ON BOUNDARY BEHAVIOUR OF OPEN, CLOSED MAPPINGS OF BOUNDED DIRICHLET INTEGRAL

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We study boundary behaviour of a class of mappings for which an inverse Poletsky modular inequality holds.

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1. INTRODUCTION

Throughout this paper, X, Y are locally compact and locally connected metric measure spaces endowed with Borel regular measures μ and ν which are finite on compact sets such that $\mu(B(x,r)) > 0$ for every ball $B(x,r) \subset X$ with $r > 0$, $\nu(B(y,r)) > 0$ for every ball $B(y,r) \subset Y$ with $r > 0$ and also X, Y have countable bases. The measure ν is doubling, $D \subset X$ is a domain and if $f : D \to Y$ is a mapping, we set for $x \in D$

$$L(x,f) = \limsup_{y \to x} \frac{d(f(y), f(x))}{d(y,x)},$$

$$l(x,f) = \liminf_{y \to x} \frac{d(f(y), f(x))}{d(y,x)},$$

$$\nu'_f(x) = \limsup_{t \to 0} \frac{\nu(f(B(x,r)))}{\mu(B(x,r))}.$$

Here, d is the distance on X and Y.

If $f : D \to Y$ is a continuous mapping, we say that f is of bounded Dirichlet integral if $\int_D L(x,f)^n d\mu < \infty$.

If $p > 1$ and $D \subset \mathbb{R}^n$ is open, we set $W^{1,p}_{\text{loc}}(D, \mathbb{R}^n)$ the Sobolev space of all mappings $f : D \to \mathbb{R}^n$ which are locally in L^p together with their weak first order partial derivatives. Usually, a mapping $f \in W^{1,n}_{\text{loc}}(D, \mathbb{R}^n)$ is of bounded Dirichlet integral if $f \in W^{1,n}_{\text{loc}}(D, \mathbb{R}^n)$ and $\int_D |f'(x)|^n dx < \infty$ and our definition is just a topological one (if f is, a.e., differentiable, then $|f'(x)| = L(x,f)$, a.e.).

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The homeomorphisms $f : D \to D_f$ such that the mappings $f \in W^{1,n}_{\text{loc}}(D, \mathbb{R}^n)$, $f^{-1} \in W^{1,n}_{\text{loc}}(D_f, D)$ and $\int_D |f'(x)|^n dx + \int_{D_f} |(f^{-1})'(y)|^n dy < \infty$ form an important class used in the mathematical models in Nonlinear Elasticity (see[15, 16]) and we study in this paper the boundary behaviour of open, discrete and closed mappings of bounded Dirichlet integral using some methods inspired by the theory of quasiregular mappings. We remind that if $n \geq 2$, $D \subset \mathbb{R}^n$ is open and $f : D \to \mathbb{R}^n$ is continuous, we say that f is quasiregular if $f \in W^{1,n}_{\text{loc}}(D, \mathbb{R}^n)$ and there exists $K \geq 1$ such that $|f'(x)|^n \leq K J_f(x)$, a.e.

Let $D \subset X$. We set $A(D)$ to be the set of all nonconstant path families in D and if $\Gamma \in A(D)$ we let $F(\Gamma) = \{\rho : D \to [0, \infty]$ Borel functions $| \int_\gamma \rho ds \geq 1$ for every $\gamma \in \Gamma$ locally rectifiable$\}$.

If $p > 1$ and $\omega : D \to [0, \infty]$ is μ-measurable and finite μ, a.e., we define the p-modulus of weight ω by

$$M^p_\omega(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_X \omega(x) \rho^p(x) d\mu \; if \; \Gamma \in A(D).$$

If $F(\Gamma) = \phi$, we set $M^p_\omega(\Gamma) = 0$. If $\omega = 1$, we set

$$M_p(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_X \rho^p(x) d\mu \; if \; \Gamma \in A(D)$$

for the classical p-modulus.

