*In memory of Lucian Bădescu,*
*who generously shared with friends what he treasured*
*the most in life: mathematics, music and Christian spirituality*

# ON THE CASTELNUOVO–MUMFORD REGULARITY OF CURVE ARRANGEMENTS

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The Castelnuovo–Mumford regularity of the Jacobian algebra and of the graded module of derivations associated to a general curve arrangement in the complex projective plane are studied. The key result is an addition-deletion type result, similar to results obtained by H. Schenck, H. Terao, Ş. O. Tohăneanu and M. Yoshinaga, but in which no quasi-homogeneity assumption is needed.

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## 1. INTRODUCTION

Let $S = \mathbb{C}[x, y, z]$ be the polynomial ring in three variables $x, y, z$ with complex coefficients, and let $C : f = 0$ be a reduced curve of degree $d \geq 3$ in the complex projective plane $\mathbb{P}^2$. If $f = f_1 \cdots f_s$ is the factorization of $f$ into a product of irreducible factors, we set $C_i : f_i = 0$ for $i = 1, \ldots, s$ for the irreducible components of $C$. Then, we regard $C$ as the curve arrangement

$$C = C_1 \cup \cdots \cup C_s$$

and denote $d = \deg C$ and $d_i = \deg C_i$. We denote by $J_f$ the Jacobian ideal of $f$, i.e., the homogeneous ideal in $S$ spanned by the partial derivatives $f_x, f_y, f_z$ of $f$, and by $M(f) = S/J_f$ the corresponding graded quotient ring, called the Jacobian (or Milnor) algebra of $f$. Consider the graded $S$-module of Jacobian syzygies of $f$ or, equivalently, the module of derivations killing $f$, namely

$$D_0(f) = \big\{(a, b, c) \in S^3 : af_x + bf_y + cf_z = 0\big\}.$$


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In this paper, we study the Castelnuovo–Mumford regularity of the graded $S$-modules $D_0(f)$ and $M(f)$. Recall that to any graded $S$-module $M$, one can associate a coherent sheaf $\widetilde{M}$ on $\mathbb{P}^2$. We say that $\widetilde{M}$ is $m$-regular if

$$H^1\big(\mathbb{P}^2, \widetilde{M}(m-1)\big) = H^2\big(\mathbb{P}^2, \widetilde{M}(m-2)\big) = 0.$$

The minimal $m$ such that $\widetilde{M}$ is $m$-regular is called the Castelnuovo–Mumford regularity of $\widetilde{M}$ and is denoted by $\operatorname{reg} \widetilde{M}$. Finally, we set $\operatorname{reg} M = \operatorname{reg} \widetilde{M}$, see for instance [15, 18] and also [16, Definition 54] for an alternative definition. Note that for a reduced singular plane curve $C$ of degree $d$, the following inequality holds

$$\operatorname{reg} D_0(f) \leq 2d - 4 \tag{1.1}$$

and the equality holds if $C$ has a unique node as its singular set, see Remark 5.1 below. On the other hand, for a line arrangement $C : f = 0$ the much stronger inequality

$$\operatorname{reg} D_0(f) \leq d - 2 \tag{1.2}$$

holds, and equality takes place if $C$ has only double points, see [15, Corollary 3.5]. The proof of this inequality is based on the following addition-deletion type result. With our notation above, assume that $s > 1$ and set

$$C' = C_1 \cup \cdots \cup C_{s-1} : f' = 0.$$

Then, when $C$ is a line arrangement, H. Schenck shows in [15] that the sheaves

$$E(f) = \widetilde{D_0(f)} \text{ and } E(f')(-1) = \widetilde{D_0(f)}(-1)$$

are related by a short exact sequence of sheaves, from which the conclusion is derived. Similar exact sequences in the case when $C$ is a conic-line arrangement having only quasi-homogeneous singularities were considered in [18], where the authors concentrate on the freeness of such arrangements. The more general situation of a curve arrangement having only quasi-homogeneous singularities was considered in [17], where an upper bound of $\operatorname{reg} E(f)$ in terms of $\operatorname{reg} E(f')$ and $d_s = \deg C_s$ when $C_s$ is smooth is given, see [17, Lemma 3.6]. These exact sequences were extended to cover the situation when non-quasi-homogeneous singularities occur, see [6, Theorem 2.3], which can be restated as follows. First, we need some notation. For an isolated hypersurface singularity $(X, 0)$, we set

