TIAN'S THEOREM FOR MOISHEZON SPACES

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Communicated by Lucian Beznea

We prove that the Fubini–Study currents associated to a sequence of singular Hermitian holomorphic line bundles on a compact normal Moishezon space distribute asymptotically as the curvature currents of their metrics.

AMS 2020 Subject Classification: Primary 32L10; Secondary 32A60, 32C20, 32U05, 32U40.

Key words: Bergman kernel function, Fubini–Study current, singular Hermitian metric, normal complex space, Moishezon space.

1. INTRODUCTION

Let (L, h) be a positive Hermitian holomorphic line bundle on a projective manifold X and set $(L^p, h^p) = (L^{\otimes p}, h^{\otimes p})$. Kodaira's embedding theorem states that for all p sufficiently large, the Kodaira map $\Phi_p : X \to \mathbb{P}\big(H^0(X, L^p)^\star\big)$ associated to (L^p, h^p) is an embedding. Hence, one can consider the Fubini–Study forms on X, $\gamma_p = \Phi_p^\star(\omega_{\rm FS})$, where $\omega_{\rm FS}$ denotes the Fubini–Study form on a projective space. A celebrated theorem of Tian (see [33]) shows that $\frac{1}{p}\gamma_p \to c_1(L, h)$ as $p \to \infty$, in the $\mathcal{C}^2$ topology on X (see also [29] for the $\mathcal{C}^\infty$ topology). Tian's theorem follows from the first term asymptotics of the Bergman kernel function associated to the space $H^0(X, L^p)$ endowed with the inner product determined by h^p and a volume form on X. We refer to the book [24] for an exposition of these topics as well as for the full asymptotic expansion of the Bergman kernel in different contexts.

In [7], we extended Tian's theorem to the case when (L, h) is a singular Hermitian holomorphic line bundle with strictly positive curvature current on a compact Kähler manifold X, the above convergence now being in the weak

D. Coman is supported by the NSF Grant DMS-2154273. X. Ma is partially supported by NSFC No. 11829102, ANR-21-CE40-0016 and funded through the Institutional Strategy of the University of Cologne within the German Excellence Initiative. G. Marinescu is partially supported by the DFG funded project CRC TRR 191 "Symplectic Structures in Geometry, Algebra and Dynamics" (Project-ID 281071066-TRR 191), the DFG Priority Program 2265 "Random Geometric Systems" (Project-ID 422743078) and the ANR-DFG project QuaSiDy (Project-ID 490843120).

REV. ROUMAINE MATH. PURES APPL. **69** (2024), *3-4*, 433–444
doi: 10.59277/RRMPA.2024.433.444

sense of currents. Later, we extended Tian's theorem further to general classes of compact Kähler spaces X [8, 9]. In all these situations, one has to replace the space $H^0(X, L^p)$ with the Bergman space $H^0_{(2)}(X, L^p)$ of square integrable holomorphic sections. See [21] for a version of Tian's theorem for CR manifolds.

In [9, Theorem 1.1], we generalized Tian's theorem by considering sequences (L_p, h_p), $p \geq 1$, of singular Hermitian holomorphic line bundles over a compact normal Kähler space X, in place of the sequence of powers (L^p, h^p) of a line bundle (L, h). Assuming that the curvature currents $c_1(L_p, h_p)$ satisfy a natural growth condition, we proved that the Fubini–Study currents γ_p associated to the Bergman spaces $H^0_{(2)}(X, L_p)$ (see (4)) distribute asymptotically like $c_1(L_p, h_p)$.

The purpose of this note is to show that the preceding result holds more generally for compact normal spaces X which are not assumed to be Kähler. The precise setting is the following:

(A) X is a compact, reduced, irreducible, normal complex space of dimension n, $X_{\rm reg}$ denotes the set of regular points of X, $X_{\rm sing}$ denotes the set of singular points of X, and ω is a Hermitian form on X.

