Dedicated to the memory of Lucian Bădescu.
He would have celebrated his 80th birthday in 2024

ON THE ARITHMETICAL SURJECTIVITY CONJECTURE OF COLLIOT-THÉLÈNE

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In this note, we extend results by Denef and Loughran, Skorobogatov, and Smeets concerning the arithmetical surjectivity conjecture of Colliot-Thélène. The question is about giving necessary and sufficient birational conditions for morphisms of varieties to be surjective on local points for almost all localizations of the base field.

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1. INTRODUCTION/MOTIVATION

The aim of this note is to shed new light on the *arithmetical surjectivity conjecture* by Colliot-Thélène, cf. [10], concerning the image of local rational points under dominant morphisms of (smooth) varieties over global fields (and beyond). The context is as follows: Let k be a global field, and $f : X \to Y$ be a morphism of k-varieties. Let $v \in \mathbb{P}(k)$ be the finite places of k, k_v be the completion of k at v, and $X(k_v)$, $Y(k_v)$ denote the k_v-rational points.

For every $v \in \mathbb{P}(k)$, the k-morphism f gives rise to a canonical map on k_v-rational points $f^{k_v} : X(k_v) \to Y(k_v)$. There are obvious examples showing that, in general, f^{k_v} is not surjective, e.g., $f : \mathbb{P}^1_{\mathbb{Q}} \to \mathbb{P}^1_{\mathbb{Q}}$ of degree two. Therefore, for $f : X \to Y$ as above, it is natural to consider the basic property

$$(\mathsf{Srj}) \qquad f^{k_v} : X(k_v) \to Y(k_v) \textit{ is surjective for almost all } v \in \mathbb{P}(k),$$

called *arithmetical surjectivity* and to ask the fundamental question:

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Q: *"Characterize" the arithmetically surjective morphisms $f : X \to Y$.* This problem was considered in a systematic way by Colliot-Thélène [10], under the following restrictive but to some extent natural hypothesis:

$(*)_{\mathsf{CT}}$ *k is a number field, X, Y are proper smooth integral k-varieties, $f : X \to Y$ is dominant morphism with geometrically integral generic fiber.*

In particular, if $L := k(Y)$ is the function field of Y, the generic fiber X_L of the morphism $f : X \to Y$ can be viewed as an L-variety. In this notation, for morphisms $f : X \to Y$ satisfying $(*)_{\mathsf{CT}}$, Colliot-Thélène considered the hypothesis (CT) and made the conjecture (CCT) below:

(CT) *For each discrete valuation k-ring R of L, and its residue field κ_R, there is a regular flat R-model $\mathfrak{X}_R$ of X_L whose special fiber $\mathfrak{X}_{\kappa_R}$ has an irreducible component $\mathfrak{X}_\mu$ which is κ_R-geometrically integral.*

Conjecture of Colliot-Thélène (CCT). *Let $f : X \to Y$ be a dominant morphism of proper smooth geometrically integral varieties over a number field k satisfying the hypotheses $(*)_{\mathsf{CT}}$ and* (CT). *Then $f : X \to Y$ is arithmetically surjective, i.e., f has the property* (Srj).

In a recent paper, Denef [12] proved a stronger form of the conjecture (CCT), by replacing the hypothesis (CT) by the weaker hypothesis (D) below. In order to explain Denef's result, we recall the following terminology: Let $f : X \to Y$ be a morphism satisfying hypothesis $(*)_{\mathsf{CT}}$. A *smooth modification* of f is any morphism $f' : X' \to Y'$ satisfying hypothesis $(*)_{\mathsf{CT}}$ such that there exist modifications (i.e., birational morphisms) $p : X' \to X$, $q : Y' \to Y$ satisfying $q \circ f' = f \circ p$, i.e., one has a commutative diagram of k-morphisms:

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow p & & \downarrow q \\ X & \xrightarrow{f} & Y \end{array}$$

Given a smooth modification $f' : X' \to Y'$ of f, for every Weil prime divisor $E' \subset Y'$, and the Weil prime divisors D' of X' above E', consider: First, the multiplicity $e(D'|E')$ of D' in $f'^*(E') \in \mathrm{Div}(X')$; second, the restriction $f'_{D'} : D' \to E'$ of f' to $D' \subset X'$, which is a morphism of integral k-varieties with generic fiber $D_{k(E')}$ a $k(E')$-variety. In this notation, f' is called *spilt,* if for every E' there is D' such that both condition below are satisfied:

(i) $e(D'|E') = 1$; (ii) $D_{k(E')}$ is a $k(E')$-geometrically integral variety.