$$\epsilon(X, 0) = \mu(X, 0) - \tau(X, 0),$$

where $\mu(X, 0)$ (respectively, $\tau(X, 0)$) is the Milnor (respectively, Tjurina) number of the singularity $(X, 0)$. We recall that $\epsilon(X, 0) \geq 0$ and the equality holds if and only if $(X,0)$ is quasi-homogeneous, see [14]. For the curves $D_1$, $D_2$ and $D = D_1 \cup D_2$ and a point $q \in D_1 \cap D_2$, we set

$$\epsilon(D_1, D_2)_q = \epsilon(D_1 \cup D_2, q) - \epsilon(D_1, q)$$

and then define

$$\epsilon(D_1, D_2) = \sum_{q \in D_1 \cap D_2} \epsilon(D_1, D_2)_q.$$

Now, we can recall our result in [6, Theorem 2.3], modulo a twist by $-1$.

THEOREM 1.1. *With the above notation, assume that $s > 1$ and $C_s$ is a smooth curve. Then there is an exact sequence of sheaves on $\mathbb{P}^2$ given by*

$$0 \to E(f')(-d_s) \xrightarrow{f_s} E(f) \to i_{2*}\mathcal{F} \to 0$$

*where $i_s : C_s \to \mathbb{P}^2$ is the inclusion and $\mathcal{F} = \mathcal{O}_{C_s}(D)$ a line bundle on $C_s$ such that*

$$\deg D = 2 - 2g_s - d_s - r - \epsilon(C', C_s),$$

*where $g_s$ is the genus of the smooth curve $C_s$ and $r$ is the number of points in the reduced scheme of $C' \cap C_s$.*

Using this result, our generalized version of [17, Lemma 3.6] is the following.

THEOREM 1.2. *With the above notation, assume that $s > 1$ and $C_s$ is a smooth curve of degree $d_s$. Then $\operatorname{reg}(D_0(f)) \leq m_0$, where*

$$m_0 = \max\Big(\operatorname{reg}\big(D_0(f')\big) + d_s, 2d_s - 3 + \Big\lfloor \frac{r + \epsilon(C', C_s)}{d_s} \Big\rfloor\Big).$$

In fact, our result also corrects a minor error in [17, Lemma 3.6], see Remark 7.2. The case when $C$ is a line arrangement was settled in [15, Theorem 3.4] and was used to prove the inequality (1.2). Theorem 1.2 has the following weaker, but much simpler version.

COROLLARY 2. *With the above notation, assume that $s > 1$ and $C_s$ is a smooth curve of degree $d_s$. Then*

$$\operatorname{reg}\big(D_0(f)\big) \leq \max\big(\operatorname{reg}\big(D_0(f')\big) + d_s, \deg(C') + 2d_s - 3\big).$$

The following result is the analog of the inequality (1.2) for the curve arrangements with all the irreducible components smooth.

THEOREM 2.1. *Let $C : f = 0$ be a curve arrangement in $\mathbb{P}^2$ with $d = \deg f$ such that the irreducible components $C_i : f_i = 0$ of $C$ are smooth curves, say of degree $d_i$, for all $i = 1, \dots, s$. Then*

$$\operatorname{reg} D_0(f) \leq d + \delta - 3,$$

*where $\delta = \max(d_i : i = 1, \dots, s)$ and the equality holds if $C$ is a nodal curve.*

Corollary 3. *Let $C : f = 0$ be a conic-line arrangement with $d = \deg f$. Then*

$$\operatorname{reg} D_0(f) \leq d - 1$$

*and the equality holds if $C$ is a nodal conic-line arrangement containing at least one smooth conic.*

*Remark* 3.1. The Castelnuovo–Mumford regularity $\operatorname{reg}(D_0(f))$ does not enjoy simple semi-continuity properties, see [7, Remark 5.3]. Hence, there seems to be no simple way to show that the maximal value of $\operatorname{reg}(D_0(f))$ in a fixed class of curve arrangements is obtained for the nodal curves in this class, as it is the case in (1.2) and Theorem 2.1. On the other hand, a line arrangement $C : f = 0$ satisfies the equality in (1.2) if and only if $C$ is not formal, see [12, Corollary 7.8], and hence, $C$ enjoys some geometric properties in this situation. One may ask whether the conic-line arrangements $C$ for which the equality holds in Corollary 3 enjoy also some special properties.

