Dedicated to the memory of Lucian Bădescu.
He would have celebrated his 80th birthday in 2024

SUCCESSIVE VANISHING ON CURVES

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Communicated by Marian Aprodu

We investigate the emptiness of adjoint linear systems associated to successive multiples of a given positive divisor with real coefficients.

AMS 2020 Subject Classification: 14E30, 14C20.

Key words: curves, vanishing.

1. INTRODUCTION

We consider in this note the following question: consider triples (X, L, N), where X is a complex projective nonsingular algebraic variety, L is a nef and big $\mathbb{R}$-divisor on X, $\operatorname{supp}(\lceil L \rceil - L)$ is a normal crossings divisor on X, and N is a positive integer such that

$$|\lceil K_X + iL \rceil| = \emptyset \text{ for all } 1 \leq i \leq N.$$

Given N (sufficiently large), can we classify the pairs (X, L) with this property? What can be said about L as N approaches $+\infty$? For example, a necessary condition on curves is $N \deg(L) \leq 1$. Note that the question does not change if we replace X by a higher model and L by its pullback. Also, we may suppose X admits no fibration such that the restriction of L to the general fiber satisfies the same properties.

Here, $\lceil L \rceil$ denotes the round up of an $\mathbb{R}$-divisor, defined componentwise. If $\lceil L \rceil - L = 0$, then $N \leq \dim X$ by Kawamata–Viehweg vanishing and the classical argument that a polynomial has no more roots than its degree. So the difficulty seems to be hidden in the fractional part $\lceil L \rceil - L$. If the coefficients of $\lceil L \rceil - L$ are rational with bounded denominators, then N is again bounded above.

Our motivation is to understand when $|iK_X| = \emptyset$ for all $1 \leq i \leq N$, where X is a complex projective nonsingular variety of general type. It is known that N is bounded above only in terms of $\dim X$ [2, 3, 7]. A positive answer to the

REV. ROUMAINE MATH. PURES APPL. **70** (2025), *1-2*, 3–17
doi: 10.59277/RRMPA.2025.3.17

question might give explicit bounds. In relation with the question above, we have the inclusions

$$\Gamma\Big(X_l, \Big\lceil K_{X_l} + (i-1)\frac{M_l}{l}\Big\rceil\Big) \subseteq \Gamma(X, iK_X)$$

where $X_l \to X$ is a sufficiently high resolution and M_l is the mobile part on X_l of the linear system $|lK_X|$. The inclusions are equalities if l is divisible by i.

In this note, we solve the question in dimension one (Theorem 3.2), and give an application when L is a positive multiple of a log divisor (Theorem 3.5). We also solve the similar problem when successive adjoint linear systems contain a given point in the base locus (Theorem 4.8). Other variants are possible, such as failure of successive adjoint linear systems to be mobile, to generate a fixed order of jets at a given point, etc.

Two observations come out from the curve case. First, for $N > 1$, only the highest two coefficients of $\lceil L\rceil - L$ matter. Second, the Farey set of given order appears naturally. In fact, effective non-vanishing properties for $\mathbb{R}$-divisors can be restated in terms of divisors with coefficients in the Farey set of a given order. A similar successive failure argument was used by Shokurov [6, Example 5.2.1] to construct lc n-complements on curves, with $n \in \{1, 2, 3, 4, 6\}$. In his setup, with $L = -K - B$ and i starting at 2, only the highest four coefficients of B matter, and it suffices to consider coefficients of B only in the Farey set of order 5.

It is very likely that our question can be solved for surfaces (cf. [1]).

2. ESTIMATES

For a positive integer N, consider the Farey set of order N defined by

$$\mathcal{F}_N = [0,1] \bigcap \bigcup_{i=1}^{N} \frac{1}{i}\mathbb{Z}.$$

The following properties hold:

- $x \in \mathcal{F}_N$ if and only if $1 - x \in \mathcal{F}_N$.
- The finite set $\mathcal{F}_N$ decomposes the interval $[0, 1)$ into finitely many disjoint intervals $[x, x')$. For $\delta \in [0, 1)$, the unique interval which contains δ is determined by the formulas $x = \max_{1\le i\le N} \frac{\lfloor i\delta\rfloor}{i}, x' = \min_{1\le i\le N} \frac{1+\lfloor i\delta\rfloor}{i}$. Denote x' by δ_N^+.

Lemma 2.1. *Let* $0 \le x < x' \le 1$. *Then* $(x, x') \cap \mathcal{F}_N = \emptyset$ *if and only if* $\lfloor ix\rfloor + \lfloor i(1 - x')\rfloor = i - 1$ *for every* $1 \le i \le N$.

Proof. We have $\lfloor ix \rfloor + \lfloor i(1-x') \rfloor \leq ix + i(1-x') < i$. Therefore,

$$\lfloor ix \rfloor + \lfloor i(1-x') \rfloor \leq i-1.$$

We notice that $(ix, ix') \cap \mathbb{Z} = \emptyset$ if and only if $\lfloor ix \rfloor + 1 \geq \lceil ix' \rceil$, that is $\lfloor ix \rfloor + \lfloor i(1-x') \rfloor \geq i-1$. So, $(ix, ix') \cap \mathbb{Z} = \emptyset$ if and only if $\lfloor ix \rfloor + \lfloor i(1-x') \rfloor = i-1$.

Finally, $(x, x') \cap \mathcal{F}_N = \emptyset$ if and only if $(ix, ix') \cap \mathbb{Z} = \emptyset$ for every $1 \leq i \leq N$. □

LEMMA 2.2. *Let $x < x'$ be two consecutive elements of $\mathcal{F}_N$. Then:*

1) $x' - x \leq \frac{1}{N}$.

2) *If $x' - x \geq \frac{1}{N+1}$, then $x = 0$ or $x' = 1$.*

Proof. We have two cases. Either the $\mathcal{F}_N$-interval is $[0, \frac{1}{N})$ or $[\frac{N-1}{N}, 1)$, of length $\frac{1}{N}$, or $x = \frac{p}{q}, x' = \frac{p'}{q'}$, where p, q, p', q' are positive integers such that $p'q - pq = 1$, $\min(q, q') \geq 2$ and $\max(q, q') \leq N < q + q'$. We have $(q-1)(q'-1) > 1$, since it is not possible that both q and q' equal 2. Therefore $qq' \geq q + q' + 1 \geq N + 2$. Then $x' - x = \frac{1}{qq'} \leq \frac{1}{N+2}$. □

LEMMA 2.3. *Let $N \geq 2$, $0 \leq b < 1$, $\frac{1}{2} \leq \delta < \frac{N-1+b}{N}$. Then there exist $1 \leq p < q \leq N$ such that $\delta < \frac{p+b}{q}$ and $\frac{p}{q} - \delta < \frac{1}{N+1}$.*

