Dedicated to Lucian Bădescu, in memoriam, a dear friend and a very esteemed colleague

ORBIFOLD QUOTIENTS OF SYMMETRIC DOMAINS OF TUBE TYPE

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In this paper, we characterize the compact orbifolds, quotients $X = \mathcal{D}/\Gamma$ of a bounded symmetric domain $\mathcal{D}$ of tube type by the action of a discontinuous group Γ, as those projective orbifolds with ample canonical divisor possessing a slope zero tensor of "orbifold type".

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INTRODUCTION

Let M be a simply connected complex manifold, and Γ be a properly discontinuous group of automorphisms (biholomorphic self-maps) of M.

Then the quotient complex analytic space $X = M/\Gamma$ is a normal complex space.

In the case where the action of Γ is **quasi-free**, namely, Γ acts freely outside of a closed complex analytic set of codimension at least 2, we just consider the normal complex space X; but, in the case where the set Σ of points $z \in M$ whose stabilizer is nontrivial has codimension 1, it is convenient to replace X by the complex global orbifold $\mathcal{X}$, consisting of the datum of X and of the irreducible Weil divisors $D_i \subset X$, whose union is the codimension 1 part of the branch locus $\mathcal{B}$ of $p : M \to X$ ($\mathcal{B} = p(\Sigma)$ is the set of critical values of p): each divisor D_i is marked with the integer m_i which is the order of the stabilizer group at a general point of the inverse image of D_i.

The more general case where Γ' is a properly discontinuous group of automorphisms of any (connected) complex manifold M' reduces to the previous one by taking M to be the universal covering of M' and letting Γ be the group of lifts to M of elements of Γ'.

One says that the above global orbifold $\mathcal{X}$ is **good** if Γ admits a finite index normal subgroup Λ which acts freely (this holds if Λ is torsion free), with quotient a compact complex manifold $Y = M/\Lambda$: in this case $X = Y/G$, where G is the finite quotient group $G := \Gamma/\Lambda$.

Particularly interesting are the cases where M is a contractible domain $M = \mathcal{D} \subset \mathbb{C}^n$: in this case Y is a classifying space $K(\Lambda, 1)$ for the group Λ, whereas (see Section 1) $\mathcal{X}$ is an orbifold classifying space for the orbifold fundamental group Γ of $\mathcal{X}$.

The easiest example is the case where $\mathcal{D} = \mathbb{C}^n$, Y is a complex torus $Y = \mathbb{C}^n/\Lambda$, and $\mathcal{X}$ is a finite quotient of a complex torus: this case was considered in [15], and can be classified by simply saying that Γ is an arbitrary abstract even crystallographic group, endowed with a complex structure on $\Lambda \otimes \mathbb{C}$.

A more difficult case is the case where $\mathcal{D} \subset\subset \mathbb{C}^n$ is a bounded symmetric domain (see [25], [24] and also [32], [17]). Again, we have a good global orbifold, by virtue of the so-called Selberg's Lemma ([35], [9]).

In the case where Γ acts freely, we have a so-called locally-symmetric manifold, and there is a vast literature devoted to their possible characterizations, some final touch with rather explicit criteria being contained in our work with Di Scala, [17], [18].

The purpose of this note is to apply the idea, as in [15], to use orbifolds in order to deal with the case of a non-free action of Γ. At least what is easier here is that necessarily X must be projective, and indeed by [30], the canonical divisor K_Y and the orbifold canonical divisor

$$K_{\mathcal{X}} := K_X + \hat{D} := K_X + \sum_i \frac{m_i - 1}{m_i} D_i$$

must be ample.

We have a priori two options for the assumptions to be made, for instance, we can consider the more general orbifolds introduced in [12] (see also [13], 5.5 and 5.8, and 6.1 of [14]) or the more special Deligne–Mostow orbifolds ([19], Section 14), locally modelled as quotients of a smooth manifold by a finite group; in the quasi-free case, where all the multiplicities $m_i = 1$, the Orbifold fundamental group $\pi_1^{\mathrm{orb}}(\mathcal{X})$ is the fundamental group $\pi_1(X^*)$ of the smooth locus X^* of X, while in general $\Gamma := \pi_1^{\mathrm{orb}}(\mathcal{X})$ is a quotient of $\pi_1(X \setminus D)$.