## 4. SOME PRELIMINARIES

We say that $C : f = 0$ is an *$m$-syzygy curve* if the module $D_0(f)$ is minimally generated by $m$ homogeneous syzygies, say $r_1, r_2, \ldots, r_m$, of degrees $\alpha_j = \deg r_j$ ordered such that

$$0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_m. \tag{4.1}$$

We call these degrees $(\alpha_1, \ldots, \alpha_m)$ the *exponents* of the curve $C$. The smallest degree $\alpha_1$ is sometimes denoted by $\operatorname{mdr}(f)$ and is called the minimal degree of a Jacobian relation for $f$.

The curve $C$ is *free* when $m = 2$, since then, $D_0(f)$ is a free module of rank 2, see for instance [3, 5, 10, 20]. In this case, $\alpha_1 + \alpha_2 = d - 1$. Moreover, there are two classes of 3-syzygy curves which are intensely studied, since they are in some sense the closest to free curves. First, we have the *nearly free curves*, introduced in [10] and studied in [2, 3, 5, 13] which are 3-syzygy curves satisfying $\alpha_3 = \alpha_2$ and $\alpha_1 + \alpha_2 = d$. Then, we have the *plus-one generated line arrangements* of level $\alpha_3$, introduced by T. Abe in [1], which are 3-syzygy line arrangements satisfying $\alpha_1 + \alpha_2 = d$. In general, a 3-syzygy curve is called a *plus-one generated curve* if it satisfies $\alpha_1 + \alpha_2 = d$.

Consider the sheafification

$$E(f) := \widetilde{D_0(f)}$$

of the graded $S$-module $D_0(f)$, which is a rank two vector bundle on $\mathbb{P}^2$, see [19] for details. Moreover, recall that

$$E(f) = T\langle C\rangle(-1), \tag{4.2}$$

where $T\langle C\rangle$ is the sheaf of logarithmic vector fields along $C$ as considered for instance in [13, 19].

*Remark* 4.1. Note that in [17, Equation (1.1)] the vector bundle $T\langle C\rangle$ is denoted by $\mathrm{Der}(-\log C)$, and hence, we have

$$\mathrm{Der}(-\log C) = T\langle C\rangle = E(f)(1). \tag{4.3}$$

In particular, this implies

$$\operatorname{reg}\big(\mathrm{Der}(-\log C)\big) = \operatorname{reg}\big(T\langle C\rangle\big) = \operatorname{reg}\big(E(f)\big) - 1. \tag{4.4}$$

On the other hand, in [15, Corollary 3.5], the vector bundle $E(f)$ is denoted by $\mathcal{D}$, and hence here, no twist is involved. Similarly, in [18, Formula (1)], the vector bundle $E(f)$ is denoted by $\mathcal{D}_0$, and hence again, no twist is involved.

We define the submodule of Koszul-type relations $KR(f)$ to be the submodule in $D_0(f)$ generated by the following 3 obvious relations of degree $d-1$, namely

$$(f_y, -f_x, 0),\ (f_z, 0, -f_x) \text{ and } (0, f_z, -f_y).$$

Finally, consider the quotient module of essential relations

$$ER(f) = D_0(f)/KR(f). \tag{4.5}$$

Note that $C : f = 0$ is smooth if and only if $ER(f) = 0$. Using this module, we define for a singular curve $C : f = 0$ the invariant

$$\mathrm{mdr}_e(f) = \min\big\{r \in \mathbb{Z} : ER(f)_r \neq 0\big\}. \tag{4.6}$$

We have $\mathrm{mdr}_e(f) = \mathrm{mdr}(f)$ when $\mathrm{mdr}(f) < d-1$.

We introduce the following invariants associated with the curve $C : f = 0$.

*Definition* 4.2. For a homogeneous reduced polynomial $f \in S_d$ one defines

(i) the *coincidence threshold*

$$\mathrm{ct}(f) = \max\big\{q : \dim M(f)_k = \dim M(g)_k \text{ for all } k \leq q\big\},$$

with $g$ a homogeneous polynomial in $S$ of the same degree $d$ as $f$ and such that $g = 0$ is a smooth curve in $\mathbb{P}^2$.

