SPECTRAL REGULARITY WITH RESPECT TO DILATIONS FOR A CLASS OF PSEUDO-DIFFERENTIAL OPERATORS

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We continue the study of the perturbation problem discussed in H. D. Cornean and R. Purice (2023) and get rid of the "slow variation" assumption by considering symbols of the form $a(x+\delta\,F(x),\xi)$ with a a real Hörmander symbol of class $S^0_{0,0}(\mathbb{R}^d\times\mathbb{R}^d)$ and F a smooth function with all its derivatives globally bounded, with $|\delta|\leq 1$. We prove that while the Hausdorff distance between the spectra of the Weyl quantization of the above symbols in a neighbourhood of $\delta=0$ is still of the order $\sqrt{|\delta|}$, the distance between their spectral edges behaves like $|\delta|^\nu$ with $\nu\in[1/2,1)$ depending on the rate of decay of the second derivatives of F at infinity.

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1. INTRODUCTION

In [2], we investigated how the spectrum of a class of ΨDO changes (seen as a subset of the real line) with respect to a family of slowly varying dilation-type perturbations. More precisely, we worked with symbols of the form $a(x+F(\delta\,x),\xi)$ with a a real Hörmander symbol of class $S^0_{0,0}(\mathbb{R}^d\times\mathbb{R}^d)$ and F a smooth function with all its derivatives globally bounded, and with $0<|\delta|\leq 1$. The motivation came from a related problem discussed in [3] in which F was an affine function. In this note, we present some results that may be obtained when one eliminates the "slow variation" hypothesis and the perturbed symbols are of the form $a(x+\delta\,F(x),\xi)$ instead of $a(x+F(\delta\,x),\xi)$. We note that when F is affine as in [3], the two problems are essentially the same.

We use the multi-index conventions of [4]. Let:

$$\nu_{n,m}(a):=\max_{|\alpha|\leq n}\max_{|\beta|\leq m}\sup_{(x,\xi)\in\mathbb{R}^{2d}}|\partial^\alpha_x\partial^\beta_\xi a|,\quad\forall(n,m)\in\mathbb{N}\times\mathbb{N}\tag{1.1}$$

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and $S^0_{0,0}(\mathbb{R}^d \times \mathbb{R}^d)$ [4] the set of smooth functions satisfying

$$\nu_{n,m}(a) < \infty, \quad \forall (n,m) \in \mathbb{N} \times \mathbb{N}. \tag{1.2}$$

We consider some real Hörmander symbol $a(x,\xi)$ of class $S^0_{0,0}(\mathbb{R}^d \times \mathbb{R}^d)$.

We denote by $(\cdot,\cdot)_{L^2(\mathbb{R}^d)}$ the scalar product in $L^2(\mathbb{R}^d)$ (considered antilinear in the first variable), with the quadratic norm denoted simply by $\|\cdot\|$ and we use the notation $\langle\cdot,\cdot\rangle_{\mathscr{S}(\mathcal{V})} : \mathscr{S}'(\mathcal{V}) \times \mathscr{S}(\mathcal{V}) \to \mathbb{C}$ for the canonical bilinear duality map for tempered distributions on the real finite dimensional Euclidean space $\mathcal{V}$.

Following Hörmander [4], we define the Weyl quantization of the symbol $a \in S^0_{0,0}(\mathbb{R}^d \times \mathbb{R}^d)$ as the operator:

$$\big(\mathfrak{Op}^w(a)\varphi\big)(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} d\eta \, e^{i<\eta, x-y>} \, a\big((x+y)/2, \eta\big)\, \varphi(y), \quad \forall \varphi \in \mathscr{S}(\mathbb{R}^d),\ \forall x \in \mathbb{R}^d. \tag{1.3}$$

Due to the Calderon–Vaillancourt Theorem (see [1] and [6], §XIII.1) this operator is bounded in $L^2(\mathbb{R}^d)$ with the following bound on the operator norm:

$$\big\|\mathfrak{Op}^w(a)\varphi\big\|_{\mathbb{B}(L^2(\mathbb{R}^d))} \le C\,\nu_{3d+4,3d+4}(a). \tag{1.4}$$

We use the same notation for its extension to the entire Hilbert space. Let $\mathfrak{K}[a] \in \mathscr{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ be the distribution kernel of $\mathfrak{Op}^w(a)$ (see [5]); it may be computed by the following formula:

$$\mathfrak{K}[a] := (2\pi)^{-d/2}\big((\mathbb{1} \otimes \mathcal{F}^-)a\big) \circ \Upsilon \tag{1.5}$$

where $\Upsilon : \mathbb{R}^d \times \mathbb{R}^d \ni (x,y) \mapsto ((x+y)/2, x-y) \in \mathbb{R}^d \times \mathbb{R}^d$ is a bijection with Jacobian -1 and:

$$(\mathcal{F}^- \varphi)(v) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} d\xi \, e^{i<\xi, v>} \, \varphi(\xi), \quad \forall \varphi \in \mathscr{S}(\mathbb{R}^d),\ \forall v \in \mathbb{R}^d \tag{1.6}$$

is the inverse Fourier transform. We also define the distribution $\widetilde{\mathfrak{K}}[a] := \mathfrak{K}[a] \circ \Upsilon^{-1} \in \mathscr{S}'(\mathbb{R}^d \times \mathbb{R}^d)$. With a slight abuse, we can write the following explicit formula:

$$\mathfrak{K}[a](z+v/2, z-v/2) \equiv \widetilde{\mathfrak{K}}[a](z,v) := (2\pi)^{-d} \int_{\mathbb{R}^d} d\eta \, e^{i<\eta, v>} \, a(z,\eta). \tag{1.7}$$

Proposition 1.1 ([4]). *The tempered distribution $\widetilde{\mathfrak{K}}[a] \in \mathscr{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ is in fact a smooth (with respect to the weak topology) distribution valued function $\mathbb{R}^d \ni z \mapsto \widetilde{\mathfrak{K}}[a](z,\cdot) \in \mathscr{S}'(\mathbb{R}^d_v)$ such that for any $z \in \mathbb{R}^d$ the distribution $\widetilde{\mathfrak{K}}[a](z,\cdot) \in \mathscr{S}'(\mathbb{R}^d_v)$ has singular support contained in $\{v = 0\}$ (possibly void) and rapid decay together with all its derivatives, in the complement of $v = 0$.*

$L^2(\mathbb{R}^d)$-boundedness criterion. Given a distribution kernel $\widetilde{\mathfrak{K}}[a]$ as in the above Proposition 1.1 and using the operator-norm estimation in the Calderon–Vaillancourt Theorem for its associated Hörmander symbol $a = (\mathbb{1}\otimes\mathcal{F})\widetilde{\mathfrak{K}}[a] \in S^0_{0,0}(\mathbb{R}^d\times\mathbb{R}^d)$, our main criterion for $L^2(\mathbb{R}^d)$-boundedness of the associated operator in $L^2(\mathbb{R}^d)$ is the boundedness of at least one of the seminorms:

$$\nu_{n,m}\big((\mathbb{1}\otimes\mathcal{F})\widetilde{\mathfrak{K}}\big), \quad \min(n,m)\geq 3d+4. \tag{1.8}$$

Notation 1.2. We use the following notations:

- $< v >:= \sqrt{1+|v|^2}$, for any $v\in\mathbb{R}^d$ and $\mathfrak{s}_p(v) :=< v >^p$ for any $p\in\mathbb{R}$.
- τ_z for the translation with $-z\in\mathbb{R}^d$ on any space of functions or distributions on $\mathbb{R}^d$.
- $C^\infty_1(\mathbb{R}^d;\mathbb{R}^d)$ defined as the space of smooth $\mathbb{R}^d$-valued functions with bounded derivatives of all strictly positive orders.

The Problem. Let $F\in C^\infty_1(\mathbb{R}^d;\mathbb{R}^d)$ and $\delta\in\mathbb{R}$ with $|\delta|\leq 1$. To any real-valued symbol $a\in S^0_{0,0}(\mathbb{R}^d\times\mathbb{R}^d)$, we associate the perturbed symbols:

$$a[F]_\delta(x,\xi) \;:=\; a\big(x+\delta\,F(x),\xi\big).$$

We are interested in the variation of the spectrum $\sigma(\mathfrak{Op}^w(a[F]_\delta))\subset\mathbb{R}$, as a set, when δ goes to 0.