One of the basic tools in studying quasiregular mappings is the modular inequality of Poletsky

$$M_n(f(\Gamma)) \leq K M_n(\Gamma) \text{ for every } \Gamma \in A(D). \tag{1.1}$$

We recommend the reader the monographs [21], [22], [30], [33] for more information about quasiregular mappings. Several generalizations of quasiregular mappings were given in the last 30 years and the most important one is the class of mappings of finite distortion (see the monographs [12] and [14] for more information) for which a Poletsky modular inequality holds in some particular cases (see [3] and [17]).

Martio has proposed the study of mappings distinguished by moduli inequalities of type

$$M_q(f(\Gamma)) \leq M^p_\omega(\Gamma) \text{ for every } \Gamma \in A(D) \tag{1.2}$$

where $p, q > 1$ and ω is measurable and finite, a.e.

In this class, Montel, Picard, Liouville type theorems, boundary extension and equicontinuity results and estimates of the modulus of continuity were given (see [1, 3], [5]–[11], [18]–[20], [23]–[26], [28]).

If $n \geq 2$ and $f : D \to \mathbb{R}^n$ is quasiregular nonconstant, an inverse Poletsky modular inequality holds, namely

$$M_n(\Gamma) \leq K \int_{\mathbb{R}^n} \rho^n(y) N(y, f, D) dy \text{ for every } \Gamma \in A(D) \tag{1.3}$$
$$\text{and for every } \rho \in F(f(\Gamma)).$$

Here,

$$N(y, f, D) = \text{Card}\, f^{-1}(y) \cap D \text{ if } y \in \mathbb{R}^n \text{ and } N(f, D) = \sup_{y \in \mathbb{R}^n} N(y, f, D).$$

Motivated by this property of quasiregular mappings, in the last years, open, discrete mappings $f : D \to Y$ for which an inverse Poletsky modular inequality holds were studied, namely

$$M_q(\Gamma) \leq M_\omega^p(f(\Gamma)) \text{ for every } \Gamma \in A(D). \tag{1.4}$$

Here, $p, q > 1$ and ω is μ-measurable and finite μ, a.e.

The weight ω in relation (1.4) is quite general, while the weight ω in relation (1.3) is a particular one, namely $\omega(y) = \text{Card } f^{-1}(y) \cap D$ for every $y \in Y$. Results concerning boundary extension, equicontinuity, lightness and discreteness of the mappings satisfying relation (1.4) were established in [4], [27]–[29].

Vuorinen, see [31]–[33], was the first who observed that the mappings of bounded Dirichlet integral satisfy some inverse Poletsky modular inequalities and applied in this class of mappings the methods used in the study of quasiregular mappings.

The following theorem is proved in [10] and shows that in very general cases the mappings of bounded Dirichlet integral satisfy a modular inequality of type (1.4) (see also [4]).

Theorem A ([10]). *Let $q > 1$, let $f : D \to Y$ be continuous such that $\mu(B_f) = 0$, $\nu(f(B_f)) = 0$, f satisfies condition (N^{-1}) and $\int_D L(x, f)^q d\mu < \infty$. Then there exists $\omega \in L^1(Y)$ such that $M_q(\Gamma) \leq M_\omega^q(f(\Gamma))$ for every $\Gamma \in A(D)$ and $\int_{f(A)} \omega(y) d\nu \leq \int_A L(z, f)^q d\mu$ for every open set $A \subset D$.*

Here, $B_f = \{x \in D | f \text{ is not a local homeomorphism at } x\}$ and we say that f satisfies condition (N^{-1}) if $\mu(f^{-1}(A)) = 0$ whenever $A \subset Y$ and $\nu(A) = 0$.

If $x \in X$, we set $\mathcal{V}(x) = \{U \subset X \text{ open } | x \in U\}$.

A mapping $f : D \to Y$ is open if it carries open sets into open sets and is closed if it carries closed sets into closed sets. We say that $f : D \to Y$ is proper if $f^{-1}(K)$ is compact whenever $K \subset Y$ is compact.