For $f : X \to Y$ satisfying $(*)_{\mathsf{CT}}$, it turns out that the hypothesis (CT) above implies the following weaker hypothesis involving *all* the smooth modification $f' : X' \to Y'$ of $f : X \to Y$ as follows.

(D) *For all $f' : X' \to Y'$, all Weil prime divisors E' of Y' are split under f'.*

THEOREM ([12, Main Theorem 1.2]). *If $f : X \to Y$ satisfies* $(*)_{\mathsf{CT}}$ *and* (D), *then f is arithmetically surjective.*

Finally, we recall the more recent results by Loughran–Skorobogatov–Smeets [16] which, for morphisms $f : X \to Y$ satisfying the hypothesis $(*)_{\mathsf{CT}}$ above, give *necessary and sufficient conditions* such that $f : X \to Y$ is arithmetically surjective, by generalizing the notion of f being split as follows. Namely, following [16], in the notation introduced above, let $f' : X' \to Y'$ be a smooth modification of $f : X \to Y$. For a Weil prime divisor E' of Y' and a Weil prime divisor D' of X' above E', let $k(D') \,|\, k(E')$ be the function field extension defined by the dominant map $f'_{D'} : D' \to E'$. One says that E' is *pseudo-split* under $f' : X' \to Y'$, if for every $\sigma \in G_{k(E')}$ in the absolute Galois group $G_{k(E')}$, there is some Weil prime divisor D' of X' above E' satisfying:

$$e(D'|E') = 1 \ \text{ and } \ k(D') \otimes_{k(E')} \overline{k(E')} \text{ has a factor stabilized by } \sigma.$$

Following Loughran–Skorobogatov–Smeets [16], consider the hypothesis involving all smooth modifications $f' : X' \to Y'$ of $f : X \to Y$ below:

(LSS) *For all $f' : X' \to Y'$, all prime $E' \in \mathrm{Div}(Y')$ are pseudo-split under f'.*

Note that if D', E' satisfy hypothesis (D), then $k(D') \,|\, k(E')$ is a regular field extension, hence $k(D') \otimes_{k(E')} \overline{k(E')}$ is a field stabilized by all $\sigma \in G_{k(E')}$, thus E' being split under f' obviously implies that E' is pseudo-split under f'. Therefore, hypothesis (D) implies hypothesis (LSS), hence leading to the following sharpening of Denef's result above:

THEOREM ([16, Theorem 1.4]). *If $f : X \to Y$ satisfies* $(*)_{\mathsf{CT}}$, *then f satisfies* (LSS) *if and only if f is arithmetically surjective.*

In this note, we provide a different approach to the basic problem (CCT) considered above, and using completely different techniques, we give wide generalizations of the results from [12], [16], see e.g., Theorems 1.4 and Theorem 1.5 below. The context and form in which these results hold and are proved is as follows:

• Instead of number fields, we consider base fields k of characteristic $\mathrm{char}(k) = 0$ satisfying the hypothesis $(\mathsf{H})_k$ below and consider the corresponding generalization $(\mathsf{Srj})_{\boldsymbol{\Omega}(k)}$ below of the arithmetical surjectivity (Srj) —which coincides with (Srj) in the case of number fields.

$(\mathsf{H})_k$ *k is of finite type over either* (i) $\mathbb{Q}$, *or* (ii) *a pseudo-finite field* k_0.[1]

Let $\boldsymbol{\Omega}(k)$ be the set of *discrete valuations* v of k with residue field kv *finite in case* (i), respectively *finite over* k_0 *in case* (ii). [2] Recall that a model of k

[1] k_0 is pseudo-finite if k_0 is perfect, PAC, and G_{k_0} is pro-cyclic free, see AX [3] for basics.
[2] By [13, Corollary 11.5.9], k_0 (being PAC) has no discrete valuations, thus $v|_{k_0}$ is trivial.

is any separated integral scheme S of finite type with function field $\kappa(S) = k$ in case (i), respectively an integral k_0-variety S with function field $k = k_0(S)$ in case (ii). For every model S of k, we denote:

$$\mathbf{\Omega}_S(k) := \big\{ v \in \mathbf{\Omega}(k) \,|\, v \text{ has a center } x_v \in S \,\big\}.$$