Proof. Suppose $\delta \geq \frac{N-1}{N}$. Then, we can take $p = N-1, q = N$. Suppose $\delta < \frac{N-1}{N}$. Let $\delta \in [x, x')$ be the unique half-open $\mathbb{Z}_N$-interval which contains it. We have $0 < x < x' = \frac{p}{q} < 1$. By Lemma 2.2.(2), $x' - x < \frac{1}{N+1}$. Then $\delta < \frac{p}{q} \leq \frac{p+b}{q}$ and $\frac{p}{q} - \delta \leq x' - x < \frac{1}{N+1}$. □

LEMMA 2.4. *Let $N \geq 1$ and $b \in [0, 1)$.*

1) $\frac{N-1+b}{N} \leq x < 1$ *if and only if $\lfloor ix - b \rfloor = i - 1$ for all $1 \leq i \leq N$.*

2) *If $N \geq 2$ and $\frac{N-2+b}{N-1} \leq x < \frac{N-1+b}{N}$, then $\lfloor Nx - b \rfloor = N - 2$.*

Proof. 1) The implication $\Longleftarrow$ is clear. Now consider the converse. Let $1 \leq i \leq N$. We have $i - \frac{i+(N-i)b}{N} \leq ix - b < i - b$. Since $b < 1$, $i + (N-i)b < i$. Therefore $i - 1 < ix - b < i$. Therefore $\lfloor ix - b \rfloor = i - 1$.

2) We have the following: $N - 1 - \frac{1-b}{N-1} \leq Nx - b < N - 1$. Therefore $\lfloor Nx - b \rfloor = N - 2$. □

PROPOSITION 2.5. *Let $0 \leq b \leq \delta < 1$, $0 \leq b' \leq \delta' < 1$, $\delta' \leq \delta$ and $\delta + \delta' \leq 1$. Let $N \geq 2$. Then*

$$\lfloor i\delta - b \rfloor + \lfloor i\delta' - b' \rfloor \geq i - 1 \text{ for all } 1 \leq i \leq N$$

if and only if one of the following holds:

a) $\delta \geq \frac{N-1+b}{N}$, *or*

b) $\frac{1+b}{2} \leq \delta < \frac{N-1+b}{N}$ *and* $\delta' \geq \max\{\frac{q-p+b'}{q}; 1 \leq p < q \leq N, \delta < \frac{p+b}{q}\}$.

Moreover, $\delta \geq 1 - \frac{1}{N}$ *in case a). In case b),* $\delta + \delta' > 1 - \frac{1}{N+1}$ *and* $\delta' \geq \frac{1}{N}$ *(in particular,* $\delta + 2\delta' > 1$*). And* $\delta' = \delta$ *if and only if* $\delta' = \delta = \frac{1}{2}, b = 0$.

Proof. Let $N = 2$. The system of inequalities becomes $\lfloor 2\delta - b \rfloor + \lfloor 2\delta' - b' \rfloor \geq 1$. That is $2\delta - b \geq 1$ or $2\delta' - b' \geq 1$. If $2\delta - b \geq 1$, we are in case a). If $2\delta' - b' \geq 1$, then $\delta' \geq \frac{1}{2}$. Then $\delta \geq \frac{1}{2}$. Then $\delta = \delta' = \frac{1}{2}$ and $b = b' = 0$. We are in case a).

Let $N \geq 3$. Suppose $\delta \geq \frac{N-1+b}{N}$. By Lemma 2.4.1), $\lfloor i\delta - b \rfloor = i - 1$ for all $1 \leq i \leq N$. The system of inequalities is satisfied. Suppose $\delta < \frac{N-1+b}{N}$. From the case $N = 2$, we obtain

$$\frac{1+b}{2} \leq \delta < \frac{N-1+b}{N}.$$

Suppose the system of inequalities is satisfied. Let $1 \leq p < q \leq N$ with $\delta < \frac{p+b}{q}$. Then $p > q\delta - b$, that is $p - 1 \geq \lfloor q\delta - b \rfloor$. The inequality for $i = q$ gives $p - 1 + \lfloor q\delta' - b' \rfloor \geq q - 1$. Therefore $\lfloor q\delta' - b' \rfloor \geq q - p$. That is $q\delta' - b' \geq q - p$. Therefore $\delta' \geq \frac{q-p+b'}{q}$. So b) holds.

Conversely, suppose b) holds. Let $1 \leq i \leq N$. Let $p = 1 + \lfloor i\delta - b \rfloor$. If $p = i - 1$, then $\lfloor i\delta - b \rfloor + \lfloor i\delta' - b' \rfloor \geq \lfloor i\delta - b \rfloor = i - 1$. If $p < i$, then $1 \leq p < i \leq N$ and $\delta < \frac{p+b}{i}$. By b) for $q = i$, we deduce $\delta' \geq \frac{i-p+b'}{i}$. Then $\lfloor i\delta' - b' \rfloor \geq i - p$. Therefore $\lfloor i\delta - b \rfloor + \lfloor i\delta' - b' \rfloor \geq p - 1 + i - p = i - 1$. We conclude that the system of inequalities holds if it is equivalent to a) or b).

In case a), $\delta \geq \frac{N-1+b}{N} \geq 1 - \frac{1}{N}$. Consider case b). We have $N \geq 3$. By Lemma 2.3, there exists $1 \leq p < q \leq N$ such that $\delta < \frac{p+b}{q}$ and $\frac{p}{q} - \delta < \frac{1}{N+1}$. By b), $\delta' \geq \frac{q-p+b'}{q}$. Therefore

$$\delta + \delta' \geq \delta + \frac{q-p}{q} > \frac{p}{q} - \frac{1}{N+1} + \frac{q-p}{q} = 1 - \frac{1}{N+1}.$$

From $\delta < \frac{N-1+b}{N}$, we deduce $\delta' \geq \frac{1+b'}{N}$. In particular, we have that $\delta' \geq \frac{1}{N}$ and $\delta + 2\delta' > 1 + \frac{1}{N(N+1)}$.

Suppose $\delta' = \delta$. Since $\delta \geq \frac{1+b}{2}$ and $\delta + \delta' \leq 1$, we deduce $\delta = \delta' = \frac{1}{2}$ and $b = 0$. □

Remark 2.6. Let $b \in [0, 1)$, let $N \geq 1$. Consider the totally ordered finite set $[0, 1) \cap \{\frac{p+b}{q}; 1 \leq q \leq N, p \in \mathbb{N}\}$. The maximal element is $\frac{N-1+b}{N}$. If $N = 1$, this is the only element. If $N \geq 2$, the next largest element is $\frac{N-2+b}{N-1}$.