On the differential geometric side, the input is simpler, see [16], [17], in the case where $\mathcal{D}$ is of tube type, which means that $\mathcal{D}$ is biholomorphic to a tube domain $\mathcal{T} = V + i\mathcal{C}$, where V is a real vector space and $\mathcal{C} \subset V$ is an open self-dual convex cone containing no lines.

In fact, the main concept in the tube case is the one of a nontrivial **slope zero tensor**

$$0 \neq \psi_Y \in H^0\big(S^{mn}(\Omega^1_Y)(-mK_Y)\big), \tag{1}$$

(here, $n := \dim(Y)$) which characterizes the locally symmetric manifolds of tube type, together with the property that K_Y is ample.

Now, ψ descends to a meromorphic section $0 \neq \psi_X$ on X of

$$S^{mn}\big(\Omega^1_X(\log D)\big)\big(-m(K_X + D)\big). \tag{2}$$

Conversely, such a tensor ψ_X on X lifts to a holomorphic tensor ψ_Y only if it is **orbifold type**, meaning that some vanishing conditions are to be imposed (see Section 1 for precise definitions).

Our first result is the following:

THEOREM 0.1. *The global compact complex orbifolds $\mathcal{X}$ of bounded symmetric domains $\mathcal{D}$ of tube type (i.e., $\mathcal{D}$ is a product of irreducible bounded symmetric domains of tube type) are the projective complex orbifolds such that:*

(1) *their orbifold fundamental group Γ admits a torsion free normal finite index subgroup Λ,*

(2) *$\mathcal{X}$ admits a meromorphic slope zero tensor $0 \neq \psi_X$ (a meromorphic section of $S^{mn}(\Omega^1_X(\log D))(-m(K_X + D))$) which is of orbifold type,*

(3) *$K_{\mathcal{X}} := K_X + \sum_i \frac{m_i - 1}{m_i} D_i$ is ample,*

and

(i) *the corresponding Galois covering $Y \to X = X/G$ ($G := \Gamma/\Lambda$) is smooth, equivalently, the orbifold universal cover of X is smooth, or*

(i') *Y has singularities which are 2-homologically connected, that is, they have a resolution of singularities $\pi : Y' \to Y$ with $\mathcal{R}^j f_*(\mathbb{Z}_{Y'}) = 0,$ for $j = 1, 2$.*

Moreover, $\mathcal{X}$ should be an orbifold classifying space.

Our main result is instead:

THEOREM 0.2. *The global compact complex orbifolds $\mathcal{X}$ of bounded symmetric domains $\mathcal{D}$ of tube type (i.e., $\mathcal{D}$ is a product of irreducible bounded symmetric domains of tube type) are the projective complex orbifolds such that:*

(1) *their orbifold fundamental group Γ admits a torsion free normal finite index subgroup Λ,*

(2) *$\mathcal{X}$ admits a meromorphic slope zero tensor $0 \neq \psi_X$ (a meromorphic section of $S^{mn}(\Omega^1_X(\log D))(-m(K_X + D)))$ of orbifold type,*

(3) $K_{\mathcal{X}} := K_X + \sum_i \frac{m_i - 1}{m_i} D_i$ *is ample,*

and

(ii) *$\mathcal{X}$ is a Deligne–Mostow complex projective orbifold, and $\mathcal{X}$ is an orbifold classifying space.*

One may speculate/ask whether condition (ii) may be replaced by the weaker assumption that X has KLT singularities.

1. COMPLEX ORBIFOLDS, DELIGNE–MOSTOW ORBIFOLDS, ORBIFOLD FUNDAMENTAL GROUPS, ORBIFOLD COVERINGS

This section is an abridged version of the corresponding section of [15], so here, we are quicker in the exposition.

Definition 3 (compare [13, Definition 5.5] and [19, Section 4]). Let Z be a normal complex space, let D be a closed analytic set of Z containing $\mathrm{Sing}(Z)$, and let $\{D_i | i \in \mathcal{I}\}$ be the irreducible components of D of codimension 1.

(1) Attaching to each D_i a positive integer $m_i \geq 1$, we obtain a **complex orbifold** $(Z, D, \{m_i | i \in \mathcal{I}\})$.