(B) (L_p, h_p), $p \geq 1$, is a sequence of holomorphic line bundles on X with singular Hermitian metrics h_p whose curvature currents verify

$$c_1(L_p, h_p) \geq a_p\, \omega \text{ on } X, \text{ where } a_p > 0 \text{ and } \lim_{p\to\infty} a_p = \infty. \tag{1}$$

We let $A_p = \int_X c_1(L_p, h_p) \wedge \omega^{n-1}$ and assume that

$$\exists\, T_0 \in \mathcal{T}(X) \text{ such that } c_1(L_p, h_p) \leq A_p T_0\,,\ \forall\, p \geq 1\,. \tag{2}$$

Condition (B) implies that L_p are big line bundles, hence X is a Moishezon space.

Let $d^c := \frac{1}{2\pi i}(\partial - \overline{\partial})$, so $dd^c = \frac{i}{\pi}\partial\overline{\partial}$. We consider currents on X in the sense of [13], and denote by $\mathcal{T}(X)$ the set of positive closed currents of bidegree $(1,1)$ on X which have local plurisubharmonic (psh) potentials, i.e., $T = dd^c v$ holds in a neighborhood of each point of X for some psh function v. We refer to [9, Section 2.1]) for a review of the notions of differential forms, psh functions and currents on complex spaces. We denote by $\mathrm{PSH}(U)$ the set of psh functions on an open set $U \subset X$. The notions of singular Hermitian metric on a line bundle over a complex space X, and its curvature current, are defined as in the case when X is smooth (see [14], [9, Section 2.2]).

Let $H^0_{(2)}(X, L_p)$ be the Bergman space of L^2-holomorphic sections of L_p relative to the metric h_p and the volume form induced by ω on X,

$$\begin{aligned} H^0_{(2)}(X, L_p) &= H^0_{(2)}(X, L_p, h_p, \omega^n) \\ &= \Big\{ S \in H^0(X, L_p) :\ \|S\|_p^2 := \int_X |S|^2_{h_p} \frac{\omega^n}{n!} < \infty \Big\}, \end{aligned} \tag{3}$$

endowed with the obvious inner product. Let P_p, γ_p be the Bergman kernel function and the Fubini–Study current of the space $H^0_{(2)}(X, L_p)$. They are defined as follows.

Let $S^p_1, \ldots, S^p_{d_p}$ be an orthonormal basis of $H^0_{(2)}(X, L_p)$. If $x \in X$ let e_p be a holomorphic frame of L_p on a neighborhood U_p of x and write $S^p_j = s^p_j e_p$ with $s^p_j \in \mathcal{O}_X(U_p)$. Then

$$P_p(x) = \sum_{j=1}^{d_p} |S^p_j(x)|^2_{h_p}\,, \ \ \gamma_p|_{U_p} = \frac{1}{2}\, dd^c \log\Big(\sum_{j=1}^{d_p} |s^p_j|^2\Big). \tag{4}$$

We have that P_p, γ_p are independent of the choice of basis. Moreover, $\gamma_p = \Phi_p^\star(\omega_{\rm FS})$, where $\Phi_p : X \dashrightarrow \mathbb{P}\big(H^0_{(2)}(X, L_p)^\star\big)$ is the (meromorphic) Kodaira map associated to the Bergman space $H^0_{(2)}(X, L_p)$.

Our main result is the following theorem.

THEOREM 1.1. *Assume that* $X, \omega, (L_p, h_p)$, $p \geq 1$, *satisfy conditions* (A)-(B). Then the following hold:

(i) $\frac{1}{A_p} \log P_p \to 0$ as $p \to \infty$, in $L^1(X, \omega^n)$.

(ii) $\frac{1}{A_p} \left(\gamma_p - c_1(L_p, h_p)\right) \to 0$ as $p \to \infty$, in the weak sense of currents on X.

Note that a complex space X that verifies (A)-(B) is a *Moishezon space.* Thus, Theorem 1.1 applies to any compact normal Moishezon space X, which is not necessarily assumed to be Kähler. Indeed, a singular Hermitian holomorphic line bundle (L_p, h_p) over X with strictly positive curvature current as in (1) is big, hence X is Moishezon (see, e.g., [9, Proposition 2.3], [4, Propositions 3.2 and 3.3]). We recall that a (reduced) compact irreducible complex space X of dimension n is called a Moishezon space if there exist n algebraically independent meromorphic functions on X (see [34, Definition 3.5], [4, Section 3]). We refer to [4, Section 3] and the references therein for the definition and some basic properties of big line bundles over complex spaces.