In particular, x_v must be a closed point of S, and conversely, for every closed point $x \in S$ there are valuations $v_x \in \mathbf{\Omega}_S(k)$ having center x on S. Further, we notice: First, since any two models S_1 and S_2 are birationally equivalent, there is a model S which has open embeddings $S \hookrightarrow S_1$ and $S \hookrightarrow S_2$, hence $\mathbf{\Omega}_S(k) \subset \mathbf{\Omega}_{S_1}(k), \mathbf{\Omega}_{S_2}(k)$. Second, $S_{\text{reg}} \subset S$ is Zariski open dense, and for $x \in S_{\text{reg}}$ there are $v \in \mathbf{\Omega}(k)$ with $x_v = x$ and $kv = \kappa(x)$. Therefore, one has:

(†) $\quad \mathcal{P}_k := \big\{ \mathbf{\Omega}_S(k) \,|\, S \text{ is regular model of } k \big\}$ is a prefilter on $\mathbf{\Omega}(k)$.

Here, recall that a prefilter $\mathcal{P}$ on a non-empty set I is any non-empty subset $\mathcal{P} \subset \mathcal{P}(I)$ of the power set $\mathcal{P}(I)$ of I satisfying:

$$\text{(i) } \emptyset \notin \mathcal{P}; \quad \text{(ii) } \forall\, A, B \in \mathcal{P}\ \exists\, C \in \mathcal{P} \text{ s.t. } C \subset A, B.$$

Finally, recall that every global field k has a unique proper regular model S_0, precisely: $S_0 = \operatorname{Spec} \mathcal{O}_k$ if k is a number field, and S_0 is the projective smooth $\mathbb{F}_p$-curve with $\kappa(S_0) = k$ if $\operatorname{char}(k) > 0$. Further, $v \in \mathbf{\Omega}(k)$ are in bijection with the closed points $x \in S_0$ via $\mathcal{O}_x = \mathcal{O}_v$. Hence $\mathbb{P}(k) = \mathbf{\Omega}_{S_0}(k)$, thus $S_0 \backslash S$ and $\mathbb{P}(k) \backslash \mathbf{\Omega}_S(k)$ are finite for k global.

This being said, a natural generalization of the property (Srj) is:

$(\mathsf{Srj})_{\mathbf{\Omega}(k)}$ *k has a model S s.t. $f^{k_v} : X(k_v) \to Y(k_v)$ is surjective $\forall\, v \in \mathbf{\Omega}_S(k)$.*

We next give the (fully) birational form of the pseudo-splitness hypothesis (LSS) from [16], and define/introduce the pseudo-splitness of *arbitrary* morphisms $f : X \to Y$ of *arbitrary* k-varieties.

• *Pseudo-splitness of prime divisors in function field extensions over k.* Let $F|k$ be a function field over an arbitrary base field k. For valuations $w \in \operatorname{Val}(F)$, we denote by wF the value group of w, by $\mathcal{O}_w, \mathfrak{m}_w$ the valuation ring/ideal of w, and by Fw the residue field of w. A *prime divisor* of $F|k$ is any w which satisfies the following equivalent conditions:

(i) $F|k$ has normal k-models Z with $x \in Z$, $\operatorname{codim}_Z(x) = 1$, $\mathcal{O}_w = \mathcal{O}_x$.

(ii) w is a k-valuation of F, i.e., w is trivial on k, and $\operatorname{td}(Fw|k) = \operatorname{td}(F|k) - 1$.

Notation. $\mathcal{D}(F|k) := \{ v \,|\, v \text{ prime divisor of } F|k \text{ or } v \text{ the trivial valuation} \}$

For k-function field extensions $E|F$, the restriction map $\mathcal{D}(E|k) \to \mathcal{D}(F|k)$, $v \mapsto w := v|_F$ is well defined and surjective. In particular, if $v \in \mathcal{D}(E|k)$ and

$w = v|_F$, then there is a canonical k-embedding of the residue function fields $Fw := \kappa(w) \hookrightarrow \kappa(v) =: Ev$, and $e(v|w) := (vE : wF)$ is finite if either v is trivial or w is non-trivial. In particular, if $w = v|_F$, then the absolute Galois group G_{Fw} of Fw acts canonically on the Fw-algebra $Ev \otimes_{Fw} \overline{Fw} = \prod_i E'_i$ by permuting the factors E'_i of $Ev \otimes_{Fw} \overline{Fw}$.