Let $D \subset X$ and $x \in \partial D$. We say that D is locally connected at x if there exists a fundamental system of neighbourhoods of x, $(U_m)_{m \in \mathbb{N}}$ such that

$U_m \cap D$ is connected for every $m \in \mathbb{N}$. We say that D is finitely connected at x if there exists a fundamental system of neighbourhoods of x, $(U_m)_{m\in\mathbb{N}}$ such that $U_m \cap D$ has a finite number of components for every $m \in \mathbb{N}$.

Usually, a domain $D \subset \mathbb{R}^n$ is weakly flat at a boundary point $x \in \partial D$ if for every $\epsilon > 0$ and every $U \in \mathcal{V}(x)$ there exists $V \in \mathcal{V}(x)$ such that $\overline{V} \subset U$ and $M_n(\Delta(E, F, D)) > \epsilon$ for every continua E, F in D which intersects ∂U and ∂V. Here, $\Delta(E, F, D) = \{\gamma : [0, 1] \to \overline{D}$ path such that $\gamma((0, 1)) \subset D$, $\gamma(0) \in E$, $\gamma(1) \in F\}$. Such a domain is locally connected at x (see [20], Lemma 3.15). Also, such a domain is "locally" weakly flat, i.e., "there exists $U_x \in \mathcal{V}(x)$ such that for every $\epsilon > 0$ and every $U \in \mathcal{V}(x)$ such that $U \subset U_x$, there exists $V \in \mathcal{V}(x)$ such that $\overline{V} \subset U$ and $M_n(\Delta(E, F, D \cap U_x)) > \epsilon$ for every continua E, F in $D \cap U_x$ which intersects ∂U and ∂V".

We give now a rather general definition of locally q-weakly flatness.

Definition 1.1. Let $D \subset X$ a domain, $q > 1$ and $x \in \partial D$. We say that D is locally q-weakly flat at x if there exist $U_x \in \mathcal{V}(x)$ and $\epsilon > 0$ such that for every $U \in \mathcal{V}(x)$ with $U \subset U_x$, there exists $V \in \mathcal{V}(x)$ such that $\overline{V} \subset U$ and $M_q(\Delta(E, F, D \cap U_x)) > \epsilon$ for every continua E, F in $D \cap U_x$ which intersects ∂U and ∂V.

The usual example of a n-weakly flat domain $D \subset \mathbb{R}^n$ at a boundary point $x \in \partial D$ is a domain such that there exists $U_x \in \mathcal{V}(x)$ and a quasiconformal homeomorphism $\Phi_x : B_r \cap H \to D \cap U_x$ such that $\Phi_x(0) = x$. Here, $B_r = B(0, r)$ in $\mathbb{R}^n$ and $H = \{z \in \mathbb{R}^n | z_n > 0\}$.

We can also see that if $D_1 \subset \mathbb{R}^n$ is q-weakly flat at 0, X is such that $\mu(B(x, r)) \geq \frac{r}{c_1}$ for every ball $B(x, r) \subset X$, $D \subset X$ is a domain, $x \in \partial D$ and there exist $m_x, M_x > 0$, $U_x \in \mathcal{V}(x)$ and a homeomorphism $\Phi_x : B_r \to U_x$ such that $\Phi_x(0) = x$, $\Phi_x(D_1 \cap B_r) = D \cap U_x$ and $L(z, \Phi_x) < M_x$, $l(z, \Phi_x) > m_x$ for every $z \in D_1 \cap B_r$, then D is q-weakly flat at x (see Lemma 2.1).

Let us give some examples of q-weakly flat domains in $\mathbb{R}^n$.