Corollary 2.7. *Let $0 \le \delta' \le \delta < 1, \delta + \delta' \le 1$, and $N \ge 2$. Then $\lfloor i\delta \rfloor + \lfloor i\delta' \rfloor \ge i-1$ for all $1 \le i \le N$ if and only if $\delta_N^+ + \delta' \ge 1$.*

Proposition 2.8. *Let $0 \le B \le \Delta$ be $\mathbb{R}$-divisors on a nonsingular curve such that $\lfloor \Delta \rfloor = 0$ and $\deg \Delta \le 1$. Let P be a point where Δ attains its maximal multiplicity, let P' be a point where $\Delta' = \Delta - \delta_P P$ attains its maximal multiplicity. Denote $\Delta'' = \Delta - \delta_P P - \delta_{P'} P'$. Let $N \ge 2$.*

Then $\deg \lfloor i\Delta - B \rfloor \ge i-1$ for all $1 \le i \le N$ if and only if $\lfloor N\Delta'' \rfloor = 0$ and $\lfloor i\delta_P - b_P \rfloor + \lfloor i\delta_{P'} - b_{P'} \rfloor \ge i-1$ for all $1 \le i \le N$.

Proof. Suffices to show $\lfloor N\Delta'' \rfloor = 0$. We use induction on N.

Let $N = 2$. Suppose by contradiction $\lfloor 2\Delta'' \rfloor \ne 0$. Then Δ'' has a coefficient $\delta'' \ge \frac{1}{2}$ and $\delta_P \ge \delta_{P'} \ge \delta'' \ge \frac{1}{2}$. Then $\deg \Delta \ge \frac{3}{2} > 1$, a contradiction. Therefore $\lfloor 2\Delta'' \rfloor = 0$.

Let $N > 2$. Suppose $\delta_P \ge \frac{N-1+b_P}{N}$. In particular, $\deg \Delta' \le \frac{1}{N}$. If $\lfloor N\Delta' \rfloor \ne 0$, then $\delta_{P'} \ge \frac{1}{N}$. Therefore $\Delta = \frac{N-1}{N}P + \frac{1}{N}P'$ and $B \le \frac{1}{N}P'$. Here, $\Delta'' = 0$. If $\lfloor N\Delta' \rfloor = 0$, then $\lfloor N\Delta'' \rfloor = 0$.

Suppose $\delta_P < \frac{N-1+b_P}{N}$.

Case $\frac{N-2+b_P}{N-1} \le \delta_P$. Then $\lfloor N\delta_P - b_P \rfloor = N-2$. Denote $B' = B - b_P P$. Our system of inequalities becomes $\deg \lfloor N\Delta' - B' \rfloor \ge 1$. That is $\lfloor N\Delta' - B' \rfloor \ne 0$. That is $N\delta_Q - b_Q \ge 1$ at some point $Q \in \operatorname{supp} \Delta'$. We have

$$\delta_P + \delta_{P'} \ge \delta_P + \delta_Q \ge \frac{N-2}{N-1} + \frac{1}{N} = 1 - \frac{1}{(N-1)N}.$$

From $(N-1)N > N$, we deduce $\delta_P + \delta_{P'} > 1 - \frac{1}{N}$. Therefore, $\deg \Delta'' < \frac{1}{N}$. Following, we have $\lfloor N\Delta'' \rfloor = 0$.

Case $\delta_P < \frac{N-2+b_P}{N-1}$. By induction, $\lfloor i\delta_P - b_P \rfloor + \lfloor i\delta_{P'} - b_{P'} \rfloor \ge i-1$ for all $1 \le i \le N-1$. By case $b)_{N-1}$ of Proposition 2.5, $\delta_P + \delta_{P'} > 1 - \frac{1}{(N-1)+1}$. Therefore, $\deg \Delta'' < \frac{1}{N}$ and $\lfloor N\Delta'' \rfloor = 0$. $\square$

Corollary 2.9. *Suppose $\Delta \ge 0$, $\lfloor \Delta \rfloor = 0$ and $\deg \Delta \le 1$. Let P be a point where Δ attains its maximal multiplicity δ. Let P' be a point where $\Delta' = \Delta - \delta P$ attains its maximal multiplicity δ'. Denote $\Delta'' = \Delta - \delta P - \delta' P'$. Let $N \ge 2$.*

Then $\deg \lfloor i\Delta \rfloor \ge i-1$ for all $1 \le i \le N$ if and only if $\lfloor N\Delta'' \rfloor = 0$ and $\delta_N^+ + \delta' \ge 1$.

2.1. Case $N = +\infty$

Let Δ be an effective $\mathbb{R}$-divisor on a nonsingular curve. Let δ be the higher multiplicity of Δ, attained at P say. Let δ' be the highest multiplicity of $\Delta' = \Delta - \delta P$. Note $\delta \ge \delta'$.

Lemma 2.10. *Suppose* $\deg \Delta \leq 1$. *Then* $\deg\lfloor i\Delta \rfloor \geq i-1$ *for all* $i \geq 1$ *if and only if* $\Delta = P$ *or* $\Delta = \delta P + (1-\delta)P'$ *for some* $\delta \in \left[\frac{1}{2}, 1\right)$.

Proof. Let $\lfloor \Delta \rfloor \neq 0$. That is $\Delta \geq P$ for some P. That is $\Delta = P$. The inequalities are satisfied. Let $\lfloor \Delta \rfloor = 0$. From above, the highest two coefficients of Δ satisfy $\delta + \delta' \geq 1 - \frac{1}{N}$, with $N \to \infty$. Therefore $\delta + \delta' \geq 1$. From $\deg \Delta \leq 1$, we deduce $\delta' = 1 - \delta$ and $\Delta = \delta P + (1-\delta)P'$. Here, $\lfloor i\delta \rfloor + \lfloor i - i\delta \rfloor = i - (\lceil i\delta \rceil - \lfloor i\delta \rfloor) \geq i - 1$. ☐

Corollary 2.11. *We consider* $\deg\lfloor i\Delta \rfloor = i-1$ *for all* $i \geq 1$ *if and only if* $\Delta = \delta P + (1-\delta)P'$ *for some* $\delta \in [\frac{1}{2}, 1] \setminus \mathbb{Q}$.

Lemma 2.12. $\deg \Delta \leq 1$, $\deg\lfloor i\Delta \rfloor \geq i-1$ $(1 \leq i \leq N)$ *if and only if* $\delta^+_N + \delta' \geq 1$.

Corollary 2.13. $\deg \Delta \leq 1$, $\deg\lfloor i\Delta \rfloor = i-1$ $(1 \leq i \leq N)$ *if and only if*

- $\delta \in [\frac{N-1}{N}, 1)$, $\lfloor N\Delta' \rfloor = 0$, *or*
- $\delta' \geq 1 - \delta^+_N$ *and either* $\deg \Delta < 1$, *or* $\deg \Delta = 1$ *and* $i\Delta \notin \mathbb{Z}$ *for all* $1 \leq i \leq N$.

