In the memory of Lucian Bădescu

FILTERED COLIMITS OF COMPLETE INTERSECTION ALGEBRAS

DORIN POPESCU

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This is mainly a small exposition on extensions of valuation rings.

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1. INTRODUCTION

After Zariski [25], a valuation ring V containing a field K of characteristic zero is a filtered union of its smooth K-subalgebras. When V contains a field K of positive characteristic this result may fail, as we believe. A possible support for this idea is given by Example 2.8.

Then we ask if V is a filtered union of its complete intersection K-subalgebras essentially of finite type. This could hold as our Theorem 3.9 hints. We remind that for a given complete intersection local ring $(R, \mathbf{m}, k)$, a minimal free resolution over R of the residue field k is described in [24, Theorem 4] and others as [3, Theorem 2.7], [4, Section 6], [7, Proposition 1.5.4], [8, Section 3], [5, Theorem 2.5], [11, Theorem 4.1]. As Oana Veliche asked, we wonder if a minimal free resolution of the residue field of V could be described using somehow Theorem 3.9.

2. HENSELIAN RINGS AND SMOOTH ALGEBRAS

A *Henselian* local ring is a local ring $(A, \mathbf{m})$ for which the Implicit Function Theorem hods, that is for every system of polynomials $f = (f_1, \ldots, f_r)$ of $A[Y]$, $Y = (Y_1, \ldots, Y_N)$, $N \geq r$ over A and every solution $y = (y_1, \ldots, y_N) \in A^N$ modulo $\mathbf{m}$ of f in A such that an $r \times r$-minor M of Jacobian matrix $(\partial f_i/\partial Y_j)_{1\leq i\leq r, 1\leq j\leq N}$ satisfies $M(y) \notin \mathbf{m}$ there exists a solution $\tilde{y}$ of $f = 0$ in A with $y \equiv \tilde{y}$ modulo $\mathbf{m}$.

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An A-algebra B is *smooth* if B is a localization of an A-algebra of type $(A[Y]/(f))_M$, where $f = (f_1, \ldots, f_r)$ is a system of polynomials in $Y = (Y_1, \ldots, Y_N)$, $N \geq r$ over A and M is an $r \times r$-minor of $(\partial f/\partial Y)$. Thus, $(A, \mathbf{m})$ is Henselian if for any smooth A-algebra B any A-morphism $B \to A/\mathbf{m}$ can be lifted to an A-morphism $w : B \to A$. So w is a retraction of the canonical morphism $u : A \to B$, that is $wu = 1_A$. A retraction of $A \to B$ maps the solutions of a system of polynomials $g = (g_1, \ldots, g_s) \in A[Y]^s$ in B in solutions of g in A.

We remind that a filtered colimit is a limit indexed by a small category that is filtered (see [22, 002V] or [22, 04AX]). A filtered union is a filtered direct limit in which all objects are subobjects of the final colimit, so that in particular, all the transition arrows are monomorphisms.

PROPOSITION 2.1. *Let $(A, \mathbf{m}) \to (A', \mathbf{m}')$ be a morhism of local rings with $A/\mathbf{m} \cong A'/\mathbf{m}'$, which is a filtered colimit of smooth A-algebras and $g \in A[Y]^s$ a system of polynomials which has solutions in A'. If $(A, \mathbf{m})$ is Henselian, then g has also solutions in A.*

Proof. Note that a solution y in A' of a system of polynomials g over A comes from a solution y' of g in a smooth A-algebra B, which is mapped in a solution of g in A by a retraction of $A \to B$. □

Examples of morphisms $u : (A, \mathbf{m}) \to (A', \mathbf{m}')$, which are filtered colimits of smooth A-algebras are given by the following theorems.

The following result is the General Neron Desingularization [13], [14], [23].

THEOREM 2.2. *Let $u : (A, \mathbf{m}) \to (A', \mathbf{m}')$ be a flat morphism of Noetherian local rings. The following statements are equivalent:*

1. *u is a filtered colimit of smooth A-algebras.*

2. *u is regular, that is for every prime ideal $\mathbf{p}$ of A and every finite field extension K of the fraction field of $A/\mathbf{p}$ the ring $K \otimes_{A/\mathbf{p}} A'$ is regular.*

Let $(A, \mathbf{m})$ be a Noetherian local ring and $\hat{A}$ its completion in the $\mathbf{m}$-adic topology. A is *excellent* if the completion map $A \to \hat{A}$ is regular. The following corollary holds by Proposition 2.1.