Theorem 1.1 is proved in Section 2. An important special case is provided by the sequence of powers $(L_p, h_p) = (L^p, h^p)$ of a singular Hermitian holomorphic line bundle (L, h) with strictly positive curvature current. See Theorem 3.1 in Section 3, which gives a full generalization of Tian's theorem to the singular setting. We recall in Section 3 a few other important applications of Theorem 1.1, in particular to the asymptotic distribution of the zeros of random sequences of holomorphic sections (see Theorem 3.3).

2. PROOF OF THEOREM 1.1

By a theorem of Moishezon ([25], [34, Theorem 3.6]), X is bimeromorphically equivalent to a projective manifold. More precisely, since X is assumed to be normal, we have that $\operatorname{codim} X_{\text{sing}} \geq 2$ and the following holds (see [4, Theorem 3.1]).

THEOREM 2.1. *If X is a compact, irreducible, normal Moishezon space then there exists a connected projective manifold $\widetilde{X}$ and a surjective holomorphic map $\pi : \widetilde{X} \to X$, given as a composition of finitely many blow-ups with smooth center, such that $\pi : \widetilde{X} \setminus \Sigma \to X \setminus Y$ is a biholomorphism, where Y is an analytic subset of X, $\operatorname{codim} Y \geq 2$, $X_{\text{sing}} \subset Y$, and $\Sigma = \pi^{-1}(Y)$ is a normal crossings divisor.*

Let $X, \omega, (L_p, h_p)$ verify assumptions (A)-(B) and $\pi : \widetilde{X} \to X$ be as in Theorem 2.1. In [9], we assumed that X is a normal Kähler space, and we showed that the desingularization $\widetilde{X}$ obtained by finitely many blow-ups with smooth centers as in [6, 19] is Kähler. This is crucial for the construction of peak sections by using methods involving $\overline{\partial}$. In our present situation, we obtain a projective desingularization $\widetilde{X}$ since X is Moishezon.

We follow the arguments from the proof of [9, Theorem 1.1], working with $\pi : \widetilde{X} \to X$ instead of the desingularization of X given in [9, Section 2.3], and using a Kähler form $\widetilde{\omega}$ on the projective manifold $\widetilde{X}$. We recall the following lemmas that are needed in the proof.

LEMMA 2.2 ([9, Lemma 2.1]). *If*

$$H^0_{(2)}(\widetilde{X}, \pi^\star L_p) = \Big\{ \widetilde{S} \in H^0(\widetilde{X}, \pi^\star L_p) : \int_{\widetilde{X}} |\widetilde{S}|^2_{\pi^\star h_p} \frac{\pi^\star \omega^n}{n!} < \infty \Big\},$$

the map $\pi^\star : H^0_{(2)}(X, L_p) \longrightarrow H^0_{(2)}(\widetilde{X}, \pi^\star L_p)$ is an isometry and the Bergman kernel function of $H^0_{(2)}(\widetilde{X}, \pi^\star L_p)$ is $\widetilde{P}_p = P_p \circ \pi$.

LEMMA 2.3 ([9, Lemma 2.2, Lemma 3.2]). *There exist $\alpha \in (0,1)$, $b_p \in \mathbb{N}$, a Hermitian form Ω on $\widetilde{X}$, and singular Hermitian metrics $\widetilde{h}_p$ on $\pi^\star L_p|_{\widetilde{X} \setminus \Sigma}$ such that $\Omega \geq \pi^\star \omega$, $b_p \to \infty$ and $b_p / A_p \to 0$ as $p \to \infty$, $\widetilde{h}_p \geq \alpha^{b_p} \pi^\star h_p$ and $c_1(\pi^\star L_p, \widetilde{h}_p) \geq b_p \Omega$ on $\widetilde{X} \setminus \Sigma$. Moreover, for every relatively compact open subset $\widetilde{U}$ of $\widetilde{X} \setminus \Sigma$ there exists a constant $\beta_{\widetilde{U}} > 1$ such that $\widetilde{h}_p \leq \beta_{\widetilde{U}}^{b_p} \pi^\star h_p$ on $\widetilde{U}$.*