Definition 1.1. In the above notation, we say that:

1) $w \in \mathcal{D}(F|k)$ is *generalized pseudo-split* (g.p.s.) in $\mathcal{D}(E|k)$, if $\forall\, \sigma \in G_{Fw}$ $\exists\, v \in \mathcal{D}(E|k)$ such that: (i) $w = v|_F$; (ii) $e(v|w) = 1$ if w is non-trivial; (iii) $Ev \otimes_{Fw} \overline{Fw}$ has a factor E' which is a *field stabilized by* σ.

2) $\mathcal{D}(F|k)$ is g.p.s. in $\mathcal{D}(E|k)$, if all $w \in \mathcal{D}(F|k)$ are g.p.s. in $\mathcal{D}(E|k)$.

The generalized pseudo-splitness relates to the hypothesis (LSS) as follows: Let $f : X \to Y$ be a dominant morphism of proper smooth varieties over a field k with $\mathrm{char}(k) = 0$, and setting $K = k(X)$, $L = k(Y)$, let $K\,|\,L$ be the corresponding k-extension of function fields. By Hironaka's Desingularization Theorem, the system of projective smooth models $(X_\mu)_\mu$ and $(Y_\mu)_\mu$ are cofinal (w.r.t. the domination relation) in the system of all the proper models of $K|k$, respectively $L|k$. Hence, if $f_\mu : X_\mu \to Y_\mu$, $\mu \in I$ is the (projective) system of all the smooth modifications of f satisfying the hypothesis $(*)_{\mathsf{CT}}$, by mere definitions one has:

Fact 1.2. *The hypothesis* (LSS) *implies that* $\mathcal{D}(L|k)$ *is g.p.s. in* $\mathcal{D}(K|k)$.

• *Generalized pseudo-splitness of morphisms of arbitrary k-varieties.*
Let $f : X \to Y$ be a morphism of *arbitrary* varieties over some base field k, and for $y \in Y$, let X_y be the reduced fiber of f at $y \in Y$. For $y \in Y$ and $x \in X_y$, we denote $k_y := \kappa(y)$, $k_x := \kappa(x)$. Hence, f defines canonically an extension of k-function fields $k_x\,|\,k_y$, and one has the restriction map $\mathcal{D}(k_x|k) \to \mathcal{D}(k_y|k)$, $v_x \mapsto v_y := (v_x)|_{k_y}$. Denoting the residue fields $\kappa_{v_y} := k_y v_y$ and $\kappa_{v_x} := k_x v_x$, it follows that $\kappa_{v_x}\,|\,\kappa_{v_y}$ is canonically a function field extension over k.

Definition 1.3. In the above notation, for $f : X \to Y$ we say that:

1) $v_y \in \mathcal{D}(k_y|k)$ is g.p.s. under f, if for every $\sigma \in G_{k_y}$ there are $x \in X_y$ and $v_x \in \mathcal{D}(k_x|k)$ satisfying: $v_y = (v_x)|_{k_y}$, $e(v_x|v_y) = 1$ if v_y is non-trivial, and $\kappa_{v_x} \otimes_{\kappa_{v_y}} \overline{\kappa_{v_y}}$ has a factor which is a *field stabilized by* σ.

   And $y \in Y$ is g.p.s. under f, if all $v_y \in \mathcal{D}(k_y|k)$ are g.p.s. under f.

2) Finally, the morphism $f : X \to Y$ is g.p.s., if all $y \in Y$ are g.p.s. under f.

This being said, the results extending/generalizing and shedding new light on the aforementioned [12, Main Theorem 1.2], and [16, Theorem 1.4], are:

THEOREM 1.4. *For k satisfying* $(\mathsf{H})_k$, $\mathrm{char}(k) = 0$, *let* $f : X \to Y$ *be a morphism of arbitrary k-varieties. Then f has property* $(\mathsf{Srj})_{\boldsymbol{\Omega}(k)}$ *if and only if f is generalized pseudo-split.*

THEOREM 1.5. *For k satisfying* $(\mathsf{H})_k$, $\mathrm{char}(k) = 0$, *let* $f : X \to Y$ *be a dominant morphism of proper smooth k-varieties, and set* $K = k(X)$, $L = k(Y)$. *Then f satisfies* $(\mathsf{Srj})_{\boldsymbol{\Omega}(k)}$ *if and only if* $\mathcal{D}(L|k)$ *is generalized pseudo-split in* $\mathcal{D}(K|k)$.