Remark 1.3. We have the inequalities:

$$\nu_{n,m}(a[F]_\delta) \;\leq\; C_n(\delta,F)\nu_{n,m}(a), \quad \forall(n,m)\in\mathbb{N}\times\mathbb{N},$$

with $C_n(\delta,F)$ depending on the sup-norm of the derivatives of F up to order $n-1$, uniformly in $\delta\in(0,1]$.

We use the short-hand notations (for $|\delta|\leq 1$):

$$K_\delta := \mathfrak{Op}^w(a[F]_\delta)\in\mathbb{B}\big(L^2(\mathbb{R}^d)\big), \quad \mathfrak{K}_\delta := \widetilde{\mathfrak{K}}\big[a[F]_\delta\big]\in\mathscr{S}'(\mathbb{R}^d\times\mathbb{R}^d); \tag{1.9}$$

$$\mathcal{E}_+(\delta) := \sup\,\sigma(K_\delta). \tag{1.10}$$

The Hausdorff distance. $\mathbf{d_H}(A,B) := \max\{\sup_{t\in A}\mathsf{dist}(t,B), \sup_{t\in B}\mathsf{dist}(t,A)\}$ for A,B subsets of $\mathbb{C}$.

2. THE MAIN RESULTS

THEOREM 2.1. *There exists $C(a,F)>0$ such that for $|\delta|\leq 1$, we have the estimation:*

$$\mathbf{d_H}\big(\sigma(K_\delta)\,,\,\sigma(K_0)\big)\;\leq\;C(a,F)\sqrt{|\delta|}.$$

Remark 2.2. Examples from the literature show that this estimation is “sharp”, i.e., spectral gaps of order $\sqrt{|\delta|}$ may be created by these type of perturbations.

From Theorem 1.5 in [2] and some other similar results from the literature, one may expect a more regular behaviour of the spectral edges seen as functions of δ. By spectral edges we understand quantities like $\sup\sigma(K_\delta)$, $\inf\sigma(K_\delta)$, or the extremities of the possible inner gaps in the spectrum. In this situation, we obtain the following result depending on the decay at infinity of the second order derivatives of the “perturbing function” $F\in C^\infty_1(\mathbb{R}^d;\mathbb{R}^d)$.

THEOREM 2.3. *Suppose that $|(\partial_{x_j}\partial_{x_k}F)(x)|\leq C<x>^{-(1+\mu)}$ for some $C>0$, $\mu>0$ and for any pair of indices (j,k). Then there exists $C(a,F)>0$ and $\delta_0>0$ such that for $|\delta|\leq\delta_0$, we have the estimation:*

$$\big|\,\mathcal{E}_\pm(\delta)\;-\;\mathcal{E}_\pm(0)\,\big|\;\leq\;C(a,F)|\delta|^{(1+\mu)/(2+\mu)}.$$

3. PROOF OF THEOREM 2.1

The main idea of the proof is to construct a “quasi-resolvent” (see (3.4)) and use the unitarity of x-translations and localization around a lattice of points in $\mathbb{R}^d_x$ in order to control the possible linear growth of F. We notice that the invariance of our arguments when changing F into $-F$ allows us to work with $\delta\geq 0$.

Let us consider some exponent $\kappa\in(0,1]$ and the discrete family of points

$$\Gamma_\delta:=\{z_\gamma(\delta):=\delta^\kappa\gamma\in\mathbb{R}^d,\ \gamma\in\mathbb{Z}^d\}.$$

We notice that for any $\mathring{z}\in\Gamma_\delta$, the bounded operator $\tau_{-\mathring{z}}K_0\tau_{\mathring{z}}$ has the integral kernel $\mathfrak{K}_0(z+\mathring{z},v)$, with $\mathfrak{K}_0(z,v)$ given in (1.9). Thus, given some $\gamma\in\mathbb{Z}^d$ let us consider the difference: $K_\delta-\tau_{-z_\gamma(\delta)}K_0\tau_{z_\gamma(\delta)}$ and its associated distribution kernel, considered as smooth distribution valued function on $\mathbb{R}^d$ and use Newton–Leibniz formula in the first variable to obtain:

(3.1)

$$\begin{aligned}\mathfrak{K}_\delta(z,\cdot)-\mathfrak{K}_0(z+\delta^\kappa\gamma,\cdot)&=\mathfrak{K}_0\big(z+\delta\,F(z),\cdot\big)-\mathfrak{K}_0(z+\delta^\kappa\gamma,\cdot)\\&=\int_0^1 ds\,\big((\nabla_z\mathfrak{K}_0)\big(z+\delta^\kappa\gamma+s(\delta\,F(z)-\delta^\kappa\gamma),\cdot\big)\big)\cdot\big(\delta\,F(z)-\delta^\kappa\gamma\big)\\&=:\;\delta^\kappa\,[\mathfrak{D}_1\mathfrak{K}_0](z,\cdot)\cdot\big(\delta^{1-\kappa}F(z)-\gamma\big)\end{aligned}$$

with the last line giving the definition of $[\mathfrak{D}_1\mathfrak{K}_0](z,\cdot)$. We can then define the mapping

$$\boldsymbol{\Psi}_s[F]^{(\delta)}_\gamma:\mathbb{R}^d\ni\ z\mapsto\ z+\delta^\kappa\gamma+s(\delta\,F(z)-\delta^\kappa\gamma)\in\mathbb{R}^d$$

and write that in the sense of tempered distributions:

$$\begin{aligned}[\mathfrak{D}_1\mathfrak{K}_0]&=\int_0^1 ds\,(\nabla_z\mathfrak{K}_0)\circ\big(\boldsymbol{\Psi}_s[F]^{(\delta)}_\gamma,\mathbb{1}\big)\\&=(2\pi)^{-d/2}\int_0^1 ds\,\Big(\big((\mathbb{1}\otimes\mathcal{F}^-)(\nabla_z a)\big)\circ\big(\boldsymbol{\Psi}_s[F]^{(\delta)}_\gamma,\mathbb{1}\big)\Big)\\&=(2\pi)^{-d/2}(\mathbb{1}\otimes\mathcal{F}^-)\Big[\int_0^1 ds\,\big((\nabla_z a)\circ\big(\boldsymbol{\Psi}_s[F]^{(\delta)}_\gamma,\mathbb{1}\big)\big)\Big],\end{aligned}$$

with $\mathbb{1}:\mathbb{R}^d\to\mathbb{R}^d$ being the identity map, denoting by (Ψ,Φ) the map

$$\mathbb{R}^d\times\mathbb{R}^d\ni(x,y)\mapsto\big(\Psi(x),\Psi(y)\big)\in\mathbb{R}^d\times\mathbb{R}^d$$

for any pair of maps $\Psi\in C^\infty(\mathbb{R}^d;\mathbb{R}^d)$ and $\Phi\in C^\infty(\mathbb{R}^d;\mathbb{R}^d)$. The above formula implies that:

$$(\mathbb{1}\otimes\mathcal{F})[\mathfrak{D}_1\mathfrak{K}_0]=(2\pi)^{-d/2}\Big[\int_0^1 ds\,\big((\nabla_z a)\circ\big(\boldsymbol{\Psi}_s[F]^{(\delta)}_\gamma,\mathbb{1}\big)\big)\Big].\tag{3.2}$$

In order to estimate the operator norm of the linear operator defined by this distribution kernel, we use our boundedness criterion (1.8) and notice that:

$$\partial^\alpha_z\partial^\beta_\xi(\mathbb{1}\otimes\mathcal{F})[\mathfrak{D}_1\mathfrak{K}_0]=(2\pi)^{-d/2}\Big[\int_0^1 ds\,\big((\nabla_z\partial^\alpha_z\partial^\beta_\xi\,a)\circ\big(\boldsymbol{\Psi}_s[F]^{(\delta)}_\gamma,\mathbb{1}\big)\big)\Big]\tag{3.3}$$

being bounded by $\nu_{|\alpha|+1,|\beta|}(a)$. Thus, if we can impose by some localization procedure, a bound uniform in $(z,\gamma)\in\mathbb{R}^d\times\mathbb{Z}^d$ for the factor $\delta^{1-\kappa}F(z)-\gamma$ and its z-derivatives then, we may obtain a decaying factor δ^κ going to 0 with $\delta\geq 0$. We are thus led to consider the following partition of unity:

- We fix a function $g\in C^\infty_0(\mathbb{R}^d;[0,1])$ such that: $\sum_{\gamma\in\mathbb{Z}^d}g(z-\gamma)^2=1$, for all $z\in\mathbb{R}^d$.
- For any $\gamma\in\mathbb{Z}^d$, we define the cut-off function:
$$g[F_\delta]_\gamma(z):=g\big(\delta^{(1-\kappa)}F(z)-\gamma\big).$$
- Given $\gamma\in\mathbb{Z}^d$, we denote by V_γ the set of all $\gamma'\in\mathbb{Z}^d$ with the property that the support of $g[F_\delta]_{\gamma'}$ has a non-empty overlap with the support of $g[F_\delta]_\gamma$, including $\gamma'=\gamma$. Denote by $\mathfrak{n}_g\in\mathbb{N}\setminus\{0\}$ the cardinal of V_γ, notice that it is clearly independent of γ and δ and that:
$$\sum_{\gamma\in\mathbb{Z}^d}\big[g[F_\delta]_\gamma(z)\big]^2=1,\ \forall z\in\mathbb{R}^d,$$

$$z \in \operatorname{supp} g[F_\delta]_\gamma \implies \exists L > 0,\ |\delta^{(1-\kappa)} F(z) - \gamma| \leq L.$$

- Finally, let us denote by $G[F_\delta]_\gamma$ the self-adjoint, bounded operator of multiplication with $g[F_\delta]_\gamma$ in $L^2(\mathbb{R}^d)$. Obviously, $G[F_\delta]_\gamma = \mathfrak{Op}^w(g[F_\delta]_\gamma)$ for $g[F_\delta]_\gamma \in S^0_{0,0}(\mathbb{R}^d \times \mathbb{R}^d)$ a symbol independent of the second variable.

The quasi-resolvent for K_δ. Let us fix any $\mathfrak{z} \notin \sigma\big(K_0\big)$ and define:
(3.4)

$$T_\gamma(\mathfrak{z};\delta) := \tau_{-z_\gamma(\delta)}\,(K_0 - \mathfrak{z}\mathbb{1})^{-1}\tau_{z_\gamma(\delta)},\quad \widetilde{T}(\mathfrak{z};\delta) := \sum_{\gamma\in\mathbb{Z}^d} G[F_\delta]_\gamma\, T_\gamma(\mathfrak{z};\delta)\, G[F_\delta]_\gamma.$$

Remark 3.1. Unitarity of translations and the functional calculus for self-adjoint operators imply that for any $\delta \in [0,1]$, we have the estimation:

$$\big\|T_\gamma(\mathfrak{z};\delta)\big\|_{\mathbb{B}(L^2(\mathbb{R}^d))} = \big\|(K_0 - \mathfrak{z}\mathbb{1})^{-1}\big\|_{\mathbb{B}(L^2(\mathbb{R}^d))} \leq \mathsf{dist}\big(\mathfrak{z},\sigma(K_0)\big)^{-1},\ \forall\gamma\in\mathbb{Z}^d.$$

LEMMA 3.2. *For any $\delta \in [0,1]$ the series in* (3.4) *is convergent in the strong operator topology and we have the estimation*

$$\begin{aligned}\big\|\widetilde{T}(\mathfrak{z};\delta)\big\|_{\mathbb{B}(L^2(\mathbb{R}^d))} &\leq \sqrt{(\mathfrak{n}_g+1)/2}\,\big\|\big(K_0 - \mathfrak{z}\mathbb{1}\big)^{-1}\big\|_{\mathbb{B}(L^2(\mathbb{R}^d))}\\ &\leq \sqrt{(\mathfrak{n}_g+1)/2}\,\mathsf{dist}\big(\mathfrak{z},\sigma(K_0)\big)^{-1}.\end{aligned}$$

Proof. For the convenience of the reader, we reproduce here our proof of Lemma 2.4 in [2]. For $\psi \in L^2(\mathbb{R}^d)$, we fix $M \in \mathbb{N}$, we define:

$$\widetilde{T}(\mathfrak{z};\delta)^{(M)} := \sum_{|\gamma|\leq M} G[F_\delta]_\gamma\, T_\gamma(\mathfrak{z};\delta)\, G[F_\delta]_\gamma$$

and compute:

$$\begin{aligned}&\sum_{|\gamma|\leq M}\sum_{\gamma'\in V_\gamma}\big\langle G[F_\delta]_\gamma\, T_\gamma(\mathfrak{z};\delta)\, G[F_\delta]_\gamma\,\psi\,,\, G[F_\delta]_{\gamma'}\, T_{\gamma'}(\mathfrak{z};\delta)\, G[F_\delta]_{\gamma'}\,\psi\big\rangle\\ &\quad\leq \frac{\nu+1}{2}\sum_{|\gamma|\leq M}\|T_{\gamma'}(\mathfrak{z};\delta)\, G[F_\delta]_{\gamma'}\,\psi\|^2\\ &\quad\leq \big\|(K_0 - \mathfrak{z}\mathbb{1})^{-1}\big\|^2_{\mathbb{B}(L^2(\mathbb{R}^d))}\frac{\mathfrak{n}_g+1}{2}\sum_{|\gamma|\leq M}\int_{\mathbb{R}^d} dz\,\big[g[F_\delta]_\gamma(z)\big]^2|\psi(z)|^2\,dx\\ &\quad\leq \big\|(K_0 - \mathfrak{z}\mathbb{1})^{-1}\big\|^2_{\mathbb{B}(L^2(\mathbb{R}^d))}\,\frac{\mathfrak{n}_g+1}{2}\,\|\psi\|^2,\end{aligned}$$

where in the last equality we used the quadratic partition of unity identity in the definition of $g \in C_0^\infty(\mathbb{R}^d;[0,1])$. The convergence and the estimation in the Lemma follow. □

Proposition 3.3. *With the above notations and assumptions, we have:* $\big\|(K_\delta-\mathfrak{z}\mathbb{1})\,\widetilde{T}(\mathfrak{z};\delta)\;-\;\mathbb{1}\big\|_{\mathbb{B}(L^2(\mathbb{R}^d))}\;\leq\;C(a,F)\,\delta^{1/2}\,\big[\mathsf{dist}\big(\mathfrak{z},\sigma(K_0)\big)\big]^{-1}.$

Proof. For any $M\in\mathbb{N}$, let $\widetilde{T}(\mathfrak{z};\delta)^{(M)}$ as in the begining of the proof of Lemma 3.2 be the partial sum approaching $\widetilde{T}(\mathfrak{z};\delta)$ in the strong operator topology and let us consider the product:

$$(K_\delta-\mathfrak{z}\mathbb{1})\,\widetilde{T}(\mathfrak{z};\delta)^{(M)}=\sum_{|\gamma|\leq M}(K_\delta-\mathfrak{z}\mathbb{1})\,G[F_\delta]_\gamma\,T_\gamma(\mathfrak{z};\delta)\,G[F_\delta]_\gamma. \tag{3.5}$$

We notice that for any $\gamma\in\mathbb{Z}^d$, we can write that:

$$\begin{aligned}
&(K_\delta-\mathfrak{z}\mathbb{1})\,G[F_\delta]_\gamma\,T_\gamma(\mathfrak{z};\delta)\,G[F_\delta]_\gamma\\
&=\tau_{-z_\gamma(\delta)}\,(K_0-\mathfrak{z}\mathbb{1})^{-1}\tau_{z_\gamma(\delta)}\,G[F_\delta]_\gamma\,T_\gamma(\mathfrak{z};\delta)\,G[F_\delta]_\gamma\\
&\quad+\big[(K_\delta-\mathfrak{z}\mathbb{1})\;-\;\tau_{-z_\gamma(\delta)}\,(K_0-\mathfrak{z}\mathbb{1})^{-1}\tau_{z_\gamma(\delta)}\big]\,G[F_\delta]_\gamma\,T_\gamma(\mathfrak{z};\delta)\,G[F_\delta]_\gamma\\
&=\big[g[F_\delta]_\gamma\big]^2\,\mathbb{1}+\big[\tau_{-z_\gamma(\delta)}\,(K_0-\mathfrak{z}\mathbb{1})^{-1}\tau_{z_\gamma(\delta)}\,,\,G[F_\delta]_\gamma\big]T_\gamma(\mathfrak{z};\delta)\,G[F_\delta]_\gamma\\
&\quad+\big[(K_\delta-\mathfrak{z}\mathbb{1})\;-\;\tau_{-z_\gamma(\delta)}\,(K_0-\mathfrak{z}\mathbb{1})^{-1}\tau_{z_\gamma(\delta)}\big]\,G[F_\delta]_\gamma\,T_\gamma(\mathfrak{z};\delta)\,G[F_\delta]_\gamma.
\end{aligned} \tag{3.6}$$