The Hermitian form Ω is obtained as $\Omega = C\pi^\star \omega + c_1(F, \theta)$, where θ is a suitable metric on $F = \mathcal{O}_{\widetilde{X}}(-\Sigma)$ and $C > 0$ is an appropriate constant. If $b_p \in \mathbb{N}$ is a sequence such that $b_p \to \infty$, $a_p \geq C b_p$, $b_p / A_p \to 0$ and if φ is a weight of θ on $\widetilde{X} \setminus \Sigma$, one defines the metric $\widetilde{h}_p = e^{-2 b_p \varphi} \pi^\star h_p$ on $\pi^\star L_p|_{\widetilde{X} \setminus \Sigma}$

and shows that it has the desired properties. In particular, the positivity of $c_1(\pi^\star L_p, \widetilde{h}_p)$ is needed to solve a $\overline{\partial}$-equation on $\widetilde{X} \setminus \Sigma$, by using the following version of Demailly's estimates for the $\overline{\partial}$-operator [12, Théorème 5.1] (see also [9, Theorem 2.5]).

THEOREM 2.4. *Let Z, $\dim Z = n$, be a complete Kähler manifold and Θ be a Kähler form on Z (not necessarily complete) such that its Ricci form $\mathrm{Ric}_\Theta \geq -2\pi B\Theta$ for some constant $B > 0$. Let (L_p, h_p) be singular Hermitian holomorphic line bundles on Z such that $c_1(L_p, h_p) \geq 2a_p\Theta$, where $a_p \geq B$. If $g \in L^2_{0,1}(Z, L_p, \mathrm{loc})$ verifies $\overline{\partial} g = 0$ and $\int_Z |g|^2_{h_p}\,\Theta^n < \infty$ then there exists $u \in L^2_{0,0}(Z, L_p, \mathrm{loc})$ such that $\overline{\partial} u = g$ and $\int_Z |u|^2_{h_p}\,\Theta^n \leq \frac{1}{a_p} \int_Z |g|^2_{h_p}\,\Theta^n$.*

Proof of Theorem 1.1. By (4), we have that $\log P_p \in L^1(X, \omega^n)$ and

$$\gamma_p - c_1(L_p, h_p) = \frac{1}{2}\, dd^c \log P_p.$$

Thus, (ii) follows at once from (i). To prove (i), we proceed in two steps.

Step 1. We prove that $\frac{1}{A_p} \log P_p \to 0$ as $p \to \infty$, in $L^1_{\mathrm{loc}}(X \setminus Y, \omega^n)$. Fix $x \in X \setminus Y \subset X_{\mathrm{reg}}$, $W \Subset X \setminus Y$ a contractible Stein coordinate neighborhood of x, $r_0 > 0$ such that the (closed) ball $V := B(x, 2r_0) \subset W$, and set $U = B(x, r_0)$. Note that the currents $\{\frac{1}{A_p}\, c_1(L_p, h_p)\}$ have uniformly bounded mass. By [17, Proposition A.16] (see also [18] and [22, Theorem 3.2.12]), we infer that there exist psh functions ψ_p on $\mathrm{int}\, V$ such that $dd^c\psi_p = c_1(L_p, h_p)$ and the sequence $\{\frac{1}{A_p}\, \psi_p\}$ is relatively compact in $L^1_{\mathrm{loc}}(\mathrm{int}\, V, \omega^n)$. Since $L_p|_W$ is holomorphically trivial, we can find holomorphic frames e_p for $L_p|_{\mathrm{int}\, V}$ such that ψ_p are the corresponding psh weights of h_p, so $|e_p|_{h_p} = e^{-\psi_p}$.

Let $\widetilde{\omega}$ be a Kähler form on $\widetilde{X}$, and Ω be the Hermitian form from Lemma 2.3. Then there exists constants $\delta_1, \delta_2 > 0$ such that

$$\Omega \geq \delta_1 \widetilde{\omega}\,,\; \widetilde{\omega} \geq \delta_2 \Omega \geq \delta_2 \pi^\star \omega. \tag{5}$$

With $\{b_p\}$ as in Lemma 2.3, we prove that there exist $C_1 > 1$ and $p_0 \in \mathbb{N}$ such that

$$-\frac{b_p \log C_1}{A_p} \leq \frac{\log P_p(z)}{A_p} \leq \frac{\log(C_1 r^{-2n})}{A_p} + \frac{2}{A_p}\Big(\max_{B(z,r)} \psi_p - \psi_p(z)\Big) \tag{6}$$

holds for all $p > p_0$, $0 < r < r_0$, and $z \in U$ with $\psi_p(z) > -\infty$. The upper bound in (6) follows from the subaverage inequality, exactly as the upper bound from [7, (7)].

