(k+1)-1)\right\}. \tag{16}$$

Then (x_n) *is asymptotically regular with rate* Σ *as defined in Theorem* 3.5.

Proof. For any $n \in \mathbb{N}$, define $T_n = \mathrm{J}_{\gamma_n A}(\mathrm{Id} - \gamma_n B)$, so that x_n can be written as

$$x_{n+1} = (1-\lambda_n)u_n + \lambda_n T_n u_n.$$

It is known [2, Proposition 26.1.(iv)] that T_n is $\frac{2\beta}{4\beta - \gamma_n}$-averaged, and hence nonexpansive. Lemma 3.2 of [4] then shows that (T_n) satisfies (9) with respect to (γ_n), and thus, by Proposition 3.4, $(\mathrm{C}\,6_q)$ holds with χ_T. All the hypotheses to apply Theorem 3.5 are thus satisfied, and hence the result follows. □

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Research Center for Logic, Optimization and Security (LOS)
Department of Computer Science
Faculty of Mathematics and Computer Science
University of Bucharest
Academiei 14 Street, 010014 Bucharest, Romania
horatiu.cheval@unibuc.ro