COROLLARY 2.3 ([13]). *Let $(A, \mathbf{m})$ be an excellent Henselian local ring and $\hat{A}$ its completion. Then every system of polynomials over A which has a solution in $\hat{A}$ has also one in A. In particular, A has the property of Artin approximation; this was conjectured by M. Artin in [2].*

THEOREM 2.4 ([25]). *A valuation ring containing a field K of characteristic zero is a filtered union of its smooth K-subalgebras.*

In the same frame, the following result holds.

THEOREM 2.5 ([18, Theorem 2]). *Let $V \subset V'$ be an extension of valuation rings containing $\mathbf{Q}$, $K \subset K'$ its fraction field extension, $\Gamma \subset \Gamma'$ the value group extension of $V \subset V'$ and $\mathbf{val} : K'^* \to \Gamma'$ the valuation of V'. Then V' is a filtered colimit of smooth V-algebras if and only if the following statements hold:*

1. *For each prime ideal q of V, the ideal qV' is prime.*

2. *For any prime ideals q_1, q_2 of V such that $q_1 \subset q_2$ and height$(q_2/q_1) = 1$ and any $x' \in q_2V' \setminus q_1'$, there exists $x \in V$ such that $\mathbf{val}(x') = \mathbf{val}(x)$, where the prime ideal q_1' of V' is the prime ideal corresponding to the maximal ideal of $V_{q_1} \otimes_V V'$, that is the maximal prime ideal of V' lying on q_1.*

An *immediate extension* of valuation rings is an extension inducing trivial extensions on residue fields and group value extensions. As a consequence of Theorem 2.5, we can prove the following.

COROLLARY 2.6 ([16, Theorem 21]). *If $V \subset V'$ is an immediate extension of valuation rings containing $\mathbf{Q}$ then V' is a filtered colimit of smooth V-algebras.*

The following corollary holds by Proposition 2.1.

COROLLARY 2.7 ([18, Proposition 18]). *Let $V \subset V'$ be an extension of valuation rings containing $\mathbf{Q}$ with the same residue field, $K \subset K'$ its fraction field extension, $\Gamma \subset \Gamma'$ the value group extension of $V \subset V'$ and $\mathbf{val} : K'^* \to \Gamma'$ the valuation of V'. Assume that V is Henselian and the statements* (1), (2) *of Theorem 2.5 hold (for example, when $V \subset V'$ is immediate). Then every system of polynomials over V which has a solution in V' has also one in V.*

Let V be a valuation ring, λ be a fixed limit ordinal and $v = \{v_i\}_{i<\lambda}$ a sequence of elements in V indexed by the ordinals i less than λ. Then v is called *pseudo-convergent* if

$$\mathbf{val}(v_i - v_{i''}) < \mathbf{val}(v_{i'} - v_{i''}) \text{ for } i < i' < i'' < \lambda$$

(see [9], [21]). A *pseudo-limit* of v is an element $w \in V$ with

$$\mathbf{val}(w - v_i) < \mathbf{val}(w - v_{i'}) \text{ (that is, } \mathbf{val}(w - v_i) = \mathbf{val}(v_i - v_{i'})) \text{ for } i < i' < \lambda.$$

The following example shows that Corollary 2.6 fails in positive characteristic.

Example 2.8 ([15, Example 3.1.3], [12], [17]). Let k be a field of characteristic $p > 0$, X a variable, $\Gamma = \mathbf{Q}$ and K the fraction field of the group algebra $k[\Gamma]$, that is the rational functions in $(X_q)_{q\in\mathbf{Q}}$. Let P be the field of all formal sums $z = \sum_{n\in\mathbf{N}} c_n X^{\alpha_n}$, where $(\alpha_n)_{n\in\mathbf{N}}$ is a monotonically increasing sequence from Γ and $c_n \in k$. Set $\mathbf{val}(z) = \alpha_s$, where $s = \min\{n \in \mathbf{N} : c_n \neq 0\}$ if $z \neq 0$ and let V be the valuation ring defined by $\mathbf{val} : P^* \to \Gamma$, $z \to \mathbf{val}(z)$.