Lemma 3.4. *For any $\gamma\in\mathbb{Z}^d$, we have the estimation:*

$$\big\|\big[(K_\delta-\mathfrak{z}\mathbb{1})\;-\;\tau_{-z_\gamma(\delta)}\,(K_0-\mathfrak{z}\mathbb{1})^{-1}\tau_{z_\gamma(\delta)}\big]\,G[F_\delta]_\gamma\big\|_{\mathbb{B}(L^2(\mathbb{R}^d))}\;\leq\;C(a,F)\delta^\kappa.$$

Proof. We consider the bounded operator:

$$H_\delta:=\big[(K_\delta-\mathfrak{z}\mathbb{1})\;-\;\tau_{-z_\gamma(\delta)}\,(K_0-\mathfrak{z}\mathbb{1})^{-1}\tau_{z_\gamma(\delta)}\big]\,G[F_\delta]_\gamma \tag{3.7}$$

appearing in the statement of Lemma 3.4 and compute its distribution kernel:

$$\mathfrak{K}[H_\delta]=\big[\mathfrak{K}\big[a[F]_\delta\big]-\big(\mathfrak{K}[a]\circ(\tau_{z_\gamma(\delta)}\otimes\tau_{z_\gamma(\delta)})\big)\big]\big(1\otimes g[F_\delta]_\gamma\big).$$

We prefer to work with the modified kernels:

$$\begin{aligned}
\check{\mathfrak{K}}_\delta&=\mathfrak{K}[H_\delta]\circ\Upsilon^{-1}\\
&=\big[\big(\mathfrak{K}\big[a[F]_\delta\big]-\big(\mathfrak{K}[a]\circ(\tau_{z_\gamma(\delta)}\otimes\tau_{z_\gamma(\delta)})\big)\big)\circ\Upsilon^{-1}\big]\big[\big(1\otimes g[F_\delta]_\gamma\big)\circ\Upsilon^{-1}\big]\\
&=\big(\mathfrak{K}_\delta-\big(\mathfrak{K}_0\circ(\tau_{z_\gamma(\delta)}\otimes 1)\big)\big)\big[\big(1\otimes g[F_\delta]_\gamma\big)\circ\Upsilon^{-1}\big]\\
&=\big[\big(\mathfrak{K}_\delta-\big(\mathfrak{K}_0\circ(\tau_{z_\gamma(\delta)}\otimes 1)\big)\big)(1\otimes\mathfrak{s}_N)\big]\big[\big[\big(1\otimes g[F_\delta]_\gamma\big)\circ\Upsilon^{-1}\big](1\otimes\mathfrak{s}_{-N})\big],
\end{aligned} \tag{3.8}$$

with the last line valid for any $N\in\mathbb{N}$ and the first factor above being bounded for any $N\in\mathbb{N}$ due to the arguments using (3.3). In fact, by (3.1) we can write:

$$\begin{aligned}
(\mathbb{1}\otimes\mathcal{F})\check{\mathfrak{K}}_\delta=\delta^\kappa\Big[\sum_{1\leq j\leq d}&\big((\mathbb{1}\otimes\mathcal{F})[\mathfrak{D}_1\mathfrak{K}_0]_j\big)\star\big((\mathbb{1}\otimes\mathcal{F})\big[(\delta^{1-\kappa}F_j-\gamma_j)\otimes\mathfrak{s}_N\big]\big)\Big]\\
&\star\big[(\mathbb{1}\otimes\mathcal{F})\big(\big[\big(1\otimes g[F_\delta]_\gamma\big)\circ\Upsilon^{-1}\big](1\otimes\mathfrak{s}_{-N})\big)\big]
\end{aligned}$$

where $\star$ denotes the bilinear operation:

$$(F \star G)(x,\xi) = \int d\eta\, F(x,\eta) G(x,\xi-\eta).$$

Concerning the second factor above, we notice that:

$$\big[(1 \otimes g[F_\delta]_\gamma) \circ \Upsilon^{-1}\big](z,v) = g\big(\delta^{(1-\kappa)}\, F(z - v/2) - \gamma\big) \tag{3.9}$$

and using the compactness of the support of the cut-off function g, we deduce that on the support of the function $(\mathbb{1} \otimes \mathcal{F})([(1 \otimes g[F_\delta]_\gamma) \circ \Upsilon^{-1}](1 \otimes \mathfrak{s}_{-N}))$ there exists some $L > 0$, depending only on the diameter of the support of g such that:

$$\begin{aligned} L &\geq \big|\delta^{(1-\kappa)} F(z - v/2) - \gamma\big| \\ &= \Big|\delta^{(1-\kappa)}\Big(F(z) - \int_0^1 ds\, \big[(v/2)\cdot(\nabla F)(z - sv/2)\big]\Big) - \gamma\Big| \\ &\geq \Big|\big|\delta^{(1-\kappa)} F(z) - \gamma\big| - \Big|\delta^{(1-\kappa)} \int_0^1 ds\, \big[(v/2)\cdot(\nabla F)(z - sv/2)\big]\Big|\Big| \end{aligned}$$

and thus we have the inequality:

$$\begin{aligned} \big|\delta^{(1-\kappa)} F(z) - \gamma\big| \;&\leq\; L + \delta^{(1-\kappa)} \Big|\int_0^1 ds\, \big[(v/2)\cdot\big(\nabla F\big)(z - sv/2)\big]\Big| \\ &\leq\; L + \delta^{(1-\kappa)}\big((1/2)\|\nabla F\|_\infty\big) < v > . \end{aligned}$$

We notice that the function $([(1 \otimes g[F_\delta]_\gamma) \circ \Upsilon^{-1}](1 \otimes \mathfrak{s}_{-N}))$ is of class $BC^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ having rapid decay in the second variable, with uniform bounds with respect to $\delta \in [0,1]$, so that its partial Fourier transform

$$(\mathbb{1} \otimes \mathcal{F})\big(\big[\big(1 \otimes g[F_\delta]_\gamma\big) \circ \Upsilon^{-1}\big](1 \otimes \mathfrak{s}_{-N})\big)$$

is a function of class $BC^\infty(\mathbb{R}^d \times \mathbb{R}^d) = S^0_{0,0}(\mathbb{R}^d \times \mathbb{R}^d)$, uniformly with respect to $\delta \in [0,1]$.

Recalling now our boundedness criterion (1.8):

$$\begin{aligned} &\big\|\big[(K_\delta - \mathfrak{z}\mathbb{1}) \,-\, \tau_{-z_\gamma(\delta)}\,(K_0 - \mathfrak{z}\mathbb{1})^{-1}\tau_{z_\gamma(\delta)}\big]\, g[F_\delta]_\gamma\big\|_{\mathbb{B}(L^2(\mathbb{R}^d))} \\ &\qquad \leq \nu_{3d+4,3d+4}\big((\mathbb{1} \otimes \mathcal{F})\check{\mathfrak{K}}_\delta\big), \end{aligned}$$

the conclusion of Lemma 3.4 follows. $\square$

Lemma 3.5. *For any $\gamma \in \mathbb{Z}^d$, we have the estimation:*

$$\big\|\big[\tau_{-z_\gamma(\delta)}\,(K_0 - \mathfrak{z}\mathbb{1})^{-1}\tau_{z_\gamma(\delta)}\,,\, G[F_\delta]_\gamma\big]\big\|_{\mathbb{B}(L^2(\mathbb{R}^d))} \leq C(a,F)\,\delta^{(1-\kappa)}.$$