(2) The **orbifold fundamental group** $\pi_1^{\mathrm{orb}}(Z \setminus D, (m_1, \ldots, m_r, \ldots))$ is defined as the quotient

$$\pi_1^{\mathrm{orb}}\big(Z \setminus D, (m_1, \ldots, m_r, \ldots)\big) := \pi_1(Z \setminus D) / \big\langle\langle(\gamma_1^{m_1}, \ldots, \gamma_r^{m_r}, \ldots\rangle\big\rangle$$

of the fundamental group of $(Z \setminus D)$ by the subgroup normally generated by simple geometric loops γ_i going each around a smooth point of the divisor D_i (and counterclockwise).

(3) The orbifold is said to be **quasi-smooth** or geometric if moreover D_i is smooth outside of $\mathrm{Sing}(Z)$.

(4) The orbifold is said to be a **Deligne–Mostow orbifold** if moreover, for each point $z \in Z$, there exists a local chart $\phi : \Omega \to U = \Omega/G$, where $0 \in \Omega \subset \mathbb{C}^n$, G is a finite subgroup of $GL(n, \mathbb{C})$, $\phi(0) = z$, U is an open neighbourhood of z, and the orbifold structure is induced by the quotient map. That is, $D \cap U$ is the branch locus of Φ, and the integers m_i are the ramification multiplicities.

(5) An orbifold is said to be **reduced** (or **impure**) if all the multiplicities $m_i = 1$.

(6) We identify two orbifolds under the equivalence relation generated by forgetting the divisors D_i with multiplicity 1.

Remark 4. (i) A D-M (= Deligne–Mostow) orbifold is quasi-smooth, and the underlying complex space Z has only quotient singularities (these are rational singularities).

(ii) In the case where we have a **reduced** orbifold, that is, there is no divisorial part, then the orbifold fundamental group is the fundamental group of $Z \setminus \mathrm{Sing}(Z)$.

(iii) If $Z = M/\Gamma$ is the quotient of a complex manifold by a properly discontinuous subgroup Γ, then Z is a D-M orbifold, since any stabilizer subgroup is finite (Γ being properly discontinuous) and by Cartan's Lemma ([6]) the action of the stabilizer subgroup becomes linear after a local change of coordinates.

(iv) one could more generally consider a wider class of orbifolds allowing also the multiplicity $m_i = \infty$: this means that the relation $\gamma_i^{m_i} = 1$ is a void condition; this more general case is useful to deal with the compactifications of finite volume quotients $X = \mathcal{D}/\Gamma$ (see, for instance, [1]).

Now, to a subgroup of the orbifold fundamental group corresponds a connected **orbifold covering** of orbifolds, (see, for instance, [19]), in particular to the trivial subgroup corresponds the orbifold universal cover

$$\big(\tilde{Z}, \tilde{D}, \{\tilde{m}_j\}\big).$$

Definition 5. We say that an orbifold $(Z, D, (m_j))$ is an orbifold classifying space if its universal covering $(\tilde{Z}, \tilde{D}, \{\tilde{m}_j\})$ satisfies the properties

(a) either $\tilde{Z}$ is contractible and the multiplicities $\tilde{m}_j$ are all equal to 1, or

(b) there is a homotopy retraction of $\tilde{Z}$ to a point which preserves the subdivisor $\tilde{D}'$ consisting of the irreducible components with multiplicity $\tilde{m}_j > 1$.

Definition 6. The orbifold canonical divisor is defined as

$$K_{\mathcal{X}} := K_X + \hat{D} := K_X + \sum_i \frac{m_i - 1}{m_i} D_i.$$

It satisfies the property that, for an orbifold covering $f : \mathcal{Y} \to \mathcal{X}$, we have

$$f^*(K_{\mathcal{X}}) = K_{\mathcal{Y}}.$$

2. LOCALLY SYMMETRIC MANIFOLDS OF TUBE TYPE AND DESCENT OF SLOPE ZERO TENSORS

As mentioned in the Introduction, a bounded symmetric domain $\mathcal{D}$ is of tube type if $\mathcal{D}$ is biholomorphic to a tube domain $\mathcal{T} = V + i\mathcal{C}$, where V is a real vector space and $\mathcal{C} \subset V$ is an open self-dual convex cone containing no lines.