Proof. $\deg\lfloor i\Delta \rfloor \leq \deg i\Delta \leq i$. The equality follows if and only if $\deg \Delta = 1$, $i\Delta \in \mathbb{Z}$. ☐

2.2. Fractions with bounded numerators

For $l \in \mathbb{Z}_{\geq 1}$, define $\mathcal{A}_l$ to be the set of rational numbers $x \in (0,1)$ which admit a representation $x = \frac{p}{q}$ with p, q positive integers and $p \leq l$.

Note that $x \in (0,1)$ belongs to $\mathcal{A}_l$ if and only if $\{\frac{1}{x}\}$ belongs to the Farey set of order l. If $x \in \mathcal{A}_l$, then $\{l'x\} \in \mathcal{A}_{l'l}$. We have inclusions $\mathcal{A}_l \subseteq \mathcal{A}_{l+1}$, and $\max \mathcal{A}_l = \frac{l}{l+1}$.

The set $1 - \mathcal{A}_l$ is related to the hyperstandard set [5] associated to $\mathcal{F}_{l+1}$.

Lemma 2.14. *Let* $x \geq y$ *belong to* $\mathcal{A}_l$, *with* $x + y = 1$. *Then* $x = \frac{p}{q}$ *and* $y = \frac{q-p}{q}$ *for some* $1 \leq p \leq l, p < q \leq 2p, \gcd(p,q) = 1$. *In particular,* $q \leq 2l$.

Lemma 2.15. *Let* $x \in \mathcal{A}_l$. *Let* N *be the unique positive integer such that* $x^+_{N+1} < 1 \leq x^+_N$. *Then* $N \leq l+1$, *and equality is attained only if* $x = \frac{l}{l+1}$.

Proof. Note that $N = \lfloor \frac{1}{1-x} \rfloor$. Let $x = \frac{p}{q}$ be the reduced form, where $p \leq l$. Since $x < 1$, $j = q - p$ is a positive integer. Then $N = \lfloor \frac{1}{1-x} \rfloor = 1 + \lfloor \frac{p}{j} \rfloor \leq 1 + p$. Equality holds if and only if $j = 1$. ☐

Lemma 2.16. *Let $1 > x \geq y > 0$ with $x + y < 1$. Let N be the unique positive integer such that $x^+_{N+1} + y < 1 \leq x^+_N + y$. Suppose x, y are rational, with reduced forms $x = \frac{p}{q}, y = \frac{p'}{q'}$. Then $N \leq (p+1)(p'+1)$, and equality is attained if and only if*

$$(x, y) = \Big(\frac{p}{p+1}, \frac{p'}{1 + p'(p+1)}\Big).$$

Proof. We have $x < 1$. Then $x = \frac{p}{p+j}$ for some positive integer j. The inequality $y < 1 - x$ is equivalent to $q' \geq 1 + p' + z$, where $z = \lfloor \frac{pp'}{j} \rfloor$. Denote $y' = \frac{p'}{1+p'+z}$. Then $y \leq y' < 1 - x$.

Note that $N + 1$ is the smallest index of a rational number contained in the interval $(y, 1 - x)$. We have two cases. If $y < y'$, then $N + 1 \leq 1 + p' + z$. If $y = y'$, then $N + 1 \leq 1 + p' + z + p + j$. We conclude that $N \leq p + p' + j + z$.

We claim that $j + z \geq 1 + pp'$ implies $j = 1$ or $pp' \leq j$. Indeed, we deduce $j + \frac{pp'}{j} \geq 1 + pp'$. If $j > 1$, we have $pp' \leq j$.

We claim that $x < \frac{1}{2}$ implies $N = 1$. Indeed, suppose $N \geq 2$. Then, assume $x^+_N \leq \frac{1}{2}$ and $y \geq \frac{1}{2}$ such that we obtain $y > x$, a contradiction!

For $x = \frac{1}{2}$, we have $p = j = 1$. We have $x > \frac{1}{2}$ if and only if $p > j$.

We conclude that $N \leq p + p' + 1 + pp'$, and the equality is attained only if $j = 1$ and $y = y'$. $\square$

Proposition 2.17. *Let $x \geq y$ in $\mathcal{A}_l$ with $x + y < 1$. Let N be the unique positive integer such that $x^+_{N+1} + y < 1 \leq x^+_N + y$. Then $N \leq (l+1)^2$, and equality is attained only for*

$$(x, y) = \Big(\frac{l}{l+1}, \frac{l}{l^2 + l + 1}\Big).$$

3. SUCCESIVE VANISHING

Let C be a complex smooth projective curve, and B, L $\mathbb{R}$-divisors on C with $B \geq 0$ and $\deg L \geq 0$.

Lemma 3.1. $|\lceil K + B + L \rceil| = \emptyset$ *if and only if one of the following holds:*

1) $C = \mathbb{P}^1, L \sim 0$, $B \leq P$ *for some point* $P \in C$.

2) $C = \mathbb{P}^1, L \sim Q - \Delta$, $\lfloor \Delta \rfloor = 0$, $B \leq \Delta$ *for some point* $Q \in C$.

3) C *is an elliptic curve,* $L \sim Q - P$, $B = 0$, *for some point* $Q \neq P$.

In particular, $\deg L \leq 1$.

Proof. By Riemann–Roch and Serre duality, we have

$$-h^0(-\lceil B+L\rceil) = g-1+\deg\lceil B+L\rceil.$$

In particular, $g \le 1$. Note $\deg\lceil B+L\rceil \ge 0$, with equality if and only if $B=0$ and L is Cartier of degree 0. If $g=1$, then $B=0$, L is Cartier of degree zero, and $h^0(-L)=0$ (case 3)).

Suppose $g=0$. Then $\deg\lceil B+L\rceil + h^0(-\lceil B+L\rceil)=1$.

If $\deg\lceil B+L\rceil=0$, then $B=0, L\sim 0$ (case 1)). Else $\deg\lceil B+L\rceil=1$. Denote $\Delta=\lceil B+L\rceil - L$. In particular, $B\le\Delta$.

Case $\lfloor\Delta\rfloor\ne 0$. That is $\Delta\ge P$ for some $P\in C$. Since $\deg L\ge 0$ and $\deg\lceil B+L\rceil=1$, we obtain $0\ne B\le\Delta=P, L\sim 0$ (case 1)).