We show next that there exist $c \in (0, 1)$ and $p_0 \in \mathbb{N}$ with the following property: if $p > p_0$ and $z \in U$ is such that $\psi_p(z) > -\infty$, then there exists

$S_{z,p} \in H^0_{(2)}(X, L_p)$ with $S_{z,p}(z) \neq 0$ and

$$c^{b_p} \|S_{z,p}\|_p^2 \leq |S_{z,p}(z)|^2_{h_p}. \tag{7}$$

This yields the lower bound in (6), since $P_p(z) \geq |S_{z,p}(z)|^2_{h_p} / \|S_{z,p}\|^2_p \geq c^{b_p}$. To this end, we work first on $\pi^\star L_p|_{\widetilde{X}\setminus\Sigma}$ using the metric $\widetilde{h}_p$ from Lemma 2.3. By estimates (5),

$$c_1\big(\pi^\star L_p|_{\widetilde{X}\setminus\Sigma}, \widetilde{h}_p\big) \geq b_p\Omega \geq \delta_1 b_p \widetilde{\omega} \text{ on } \widetilde{X}\setminus\Sigma, \text{ where } b_p \to \infty.$$

We have that $\widetilde{X}\setminus\Sigma$ has a complete Kähler metric [12, 26], and that $\mathrm{Ric}_{\widetilde{\omega}} \geq -2\pi B\widetilde{\omega}$ on $\widetilde{X}$ for some $B > 0$. Using ideas from [15, Proposition 3.1], [16, Section 9], we apply the Ohsawa–Takegoshi extension theorem [27] and Theorem 2.4 as in the proof of [7, Theorem 5.1] to show that there exist $C_2 > 1$, $p_0 \in \mathbb{N}$, such that if $p > p_0$ and $\widetilde{z} \in \pi^{-1}(U)$, $\psi_p \circ \pi(\widetilde{z}) > -\infty$, then there is $\widetilde{S} \in H^0(\widetilde{X}\setminus\Sigma, \pi^\star L_p)$ verifying $\widetilde{S}(\widetilde{z}) \neq 0$ and

$$\int_{\widetilde{X}\setminus\Sigma} |\widetilde{S}|^2_{\widetilde{h}_p} \frac{\widetilde{\omega}^n}{n!} \leq C_2 |\widetilde{S}(\widetilde{z})|^2_{\widetilde{h}_p}.$$

By Lemma 2.3 and (5), we obtain

$$\delta_2^n \alpha^{b_p} \int_{\widetilde{X}\setminus\Sigma} |\widetilde{S}|^2_{\pi^\star h_p} \frac{\pi^\star\omega^n}{n!} \leq C_2 \beta^{b_p} |\widetilde{S}(\widetilde{z})|^2_{\pi^\star h_p}, \tag{8}$$

where $\beta > 1$ is so that $\widetilde{h}_p \leq \beta^{b_p}\pi^\star h_p$ on $\pi^{-1}(U)$. As $\pi : \widetilde{X}\setminus\Sigma \to X\setminus Y$ is a biholomorphism, we let $z = \pi(\widetilde{z})$ and $S_{z,p}$ be the section of $L_p|_{X\setminus Y}$ induced by $\widetilde{S}$. Since X is normal and $\operatorname{codim} Y \geq 2$, $S_{z,p}$ extends to a holomorphic section on X and (7) follows from (8).