Let

$$\rho_n = (p^{n+1} - 1)/(p-1)p^{n+1}, \quad y = -1 + \sum_{n\geq 0} (-1)^n X^{\rho_n}$$
$$\text{and } a_i = -1 + \sum_{0\leq n\leq i} (-1)^n X^{\rho_n}.$$

We have $1 + \rho_n = p(\rho_{n+1})$ for $n \geq 0$ and $p\rho_0 = 1$ and y is a pseudo-limit of the pseudo-convergent sequence $a = (a_i)_{i\in\mathbf{N}}$, which has no pseudo-limit in K. Then y is a root of the separable polynomial $g = Y^p + XY + 1 \in K[Y]$ and the algebraic separable extension $V_0 = V \cap K \subset V_1 = V \cap K(y)$ is not dense, in particular V_1 is not a filtered colimit of smooth V_0-algebras (apply [15, Theorem 6.9], or [17, Theorem 2]).

However, there exists an important result when char $k = p > 0$.

THEOREM 2.9 ([1, Theorem 4.1.1]). *Every perfect valuation ring of characteristic $p > 0$ is a filtered union of its smooth $\mathbf{F}_p$-subalgebras.*

We mention also the following result.

THEOREM 2.10 ([19]). *Let V be a one dimensional valuation ring with value group Γ containing its residue field and U a set with* $\text{card}(U) > \text{card}(\Gamma)$. *Then there exists an ultrafilter $\mathcal{U}$ on U such that taking the ultrapower $\tilde{V}$ of V with respect to $\mathcal{U}$ and $\bar{V} = \tilde{V}/\cap_{z\in V, z\neq 0} z\tilde{V}$, the following assertions hold:*

1. *$\bar{V}$ contains the completion of V.*

2. *If V' is an immediate extension of V contained in $\bar{V}$ with V'/V separable then V' is a filtered colimit of smooth V-algebras.*

3. COMPLETE INTERSECTION ALGEBRAS

For a general immediate extension of valuation rings $V \subset V'$, we should expect that V' is a filtered union of its complete intersection V-subalgebras. In the Noetherian case, a morphism of rings is a filtered direct limit of smooth algebras if and only if it is a regular morphism (see Theorem 2.2).

A *complete intersection* V-algebra *essentially of finite type* is a local V-algebra of type $C/(P)$, where C is a localization of a polynomial V-algebra of finite type and P is a regular sequence of elements of C. Theorem 3.8 stated below says that V' is a filtered union of its V-subalgebras of type $C/(P)$. Since V' is local, it is enough to say that V' is a filtered union of its V-subalgebras of type $T_h/(P)$, T being a polynomial V-algebra of finite type, $0 \neq h \in T$ and P is a regular sequence of elements of T. Clearly, T_h is a smooth V-algebra and, in fact, it is enough to say that V' is a filtered union of its V-subalgebras of type $G/(P)$, where G is a smooth V-algebra of finite type and P is a regular sequence of elements of G. Conversely, a V-algebra of such type $G/(P)$ has the form $T_h/(P)$ for some T, h, P using [23, Theorem 2.5]. By abuse, we understand by a *complete intersection* V-algebra of finite type a V-algebra of such type $G/(P)$, or $T_h/(P)$ which are not assumed to be flat over V.

The following lemma is a variant of Ostrowski ([12, Chapters IV and III], see also [21, (II,4), Lemma 8] and [20, Lemma 2]).

LEMMA 3.1 ([12]). *Let $\beta_1, \dots, \beta_m$ be any elements of an ordered abelian group G, λ a limit ordinal and let $\{\gamma_s\}_{s<\lambda}$ be a well-ordered, monotone increasing set of elements of G, without a last element. Let $t_1, \dots, t_m$, be distinct integers. Then there exists an ordinal $\nu < \lambda$ such that $\beta_i + t_i\gamma_s$ are different for all $s > \nu$. Also, there exists an integer $1 \leq r \leq m$ such that*

$$\beta_i + t_i\gamma_s > \beta_r + t_r\gamma_s$$

for all $i \neq r$ and $s > \nu$.

The following two results use the above lemma.