COROLLARY 1.6. *The property* $(\mathsf{Srj})_{\boldsymbol{\Omega}(k)}$ *is a fully birational property of dominant morphisms* $f : X \to Y$ *of proper smooth k-varieties, i.e., it depends on properties of the function field extension* $k(X)|k(Y)$ *only.*

The main point in our approach is to use Ax–Kochen–Ershov Principle (AKE) type results (together with some general model-theoretical facts about rational points and ultraproducts of local fields), as originating from [3, 4, 5], see, e.g., [18] for details on AKE.

Finally, one should mention that [12, Subsection 6.3], gives a sketch of a quite short proof of (CCT) – as initially stated by Colliot-Thélène – using the AKE Principle, but not of the stronger final results from in [12]. Actually, the main results of both [12] and [16] are based on quite deep desingularization facts, e.g., [1, 2], and build on previous results and ideas by the authors, cf. [11, 17, 21], aimed at – among other things – giving arithmetic geometry proofs of the AKE. We should also mention that using methods similar to the ones introduced here, Z. Cai [8] reproved/improved and shed completely new light on the birationality of the main results of Gvirtz [14].

Here is an example – resulting from discussions with Daniel Loughran, showing the relation between Theorem 1.4 above, and the previous results.

Example 1.7. Let

$$Y = \mathbb{P}^1_t, X = V(T_0^2 + T_1^2 - t^2T_2^2) \subset Y \times_k \operatorname{Proj} k[T_0, T_1, T_2].$$

One checks directly that for $k = \mathbb{Q}$, the canonical projection $f : X \to Y$ has the property (Srj), and f is smooth and split above all points $y \in Y$ satisfying $y \neq (1:0)$. Further, for the k-rational point $y = (1:0) \in Y$ one has: The fiber X_y above $(1:0) \in Y$ is smooth, but not pseudo-split. In particular, the previous results do not apply. On the other hand, f is generalized pseudo-split: Namely, all $y \neq (1:0)$ are split under f, thus pseudo-split under f; and for $y = (1:0)$, one has $X_y \ni x = (0:0:1) \mapsto (1:0) = y \in Y$, $K_x = k = L_y$, and $\mathcal{D}(K_x|k) = \{v_k^0\} = \mathcal{D}(L_y|k)$ with v_k^0 the trivial valuation of k. Hence, y is pseudo-split under f in the sense defined above.

2. ULTRAPRODUCTS AND RATIONAL POINTS / GENERALIZED PSEUDO-SPITNESS

2.1. Ultraproducts and approximation results for points

We begin by recalling a few facts, which are/should be well known to experts; see, e.g., [6], [9], [13, Chapter 7], for details on ultraproducts and other model theoretical facts. The fact below is a (very) special case of Łoś Ultraproducts Theorem (but one can give easily a direct proof using just definitions). Namely, in the class of field extensions $\tilde{k}|k$, consider the following $\forall\,\exists$ formula in the language $\mathcal{L}_{\rm rings}$ augmented with constants for k:

$$(*) \qquad \forall\, y \in Y(\tilde{k}) \;\; \exists\, x \in X(\tilde{k}) \;\text{ such that }\; f(x) = y\,.$$

Then Łoś Ultraproducts Theorem instantly gives the following.

Fact 2.1. *Let $(k_i|k)_{i\in I}$ be a family of field extensions, $\mathcal{P}_I$ be a fixed prefilter on I, and for every ultrafilter $\mathcal{U}$ on I with $\mathcal{P}_I \subset \mathcal{U}$, let ${}^*k_{\mathcal{U}} := \prod_{i\in I} k_i/\mathcal{U}$ be the corresponding ultraproduct. Then, for every morphism $f: X \to Y$ of k-varieties, the following are equivalent:*

(i) *There is $I_0 \in \mathcal{P}_I$ such that $f^{k_i} : X(k_i) \to Y(k_i)$ is surjective $\forall\, i \in I_0$.*

(ii) *The map $f^{{}^*k_{\mathcal{U}}} : X({}^*k_{\mathcal{U}}) \to Y({}^*k_{\mathcal{U}})$ is surjective for all ultrafilters $\mathcal{U} \supset \mathcal{P}_I$.*

Thus if I is infinite, $f^{k_i} : X(k_i) \to Y(k_i)$ is surjective for almost all $i \in I$ iff $f^{{}^*k_{\mathcal{U}}} : X({}^*k_{\mathcal{U}}) \to Y({}^*k_{\mathcal{U}})$ is surjective for all non-principal ultrafilters $\mathcal{U}$ in I.