Recall that $\{\frac{1}{A_p}\psi_p\}$ is relatively compact in $L^1_{\mathrm{loc}}(\operatorname{int} V, \omega^n)$, hence it is locally uniformly upper bounded in $\operatorname{int} V$. It follows from (6) that there is a constant $C_3 > 0$ such that

$$\Big|\frac{1}{A_p}\log P_p\Big| \leq C_3 - \frac{2}{A_p}\psi_p \ \text{ a.e. on } U,\ \forall\, p > p_0. \tag{9}$$

Moreover, if a subsequence $\frac{1}{A_{p_j}}\psi_{p_j} \to \psi$ in $L^1_{\mathrm{loc}}(\operatorname{int} V, \omega^n)$ and, a.e., on $\operatorname{int} V$, where ψ is psh on $\operatorname{int} V$, we infer from equation (6) and the Hartogs lemma [22, Theorem 3.2.13] that

$$0 \leq \liminf \frac{\log P_{p_j}(z)}{A_{p_j}} \leq \limsup \frac{\log P_{p_j}(z)}{A_{p_j}} \leq 2\Big(\max_{B(z,r)} \psi - \psi(z)\Big)$$

holds for, a.e., $z \in U$ and every $r < r_0$. Thus, $\frac{1}{A_{p_j}}\log P_{p_j} \to 0$, a.e., on U, and hence in $L^1(U, \omega^n)$ by (9) and the generalized Lebesgue dominated convergence theorem. We conclude that $\frac{1}{A_p}\log P_p \to 0$ as $p \to \infty$ in $L^1_{\mathrm{loc}}(X\setminus Y, \omega^n)$.

Step 2. We finish the proof of (i) by showing that there exists a compact set $K \subset X$ such that $Y \subset \operatorname{int} K$ and $\frac{1}{A_p} \log P_p \to 0$ in $L^1(K, \omega^n)$. Let $H^0_{(2)}(\widetilde{X}, \pi^\star L_p)$ be the Bergman spaces from Lemma 2.2. It follows by (2) that there exists $M > 0$ such that

$$\int_{\widetilde{X}} c_1(\pi^\star L_p, \pi^\star h_p) \wedge \Omega^{n-1} \leq M A_p\,, \quad \forall\, p \geq 1\,. \tag{10}$$

Let $y \in \Sigma$. By (10), we can proceed as in Step 1 to find an open neighborhood $\widetilde{W}$ of y and holomorphic frames $\widetilde{e}_p$ of $\pi^\star L_p|_{\widetilde{W}}$ with corresponding psh weights $\widetilde{\psi}_p$ of $\pi^\star h_p$, such that the sequence $\{\frac{1}{A_p}\, \widetilde{\psi}_p\}$ is relatively compact in $L^1_{\text{loc}}(\widetilde{W}, \Omega^n)$. Let $\{\widetilde{S}^p_j : 1 \leq j \leq d_p\}$ be an orthonormal basis of $H^0_{(2)}(\widetilde{X}, \pi^\star L_p)$ and $\widetilde{S}^p_j = \widetilde{s}^p_j \widetilde{e}_p$, with $\widetilde{s}^p_j \in \mathcal{O}_{\widetilde{X}}(\widetilde{W})$. By Lemma 2.2,

$$\frac{1}{A_p}\, \widetilde{v}_p - \frac{1}{A_p}\, \widetilde{\psi}_p = \frac{1}{2A_p} \log P_p \circ \pi, \text{ where } \widetilde{v}_p = \frac{1}{2} \log\Bigg(\sum_{j=1}^{d_p} |\widetilde{s}^p_j|^2 \Bigg) \in \mathrm{PSH}(\widetilde{W}).$$

We claim that $\frac{1}{A_p} \log P_p \circ \pi \to 0$ in $L^1_{\text{loc}}(\widetilde{W}, \Omega^n)$. Indeed, assume that a subsequence $\{\frac{1}{A_{p_j}}\, \widetilde{\psi}_{p_j}\}$ converges in $L^1_{\text{loc}}(\widetilde{W}, \Omega^n)$ to a psh function $\widetilde{\psi}$ on $\widetilde{W}$. By Step 1, $\frac{1}{A_p} \log P_p \circ \pi \to 0$ in $L^1_{\text{loc}}(\widetilde{W} \setminus \Sigma, \Omega^n)$, hence $\frac{1}{A_{p_j}}\, \widetilde{v}_{p_j} \to \widetilde{\psi}$ in $L^1_{\text{loc}}(\widetilde{W} \setminus \Sigma, \Omega^n)$. It follows that $\{\frac{1}{A_{p_j}}\, \widetilde{v}_{p_j}\}$ is locally uniformly upper bounded in $\widetilde{W}$ and $\frac{1}{A_{p_j}}\, \widetilde{v}_{p_j} \to \widetilde{\psi}$ in $L^1_{\text{loc}}(\widetilde{W}, \Omega^n)$. This proves our claim.