(ii) the *stability threshold*

$$\mathrm{st}(f) = \min\big\{q : \dim M(f)_k = \tau(C) \text{ for all } k \geq q\big\}.$$

In particular, for a smooth curve $C : f = 0$ one has $\mathrm{ct}(f) = \infty$ and $\mathrm{st}(f) = 3(d-2)+1$. It is clear that for a singular curve $C : f = 0$ one has

$$\mathrm{ct}(f) = \mathrm{mdr}_e(f) + d - 2. \tag{4.7}$$

These new invariants $\mathrm{ct}(f)$ and $\mathrm{st}(f)$ enter into the following result, see [5, Corollary 1.7], where $T = 3(d-2)$.

THEOREM 4.1. *Let $C : f = 0$ be a degree $d$ reduced curve in $\mathbb{P}^2$. Then $C$ is a free (respectively, nearly free) curve if and only if*

$$\mathrm{ct}(f) + \mathrm{st}(f) = T \text{ (respectively, } \mathrm{ct}(f) + \mathrm{st}(f) = T + 2).$$

*In the remaining cases, one has* $\mathrm{ct}(f) + \mathrm{st}(f) \geq T + 3$.

To state the following result, we recall some more notation. Let $J = J_f$ be the Jacobian ideal of $f$ and $I = I_f$ be its saturation with respect to the maximal ideal $(x, y, z)$. Then, the singular subscheme $\Sigma_f$ of the reduced curve $C : f = 0$ is the 0-dimensional scheme defined by the ideal $I$ and we consider the following sequence of defects

$$\mathrm{def}_k \Sigma_f = \tau(C) - \dim \frac{S_k}{I_k}. \tag{4.8}$$

With this notation, one has the following result, see [4, Theorem 1], where $g$ is as in Definition 4.2 (i).

THEOREM 4.2. *Let $C : f = 0$ be a degree $d$ reduced curve in $\mathbb{P}^2$. If $\Sigma_f$ denotes its singular locus subscheme, then*

$$\dim M(f)_{T-k} = \dim M(g)_k + \mathrm{def}_k \Sigma_f$$

*for $0 \leq k \leq 2d - 5$. In particular, if* $\mathrm{indeg}(I_f) \leq d - 2$, *then*

$$\mathrm{st}(f) = T - \mathrm{indeg}(I_f) + 1.$$

The second claim in Theorem 4.2 follows by taking

$$k = \mathrm{indeg}(I_f) - 1 \leq d - 3$$

in the first claim and using the obvious equality $M(g)_j = S_j$ for $j < d - 1$.

LEMMA 5. *Let $C : f = 0$ be a reduced plane curve of degree $d$. Then*

$$\mathrm{reg}\, J_f = \mathrm{reg}\big(D_0(f)\big) + d - 2 \text{ and } \mathrm{reg}\big(M(f)\big) = \mathrm{reg}\big(D_0(f)\big) + d - 3.$$

*Proof.* The first claim follows from the obvious exact sequence

$$0 \to D_0(f) \to S^3 \to J_f(d-1) \to 0.$$

The second claim follows from the obvious exact sequence

$$0 \to J_f \to S \to M(f) \to 0$$

which implies $\mathrm{reg}(M(f)) = \mathrm{reg}(J_f) - 1$. □

The next result gives the relation between these invariants, see [7, Theorem 3.3].

THEOREM 5.1. *Let $C : f = 0$ be a reduced singular plane curve of degree $d$. Then the equality*

$$\operatorname{reg}\big(M(f)\big) = \operatorname{st}(f)$$

*holds if and only if $C : f = 0$ is a free curve. Otherwise, one has*

$$\operatorname{reg}\big(M(f)\big) = \operatorname{st}(f) - 1.$$

*Remark* 5.1. It was shown in [9, Theorem 1.5 and Example 4.3 (i)] that for a reduced singular plane curve of degree $d$, one has $\operatorname{st}(f) \leq 3(d-2)$, and that equality holds when $C$ has a unique node as its singular set. Such a curve is not free when $d \geq 3$. It follows that for $d \geq 3$, one has

$$\operatorname{reg}\big(M(f)\big) \leq 3d-7 \text{ and } \operatorname{reg}\big(D_0(f)\big) \leq 2d-4$$

with equalities when $C : f = 0$ is a uninodal curve.

*Example* 5.2. (i) If $C : f = 0$ is a free curve of degree $d$ with exponents $(\alpha_1, \alpha_2)$ with $\alpha_1 \leq \alpha_2$, then one has $1 \leq \alpha_1 \leq (d-1)/2$ and hence

$$\operatorname{ct}(f) = \alpha_1 + d - 2,$$
$$\operatorname{reg}\big(M(f)\big) = \operatorname{st}(f) = 2(d-2) - \alpha_1 = d - 3 + \alpha_2 \text{ and } \operatorname{reg}\big(D_0(f)\big) = \alpha_2.$$

This follows from relation (4.7), Theorems 4.1 and 5.1 and Lemma 5.