Lemma 1.2. *Let $D \subset \mathbb{R}^n$ be a domain such that $0 \in \partial D$, let $A_\rho = \{t \in (0, \rho) | D \cap S(0, t)$ is a cap of a sphere$\}$ and suppose that one of the following conditions hold:*

$$\liminf_{\rho \to 0} \frac{\mu_1(A_\rho)}{\rho} > \epsilon > 0. \tag{1.5}$$

There exists $n - 1 < q < n$ such that

$$\liminf_{\rho \to 0} \frac{\mu_1(A_\rho)}{\rho^{q-n+1}} > \epsilon > 0. \tag{1.6}$$

Then, if condition (1.5) holds, it results that D is locally n-weakly flat at x and if condition (1.6) holds, then D is locally q-weakly flat at x. Here,

$D \cap S(0,t)$ is a cap of a sphere if $D \cap S(0,t) = H \cap S(0,t)$, where H is an open half space in $\mathbb{R}^n$.

The following theorem extends some results from [31] and [29] given for open, discrete and closed mappings in $\mathbb{R}^n$.

THEOREM 1.3. *Let $q > 1$, $D \subset X$, $G \subset Y$ be domains, $x \in \partial D$, let Y be locally pathwise connected and G finitely connected at every point $y \in \partial G$ and suppose that D is locally q-weakly flat at x. Let $f : D \to G$ be continuous, open, discrete and closed satisfying condition (N^{-1}) such that $\mu(B_f) = 0$, $\nu(f(B_f)) = 0$ and there exists $U_x \in \mathcal{V}(x)$ such that $\int_{D \cap U_x} L(z,f)^q d\mu < \infty$. Then there exists $\lim_{y \to x} f(y) \in \overline{Y}$.*

Here, $\overline{Y}$ is the Alexandrov's compactification of Y.

The following theorems result immediately from Theorem 1.3.

THEOREM 1.4. *Let Y be locally pathwise connected and G finitely connected at every point $y \in \partial G$, let $D \subset X$, $G \subset Y$ be domains and let $f : D \to G$ be continuous, open, discrete and closed satisfying condition (N^{-1}) and such that $\mu(B_f) = 0$, $\nu(f(B_f)) = 0$. Suppose that for every $x \in \partial D$ there exists $q_x > 1$ and $U_x \in \mathcal{V}(x)$ such that D is locally q_x-weakly flat at x and $\int_{D \cap U_x} L(z,f)^{q_x} d\mu < \infty$. Then there exists $g : \overline{D} \to \overline{G}$ continuous such that $g|D = f$.*

THEOREM 1.5. *Let $n \geq 2$, $D \subset \mathbb{R}^n$ a domain, let Y be locally pathwise connected and $G \subset Y$ a domain finitely connected at every point $y \in \partial G$ and let $f : D \to G$ be continuous, open, discrete and closed satisfying condition (N^{-1}) and such that $\mu_n(B_f) = 0$, $\nu(f(B_f)) = 0$. Suppose that for every $x \in \partial D$ there exists $U_x \in \mathcal{V}(x)$, $r_x > 0$, $n - 1 < q_x \leq n$ such that $D \cap S(x,t)$ is a cap of a sphere for every $0 < t < r_x$ and $\int_{D \cap U_x} L(z,f)^{q_x} d\mu_n < \infty$. Then there exists $g : \overline{D} \to \overline{G}$ continuous such that $g|D = f$.*

Here, μ_n is the Lebesgue measure in $\mathbb{R}^n$.

2. PROOFS OF THE RESULTS

LEMMA 2.1. *Let Y be such that there exists a constant $c_1 > 0$ such that $\nu(B(y,r)) \geq \frac{r^n}{c_1}$ for every ball $B(y,r) \subset Y$ and let $c_2 > 0$ be such that $\mu(B(x,r)) \leq c_2 r^n$ for every ball $B(x,r) \subset X$. Let $D \subset X$, $G \subset Y$ be domains and $f : D \to G$ a homeomorphism such that there exist $m, M > 0$ such that $L(z,f) \leq M$, $l(z,f) \geq m$ for every $z \in D$ and let $q > 1$. Then*
$M_q(\Gamma) \leq C M_q(f(\Gamma))$ for every $\Gamma \in A(D)$.