Proof. In a similar way, with the proof of our previous Lemma 3.4, we consider the linear operator:

$$\begin{aligned}(3.10)\quad &\big[\tau_{-z_\gamma(\delta)}\,(K_0-\mathfrak{z}\mathbb{1})^{-1}\tau_{z_\gamma(\delta)}\,,\,G[F_\delta]_\gamma\big]\\ &\quad=\tau_{-z_\gamma(\delta)}\,(K_0-\mathfrak{z}\mathbb{1})^{-1}\tau_{z_\gamma(\delta)}\,,\,G[F_\delta]_\gamma\;-\;G[F_\delta]_\gamma\,\tau_{-z_\gamma(\delta)}\,\big(K_0-\mathfrak{z}\mathbb{1}\big)^{-1}\tau_{z_\gamma(\delta)}\end{aligned}$$

and its distribution kernel:

$$\mathfrak{K}_{C,\delta}\;:=\;\big(\mathfrak{K}[a]\circ(\tau_{z_\gamma(\delta)}\otimes\tau_{z_\gamma(\delta)})\big)\big[\big(1\otimes g[F_\delta]_\gamma\big)\;-\;\big(g[F_\delta]_\gamma\otimes 1\big)\big]$$

with the modified form:

$$\begin{aligned}\widetilde{\mathfrak{K}}_{C,\delta}:&=\mathfrak{K}_{C,\delta}\circ\Upsilon^{-1}\\ &=\big[\mathfrak{K}_0\circ(\tau_{z_\gamma(\delta)}\otimes\mathbb{1})\big]\big[\big(\big(1\otimes g[F_\delta]_\gamma\big)\circ\Upsilon^{-1}\big)\;-\;\big(\big(g[F_\delta]_\gamma\otimes 1\big)\circ\Upsilon^{-1}\big)\big].\end{aligned}$$

Let us analyse the smooth function in the second factor above:

$$\begin{aligned}(3.11)\quad &\big[\big(\big(1\otimes g[F_\delta]_\gamma\big)\circ\Upsilon^{-1}\big)\;-\;\big(\big(g[F_\delta]_\gamma\otimes 1\big)\circ\Upsilon^{-1}\big)\big](z,v)\\ &=g\big(\delta^{(1-\kappa)}F(z-v/2)-\gamma\big)\;-\;g\big(\delta^{(1-\kappa)}F(z+v/2)-\gamma\big)\\ &=-\int_0^1 ds\,\big((\nabla g)\big(\delta^{(1-\kappa)}F(z-v/2)-\gamma+s\delta^{(1-\kappa)}\\ &\qquad\times\big(F(z+v/2)-F(z-v/2)\big)(\delta^{(1-\kappa)})\big(F(z+v/2)-F(z-v/2)\big)\\ &=-\delta^{(1-\kappa)}\sum_{1\leq j,k\leq d}\int_0^1 ds\int_{-1/2}^{1/2}dt\,v_k\,\partial_k F_j(z+sv)\,r\\ &\qquad\times\;\big(\partial_j\,g\big)\big(\delta^{(1-\kappa)}F(z-v/2)-\gamma+s\delta^{(1-\kappa)}\big(F(z+v/2)-F(z-v/2)\big)\end{aligned}$$

and our boundedness criterion clearly implies the conclusion of Lemma 3.5. □

Putting together (3.6), Remark 3.1 and the above two lemmas, and optimizing the estimation by taking $\kappa=1-\kappa=1/2$, we conclude that:

$$(3.12)\qquad \begin{aligned}&(K_\delta-\mathfrak{z}\mathbb{1})\,G[F_\delta]_\gamma\,T_\gamma(\mathfrak{z};\delta)\,G[F_\delta]_\gamma-\big[g[F_\delta]_\gamma\big]^2\,\mathbb{1}\;=\;X^{(\delta)}_\gamma\,G[F_\delta]_\gamma,\\ &\big\|X^{(\delta)}_\gamma\big\|_{\mathbb{B}(L^2(\mathbb{R}^d))}\;\leq\;C(a,F)\,\delta^{1/2}\,\big(\mathsf{dist}\big(\mathfrak{z},\sigma(K_0)\big)\big)^{-1}.\end{aligned}$$

Finally, we have to use the fact that

$$\sum_{\gamma\in\mathbb{Z}^d}g(x-\gamma)^2=1\text{ and }\sum_{\gamma\in\mathbb{Z}^d}g(x-\gamma)\in[0,\mathfrak{n}_g],$$

both series being locally finite, so that the finite sums in (3.5) are convergent and summing up over $\gamma\in\mathbb{Z}^d$ using the estimation (3.12) clearly implies the conclusion of Proposition 3.3. □

End of the proof of Theorem 2.1. If $\mathsf{dist}(\mathfrak{z},\sigma(\mathfrak{Op}(a)))\geq C\delta^{1/2}$, the conclusion of Proposition 3.3 implies that $\mathfrak{z}\notin\sigma(K_\delta)$.

Finally, replacing $\widetilde{T}(\mathfrak{z};\delta)$ in (3.4) by:

$$\widetilde{S}(\mathfrak{z};\delta):=\sum_{\gamma\in\mathbb{Z}^d}G[F_\delta]_\gamma\,\tau_{z_\gamma(\delta)}\,(K_\delta-\mathfrak{z}\mathbb{1})^{-1}\tau_{-z_\gamma(\delta)}\,G[F_\delta]_\gamma \tag{3.13}$$

all the arguments above can still be applied in order to obtain the following estimation similar to the conclusion of Proposition 3.3:

$$\big\|\big(K_0-\mathfrak{z}\mathbb{1}\big)\,\widetilde{S}(\mathfrak{z};\delta)\;-\;\mathbb{1}\big\|_{\mathbb{B}(L^2(\mathbb{R}^d))}\;\leq\;C(a,F)\,\delta^{1/2}\,\big(\mathsf{dist}\big(\mathfrak{z},\sigma(K_\delta)\big)\big)^{-1}.$$

It follows that if $\mathsf{dist}(\mathfrak{z},\sigma(\mathfrak{Op}^w\big(a[F]_\delta)\big))\geq C\delta^{1/2}$ then, $\mathfrak{z}\notin\sigma(K_0)$ and Theorem 2.1 is proven.

4. PROOF OF THEOREM 2.3

In this case, we no longer estimate norms, but rather quadratic forms. The main idea is to replace the perturbation $x\mapsto x+\delta\,F(x)$ with a similar one in a new variable $u\in\mathbb{R}^d$, namely $x\mapsto x+\delta\,F(u)$, and use the unitarity of translations in estimating the modified quadratic form. In order to control the distance between the new variable u and $z:=(x+y)/2$, we use a scaled weight function $\mathfrak{W}_\kappa(z-u)$ as in [2] (see (4.2)). We treat only the case $\mathcal{E}_+(\delta)-\mathcal{E}_+(0)$, the other one, i.e., $\mathcal{E}_-(\delta)-\mathcal{E}_-(0)$ following by a quite similar argument.

We intend to estimate the difference $\mathcal{E}_+(\delta)-\mathcal{E}_+(0)$ for $\delta>0$ small enough, and begin by making explicit the defining formula (1.10):

$$\begin{aligned}\mathcal{E}_+(\delta)&=\sup_{\|\phi\|_{L^2(\mathbb{R}^d)}=1}\big(\phi\,,\,\mathfrak{Op}\big(a[F]_\delta\big)\,\phi\big)_{L^2(\mathbb{R}^d)}\\&=\sup_{\|\phi\|_{L^2(\mathbb{R}^d)}=1}\big\langle\mathfrak{K}_\delta\,,\,(\overline{\phi}\otimes\phi)\circ\Upsilon^{-1}\big\rangle_{\mathscr{S}(\mathbb{R}^d\times\mathbb{R}^d)}.\end{aligned}\tag{4.1}$$

The weight function. Let us consider the functions:

$$\mathfrak{W}(z):=(4\pi)^{-d/2}\,e^{-\frac{|z|^2}{4}},\qquad\mathfrak{W}_\kappa(z):=\kappa^{d/2}\mathfrak{W}(\kappa\,z),\;\forall\kappa\in(0,1]\tag{4.2}$$

and the following identity:

$$2^{-1}\big(|w+v/2|^2+|w-v/2|^2\big)=|w|^2+|v|^2/4,\quad\forall(w,v)\in\mathbb{R}^d\times\mathbb{R}^d.\tag{4.3}$$

We deduce that:

$$\begin{aligned}&\int_{\mathbb{R}^d}dz\,\mathfrak{W}_\kappa(z)\;=\;1,\quad\forall\kappa\in(0,1],\\&\mathfrak{W}_\kappa(z-u)\\&\quad=\big((\kappa/4\pi)^{-d/2}\mathfrak{W}_\kappa(v)\big)^{-1/4}\mathfrak{W}_\kappa(z-u+v/2)^{1/2}\mathfrak{W}_\kappa(z-u-v/2)^{1/2},\end{aligned}\tag{4.4}$$

for any $(z,u,v,\kappa)\in[\mathbb{R}^d]^3\times(0,1]$.