(ii) If $C : f = 0$ is a plus-one generated curve with exponents $(\alpha_1, \alpha_2, \alpha_3)$, then $\alpha_1 + \alpha_2 = d$ and one has

$$\operatorname{st}(f) = d - 2 + \alpha_3$$

see [11, Proposition 2.1]. It follows as above that

$$\operatorname{ct}(f) = \alpha_1 + d - 2,\ \operatorname{reg}\big(M(f)\big) = \operatorname{st}(f) - 1 = d - 4 + \alpha_3 \text{ and } \operatorname{reg}\big(D_0(f)\big) = \alpha_3 - 1.$$

We conclude this section with a local result, needed in the proofs in the next section.

LEMMA 6. *Let us consider a reduced plane curve singularity $(D_1, 0)$ and a smooth germ $(D_2, 0)$ which is not an irreducible component of $(D_1, 0)$. Then*

$$(D_1, D_2)_0 - \epsilon(D_1, D_2)_0 - 1 \geq 0,$$

*where $(D_1, D_2)_0$ denotes the intersection multiplicity of $(D_1, 0)$ and $(D_2, 0)$.*

*Proof.* We have

$$\epsilon(D_1, D_2)_0 = \mu(D_1 \cup D_2, 0) - \tau(D_1 \cup D_2, 0) - \big(\mu(D_1, 0) - \tau(D_1, 0)\big).$$

Using the formula

$$\mu(D_1 \cup D_2, 0) = \mu(D_1, 0) + \mu(D_2, 0) + 2(D_1, D_2)_0 - 1,$$

see [21, Theorem 6.5.1], the claim in Lemma 6 is equivalent to the much simpler inequality

$$\tau(D_1 \cup D_2, 0) \geq \tau(D_1, 0) + (D_1, D_2)_0.$$

Choose local coordinates at $0 \in \mathbb{C}^2$ such that the smooth germ $(D_2, 0)$ is given by $u = 0$ and the singularity $(D_1, 0)$ by $g = 0$. Let $R = \mathbb{C}\{u, v\}$ be the convergent power series local ring with $\mathbb{C}$ coefficients and variables $u$ and $v$. Then $g \in R$ is reduced and nondivisible by $u$. The singularity $(D_1, 0)$ has an associated Tjurina algebra

$$T(g) = R/I_g,$$

where $I_g = (g, g_u, g_v)$, such that $\tau(D_1, 0) = \dim T(g)$. Similarly, we consider $T(ug) = R/I_{ug}$ with $I_{ug} = (ug, g + ug_u, ug_v)$ and $\tau(D_1 \cup D_2, 0) = \dim T(ug)$. We have the following exact sequence

$$0 \to T(g) \to \frac{R}{(g, ug_u, ug_v)} \to \frac{R}{(g, u)} \to 0,$$

where the second map is multiplication by $u$ and the third map is the obvious projection. To show that the second map is injective, assume that for $h \in R$ we have $uh \in (g, ug_u, ug_v)$. It follows that there are germs $a, b, c \in R$ such that

$$uh = ag + bug_u + cug_v.$$

This equality implies that $a$ is divisible by $u$, which is not a factor of $g$, and hence $h \in I_g$. This exact sequence implies that

$$\dim \frac{R}{(g, ug_u, ug_v)} = \dim T(g) + \dim \frac{R}{(g, u)} = \tau(D_1, 0) + (D_1, D_2)_0.$$

Since $I_{ug} \subset (g, ug_u, ug_v)$, we have

$$\tau(D_1 \cup D_2, 0) \geq \dim \frac{R}{(g, ug_u, ug_v)}$$

and this completes our proof. □

## 7. THE PROOFS OF THE MAIN RESULTS

### 7.1. Proof of Theorem 1.2

If we twist the exact sequence of sheaves in Theorem 1.1 by $\mathcal{O}_{\mathbb{P}^2}(t)$ and take the associated long cohomology sequence, we get

$$H^1\big(\mathbb{P}^2, E(f')(t - d_s)\big) \to H^1\big(\mathbb{P}^2, E(f)(t)\big) \to H^1\big(C_s, \mathcal{O}_{C_s}(D_t)\big) \to$$
$$\to H^2\big(\mathbb{P}^2, E(f')(t - d_s)\big) \to H^2\big(\mathbb{P}^2, E(f)(t)\big) \to 0.$$