Proof. We see that $\frac{\nu(f(B(x,r)))}{\mu(B(x,r))} \geq \frac{\nu(B(f(x),mr))}{c_2 r^n} \geq \frac{m^n r^n}{c_1 c_2 r^n} = \frac{m^n}{c_1 c_2}$ for every ball $B(x,r) \subset D$ and hence, $\nu'_f(x) \geq \frac{m^n}{c_1 c_2}$ for every $x \in D$.

Let $\Gamma \in A(D)$ and $\rho \in F(f(\Gamma))$ and let $\Delta = \{\gamma \in \Gamma | f \circ \gamma^0$ is absolutely continuous$\}$. Here, if $\gamma : [a,b] \to D$ is a rectifiable path, then $\gamma^0 : [0, l(\gamma)] \to D$ is given by the relation $\gamma = \gamma \circ s_\gamma$ and s_γ is the length function of γ. We see from [7] that $M_q(\Gamma) = M_q(\Delta)$.

Let $\gamma \in \Delta$. Using Lemma 2.3 from [7], we have

$$1 \leq \int_{f\circ\gamma} \rho ds \leq \int_\gamma \rho(f(x)) L(x,f) ds \leq M \int_\gamma \rho(f(x)) ds$$

and hence $M\rho \circ f \in F(\Delta)$. Using the change of variable formulae, we have

$$M_q(\Gamma) = M_q(\Delta) \leq M^q \int_D \rho^q(f(x)) d\mu \leq \frac{M^q c_1 c_2}{m^n} \int_D \rho^q(f(x)) \nu'_f(x) d\mu$$

$$\leq \frac{M^q c_1 c_2}{m^n} \int_{f(D)} \rho^q(y) d\nu.$$

Since $\rho \in F(f(\Gamma))$ was arbitrarily chosen and taking $C = \frac{M^q c_1 c_2}{m^n}$, we proved that $M_q(\Gamma) \leq C M_q(f(\Gamma))$. $\square$

Proof of Lemma 1.2. If Γ is a path family in $S(0,t)$ and $q > 1$, we set

$$M^q_{S(0,t)}(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{S(0,t)} \rho^q(z) d_{S(0,t)}.$$

Using [30] (Theorem 10.12, p. 28) and [2] (Theorem 3), we see that if C is a cap of sphere in $\mathbb{R}^n$ and $a, b \in C$, $a \neq b$, then

$$M^n_{S(0,t)}(\Delta(a,b,C)) \geq \frac{C(n)}{t},$$

$$M^q_{S(0,t)}(\Delta(a,b,C)) \geq \frac{C(n,q)}{t^{q-n+1}} \; if \; n-1 < q < n.$$

We can suppose that $0 < \epsilon < 1$. Let $0 < r < 1$ be such that $\mu_1(A_\rho) > \epsilon\rho$ if $q = n$, $\mu_1(A_\rho) > \epsilon\rho^{q-n+1}$ if $n-1 < q < n$ for every $0 < \rho \leq r$ and let $U \in \mathcal{V}(0)$ be such that $U \subset B(0,r)$, let $\delta = \inf_{z \in \partial U} |z| > 0$ and we can suppose that $0 < \delta < r$. Let $V = B(0, \frac{\delta\epsilon}{4})$ and $K_\delta = A_\delta \cap (\frac{\delta\epsilon}{4}, \delta)$. Then $\mu_1(K_\delta) \geq \mu_1(A_\delta) - \frac{\delta\epsilon}{4} > \frac{\mu_1(A_\delta)}{2}$. We can find points $a_t \in D \cap S(0,t) \cap E$, $b_t \in D \cap S(0,t) \cap F$ for every $t \in K_\delta$, where E, F are continua in $D \cap B_r$ which intersects ∂U and ∂V. Let $\Gamma \in \Delta(E, F, D \cap B_\delta)$ and $\rho \in F(\Gamma)$. Then, if $\Gamma_t = \Delta(a_t, b_t, D \cap S(0,t))$ for every $t \in K_\delta$, it results that $\rho | S(0,t) \in F(\Gamma_t)$ for every $t \in K_\delta$.