Our strategy is to replace in formula (4.1) the distribution:

$$\mathfrak{K}_\delta = \Big(\int_{\mathbb{R}^d} du\,\big((\tau_u\mathfrak{W}_\kappa)\otimes 1\big)\Big)\,\mathfrak{K}_\delta, \tag{4.5}$$

with the distribution:

$$\mathfrak{W}_\kappa[\mathfrak{K}_\delta] := \int_{\mathbb{R}^d} du\,\big((\tau_u\mathfrak{W}_\kappa)\otimes 1\big)\big(\big(\tau_{-\delta\,F(u)}\otimes \mathbb{1}\big)\mathfrak{K}_0\big) \tag{4.6}$$

and estimate:

$$\widetilde{\mathcal{E}_+}(\kappa,\delta) := \sup_{\|\phi\|_{L^2}=1}\big\langle \mathfrak{W}_\kappa[\mathfrak{K}_\delta]\,,\,(\overline{\phi}\otimes\phi)\circ\Upsilon^{-1}\big\rangle_{\mathscr{S}(\mathbb{R}^d\times\mathbb{R}^d)}. \tag{4.7}$$

Proposition 4.1. *With the above notations and hypothesis, for any $\phi\in\mathscr{S}(\mathbb{R}^d)$ there exists some $C(a,F)>0$ such that the following estimation is true:*

$$\big\langle \mathfrak{W}_\kappa[\mathfrak{K}_\delta]\,,\,(\overline{\phi}\otimes\phi)\circ\Upsilon^{-1}\big\rangle_{\mathscr{S}(\mathbb{R}^d\times\mathbb{R}^d)} = \big(\phi\,,\,\mathfrak{Op}(a)\,\phi\big)_{L^2(\mathbb{R}^d)} + C(a,F)\,\kappa^2\,\|\phi\|^2_{L^2}.$$

Proof. Starting from (4.7), we have to estimate the following iterated integrals:

$$\Big|\int_{\mathbb{R}^d} du\int_{\mathbb{R}^d} dz\,\mathfrak{W}_\kappa(z-u)\int_{\mathbb{R}^d} dv\,\overline{\phi(z+v/2)}\phi(z-v/2)\,\mathfrak{K}_0\big(z+\delta F(u),v\big)\Big|. \tag{4.8}$$

We use the rapid decay in $v\in\mathbb{R}^d$ of the kernel $\mathfrak{K}_0(z+\delta F(u),v)$ by breaking the integral in $v\in\mathbb{R}^d$ in a bounded region and its complementary. In fact, we choose some function $\chi\in C_0^\infty(\mathbb{R}^d)$, taking values in $[0,1]$, having support in the ball $|v|\le R$ and being equal to 1 on the ball $|v|\le r$, for some strictly positive $r<R$.

Let us first estimate the integral on the unbounded region, for any $\kappa\in(0,1)$ and $N\in\mathbb{N}$:

$$\begin{aligned}&\Big|\int_{\mathbb{R}^d} du\int_{\mathbb{R}^d} dz\,\mathfrak{W}_\kappa(z-u)\\ &\qquad\times\int_{\mathbb{R}^d} dv\,\overline{\phi(z+v/2)}\phi(z-v/2)\,\mathfrak{K}_0\big(z+\delta F(u),v\big)\big[1-\chi(\kappa\,v)\big]\Big|\\ &\quad\le \kappa^N\Big(r^{-N}\sup_{z\in\mathbb{R}^d}\sup_{|v|\ge r} <v>^N\,\big|\mathfrak{K}_0(z,v)\big|\Big)\,\|\phi\|^2_{L^2(\mathbb{R}^d)}\;\le\; C_r(a)\,\kappa^N\,\|\phi\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

On the support of χ, we use the second formula in (4.4) and write that:

$$\int_{\mathbb{R}^d}\!du\!\int_{\mathbb{R}^d}\!dz\,\mathfrak{W}_\kappa(z-u)\int_{\mathbb{R}^d}dv\,\overline{\phi(z+v/2)}\phi(z-v/2)\,\mathfrak{K}_0\big(z+\delta F(u),v\big)\,\chi(\kappa\,v)$$

$$= \Big(\widetilde{\phi} \,,\, \big(\mathbb{1}_{\mathbb{R}^d_u} \otimes \tau_{-\delta F(u)} \big) \big(\mathbb{1}_{\mathbb{R}^d_u} \otimes \mathfrak{Op}^w(a_\kappa) \big) \big(\mathbb{1}_{\mathbb{R}^d_u} \otimes \tau_{\delta F(u)} \big) \widetilde{\phi} \Big)_{L^2(\mathbb{R}^d; L^2(\mathbb{R}^d))}$$

with:

$$\begin{aligned}
&\big(\mathfrak{K}[a_\kappa] \circ \Upsilon\big)(z,v) \\
&\quad := \big((\kappa/4\pi)^{-d/2} \mathfrak{W}_\kappa(v) \big)^{-1/4} \mathfrak{K}_0(z,v)\, \chi(\kappa\, v) = e^{|\kappa\, v|^2/16}\, \chi(\kappa\, v)\, \mathfrak{K}_0(z,v), \\
&\quad = \mathfrak{K}_0(z,v)\, \chi(\kappa\, v) \;+\; \kappa^2 \int_0^1 ds\, (|v|^2/16)\, e^{s|\kappa\, v|^2/16}\, \chi(\kappa\, v)\, \mathfrak{K}_0(z,v), \\
&\widetilde{\phi} := (\tau_{-u} \mathfrak{W}_\kappa)\, \phi \in\, L^2\big(\mathbb{R}^d_u; L^2(\mathbb{R}^d_z)\big).
\end{aligned}$$

We notice that we have a unitary map

$$L^2(\mathbb{R}^d) \ni \phi \,\mapsto\, \widetilde{\phi} := (\tau_{-u} \mathfrak{W}_\kappa)\, \phi \in\, L^2\big(\mathbb{R}^d_u; L^2(\mathbb{R}^d_z)\big) \tag{4.9}$$

and the equality (taking into account the unitarity of translations):

$$\big(\widetilde{\phi} \,,\, \big(\mathbb{1}_{\mathbb{R}^d_u} \otimes \tau_{-\delta F(u)} \cdot \mathfrak{Op}^w(a_\kappa) \cdot \tau_{\delta F(u)} \big) \widetilde{\phi} \big)_{L^2(\mathbb{R}^d; L^2(\mathbb{R}^d))} = \big(\phi \,,\, \mathfrak{Op}^w(a_\kappa) \phi \big)_{L^2(\mathbb{R}^d)}.$$

Thus, if we change the Hilbert space $L^2(\mathbb{R}^d)$ with $L^2(\mathbb{R}^d; L^2(\mathbb{R}^d))$ via the above unitary map, we may conclude that:

$$\begin{aligned}
&\int_{\mathbb{R}^d} du \int_{\mathbb{R}^d} dz\, \mathfrak{W}_\kappa(z-u) \int_{\mathbb{R}^d} dv\, \overline{\phi(z+v/2)} \phi(z-v/2)\, \mathfrak{K}_0\big(z+\delta F(u), v\big)\, \chi(\kappa\, v) \\
&\quad = \big(\phi \,,\, \mathfrak{Op}^w(a_\kappa) \phi \big)_{L^2(\mathbb{R}^d)} = \big(\phi \,,\, \mathfrak{Op}^w(a_\chi) \phi \big)_{L^2(\mathbb{R}^d)} + \kappa^2 \int_{\mathbb{R}^d} dz \int_{\mathbb{R}^d} dv \\
&\qquad\qquad \times\, \overline{\phi(z+v/2)} \phi(z-v/2) \Big(\int_0^1 ds\, e^{s \frac{|\kappa\, v|^2}{16}} \Big) \frac{|v|^2}{16}\, \mathfrak{K}_0\big(z+\delta F(u), v\big)\, \chi(\kappa\, v)
\end{aligned}$$