Since Σ is compact, we infer by the above that there exists a compact set $K \subset X$ such that $Y \subset \operatorname{int} K$ and $\frac{1}{A_p} \log P_p \circ \pi \to 0$ in $L^1(\pi^{-1}(K), \Omega^n)$. Then

$$\begin{aligned}\frac{1}{A_p} \int_K |\log P_p|\, \omega^n &= \frac{1}{A_p} \int_{\pi^{-1}(K)} |\log P_p \circ \pi|\, \pi^\star \omega^n \\ &\leq \frac{1}{A_p} \int_{\pi^{-1}(K)} |\log P_p \circ \pi|\, \Omega^n \to 0\end{aligned}$$

as $p \to \infty$, and the proof is finished. □

3. APPLICATIONS

In the case of the sequence of powers of a single line bundle, Theorem 1.1 yields the following generalization of Tian's theorem to the setting of big line bundles on Moishezon spaces.

THEOREM 3.1. *Let X be a compact, reduced, irreducible, normal complex space of dimension n and (L, h) be a singular Hermitian holomorphic line* *bundle on X such that $c_1(L,h) \geq \varepsilon\omega$, where $\varepsilon > 0$ is a constant and ω is a Hermitian form on X. If P_p, γ_p are the Bergman kernel function and Fubini–Study current of $H^0_{(2)}(X, L^p, h^p, \omega^n)$ then, as $p \to \infty$,*

$$\frac{1}{p}\log P_p \to 0 \text{ in } L^1(X,\omega^n),\ \frac{1}{p}\gamma_p \to c_1(L,h) \text{ weakly on } X.$$

Proof. If $\|c_1(L,h)\| = \int_X c_1(L,h) \wedge \omega^{n-1}$, then the assumptions (A)-(B) hold with

$$a_p = p\,\varepsilon,\ A_p = p\,\|c_1(L,h)\|,\ T_0 = c_1(L,h)/\|c_1(L,h)\|. \qquad \square$$

Recall that a Kähler current is a positive closed current T of bidegree $(1,1)$ on X such that $T \geq \varepsilon\omega$ for some constant $\varepsilon > 0$. Let (L,h) be a singular Hermitian holomorphic line bundle on X with positive curvature current $c_1(L,h) \geq 0$, and such that L has a singular Hermitian metric h_0 whose curvature is a Kähler current. As in [9, Corollary 5.2], Theorem 1.1 can be applied to the sequence of line bundles $(L^p, h^{p-n_p} \otimes h_0^{n_p})$, where $n_p \in \mathbb{N}$ and $n_p \to \infty$, $n_p/p \to 0$ as $p \to \infty$. One can also apply Theorem 1.1 to the sequence of tensor products of powers of several line bundles as in [9, Corollary 5.11]. We refer to [9, Section 5] for the details.

Let us consider now the special case when X is smooth, i.e., a connected compact complex manifold of dimension n. If X is assumed to be Kähler then the domination condition (2) is not needed as one can work directly on X without the use of a modification π. More precisely, in [9] we proved the following.

THEOREM 3.2 ([9, Theorem 1.2]). *Let (X,ω) be a compact Kähler manifold of dimension n and (L_p, h_p), $p \geq 1$, be a sequence of singular Hermitian holomorphic line bundles on X which satisfy $c_1(L_p,h_p) \geq a_p\,\omega$, where $a_p > 0$ and $a_p \to \infty$. If P_p, γ_p are the Bergman kernel function and Fubini–Study current of $H^0_{(2)}(X, L_p)$, and if $A_p = \int_X c_1(L_p,h_p) \wedge \omega^{n-1}$, then $\frac{1}{A_p}\log P_p \to 0$ in $L^1(X,\omega^n)$ and $\frac{1}{A_p}(\gamma_p - c_1(L_p,h_p)) \to 0$ weakly on X.*

However, if X is a Moishezon manifold which is not Kähler, and hence not projective, we still have to use in our proof of Theorem 1.1 the modification $\pi : \widetilde{X} \to X$ provided in Theorem 2.1. So, we have to require the domination condition (2) in assumption (B).