Recall the notation for the classical domains:

- $I_{n,p}$ is the domain $\mathcal{D} = \{Z \in M_{n,p}(\mathbb{C}) : \mathrm{I}_p -^t Z \cdot \overline{Z} > 0\}$.
- II_n is the intersection of the domain $I_{n,n}$ with the subspace of skew symmetric matrices.
- III_n is the intersection of the domain $I_{n,n}$ with the subspace of symmetric matrices.
- IV_n is the Lie Ball in $\mathbb{C}^n$,
  $$\big\{z||z_1^2 + \cdots + z_n^2| < 1,\ 1 + |z_1^2 + \cdots + z_n^2|^2 - 2\big(|z_1|^2 + \cdots + |z_n|^2\big) > 0\big\}.$$

Moreover, there are the exceptional domains $\mathcal{D}_{16}$ of dimension 16 and $\mathcal{D}_{27}$ of dimension 27 (related to 2×2 and 3×3 matrices over the Cayley algebra).

The tube domain condition excludes the domains $I_{n,p}$ with $n \neq p$.

The following result was shown in [17], based on the concept of a slope zero tensor mentioned in the Introduction, see (1):

$$0 \neq \psi_Y \in H^0\big(S^{mn}(\Omega^1_Y)(-mK_Y)\big).$$

THEOREM 2.1. *Let X be a compact complex manifold of dimension n. Then the following two conditions hold:*

(1) *K_X is ample*

(2) *X admits a nontrivial slope zero tensor $\psi_X \in H^0(S^{mn}(\Omega^1_X)(-mK_X))$ (here, m is a positive integer)*

if and only if $X \cong \Omega/\Gamma$, where Ω is a bounded symmetric domain of tube type and Γ is a cocompact discrete subgroup of $\mathrm{Aut}(\Omega)$ acting freely.

Moreover, the degrees and the multiplicities of the irreducible factors of the polynomial ψ_p (the evaluation of ψ at a point p) determine uniquely the universal covering $\widetilde{X} = \Omega$.

In particular, for $m = 2$, we get that the universal covering $\widetilde{X}$ is a polydisk if and only if ψ_p is the square of a squarefree polynomial (indeed, of a product of linear forms).

Now, if X is a global orbifold of a symmetric bounded domain $\mathcal{D}$ of tube type, associated to a properly discontinuous subgroup Γ of biholomorphisms of $\mathcal{D}$, then by Selberg's Lemma the group Γ admits a finite index normal subgroup Λ and the compact complex manifold $Y := \mathcal{D}/\Lambda$ fulfils conditions (1) and (2) of Theorem 2.1, in particular, it admits a nontrivial slope zero tensor ψ_Y.

Since $X = Y/G$, $G = \Gamma/\Lambda$, we want to show that ψ_Y, which is clearly G-invariant, descends to some meromorphic tensor ψ^*_X on the smooth locus X^* of X. Then, we define

$$\psi_X := i_*(\psi^*_X), \text{ for } i : X^* \to X \text{ being the inclusion.}$$

In order to achieve our goal, let us consider now the following local situation.

Lemma 2.2. *Consider the action of the cyclic group μ_q of roots of unity at the origin in $Y := \mathbb{C}^n$, via the action, for $\zeta \in \mu_q$:*

$$(x, z) := (x_1, \ldots, x_{n-1}, z) \mapsto (x, \zeta z),$$

with quotient map $\pi : Y \to X$,

$$\pi : (x, z) := (x_1, \ldots, x_{n-1}, z) \mapsto (x, y), \ y := z^q.$$

Then, a G-invariant slope zero holomorphic tensor

$$0 \neq \psi \in H^0\big(S^{mn}(\Omega^1_Y)(-mK_Y)\big)$$

descends to a tensor $0 \neq \phi$, a meromorphic section of (here $D = div(y)$)

$$S^{mn}\big(\Omega^1_X(\log D)\big)\big(-m(K_X + D)\big).$$

Proof. Write ψ as (here, $r + |I| = mn$)

$$\begin{aligned}
&\sum_{r=0}^{mn} \sum_{|I|=mn-r} A_I(x, z) \frac{(dx)^I dz^r}{(dx_1 \wedge \cdots \wedge dx_{n-1} \wedge dz)^m} \\
&\quad = \sum_{r=0}^{mn} \sum_{I,h} B_{I,h}(x) z^h \frac{(dx)^I dz^r}{(dx_1 \wedge \cdots \wedge dx_{n-1} \wedge dz)^m} \\
&\quad = \sum_{r=0}^{mn} \sum_{I,h} B_{I,h}(x) q^{-r+m} z^{h+r-m} \frac{(dx)^I d(\log y)^r}{\big(dx_1 \wedge \cdots \wedge dx_{n-1} \wedge d(\log y)\big)^m}.
\end{aligned}$$