LEMMA 3.2 ([20, Lemma 3]). *Let $V \subset V'$ be an immediate extension of valuation rings, $K \subset K'$ its fraction field extension,* **val** *the valuation of V' and $(v_i)_{i<\lambda}$ an algebraic pseudo-convergent sequence in V, which has a pseudo-limit x in V', but no pseudo-limit in K. Set $x_i = (x-v_i)/(v_{i+1}-v_i)$. Let $s \in \mathbf{N}$ be the minimal degree of the polynomials $f \in V[Y]$ such that $\mathbf{val}(f(v_i)) < \mathbf{val}(f(v_j))$ for large $i < j < \lambda$ and $g \in V[Y]$ a polynomial with $\deg g < s$. Then there exist $d \in V \setminus \{0\}$ and $u \in V[x_i]$ for some $i < \lambda$ with $g(x) = du$ and $\mathbf{val}(u) = 0$.*

LEMMA 3.3 ([20, Lemma 4]). *Let $V \subset V'$ be an immediate extension of valuation rings, $K \subset K'$ its fraction field extension,* **val** *the valuation of V' and $(v_i)_{i<\lambda}$ an algebraic pseudo-convergent sequence in K, which has a pseudo-limit x in V', algebraic over V, but no pseudo-limit in K. Assume that $h =$Irr(x, K) is from $V[X]$. Then*

1. $\mathbf{val}(h(v_i)) < \mathbf{val}(h(v_j))$ *for large $i < j < \lambda$,*

2. *if h has minimal degree among the polynomials $f \in V[X]$ such that $\mathbf{val}(f(v_i)) < \mathbf{val}(f(v_j))$ for large $i < j < \lambda$, then $V'' = V' \cap K(x)$ is a filtered union of its complete intersection V-subalgebras.*

Remark 3.4. The extension $V \subset V''$ from (2) of the above lemma is isomorphic with the one constructed in [9, Theorem 3].

LEMMA 3.5 ([10, Lemma 5.2]). *Assume that $(v_j)_j$ is a pseudo-convergent sequence in V with a pseudo-limit x in an extension V' of V but with no pseudo-limit in V. Let $f \in V[X]$ be a polynomial. Then $(f(v_j))_j$ is ultimately pseudo-convergent and $f(x)$ is a pseudo-limit of it.*

An interesting situation is given below.

Example 3.6 (Anonymous Referee of a previous paper). Let $R = \mathbf{F}_2[[t]]$ be the formal power series ring in t over $\mathbf{F}_2 = \mathbf{Z}/2\mathbf{Z}$. Then the fraction field $\mathbf{F}_2((t))$ of R has a t-adic valuation which has a canonical extension denoted by $\mathbf{val}$ to the perfect hull $K = \mathbf{F}_2((t))^{1/2^\infty}$ of $\mathbf{F}_2((t))$. Let V be the valuation ring defined by $\mathbf{val}$ on K. Let a be a root of the polynomial $f = X^2 - X - (1/t) \in K[X]$. Then the extension of $\mathbf{val}$ to $K(a)$ gives an immediate extension $K \subset K(a)$ with a value group Γ. This extension is closed to an example of [12] (see Example 2.8). We can assume that the root a is a pseudo-limit of the pseudo-convergent sequence $(a_n)_{n\in\mathbf{N}}$ given by

$$a_n = \sum_{i=1}^{n} t^{-2^{-i}},$$

which has no pseudo-limit in K. Note that $\mathbf{val}(a_n) < 0$ and so $a_n \notin V$ for all $n \in \mathbf{N}$.

Let $0 < \alpha \in \Gamma$ and $b = (b_j)_{j<\lambda}$ be a transcendental pseudo-convergent sequence over $K(a)$ such that $\mathbf{val}(b_j) = \alpha$ for high enough j. By [9, Theorem 2], there exists an unique immediate transcendental extension $K(a) \subset L = K(a, z)$ such that z is a pseudo-limit of b (the valuation of L is still denoted by $\mathbf{val}$). Then $\mathbf{val}(z) = \alpha > 0$. Set $x = z + a \in L$. We have

$$\begin{aligned}\mathbf{val}\big(f(x)\big) &= \mathbf{val}\big((x-a)^2 - (x-a) + a^2 - a - (1/t)\big)\\ &= \mathbf{val}(x-a) + \mathbf{val}(x-a-1) = \alpha + 0 > 0.\end{aligned}$$