Case $\lfloor\Delta\rfloor=0$. Choose any point $Q\in C$. Then $L\sim Q-\Delta$ (case 2)). □

THEOREM 3.2. *Let $N\ge 2$. Then $|\lceil K+B+iL\rceil|=\emptyset$ for all $1\le i\le N$ if and only if one of the following holds:*

1) *$C=\mathbb{P}^1, L\sim 0$, $B\le P$ for some point $P\in C$.*

2) *$C=\mathbb{P}^1, L\sim Q-\Delta$, $\lfloor\Delta\rfloor=0$, $B\le\Delta$ and $\deg\lfloor i\Delta-B\rfloor\ge i-1$ for all $1\le i\le N$.*

3) *C is an elliptic curve, $L\sim Q-P$, $P\notin \operatorname{Bs}|iQ-(i-1)P|$ for all $1\le i\le N$, and $B=0$.*

Proof. From $N=1$, we have three cases.

In case 1), $K+B+iL\sim -P$ for all $i\ge 1$. Therefore $|\lceil K+B+iL\rceil|=\emptyset$ for all $i\ge 1$.

In case 2), C is a rational curve, $L\sim Q-\Delta$ for some point Q, $\lfloor\Delta\rfloor=0$ and $B\le\Delta$. Note that $\Delta=\lceil L\rceil - L$, so Δ is an intrinsic invariant of the linear equivalence class of L. We have

$$K+B+iL\sim K+iQ-(i\Delta-B)\sim(i-2)Q-(i\Delta-B).$$

So $|\lceil K+B+iL\rceil|=\emptyset$ if and only if $\deg\lfloor i\Delta-B\rfloor\ge i-1$.

Note $\lceil K+\Delta+2L\rceil\sim 0$.

In case 3), C is an elliptic curve, $L\sim Q-P$ and $|iL|=\emptyset$ for every $1\le i\le N$. The linear system $|iQ-(i-1)P|$ is fixed, so the last property is equivalent to $P\notin\operatorname{Bs}|iQ-(i-1)P|$ for all $1\le i\le N$. □

The divisor L may have non-integer coefficients only in case 2). And in this case, $\Delta=\lceil L\rceil - L$.

Corollary 3.3. *Let $C = \mathbb{P}^1$ and L an $\mathbb{R}$-divisor with* $\deg L \ge 0$. *Then* $\lceil K + iL \rceil = \emptyset$ *for all* $1 \le i \le N$ *if and only if there exists* $a \in k(C)^\times$ *such that*

$$(a) + L \le x'P' - xP,$$

where $P \ne P'$ *and* $x < x'$ *are consecutive elements of* $\mathcal{F}_N$. *For each* N, *the maximal elements are rational, and finitely many. For* $N = 1$, *the maximal element is unique, equal to* P'. *For* $N \ge 2$, *there exists an integer* $1 \le i \le (N-1)N$ *such that the linear system* $|iL^{\max}|$ *is free of degree* 1.

Theorem 3.4. $|\lceil K + B + iL \rceil| = \emptyset$ *for all* $i \ge 1$ *if and only if one of the following holds:*

- $C = \mathbb{P}^1$, $L \sim 0$, $B \le P$ *for some* P.
- $C = \mathbb{P}^1$, $L \sim \epsilon(P_1 - P_2)$, $P_1 \ne P_2, \epsilon \in (0,1)$, *and either* $B = 0$, *or* ϵ *is rational of index* l *and* $0 \ne B \le \frac{1}{l}P_1$.
- C *is an elliptic curve,* $L \sim P_1 - P_2$, $|iP_1 - (i-1)P_2| \ne P_2$ *for all* $i \ge 1$, $B = 0$.

Proof. We may suppose $C = \mathbb{P}^1$.

Case $\lfloor \Delta \rfloor \ne 0$. That is $\delta \ge 1$. Then $\delta = 1$. Then $\Delta = P, L \sim 0$ (first case).

Case $\lfloor \Delta \rfloor = 0$. Let P be the point of maximal multiplicity for Δ (assumed non-zero). Then $\delta < 1$. Let $N(1-\delta) > 1$. Then case a)$_N$ does not occur. So we are in case b)$_N$. Therefore

$$\delta + \delta' > 1 - \frac{1}{N}.$$

Letting $N \to \infty$, we obtain $\delta + \delta' \ge 1$. Then the equality holds. Then

$$\Delta = \delta P + (1-\delta)P', L \sim (1-\delta)(P - P').$$

We have $K + B + iL \sim K + iQ - (i\Delta - B)$. Therefore, vanishing holds up to N if and only if

$$\lfloor i\delta - b \rfloor + \lfloor i(1-\delta) - b' \rfloor \ge i - 1 \ (i \ge 1).$$

This is equivalent to

$$b + b' \le 1 - \{i\delta - b\} \ (i \ge 1).$$

From $i = 1$, we deduce that $b' = 0$. The system of inequalities becomes $1 - b \ge \{i\delta - b\}$ for all $i \ge 1$. If $b = 0$, it is satisfied. If $b > 0$, it is satisfied if and only if δ is rational and $b \le \frac{1}{q}$ where q is the index of δ. □

3.1. An application

Let C be a proper smooth curve, let B be an effective $\mathbb{R}$-divisor on C such that $\deg(K+B) \geq 0$.

– Suppose $\deg\lfloor B \rfloor \geq 1$. By Lemma 3.1, exactly one of the following holds:

a) $\deg\lfloor B \rfloor = 1$ and $r(K+B) \sim 0$ for some positive integer r, or

b) $|\lceil K + \lfloor B \rfloor + n(K+B) \rceil| \neq \emptyset$ for all $n \geq 1$.

In case a), we may choose r minimal with this property, and then $|\lceil K + \lfloor B \rfloor + n(K+B) \rceil| \neq \emptyset$ if and only if r does not divide n.

– For the rest of this section, suppose $\lfloor B \rfloor = 0$. That is, B has coefficients in $[0,1)$. Let $m \in \mathbb{Z}_{\geq 1}$. Suppose $N \geq 2$ is an integer such that $|\lceil K + im(K+B) \rceil| = \emptyset$ for every $1 \leq i \leq N$.

By Theorem 3.2, $C = \mathbb{P}^1$ and there are two possibilities:

1) $m(K+B) \sim 0$, or

2) $m(K+B) \sim Q - \Delta$ where $\lfloor \Delta \rfloor = 0$ and $\deg\lfloor i\Delta \rfloor \geq i-1$ for $1 \leq i \leq N$.

Consider case 2). Then $\Delta = \lceil mB \rceil - mB$. Write $\Delta = \delta P + \delta' P' + \Delta''$, where P, P' are distinct points not contained in the support of Δ'', $1 > \delta \geq \delta' \geq 0$ and δ' is greater or equal to the coefficients of Δ''. Since $K+B$ is nef, we have $\deg \Delta \leq 1$. In particular, $\delta + \delta' \leq 1$. We have two cases:

2a) Suppose $\delta + \delta' = 1$. Then $\Delta = \delta P + (1-\delta)P'$ and $\frac{1}{2} \leq \delta < 1$. Since $Q \sim P$, we obtain $m(K+B) \sim (1-\delta)(P - P')$.