One of the main applications of Tian's theorem is to the study of the asymptotic distribution of zeros of random sequences of sections in $H^0(Z, L^p)$ as $p \to \infty$, where L is a holomorphic line bundle over a compact complex manifold Z. This started with the pioneering work of Shiffman and Zelditch [31] in the case of a positive line bundle (L,h) over a projective manifold Z (see also [32, 30]). It is shown in [31] that for almost all sequences $\{\sigma_p \in H^0(Z, L^p)\}_{p\geq 1}$, with respect to the spherical measure, one has that $\frac{1}{p}\,[\sigma_p = 0] \to c_1(L,h)$ weakly on Z, where $[s = 0]$ denotes the current of integration over the zero divisor of a holomorphic section s. In the case of singular Hermitian holomorphic line bundles, we proved that similar results hold in different contexts [7, 8, 10, 11].

The study of the asymptotic distribution of zeros of random sections in the Bergman spaces $H^0_{(2)}(X, L_p)$ for an arbitrary sequence of singular Hermitian holomorphic line bundles (L_p, h_p) over a compact normal Kähler space X was pursued in [9, 3]. In particular, we considered in [3, Theorem 1.1] very general probability measures on the spaces $H^0_{(2)}(X, L_p)$, as follows. We identify the spaces $H^0_{(2)}(X, L_p) \simeq \mathbb{C}^{d_p}$ using fixed orthonormal bases $S^p_1, \dots, S^p_{d_p}$ and we endow them with probability measures σ_p such that the following holds:

(C) There exist a constant $\nu \geq 1$ and for every $p \geq 1$ constants $C_p > 0$ such that

$$\int_{\mathbb{C}^{d_p}} \big|\log|\langle a, u\rangle|\,\big|^\nu\, d\sigma_p(a) \leq C_p\,, \text{ for any } u \in \mathbb{C}^{d_p} \text{ with } \|u\| = 1\,.$$

Note that [3, Theorem 1.1] holds in our present context. Indeed, we can apply the general equidistribution result [3, Theorem 4.1] together with Theorem 1.1. We recall one of its assertions here.

THEOREM 3.3. *Assume that* $X, \omega, (L_p, h_p), \sigma_p$ *verify* (A), (B), (C) *and consider the product probability space*

$$(\mathcal{H}, \sigma) = \left(\prod_{p=1}^{\infty} H^0_{(2)}(X, L_p), \prod_{p=1}^{\infty} \sigma_p\right).$$

If $\sum_{p=1}^{\infty} C_p A_p^{-\nu} < \infty$ *then for* σ*-a.e. sequence* $\{s_p\} \in \mathcal{H}$ *we have, as* $p \to \infty$,

$$\frac{1}{A_p}\log|s_p|_{h_p} \to 0 \text{ in } L^1(X, \omega^n),\ \ \frac{1}{A_p}\big([s_p = 0] - c_1(L_p, h_p)\big) \to 0 \text{ weakly on } X.$$

We refer to [3, 2] for general classes of measures σ_p that satisfy condition (C), including Gaussians, Fubini–Study volumes, and area measure of spheres. Note that if the measures σ_p verify (C) with constants $C_p = \Gamma_\nu$ independent of p (like the Gaussians and the Fubini–Study volumes) then the hypothesis of Theorem 3.3 becomes $\sum_{p=1}^{\infty} A_p^{-\nu} < \infty$.

We close the paper with some remarks on Moishezon manifolds. By a theorem of Moishezon, a Moishezon manifold is projective if and only if it carries a Kähler metric, see [25] and [24, Theorem 2.2.26]. Moreover, any Moishezon manifold of dimension two is projective, by Theorem 2.1. Indeed, in dimension two we can blow up only points and the blow-up $\widehat{X}$ at a point of a compact manifold X is projective if and only if X is projective. Hence, non-projective Moishezon manifolds have dimension greater than two. The first example of this kind was obtained by Hironaka in his thesis (1961) and is described in [20, Appendix B, Example 3.4.1]. It is a manifold which contains a curve which is homologous to zero, which is impossible on a Kähler manifold. Further examples can be found in [1, 5, 23, 28], see also [24, Section 2.3.4].

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