Suppose that condition (1.5) holds. Then

$$\int_{\mathbb{R}^n} \rho^n(z)d\mu_n \geq \int_{B_r} \rho^n(z)d\mu_n = \int_0^r \Big(\int_{S(0,t)} \rho^n(z)d_{S(0,t)} \Big) dt$$

$$\geq \int_{K_\delta} \frac{C(n)}{t} dt \geq \int_{K_\delta} \frac{C(n)}{\delta} dt = \frac{C(n)\mu_1(K_\delta)}{\delta} \geq \frac{C(n)\mu_1(A_\delta)}{2\delta} \geq \frac{C(n)\epsilon}{2}.$$

If condition (1.6) holds, then

$$\int_{\mathbb{R}^n} \rho^q(z)d\mu_n \geq \int_{B_r} \rho^q(z)d\mu_n = \int_0^r \Big(\int_{S(0,t)} \rho^q(z)d_{S(0,t)} \Big) dt$$

$$\geq \int_{K_\delta} \frac{C(n,q)}{t^{q-n+1}} dt \geq \int_{K_\delta} \frac{C(n,q)}{\delta^{q-n+1}} dt = \frac{C(n,q)\mu_1(K_\delta)}{\delta^{q-n+1}} \geq \frac{C(n,q)\mu_1(A_\delta)}{2\delta^{q-n+1}} \geq \frac{C(n,q)}{2}\epsilon.$$

Since $\rho \in F(\Gamma)$ was arbitrarily chosen, we proved that D is locally q-weakly flat at 0. □

Proof of Theorem 1.3. Let us show that $f : D \to G$ is a proper mapping. Let $K \subset G$ be compact and let $x_k \in f^{-1}(K)$, $k \in \mathbb{N}$. Taking eventually a subsequence, we can suppose that there exists $y \in K$ such that $y_k = f(x_k) \to y$. Let $A = \{x_k\}_{k\in\mathbb{N}}$. If $A' \cap D = \phi$, then A is closed in D and we can find $r_k \to 0$ and $a_k \in B(x_k, r_k)$ such that $f(a_k) \neq y$ for every $k \in \mathbb{N}$ and $f(a_k) \to y$ and $\overline{B}(x_k, r_k) \cap \overline{B}(x_p, r_p) = \phi$ for $k \neq p$, $k, p \in \mathbb{N}$. Let $B = \{a_k\}_{k\in\mathbb{N}}$. Then $B' \cap D = \phi$ and hence B is closed in D and $y \in \overline{f(B)} \setminus f(B)$ and this contradicts the fact that $f : D \to G$ is a closed mapping. It results that $A' \cap D \neq \phi$ and taking eventually a subsequence, we find $a \in D$ such that $x_k \to a$ and hence $f^{-1}(K)$ is compact. We proved that f is proper and hence $f^{-1}(y)$ is a finite set for every $y \in G$.

Suppose that $\lim_{y\to x} f(y) \in \overline{Y}$ does not exist. Then there exist $x_k, y_k \in D \cap U_x$, $x_k \to x$, $y_k \to x$ such that $f(x_k) \to z_1$, $f(y_k) \to z_2$ with $z_1 \neq z_2$ and since f is a closed mapping, we see that $z_1, z_2 \in \partial G$. Since G is finitely connected at z_1, z_2, we find $k_0 \in \mathbb{N}$, $R > 0$, $U_1 \in \mathcal{V}(z_1)$, $U_2 \in \mathcal{V}(z_2)$ such that $d(\overline{U_1}, \overline{U_2}) > R$ and components E_p of $U_p \cap G$, $p = 1, 2$ such that $f(x_k) \in E_1 \cap G$, $f(y_k) \in E_2 \cap G$ for every $k \geq k_0$ and we can take $k_0 = 1$. Since E_1 and E_2 are domains and Y is locally pathwise connected, we see that E_1, E_2 are pathwise connected.