Here

$$\deg D_t = \deg D + td_s = (2 + t)d_s - d_s^2 - r - \epsilon(C', C_s).$$

Here, the vanishing $H^1(\mathbb{P}^2, E(f')(t-d_s)) = 0$ takes place for any $t - d_s \geq \operatorname{reg}(D_0(f')) - 1$, hence for any

$$t \geq \operatorname{reg}\bigl(D_0(f')\bigr) + d_s - 1. \tag{7.1}$$

Next, we have

$$\dim H^1\bigl(C_s, \mathcal{O}_{C_s}(D_t)\bigr) = \dim H^0\bigl(C_s, \mathcal{O}_{C_s}(K - D_t)\bigr),$$

where $K$ is the canonical divisor of the curve $C_s$. It follows that

$$\deg(K - D_t) = \deg(K) - \deg(D_t) = 2d_s^2 - 5d_s + r + \epsilon(C', C_s) - td_s,$$

since $\deg K = d_s(d_s - 3)$. Next, $H^1(C_s, \mathcal{O}_{C_s}(D_t)) = 0$ if $\deg(K - D_t) < 0$, in other words if

$$t > 2d_s - 5 + \frac{r + \epsilon(C', C_s)}{d_s}. \tag{7.2}$$

This strict inequality is easily seen to be equivalent to the following non-strict inequality.

$$t \geq 2d_s - 4 + \left\lfloor \frac{r + \epsilon(C', C_s)}{d_s} \right\rfloor. \tag{7.3}$$

The inequalities (7.1) and (7.3) imply that the integer $m_0$ defined in Theorem 1.2 satisfies $H^1(\mathbb{P}^2, E(f)(m_0 - 1)) = 0$. To complete the proof of Theorem 1.2 it remains to show that $H^2(\mathbb{P}^2, E(f)(m_0 - 2)) = 0$. To get this vanishing, we take $t = m_0 - 2$ in the above exact sequence and see that

$$H^2\bigl(\mathbb{P}^2, E(f')(m_0 - 2 - d_s)\bigr) = 0.$$

Indeed, one has

$$m_0 - 2 - d_s \geq \bigl(\operatorname{reg}\bigl(D_0(f')\bigr) + d_s\bigr) - 2 - d_s = \operatorname{reg}\bigl(D_0(f')\bigr) - 2$$

and we know that $H^2(\mathbb{P}^2, E(f')(\operatorname{reg}(D_0(f')) - 2) = 0$. In fact, for any coherent sheaf $\mathcal{F}$ of $\mathbb{P}^2$, the vanishing $H^2(\mathbb{P}^2, \mathcal{F}(m)) = 0$ implies the vanishing $H^2(\mathbb{P}^2, \mathcal{F}(m+1)) = 0$ as the obvious exact sequence

$$0 \to \mathcal{F}(m) \to \mathcal{F}(m+1) \to \mathcal{G} \to 0$$

shows. Here, the morphism $\mathcal{F}(m) \to \mathcal{F}(m+1)$ is induced by multiplication by a linear form $\ell \in S_1$, $L$ is the line $\ell = 0$ and $\mathcal{G}$ is a coherent sheaf supported on $L$. This completes our proof.

*Remark* 7.2. Even in the case when all the singularities in the intersection $C' \cap C_s$ are quasi-homogeneous, our result is slightly different from [17, Lemma 3.6]. First of all, taking into account the twist explained in Remark 4.1 and equation (4.4), Lemma 3.6 in [17] can be restated as

$$\operatorname{reg}\bigl(D_0(f)\bigr) \leq \max\Bigl(\operatorname{reg}\bigl(D_0(f')\bigr) + d_s, 2d_s - 4 + \left\lfloor \frac{r}{d_s} \right\rfloor\Bigr).$$