where we have put into evidence the symbol $a_\chi \in S^0_{0,0}(\mathbb{R}^d \times \mathbb{R}^d)$ associated to the integral kernel $\mathfrak{K}[a](x,y)\chi(\kappa(x-y))$. Then, we may control the factor $(|v|^2/16)$ using the rapid decay of $\mathfrak{K}_0$ with respect to the variable $v \in \mathbb{R}^d$ and write that $\exp(s|\kappa\, v|^2/16)\chi(\kappa\, v) \leq \exp(R^2/16)$. Finally, we use once again the estimation on the support of $1-\chi$:

$$\begin{aligned}
&\big(\phi \,,\, \mathfrak{Op}(a_\chi)\, \phi \big)_{L^2(\mathbb{R}^d)} \\
&\quad = \big(\phi \,,\, \mathfrak{Op}(a) \phi \big)_{L^2(\mathbb{R}^d)} - \int_{\mathbb{R}^d} dz \int_{\mathbb{R}^d} dv\, \overline{\phi(z+v/2)} \phi(z-v/2) \mathfrak{K}_0(z,v) \big[1 - \chi(\kappa\, v) \big] \\
&\Big| \big(\phi \,,\, \mathfrak{Op}(a)\, \phi \big)_{L^2(\mathbb{R}^d)} - \int_{\mathbb{R}^d} dz \int_{\mathbb{R}^d} dv\, \overline{\phi(z+v/2)} \phi(z-v/2)\, \mathfrak{K}_0(z,v) \big[1 - \chi(\kappa\, v) \big] \Big| \\
&\qquad \leq\, \kappa^N \Big(r^{-N} \sup_{z \in \mathbb{R}^d} \sup_{|v| \geq r} < v >^N \big| \mathfrak{K}_0(z,v) \big| \Big) \, \|\phi\|^2_{L^2(\mathbb{R}^d)} \,\leq\, C(a)\, \kappa^N\, \|\phi\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

□

Proposition 4.1 clearly implies the estimation:

$$\big|\widetilde{\mathcal{E}_+}(\delta) \;-\; \mathcal{E}_+(0)\big| \;=\; \mathcal{O}(\kappa^2). \tag{4.10}$$

PROPOSITION 4.2. *There exists $C(a,F)>0$ such that for any $\phi\in\, L^2(\mathbb{R}^d)$ and for any $(\kappa,\theta)\in(0,1)\times(0,1)$:*

$$\begin{aligned}&\big\langle \mathfrak{K}_\delta\,,\,(\overline{\phi}\otimes\phi)\circ\Upsilon^{-1}\big\rangle_{\mathscr{S}(\mathbb{R}^d\times\mathbb{R}^d)} \;-\; \big\langle \mathfrak{W}[\mathfrak{K}_\delta]\,,\,(\overline{\phi}\otimes\phi)\circ\Upsilon^{-1}\big\rangle_{\mathscr{S}(\mathbb{R}^d\times\mathbb{R}^d)}\\ &\qquad=\; C(a,F)\,\|\phi\|^2_{L^2(\mathbb{R}^d)}\,(\delta/\theta+\delta\kappa^{-2}\theta^{1+\mu}+\delta^2\kappa^{-2}).\end{aligned} \tag{4.11}$$

Proof. We compute

$$\begin{aligned}&\big\langle \mathfrak{K}_\delta\,,\,(\overline{\phi}\otimes\phi)\circ\Upsilon^{-1}\big\rangle_{\mathscr{S}(\mathbb{R}^d\times\mathbb{R}^d)} \;-\; \big\langle \mathfrak{W}[\mathfrak{K}_\delta]\,,\,(\overline{\phi}\otimes\phi)\circ\Upsilon^{-1}\big\rangle_{\mathscr{S}(\mathbb{R}^d\times\mathbb{R}^d)}\\
&\quad=\int_{\mathbb{R}^d}du\int_{\mathbb{R}^d}dz\,\mathfrak{W}_\kappa(z-u)\int_{\mathbb{R}^d}dv\,\overline{\phi(z+v/2)}\phi(z-v/2)\\
&\qquad\qquad\times\;\big[\mathfrak{K}_0\big(z+\delta F(z),v\big)\;-\;\mathfrak{K}_0\big(z+\delta F(u),v\big)\big]\\
&\quad=-\delta\!\int_{\mathbb{R}^d}du\int_{\mathbb{R}^d}dz\,\mathfrak{W}_\kappa(z-u)\!\int_{\mathbb{R}^d}dv\,\overline{\phi(z+v/2)}\phi(z-v/2)\\
&\qquad\qquad\times\;(\nabla_z\mathfrak{K}_0)\big(z+\delta F(z)+s\delta\big(F(u)-F(z)\big),v\big)\\
&\qquad\qquad\times\;\Big[\int_0^1 ds\,\big((z-u)\cdot\nabla F\big)\big(z+s(u-z)\big)\Big].\end{aligned}$$

We need a second cut-off, this time on the perturbing field $F\in C^\infty_1(\mathbb{R}^d)$. Let us consider the same function $\chi\in C^\infty_0(\mathbb{R}^d)$ as in the proof above and the weighted one $\chi_\theta(z):=\chi(\theta\,z)$ for some cut-off parameter $\theta\in(0,1]$. Then, we define:

$$F_\theta:=\chi_\theta\,F,\qquad F_\theta^\perp:=(1-\chi_\theta)\,F \tag{4.12}$$

and the corresponding integral kernels $\mathfrak{K}^\circ_\delta$ and $\mathfrak{W}[\mathfrak{K}_\delta]^\circ$ with F replaced by F_θ and respectively, $\mathfrak{K}^\perp_\delta$ and $\mathfrak{W}[\mathfrak{K}_\delta]^\perp$ with F replaced by $F^\perp_\theta$.

We have the evident estimations:

$$\begin{aligned}&\big\langle \mathfrak{K}^\circ_\delta\,,\,(\overline{\phi}\otimes\phi)\circ\Upsilon^{-1}\big\rangle_{\mathscr{S}(\mathbb{R}^d\times\mathbb{R}^d)} - \big\langle \mathfrak{K}_0\,,\,(\overline{\phi}\otimes\phi)\circ\Upsilon^{-1}\big\rangle_{\mathscr{S}(\mathbb{R}^d\times\mathbb{R}^d)}\\
&\quad=\int_{\mathbb{R}^d}dz\int_{\mathbb{R}^d}dv\,\overline{\phi(z+v/2)}\phi(z-v/2)\,\big[\mathfrak{K}_0\big(z+\delta F_\theta(z),v\big)-\mathfrak{K}_0(z,v)\big]\\
&\quad=\delta\!\int_{\mathbb{R}^d}dz\int_{\mathbb{R}^d}dv\,\overline{\phi(z+v/2)}\phi(z-v/2)\int_0^1\!ds\,\big[F_\theta(z)\cdot(\partial_z\mathfrak{K}_0)\big(z+s\delta\,F_\theta(z),v\big)\big],\end{aligned}$$

and

$$\begin{aligned}&\big|\big\langle \mathfrak{W}[\mathfrak{K}_\delta]^\circ\,,\,(\overline{\phi}\otimes\phi)\circ\Upsilon^{-1}\big\rangle_{\mathscr{S}(\mathbb{R}^d\times\mathbb{R}^d)} - \big\langle \mathfrak{K}_0\,,\,(\overline{\phi}\otimes\phi)\circ\Upsilon^{-1}\big\rangle_{\mathscr{S}(\mathbb{R}^d\times\mathbb{R}^d)}\big|\\
&\quad=\Big|\int_{\mathbb{R}^d}du\int_{\mathbb{R}^d}dz\,\mathfrak{W}_\kappa(z-u)\int_{\mathbb{R}^d}dv\,\overline{\phi(z+v/2)}\phi(z-v/2)\end{aligned}$$

$$
\times \left[\mathfrak{K}_0(z+\delta F_\theta(u),v)-\mathfrak{K}_0(z,v)\right]\Big|
$$

$$
=\delta\int_{\mathbb{R}^d}dz\int_{\mathbb{R}^d}dv\,\overline{\phi(z+v/2)}\phi(z-v/2)\left[\int_{\mathbb{R}^d}du\,\mathfrak{W}_\kappa(z-u)\right.
$$

$$
\left.\times\left(\int_0^1 ds\left[F_\theta(u)\cdot(\partial_z\mathfrak{K}_0)\big(z+s\delta\,F_\theta(u),v\big)\right]\right)\right]
$$

We estimate the above two differences in the next two lemmas.