2b) Suppose $\delta + \delta' < 1$. By Corollary 2.9, $\deg\lfloor i\Delta \rfloor \geq i-1$ for all $1 \leq i \leq N$ if and only if $1 \leq \delta_N^+ + \delta'$ and $\lfloor N\Delta'' \rfloor = 0$.

By Lemma 2.15 and Proposition 2.17, we obtain.

Theorem 3.5. *Suppose the smallest two (possibly equal) non-zero coefficients of $\{mB\}$ are of the form $1 - \frac{p}{q}$, for some positive integers p, q with $p \leq l$. Then exactly one of the following holds:*

a) $nm(K+B) \sim 0$ *for some* $1 \leq n \leq 2l$, *or*

b) $|\lceil K + nm(K+B) \rceil| \neq \emptyset$ *for some* $1 \leq n \leq (l+1)^2 + 1$.

The inequality in b) is attained for $C = \mathbb{P}^1$, $m = 1$, $B = (1 - \frac{l}{1+l})P + (1 - \frac{l}{1+l+l^2})P'$. This resembles examples considered in [4].

Corollary 3.6. *Suppose the smallest two (possibly equal) non-zero coefficients of B are of the form $1-\frac{1}{q}$, for some positive integers q (i.e., standard coefficients). Then exactly one of the following holds:*

a) $nm(K+B) \sim 0$ *for some* $1 \le n \le 2$, *or*

b) $|\lceil K+nm(K+B)\rceil| \ne \emptyset$ *for some* $1 \le n \le 5$.

4. SUCCESSIVE BASE POINT

Let C/k be a nonsingular projective algebraic curve, let B, L be $\mathbb{R}$-divisors such that $B \ge 0$ and $\deg L \ge 0$. Let $Q \in C$ be a closed point.

Proposition 4.1. *Let D be a divisor on C. Then $Q \in \operatorname{Bs}|K+D|$ if and only if $Q \notin \operatorname{Bs}|Q-D|$.*

Proof. From the short exact sequence $0 \to \mathcal{I}_Q(K+D) \to \mathcal{O}_C(K+D) \to \mathcal{O}_Q \to 0$, we deduce that $Q \in \operatorname{Bs}|K+D|$ if and only if the homomorphism $H^1(K+D-Q) \to H^1(K+D)$ is not injective. By Serre duality, this means that $\Gamma(-D) \to \Gamma(-D+Q)$ is not surjective. That is $Q \notin \operatorname{Bs}|-D+Q|$. □

Theorem 4.2. $Q \in \operatorname{Bs}|\lceil K+B+L\rceil|$ *if and only if one of the following holds:*

1) $L \sim Q-P$ $(Q \ne P), B=0$.

2) $L \sim Q-P$, $0 \ne B \le P$.

3) $L \sim Q-\Delta, \lfloor \Delta \rfloor = 0, B \le \Delta$.

Proof. Our assumption is equivalent to $Q \notin \operatorname{Bs}|Q-\lceil B+L\rceil|$. In particular, $\deg\lceil B+L\rceil \le 1$.

Case $\deg\lceil B+L\rceil = 0$. Then $B=0$, L has integer coefficients and has degree zero. Following, $D=P$ for some point $P \ne Q$. We are in case 1).

Case $\deg\lceil B+L\rceil = 1$. Then $D=0$, so $\lceil B+L\rceil \sim Q$ and we denote $\Delta = \lceil B+L\rceil - L$. Then $B \le \Delta$, $L \sim Q-\Delta$. The property $\lceil B+L\rceil \sim Q$ translates into $\lfloor \Delta - B\rfloor = 0$. If $\lfloor \Delta \rfloor = 0$, this property holds (case 3)). If $\lfloor \Delta \rfloor \ne 0$, we deduce from $\deg \Delta \le 1$ that $\Delta = P$ for some point $P \in C$ and $0 \ne B \le P$ (case 2)). □

Remarks:

– If $L_1 \sim L_2$, then $\lceil L_1\rceil - L_1 = \lceil L_2\rceil - L_2$ and $L_1 - \lfloor L_1\rfloor = L_2 - \lfloor L_2\rfloor$;

– $\deg\lceil L\rceil$ is 0 in cases 1),2), and 1 in case 3). In case 3), $\Delta = \lceil L\rceil - L$;

– We have $\lceil K+B+L\rceil \sim K+Q-P$ $(Q \neq P)$ in case 1), and $\lceil K+B+L\rceil \sim K+Q$ in cases 2), 3);

– $|K+Q|$ is empty if $C=\mathbb{P}^1$, $|K|+Q$ if $g \geq 1$;

– $|K+Q-P|$ $(Q \neq P)$ is empty if $g \leq 1$, $|K-P-P'|+Q+P'$ if C is hyperelliptic of genus $g \geq 2$ (where $P+P'$ is the fiber of $C \to \mathbb{P}^1$) , $|K-P|+Q$ if C is non-hyperelliptic of genus $g \geq 2$.

In particular, if the linear system $|\lceil K+B+L\rceil|$ is not empty, its fixed part has degree 0, 1 or 2.

– It follows that B is a boundary and $\lfloor B\rfloor \neq 0$ if and only if $B=P, L \sim Q-P$ (so P is uniquely determined by B).

Lemma 4.3. *$Q \in \operatorname{Bs}|\lceil K+B+iL\rceil|$ for $1 \leq i \leq 2$ if and only if one of the following holds:*

1) $L \sim Q-P$ $(Q \neq P), Q \notin \operatorname{Bs}|2P-Q|, B=0.$

2) $L \sim 0, 0 \neq B \leq P \sim Q.$

3) $L \sim Q-\Delta, \lfloor\Delta\rfloor=0, B \leq \Delta, Q \notin \operatorname{Bs}|\lfloor 2\Delta-B\rfloor-Q|.$

Proof. From $N=1$, we have three cases:

1) $L \sim Q-P$, $Q \neq P$, $B=0$. The new condition is $Q \notin \operatorname{Bs}|Q-2(Q-P)|$. That is $Q \notin \operatorname{Bs}|2P-Q|$.

2) $L \sim Q-P$, $0 \neq B \leq P$. The new condition is $Q \notin \operatorname{Bs}|Q-\lceil B+2(Q-P)\rceil|$. That is $Q \notin \operatorname{Bs}|\lfloor 2P-B\rfloor-Q|$. In particular, $\deg\lfloor 2P-B\rfloor \geq 1$. That is $0 \neq B \leq \frac{1}{2}P$. Then $\lfloor 2P-B\rfloor=P$. The condition becomes $Q \notin \operatorname{Bs}|P-Q|$. Therefore $P \sim Q$. Therefore $L \sim 0$.