This difference with our Theorem 1.2 comes from the fact that in [17] the strict inequality (7.2) is not replaced by the non-strict inequality (7.3). When $r/d_s$ is an integer and if

$$\operatorname{reg}\big(D_0(f')\big) + d_s \leq 2d_s - 4 + \frac{r}{d_s}$$

then the claim in [17, Lemma 3.6] is false. Such situations really do occur. Indeed, let $C'$ be a free curve of degree $d'$ and exponents $(\alpha_1, \alpha_2)$ with inequality $\alpha_2 \leq d' - 3$. Let $C_s$ be a smooth curve meeting $C'$ transversally in $d_s d'$ points, which are all nodes for $C$. Then Example 5.2 (i) implies

$$\operatorname{reg}\big(D_0(f')\big) + d_s = \alpha_2 + d_s \leq d' - 3 + d_s \leq 2d_s - 4 + d'.$$

Hence, such examples exist even in the class of line arrangements. On the other hand, Theorem 3.4 in [15] which covers the case of $C$ a line arrangement is correctly stated.

### 7.3. **Proof of Corollary 2**

Note that using Lemma 6, we have the following

$$r + \epsilon(C', C_s) = \sum_{p \in C' \cap C_s} \big(1 + \epsilon(C', C_s)_p\big) \leq \sum_{p \in C' \cap C_s} (C', C_s)_p = \deg(C') d_s.$$

This clearly proves Corollary 2.

### 7.4. **Proof of Theorem 2.1 and Corollary 3**

We can assume in this proof that $\delta = d_1 > 1$, since the case of line arrangements is clear by [15]. Then, we have

$$\operatorname{reg}\big(D_0(f_1)\big) = 2d_1 - 3,$$

using for instance Theorem 5.1. Hence, the first claim holds for $k = 1$. Now assume that this claim holds for $k = s - 1$ and apply Corollary 2. We get

$$\operatorname{reg}\big(D_0(f)\big) \leq \max(d + d_1 - 3, d_1 + \cdots + d_{s-1} + 2d_s - 3).$$

This inequality yields the first claim for $k = s$ since $d_1 \geq d_s$.

Now, we consider the second claim, when $C$ is in addition a nodal curve. Such a curve $C$ cannot be free, see for instance [8]. Hence, the equality $\operatorname{reg}(D_0(f)) = d + d_1 - 3$ is equivalent to the equality

$$\operatorname{st}(f) = 2d - 5 + d_1, \tag{7.4}$$

in view of Lemma 5 and Theorem 5.1. Hence, it remains to prove the following.

LEMMA 8. *Let* $C : f = 0$ *be a nodal curve arrangement in* $\mathbb{P}^2$ *with* $d = \deg f$ *such that the irreducible components* $C_i : f_i = 0$ *of* $C$ *are smooth curves, say of degree* $d_i$*, for all* $i = 1, \ldots, s$*. Then, if* $\delta = \max(d_i : i = 1, \ldots, s) > 1$*, one has the following equalities*

$$\operatorname{st}(f) = 2d - 5 + \delta \text{ and } \operatorname{indeg} I_f = d - \delta.$$

*Proof.* Assume again that $\delta = d_1$. First, we use Theorem 4.2 and see that (7.4) is equivalent to $\operatorname{indeg} I_f = d - d_1$, in other words to the two relations

$$I_{f,d-d_1-1} = 0 \text{ and } I_{f,d-d_1} \neq 0.$$

Since $C$ is a nodal curve, then $I_f$ consists of all the polynomials vanishing at all the nodes of $C$. In particular,

$$f_2 f_3 \cdot \ldots \cdot f_s \in I_{f,d-d_1}$$

and hence $I_{f,d-d_1} \neq 0$.

Finally, we prove that $I_{f,d-d_1-1} = 0$. Let $h \in I_{f,d-d_1-1}$ and assume first that the curve $H : h = 0$ is reduced. For any $1 \leq k \leq s$, we consider the intersection $H \cap C_k$. Note that on $C_k$ there are exactly $d_k(d - d_k)$ nodes of the curve $C$. The inequality

$$d_k(d - d_k) > d_k(d - d_1 - 1) = \deg C_1 \deg H$$

implies that $C_k$ is an irreducible component of $H$ for all $k = 1, \ldots, s$. This is impossible since

$$\deg C = d > d - d_1 - 1 = \deg H.$$

If the curve $H$ is not reduced, we apply the above argument to the associated reduced curve $H^{\mathrm{red}}$ and get again a contradiction since $\deg H^{\mathrm{red}} \leq \deg H$. This completes the proof of Lemma 8 and also of Theorem 2.1. □

Corollary 3 is an obvious consequence of Theorem 2.1 for $\delta = 2$.

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