Note that

$$\mathbf{val}(x - a_n) = \min\big\{\mathbf{val}(x-a), \mathbf{val}(a-a_n)\big\} = \mathbf{val}(a_{n+1} - a_n)$$

because $\mathbf{val}(x-a) = \alpha > 0$ and $\mathbf{val}(a - a_n) = \mathbf{val}(a_{n+1} - a_n) < 0$, that is x is also a pseudo-limit of $(a_n)_n$. Also, we see that $\mathbf{val}(a_n^2 - a_m - (1/t)) < 0$ for all $n, m \in \mathbf{N}$ with $m \neq n-1$ and $a_n^2 - a_m - (1/t) = 0$ when $m = n-1$.

Consequently, $\mathbf{val}(a_n^2 - a_m - (1/t))$ is either < 0, or it is ∞ for all $n, m \in \mathbf{N}$ and it follows that $\mathbf{val}(a_n^2 - a_m - (1/t))$ is never $\mathbf{val}(f(x))$.

We see that the ultimately pseudo-convergent sequence $(a_n^2)_{n\in\mathbf{N}}$ (using Lemma 3.5) has no pseudo-limit in K because $K^2 = K$ and a pseudo-limit of (a_n^2) in K would give a pseudo-limit of (a_n) in K, which is false. Set $y_0 = x$, $y_1 = y_0^2$, $g = tY_1 - tY_0 - 1 \in K[Y_0, Y_1]$. We have $g(y_0, y_1) = tf(x)$. Then $\mathbf{val}(g(a_n^2, a_m))$, $n, m \in \mathbf{N}$ is never $\mathbf{val}(g(y_0, y_1))$ because $\mathbf{val}(a_n^2 - a_m - (1/t))$ is never $\mathbf{val}(f(x))$ as we have seen. So Lemma 3.2 does not work for polynomials in two variables.

Lemma 3.7 ([20, Lemma 6]). *Let B be a complete intersection algebra over a ring A and C a complete intersection algebra over B. Then C is a complete intersection algebra over A.*

Using Lemma 3.3 and the above lemma, we proved the following result.

Theorem 3.8 ([20, Theorem 1]). *Let V' be an immediate extension of a valuation ring V and $K \subset K'$ the fraction field extension. If K'/K is algebraic then V' is a filtered union of its complete intersection V-subalgebras of finite type.*

As a consequence, the next theorem follows.

Theorem 3.9. *Let V be a valuation ring containing its residue field k with a free value group (for example, when it is finitely generated) Γ and the valuation $\mathbf{val} : K^* \to \Gamma$. Assume that y is a system of elements of V such that $\mathbf{val}(y)$ is a basis of Γ and the fraction field of V is an algebraic extension of $k(y)$. Then V is a filtered union of its complete intersection k-subalgebras of finite type.*

Proof. By [16, Lemma 27 (1)], we see that V is an immediate extension of the valuation ring $V_0 = V \cap k(y)$, which is a filtered union of localizations of its polynomial k-subalgebras (see [6, Theorem 1, in VI (10.3)], or [16, Lemma 26 (1)]). In fact, important here is that there exists a cross-section $s : \Gamma \to K^*$ (a *cross-section* of V is a section in the category of abelian groups of the valuation map $\mathbf{val}$). Since Γ is free, we define s by $\mathbf{val}(y) \to y$. Now it is enough to apply Theorem 3.8 for $V_0 \subset V$. □

The above theorem connects the theory of valuation rings with the theory of complete intersection local rings. We remind that for a given complete intersection local ring $(R, \mathbf{m}, k)$ a minimal free resolution over R of the residue field k is described in [24, Theorem 4] and others as [3, Theorem 2.7], [4, Section 6], [7, Proposition 1.5.4], [8, Section 3], [5, Theorem 2.5], [11, Theorem 4.1].

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Simion Stoilow Institute of Mathematics
of the Romanian Academy
P.O. Box 1-764, RO-014700 Bucharest, Romania
dorin.m.popescu@gmail.com
and
University of Bucharest
Faculty of Mathematics and Computer Science
Str. Academiei 14, Bucharest 1, RO-010014, Romania