3) $L \sim Q-\Delta$, $\lfloor\Delta\rfloor=0$, $B \leq \Delta$. The new condition is $Q \notin \operatorname{Bs}|Q-\lceil B+2Q-2\Delta\rceil|$. That is $Q \notin \operatorname{Bs}|\lfloor 2\Delta-B\rfloor-Q|$. □

Theorem 4.4. *Let $N \geq 2$. Then $Q \in \operatorname{Bs}|\lceil K+B+iL\rceil|$ for $1 \leq i \leq N$ if and only if one of the following holds:*

1) $L \sim Q-P$ $(Q \neq P), Q \notin \cup_{i=2}^N \operatorname{Bs}|iP-(i-1)Q|, B=0.$

2) $L \sim 0, 0 \neq B \leq P \sim Q.$

3) $L \sim Q-\Delta, \lfloor\Delta\rfloor=0, B \leq \Delta, Q \notin \cup_{i=2}^N \operatorname{Bs}|\lfloor i\Delta-B\rfloor-(i-1)Q|.$

It remains to classify case 3): suppose $L \sim Q-\Delta, \lfloor\Delta\rfloor=0, B \leq \Delta$.

Note $\lfloor i\Delta-B\rfloor \leq i\Delta-B \leq i\Delta$ and $\deg\Delta \leq 1$. Therefore $\deg\lfloor i\Delta-B\rfloor \leq i$, and equality holds if and only if $B=0, \deg\Delta=1, iB \in \mathbb{Z}$. Therefore $Q \notin \operatorname{Bs}|\lfloor i\Delta-B\rfloor-(i-1)Q|$ if and only if

- $B=0, \deg\Delta=1, i\Delta\in\mathbb{Z}, Q\notin \mathrm{Bs}\,|i\Delta-(i-1)Q|$, or
- $\lfloor i\Delta - B\rfloor \sim (i-1)Q$.

LEMMA 4.5. *Let $P_1\neq P_2, P_1\sim P_2$. Then $C\simeq\mathbb{P}^1$.*

Below, if we write $\Delta=\delta_1P_1+\delta_2P_2$, we mean $P_1\neq P_2$ too.

LEMMA 4.6. *$Q\in \mathrm{Bs}\,|\lceil K+B+2L\rceil|$ if and only if one of the following holds:*

- $\Delta=\frac{1}{2}P_1+\frac{1}{2}P_2, Q\notin \mathrm{Bs}\,|P_1+P_2-Q|, B=0.$
- $\Delta=\frac{1}{2}P_1+\frac{1}{2}P_2, Q\sim P_1, 0\neq B\leq P_2.$
- $\lfloor 2\Delta\rfloor\sim Q, B\leq \min(\Delta,\{2\Delta\}).$

Proof. Go through cases 1, 2, 3 of Theorem 4.3. □

LEMMA 4.7. *$Q\in\cap_{i=2}^3 \mathrm{Bs}\,|\lceil K+B+iL\rceil|$ if and only if one of the following holds:*

- $C\simeq\mathbb{P}^1$, $\Delta=\frac{1}{2}P_1+\frac{1}{2}P_2, B\leq\frac{1}{2}P_2$. $N=+\infty$.
- $C\simeq\mathbb{P}^1, \Delta=\frac{2}{3}P_1+\frac{1}{3}P_2, 0\neq B\leq\frac{1}{3}P_1$. $N=+\infty$.
- $\Delta=\frac{2}{3}P_1+\frac{1}{3}P_2, Q\sim P_1, B\leq\frac{1}{3}P_2$.
- $\lfloor i\Delta\rfloor\sim(i-1)Q\ (1\leq i\leq 3), B\leq\min_{i=1}^3\{i\Delta\}$.

Proof. For each solution in Lemma 4.6, go through cases 1, 2, 3 of Theorem 4.4. □

THEOREM 4.8. *Let $L\sim Q-\Delta, \lfloor\Delta\rfloor=0, B\leq\Delta$, and $N\geq 3$. Then $Q\in\cap_{i=1}^N \mathrm{Bs}\,|\lceil K+B+iL\rceil|$ if and only if one of the following holds:*

1) $C\simeq\mathbb{P}^1$, $\Delta=\delta P_1+(1-\delta)P_2, \delta\in\mathbb{Z}_N\cap[\frac{1}{2},\frac{N-1}{N})$, *and* $B\leq\frac{1}{l}P_j$ *for* $j=1$ *or* 2, *where* $l=\mathrm{index}(\delta)$.
2) $C\simeq\mathbb{P}^1, \Delta=\frac{N-1}{N}P_1+\frac{1}{N}P_2, 0\neq B\leq\frac{1}{N}P_1$.
3) $\Delta=\frac{N-1}{N}P_1+\frac{1}{N}P_2, Q\sim P_1, B\leq\frac{1}{N}P_2$.
4) $\lfloor i\Delta\rfloor\sim(i-1)Q\ (1\leq i\leq N), B\leq\min_{i=1}^N\{i\Delta\}$.

Proof. We use induction on N. We do case $N=3$ by hand. Let $N>3$. We consider separately the solutions for $N-1$, and impose the new condition $Q \in \operatorname{Bs}|\lceil K+B+NL\rceil|$, using Theorem 4.2.

Case $(1)_{N-1}, (2)_{N-1}$: here, one must show $N \to \infty$ (must write down).

Case $(3)_{N-1}$: Let $\Delta = \frac{N-2}{N-1}P_1 + \frac{1}{N-1}P_2, Q \sim P_1, B \le \frac{1}{N-1}P_2$. Since $N\Delta \notin \mathbb{Z}$, only case 3) of Theorem 4.2 may apply for NL. The new condition is $NL \sim Q - \Delta_N$, $\lfloor \Delta_N \rfloor = 0, B \le \Delta_N$. We obtain $\Delta_N \sim N\Delta - (N-1)Q$. That is $\Delta_N = \{N\Delta\}$ and $\lfloor N\Delta \rfloor \sim (N-1)Q$. We have

$$N\Delta = \Big(N-2+\frac{N-2}{N-1}\Big)P_1 + \Big(1+\frac{1}{N-1}\Big)P_2.$$

Therefore $\lfloor N\Delta \rfloor = (N-2)P_1 + P_2$. Then $(N-2)P_1 + P_2 \sim (N-1)Q$. Then $P_2 \sim P_1$. Since $P_1 \ne P_2$, we obtain $C \simeq \mathbb{P}^1$. The condition $B \le \Delta_N$ is already satisfied. We obtain $C \simeq \mathbb{P}^1, \Delta = \frac{N-2}{N-1}P_1 + \frac{1}{N-1}P_2, B \le \frac{1}{N-1}P_2$, which belongs to case $(1)_N$.