G-invariance is equivalent to the condition that q divides $h + r - m$, hence $h = bq + m - r$, and we have, downstairs on X, the tensor

$$\phi := \sum_{r=0}^{mn} \sum_{I,h} q^{-r+m} B_{I,h}(x) \ y^b \frac{(dx)^I d(\log y)^r}{\big(dx_1 \wedge \cdots \wedge dx_{n-1} \wedge d(\log y)\big)^m}.$$

The holomorphic part of the tensor ϕ is the sum of the series where $b \geq 0$, that is, corresponding to the terms with

$$h + r \geq m.$$

Its order of pole on D is at most $-b = [\frac{m}{q}]$. $\square$

Definition 7. We say that the meromorphic tensor ϕ is of **orbifold type** if conversely its pull back ψ to any orbifold covering with multiplicities ≤ 1 is holomorphic: this means that we only have terms with b such that $bq + m \geq r$.

Remark 8. (i) Since in the case of a good global orbifold quotient the slope tensor of Y cannot vanish on a divisor, the case $h = 0$ must occur, hence, it follows that $q|(m-r)$ if we have a nonzero term with $h = 0$ and given r.

(ii) In the case of a polydisk $\mathcal{D}$, we have semispecial tensors (see [17]), hence, we may assume that $m = 1$ in the previous lemma, and it follows that these descend to the quotient as holomorphic tensors.

3. PROOF OF THE MAIN THEOREMS

We begin with some general observations, valid also for the speculation made in the Introduction.

First of all, we concentrate especially on the necessity parts of the statements; observe that $\mathcal{X} = Y/G$ is a Deligne–Mostow orbifold and the singularities of X are quotient singularities, since, at any point $y \in Y$ having a nontrivial stabilizer $G_y < G$, the group G_y acts linearly by Cartan's Lemma [6]. Since $\mathcal{D}$ is contractible, $\mathcal{X}$ is an orbifold classifying space.

By [31] (Proposition 5.15, p. 158), quotient singularities (X, x) are rational singularities, that is, they are normal and, if $f : Z \to X$ is a local resolution, then $\mathcal{R}^i f_* \mathcal{O}_Z = 0$ for $i \geq 1$. They enjoy also the stronger property of being KLT (Kawamata Log Terminal) singularities.

Indeed, Proposition 5.22 of [31] (where dlt=KLT if there is no boundary divisor Δ, Δ') says that KLT singularities are rational singularities, while Proposition 5.20, p. 160, says that if we have a finite morphism between normal varieties, $F : Y \to X$, then X is KLT if and only if Y is KLT).

Hence, conversely, if we start with a Deligne–Mostow orbifold X, the corresponding finite covering Y is Deligne–Mostow, and if X is KLT, then also Y is KLT.

First important principle. In both cases (ii), (iii), the normal complex space Y has rational singularities.

Moreover, Y is projective if and only if X is projective (by averaging, we can find on Y a G-invariant very ample divisor).

We pass now to the converse implications.

Key argument. We consider the orbifold covering Y associated to the normal torsion free subgroup

$$\Lambda < \Gamma := \pi_1^{\text{orb}}(X),$$

and we show that Y is a locally symmetric manifold.

LEMMA 3.1. *The orbifold Y is just a normal complex space, that is, there are no marked divisors with multiplicity $m_i \geq 2$.*

Proof. Consider the exact sequence

$$1 \to \Lambda \to \Gamma \to G \to 1.$$

Then the generators γ_i have finite order m_i, hence, their image in G has order exactly m_i, because Λ is torsion free.

This means that the covering $Y \to X$ is ramified with multiplicity m_i at the divisor D_i, and therefore, its inverse image in Y is a reduced divisor with multiplicity 1. □

3.1. **Proof of Theorem 0.1**

Case (i): if we assume that Y is smooth, then Y admits a nontrivial slope zero tensor, and we may directly invoke Theorem 2.1, using that by (3), $\mathcal{X}$ and Y have ample canonical divisor, since $K_Y = \pi^*(K_{\mathcal{X}})$.

Similarly, if the universal covering is smooth, also Y is smooth, because by assumption Λ is torsion free and the stabilizers are finite, whence Λ acts freely.

Case (i'): as in [15], we prove that Y must be smooth.