Lemma 4.3. *The symbol $a'_{\delta,\theta}(z,\eta)$ associated to the kernel*

$$
\mathfrak{K}'_{\delta,\theta}(z,v):=\int_0^1 ds\left[F_\theta(z)\cdot(\partial_z\mathfrak{K}_0)\big(z+s\delta\,F_\theta(z),v\big)\right]
$$

belongs to $S^0_{0,0}(\mathbb{R}^d\times\mathbb{R}^d)$ with seminorms bounded by $C\theta^{-1}$ for $(\delta,\theta)\in(0,1]^2$.

Proof. We can write that:

$$
\begin{aligned}
a'_{\delta,\theta}(z,\eta)&=(2\pi)^d\int_{\mathbb{R}^d}dv\,e^{-i<\eta,v>}\,\mathfrak{K}'_{\delta,\theta}(z,v)\\
&=(2\pi)^{d/2}\int_{\mathbb{R}^d}dv\,e^{-i<\eta,v>}\int_0^1 ds\left[F_\theta(z)\cdot\big(\partial_z(\mathbb{1}\otimes\mathcal{F}^-)a\big)\big(z+s\delta\,F_\theta(z),v\big)\right]\\
&=\int_0^1 ds\left[F_\theta(z)\cdot(\partial_z a)\big(z+s\delta\,F_\theta(z),\eta\big)\right].
\end{aligned}
$$

As in Remark 1.3, we notice that

$$
\begin{aligned}
\nu_{n,m}(a'_{\delta,\theta})&\leq\big(\sup_{z\in\mathbb{R}^d}\big|F_\theta(z)\big|\big)\sup_{0\leq s\leq 1}\nu_{n+1,m}(a[F]_s)\\
&\leq M_F\,\theta^{-1}\sup_{0\leq s\leq 1}\nu_{n+1,m}\big(a[F]_s\big)\quad\square
\end{aligned}
\tag{4.13}
$$

Lemma 4.4. *The symbol $a''_{\delta,\theta}(z,\eta)$ associated to the kernel*

$$
\mathfrak{K}''_{\delta,\theta}(z,v):=\int_{\mathbb{R}^d}du\,\mathfrak{W}_\kappa(z-u)\left(\int_0^1 ds\left[F_\theta(u)\cdot(\partial_z\mathfrak{K}_0)\big(z+s\delta\,F_\theta(u),v\big)\right]\right)
$$

belongs to $S^0_{0,0}(\mathbb{R}^d\times\mathbb{R}^d)$ with seminorms bounded by $C\theta^{-1}$ for $(\delta,\theta)\in(0,1]^2$.

Proof. We can write that:

$$
\begin{aligned}
a''_{\delta,\theta}(z,\eta)&=(2\pi)^d\int_{\mathbb{R}^d}dv\,e^{-i<\eta,v>}\,\mathfrak{K}''_{\delta,\theta}(z,v)\\
&=\int_0^1 ds\int_{\mathbb{R}^d}du\,\mathfrak{W}_\kappa(z-u)\left(\left[F_\theta(u)\cdot(\partial_z\,a)\big(z+s\delta\,F_\theta(u),\eta\big)\right]\right).
\end{aligned}
$$

As in Remark 1.3, we notice that

$$\begin{aligned}\nu_{n,m}(a'_{\delta,\theta}) &\leq \big(\sup_{u\in\mathbb{R}^d}\big|F_\theta(u)\big|\big)\sup_{0\leq s\leq 1}\nu_{n+1,m}(a[F]_s)\\ &\leq M_F\,\theta^{-1}\sup_{0\leq s\leq 1}\nu_{n+1,m}(a[F]_s)\quad\square\end{aligned}\tag{4.14}$$

Putting the above results together, we conclude that:

$$\begin{aligned}&\big|\big\langle\mathfrak{K}^\circ_\delta\,,\,(\overline{\phi}\otimes\phi)\circ\Upsilon^{-1}\big\rangle_{\mathscr{S}(\mathbb{R}^d\times\mathbb{R}^d)}\,-\,\big\langle\mathfrak{W}[\mathfrak{K}^\circ_\delta]\,,\,(\overline{\phi}\otimes\phi)\circ\Upsilon^{-1}\big\rangle_{\mathscr{S}(\mathbb{R}^d\times\mathbb{R}^d)}\big|\\ &\qquad\leq\,C(F)\delta\theta^{-1}.\end{aligned}\tag{4.15}$$

Let us consider now the “outer region” integrals:

$$\begin{aligned}&\big\langle\mathfrak{K}^\perp_\delta\,,\,(\overline{\phi}\otimes\phi)\circ\Upsilon^{-1}\big\rangle_{\mathscr{S}(\mathbb{R}^d\times\mathbb{R}^d)}\,-\,\big\langle\mathfrak{W}[\mathfrak{K}^\perp_\delta]\,,\,(\overline{\phi}\otimes\phi)\circ\Upsilon^{-1}\big\rangle_{\mathscr{S}(\mathbb{R}^d\times\mathbb{R}^d)}\\ &\quad=\,\delta\mathcal{I}_1[\phi](\delta,\theta,\kappa)\,+\,\delta^2\mathcal{I}_2[\phi](\delta,\theta,\kappa)\end{aligned}\tag{4.16}$$

where:

$$\begin{aligned}\mathcal{I}_1[\phi](\delta,\theta,\kappa)&:=\int_{\mathbb{R}^d}du\int_{\mathbb{R}^d}dz\,\mathfrak{W}_\kappa(z-u)\int_{\mathbb{R}^d}dv\,\overline{\phi(z+v/2)}\phi(z-v/2)\\ &\times\,(\nabla_z\mathfrak{K}_0)\big(z+\delta F^\perp_\theta(z),v\big)\Big[\big((z-u)\cdot\nabla F^\perp_\theta\big)(z)\\ &+\,(1/2)\int_0^1 ds\,(1-s)\big((z-u)\otimes(z-u)(\nabla\otimes\nabla)\,F^\perp_\theta\big)\big(z+s(u-z)\big)\Big]\end{aligned}$$

$$\begin{aligned}\mathcal{I}_2[\phi](\delta,\theta,\kappa)&:=\,(1/2)\int_{\mathbb{R}^d}du\int_{\mathbb{R}^d}dz\,\mathfrak{W}_\kappa(z-u)\int_{\mathbb{R}^d}dv\,\overline{\phi(z+v/2)}\phi(z-v/2)\\ \times&\int_0^1(1-s)ds\big((\nabla_z\otimes\nabla_z)\mathfrak{K}_0\big)\big(z+\delta F^\perp_\theta(z)+s\delta(F^\perp_\theta(u)-F^\perp_\theta(z)),v\big)\\ \times&\int_0^1 dt\,\big((z-u)\cdot\nabla F^\perp_\theta\big)(z+t(u-z))\int_0^1 dt'\,\big((z-u)\cdot\nabla F^\perp_\theta\big)(z+t'(u-z)).\end{aligned}$$

We may conclude that:

$$\big|\mathcal{I}_1[\phi](\delta,\theta,\kappa)\big|\,\leq\,\kappa^{-2}\theta^{1+\mu}\|\phi\|^2_{L^2},\qquad\big|\mathcal{I}_2[\phi](\delta,\theta,\kappa)\big|\,\leq\,\kappa^{-2}\|\phi\|^2_{L^2}.\qquad\square\tag{4.17}$$

The result of Proposition 4.2 clearly implies that:

$$\big|\widetilde{\mathcal{E}_+}(\delta)\,-\,\mathcal{E}_+(\delta)\big|\,=\,\mathcal{O}(\delta/\theta+\delta\kappa^{-2}\theta^{1+\mu}+\delta^2\kappa^{-2}).\tag{4.18}$$

Taking into account (4.10) and (4.18), in order to finish the proof of Theorem 2.3, we only have to make the following choices for our scaling parameters:

- $\theta=\delta^{1-\rho}$ for some $\rho\in(0,1)$, so that $\delta\theta^{-1}=\delta^\rho$;
- $\kappa^2=\delta^\rho$ so that $\delta\kappa^{-2}\theta^{1+\mu}=\delta^{(1-\rho)+(1-\rho)(1+\mu)}=\delta^{(2+\mu)(1-\rho)}$ and $\delta^2\kappa^{-2}=\delta^{(2-\rho)}$;
- imposing $\rho=(2+\mu)(1-\rho)\in(0,1)$ means taking $\rho=(1+\mu)/(2+\mu)$.

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