Let $\epsilon > 0$ and $Q \in \mathcal{V}(x)$ with $\overline{Q} \subset U_x$ such that for every $U \in \mathcal{V}(x)$ with $\overline{U} \subset Q$ there exists $V \in \mathcal{V}(x)$ such that $\overline{V} \subset U$ and $M_q(\Delta(E, F, D \cap Q) > \epsilon$ for every continua E, F in $D \cap Q$ which intersects ∂U and ∂V. Let $U \in \mathcal{V}(x)$

be such that $\overline{U} \subset Q$ and $\int_{D\cap U} L(z,f)^q d\mu < \epsilon R^q$ and take $V \in \mathcal{V}(x)$ such that $\overline{V} \subset U$ and $M_q(\Delta(E,F,D\cap Q) > \epsilon$ for every continua E,F in $D\cap Q$ which intersects ∂U and ∂V.

Since $f^{-1}(f(x_1))$ is a finite set, we can suppose that $f^{-1}(f(x_1)) \cap \overline{U} = \phi$. Let $p_k : [0,1] \to E_1$ be a path such that $p_k(0) = f(x_k)$ and $p_k(1) = f(x_1)$. Since $f : D \to G$ lifts the paths (see [34], p. 186), we find a path $\tilde{q}_k : [0,1] \to D$ such that $\tilde{q}_k(0) = x_k$ and $f \circ \tilde{q}_k = p_k$ and taking x_k close enough to x, we can suppose that $x_k \in V$. Then $\tilde{q}_k(1) \notin \overline{U}$ and then $|\tilde{q}_k| \cap \partial U \neq \phi$, $|\tilde{q}_k| \cap \partial V \neq \phi$. We take q_k a subpath of $\tilde{q}_k$ such that $q_k(0) = x_k$, $|q_k| \subset \overline{U}$, $|q_k| \cap \partial U \neq \phi$, $|q_k| \cap \partial V \neq \phi$ and $|f \circ q_k| \subset |p_k|$.

In the same way, we find a path $\gamma_k : [0,1] \to D$ such that $\gamma_k(0) = y_k$, $|\gamma_k| \cap \partial U = \phi$, $|\gamma_k| \cap \partial V \neq \phi$ and $|f \circ \gamma_k| \subset |\lambda_k|$, where $\lambda_k : [0,1] \to E_2$ is a path such that $\lambda_k(0) = f(y_k)$ and $\lambda_k(1) = f(y_1)$. Let $\Gamma_k = \Delta(|q_k|, |\gamma_k|, D\cap U)$. We see from Theorem A that there exists $\omega_x \in L^1(Y)$ such that $M_q(\Gamma) \leq M^q_{\omega_x}(f(\Gamma))$ for every $\Gamma \in A(D\cap U)$ and $\int_{f(A)} \omega_x(z)d\nu \leq \int_A L(z,f)^q d\mu$ for every open set $A \subset D \cap U$. We also see that

$$\eta = \frac{1}{d(E_1,E_2)} \chi_{f(D\cap U)} \in F\big(f(\Gamma_k)\big).$$

We have

$$\epsilon < M_q(\Gamma_k) \leq M^q_{\omega_x}\big(f(\Gamma_k)\big) \leq \frac{1}{R^q} \int\limits_{f(D\cap U)} \omega_x(y)d\nu \leq \frac{1}{R^q} \int\limits_{D\cap U} L(z,f)^q d\mu < \epsilon$$

and we reached a contradiction.

We, therefore, proved that there exists $\lim_{y\to x} f(y) \in \overline{Y}$. □

Proof of Theorem 1.5. We see from Lemma 1.2 that D is locally q_x-weakly flat at every point $x \in \partial D$ and we apply Theorem 1.3. □

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