Let Y' be a resolution of Y. Since by assumption $\mathcal{R}^1 f_*(\mathbb{Z}_{Y'}) = 0$ (this is true, for instance, if Y has rational singularities) and $\mathcal{R}^2 f_*(\mathbb{Z}_{Y'}) = 0$, we have an isomorphism

$$H^j(Y', \mathbb{Q}) \cong H^j(Y, \mathbb{Q}), j = 1, 2.$$

Hence, the degree 1 morphism $\alpha : Y' \to Y$ yields an isomorphism of first and second cohomology groups.

We follow a similar argument to the one used in [10], proof of Proposition 4.8: it suffices to show that α is finite, because then α, being finite and birational, is an isomorphism $Y' \cong Y$ by normality, hence, we have shown that Y is smooth and we proceed as for case (i).

Now, if α is not finite, there is a curve C which is contracted by α, hence, its homology class $c \in H_2(Y', \mathbb{Q})$ maps to zero in $H_2(Y, \mathbb{Q})$. And, since $H^2(Y, \mathbb{Q}) \cong H^2(Y', \mathbb{Q})$, the class c of C, by the projection formula, is orthogonal to the whole of $H^2(Y', \mathbb{Q})$, which is the pull back of $H^2(Y, \mathbb{Q})$.

This is a contradiction because, Y' being projective, the class c of C cannot be trivial.

3.2. **Proof of Theorem 0.2**

By (ii), X is a Deligne–Mostow orbifold, hence, also Y is a Deligne–Mostow orbifold, therefore, Y is a normal space with quotient singularities (and these are rational singularities)[1].

We need now to mimic the proof in the case where Y is smooth, for instance, the proof of Theorem 2.1, extending it to the case of a normal space Y with quotient singularities [2].

The first ingredient is: the existence of a complete Kähler–Einstein metric on the orbifold Y with ample K_Y. This was first proven in dimension 2 in [29] (see also [36] for the techniques used) and was proven later on in a more general situation in [7] and [20].

The second ingredient is Proposition 5.4 of [8]: take the orbifold universal covering $\tilde{Y}$ of Y, and let Y' be its smooth part. Then there exists a De Rham decomposition of Riemannian manifolds

$$Y' = \prod_i Y_i',$$

where the holonomy action on each factor is irreducible.

Now, the slope zero tensor is parallel for the Levi-Civita connection (by the Bochner principle, since the slope is zero), as proven by S. Kobayashi in [26], hence all the factors have holonomy different from the Unitary group.

Since K_Y is ample, there are no flat factors, and by the Theorem of Berger [3] and Simons the holonomy of each factor is the holonomy of an irreducible bounded symmetric domain.

Now, the orbifold metric on $\tilde{Y}$ is complete, since the metric on the orbifold Y is complete; and for each i, we can take the completion $\tilde{Y}_i$, hereby obtaining a decomposition for $\tilde{Y}$.

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[1]Similarly, under assumption (iii), Y has KLT singularities, which are also rational singularities.

[2]Pay attention: the orbifolds of [8] are the D-M orbifolds with all $m_i \leq 1$, that we call here of reduced type!

Now, $Y_i' \subset \tilde{Y}_i$ admits a holomorphic map f_i' to an irreducible bounded symmetric domain $\mathcal{D}_i$: by the Hartogs property, f_i extends to a holomorphic map

$$f_i : \tilde{Y}_i \to \mathcal{D}_i,$$

which is an isometry when restricted to Y_i'.

Because of completeness, f_i is surjective, and since $\tilde{Y}_i$ is normal, its singular locus has codimension 2, hence $f_i' : Y_i' \to f_i'(Y_i')$ must be an isomorphism as the target is simply connected, and we have a covering space.

Again by Hartogs, the inverse of f_i, defined on $f_i'(Y_i')$, extends to yield an isomorphism; hence, f_i is an isomorphism.

In particular, it follows that Y is smooth, hence, we only need now to invoke Theorem 2.1.

3.3. **Final remarks**

Remark 9. (a) As already discussed, if X has KLT singularities, by Proposition 5.20 of [31], Y also has KLT singularities (these are also rational singularities). In this case, we need again to find a Kähler–Einstein metric on Y, and to use the Bochner principle.

(b) One may also ask whether one can replace the condition of KLT singularities for X by the condition that X has rational singularities, proving then that Y also has rational singularities.

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