Case $(4)_{N-1}$: suppose case 3) of Theorem 4.2 applies to NL. That is $NL \sim Q - \Delta_N$, $\lfloor \Delta_N \rfloor = 0, B \le \Delta_N$. As above, we obtain $\Delta_N = \{N\Delta\}$, $\lfloor N\Delta \rfloor \sim (N-1)Q$. We obtain case $(4)_N$.

Suppose now that cases 1) or 2) of Theorem 4.2 apply to NL. That is $N\Delta \sim (N-1)Q + P_N$, and either $Q \ne P_N, B = 0$, or $0 \ne B \le P_N$. We have

$$N\Delta \sim (N-1)Q + P_N.$$

Now Δ has degree one. It has at least two coefficients.

Case $\delta \ge \frac{N-1}{N}$. Then $\delta \ge \frac{N-1}{N}$. Then $\Delta = \frac{N-1}{N}P_1 + \frac{1}{N}P_2$. The conditions become $P_1 \sim Q, P_N \sim P_2$.

Case $\delta < \frac{N-1}{N}$. Then $\delta + \delta' > 1 - \frac{1}{N+1}$. Since $N\Delta \in \mathbb{Z}$, we deduce $\Delta'' = 0$. Therefore $\Delta = \delta P_1 + (1-\delta)P_2$, and $i\delta \notin \mathbb{Z}$ for every $1 \le i \le N-1$. Since $2\Delta \notin \mathbb{Z}$, we have $\delta \ne \frac{1}{2}$. Therefore $\delta \in (\frac{1}{2}, \frac{N-1}{N})$. Consequently, $\lfloor 2\Delta' \rfloor = 0$. The condition $\lfloor 2\Delta \rfloor \sim Q$ becomes $P_1 \sim Q$. Set $j = \lceil \frac{1}{1-\delta} \rceil$, so that $\lfloor j(1-\delta) \rfloor = 1$. We have $\delta < \frac{N-1}{N}$. Since $\delta \in \mathbb{Z}_N$, we obtain $\delta \le \frac{N-2}{N-1}$. Since $(N-1)\delta \notin \mathbb{Z}$, we obtain $\delta < \frac{N-2}{N-1}$. Therefore $j \le N-1$. Then $\lfloor j\Delta \rfloor \sim (j-1)Q$ and $P_1 \sim Q$ imply $P_2 \sim Q$. Then $P_1 \sim P_2$. Therefore $C \simeq \mathbb{P}^1$. We obtained two cases:

- $\Delta = \frac{N-1}{N}P_1 + \frac{1}{N}P_2, P_1 \sim Q, P_N \sim P_2$.
- $C = \mathbb{P}^1$, $\Delta = \delta P_1 + (1-\delta)P_2$, and $\delta < \frac{N-1}{N}$, $\delta \in \mathbb{Z}_N \setminus \mathbb{Z}_{N-1}$.

It remains to understand the condition on B too. First case: $(3)_N$ or $(2)_N$. Second $(1)_N$. □

Corollary 4.9. *$Q \in \operatorname{Bs}|\lceil K+B+iL\rceil|$ for every $i \ge 1$ if and only if one of the following holds:*

1) $L \sim Q - P \ (Q \neq P), Q \notin \cup_{i \geq 2} \operatorname{Bs} |iP - (i-1)Q|, B = 0$.

2) $L \sim 0, 0 \neq B \leq P \sim Q$.

3) $C \simeq \mathbb{P}^1$, $L \sim \epsilon(P_1 - P_2) \ (P_1 \neq P_2), \epsilon \in (0, \frac{1}{2}]$, *and either* $\epsilon \notin \mathbb{Q}$ *and* $B = 0$, *or* $\epsilon \in \mathbb{Q}$ *and* $B \leq \frac{1}{l} P_j$ *for* $j = 1$ *or* 2, *where* $l \geq 1$ *is minimal such that* $l\epsilon \in \mathbb{Z}$.

Proof. We use Theorem 4.8 for every N. The first two cases are valid for all N. Two cases remain:

3a) $C = \mathbb{P}^1$, $L \sim Q - (\delta P_1 + (1-\delta)P_2)$, $\delta \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and $B \leq \frac{1}{l} P_j$ for some $j = 1, 2$, where l is the index of δ.

3b) $L \sim Q - \Delta$, $\lfloor \Delta \rfloor = 0$, $\lfloor i\Delta \rfloor \sim (i-1)Q$ for all $i \geq 1$, $B \leq \inf_{i \geq 1} \{i\Delta\}$. By diophantine approximation, the last condition becomes $B = 0$. By Corollary 2.12, $\Delta = \delta P_1 + (1-\delta)P_2$ with $\delta \in [\frac{1}{2}, 1) \setminus \mathbb{Q}$. Finally,

$$\lfloor i\delta \rfloor P_1 + \lfloor i - i\delta \rfloor P_2 \sim (i-1)Q \ (i \geq 1).$$

Since $\delta \neq \frac{1}{2}$, we have $\delta' < \frac{1}{2}$. Therefore, the condition for $i = 2$ becomes $P_1 \sim Q$. Let $j = \lceil \frac{1}{1-\delta} \rceil$. Then $\lfloor j - j\delta \rfloor = 1$. The condition for j becomes $P_2 \sim Q$. Therefore $P_1 \sim P_2$, so we have $C \simeq \mathbb{P}^1$.

Note that $L \sim Q - (\delta P_1 + (1-\delta)P_2) \sim (1-\delta)(P_1 - P_2)$. □

Acknowledgments. This work was finalized while visiting Yau Mathematical Sciences Center, Tsinghua University. I would like to thank Caucher Birkar for the kind invitation.

REFERENCES

[1] J.A. Chen, M. Chen, and D.-Q. Zhang, *A non-vanishing theorem for $\mathbb{Q}$-divisors on surfaces*. J. Algebra **293** (2005), *2*, 363–384.

[2] C.D. Hacon and J. McKernan, *Boundedness of pluricanonical maps of varieties of general type*. Invent. Math. **166** (2006), *1*, 1–25.

[3] C.D. Hacon, J. McKernan, and C. Xu, *On the birational automorphisms of varieties of general type*. Ann. of Math. (2) **177** (2013), *3*, 1077–1111.

[4] J. Kollár, *Log surfaces of general type; some conjectures*. Contemp. Math. **162** (1994), 261–275.

[5] Yu. G. Prokhorov and V.V. Shokurov, *Towards the second main theorem on complements*. J. Algebraic Geom. **18** (2009), *1*, 151–199.

[6] V.V. Shokurov, 3-*fold log flips*. Russian Acad. Sci. Izv. Math. **40** (1993), *1*, 95–202.

[7] S. Takayama, *Pluricanonical systems on algebraic varieties of general type*. Invent. Math. **165** (2006), *3*, 551–587.

Received 31 May 2024

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