Definition 2.2. A field k-extension $k' \to l'$ is called *quasi-elementary*, if for every $\forall\,\exists$ formula ϕ in the language $\mathcal{L}_{\rm rings}$ augmented with constant from k, one has: ϕ holds over k' iff ϕ holds in l'.[3]

Fact 2.3. *Let $f: X \to Y$ be a morphism of k varieties, and $\mathcal{C}_f$ be the class of field extensions $k'|k$ with $f^{k'} : X(k') \to Y(k')$ surjective. One has:*

1) *$\mathcal{C}_f$ is closed w.r.t. ultraproducts and sub-ultrapowers, i.e., $\mathcal{C}_f$ satisfies: If $k_i \in \mathcal{C}_f$, $i \in I$, then $\prod_i k_i/\mathcal{U} \in \mathcal{C}_f$, and if $k'^I/\mathcal{U} \in \mathcal{C}_f$, then $k' \in \mathcal{C}_f$.*

2) *$\mathcal{C}_f$ is closed under quasi-elementary k-field extensions, i.e., if $k' \hookrightarrow l'$ is a quasi-elementary k-field extension, then $k' \in \mathcal{C}_f$ iff $l' \in \mathcal{C}_f$.*

Proof. Assertion 1) follows from Fact 2.1 by mere definition. For 2): The proof follows immediately using the formula $(*)$ from the proof of Fact 2.1. $\square$

[3] Thanks to the Referee for this reformulation of my old definition of “quasi-elementary.”

2.2. Ultraproducts of localizations of arithmetically significant fields

We introduce notation and recall well-known facts and generalize the context in which the conclusion of Theorems 1.4, 1.5 hold, finally allowing to announce Theorems 4.1, 5.1 below. We first collect basic facts in a general setting and subsequently discuss the more special situation of fields satisfying Hypothesis $(\mathrm{H})_k$ as stated in the Introduction.

2.2.1 Basics and Notation

Notations/Remarks 2.4. For *arbitrary fields* k, let $A \subset k^\times$ denote finite subsets, and consider sets $\Sigma_k \subset \mathrm{Val}(k)$ of *discrete valuations* v, with perfect residue field kv if $\mathrm{char}(k) = p > 0$, satisfying:

$$(\mathcal{P}) \qquad \Sigma_A := \big\{v \in \Sigma_k \,|\, A \subset \mathcal{O}_v^\times\big\} \neq \emptyset \ \forall\, A \subset k^\times \text{ finite.}$$

In particular, $\mathcal{P}_{\Sigma_k} := \{\Sigma_A\}_A$ *is a prefilter* on the set of valuations Σ_k.

For $v \in \Sigma_k$, let k_v be the completion of k at $v \in \Sigma_k$, and $\mathcal{U}$ *always* be ultrafilters on Σ_k with $\mathcal{P}_{\Sigma_k} \subset \mathcal{U}$. Thus, $\mathcal{P}_{\Sigma_k}$ and $\mathcal{U}$ are non-principal. Given $\mathcal{U}$, consider the ultraproducts:

$$^*k_{\mathcal{U}} := \textstyle\prod_v k_v/\mathcal{U}, \quad ^*\mathcal{O}_{\mathcal{U}} := \prod_v \mathcal{O}_v/\mathcal{U}, \quad ^*\mathfrak{m}_{\mathcal{U}} := \prod_v \mathfrak{m}_v/\mathcal{U}, \quad ^*\kappa_{\mathcal{U}} := \prod_v kv/\mathcal{U}.$$

Then $^*\mathcal{O}_{\mathcal{U}}$ is the valuation ring of $^*k_{\mathcal{U}}$, say $^*\mathcal{O}_{\mathcal{U}} = \mathcal{O}_{^*v_{\mathcal{U}}}$ with valuation $^*v_{\mathcal{U}}$ having valuation ideal $\mathfrak{m}_{^*v_{\mathcal{U}}} = {^*\mathfrak{m}_{\mathcal{U}}}$, residue field $^*k_{\mathcal{U}}{^*v_{\mathcal{U}}} = {^*\kappa_{\mathcal{U}}}$, and value group $^*v_{\mathcal{U}}{^*k_{\mathcal{U}}} = \prod_v vk/\mathcal{U} = \mathbb{Z}^{\Sigma_k}/\mathcal{U} = {^*\mathbb{Z}_{\mathcal{U}}}$.

1) One has the (canonical) diagonal field embedding $^*\imath_{\mathcal{U}} : k \hookrightarrow {^*k_{\mathcal{U}}}$, and since $\mathcal{P}_{\Sigma_k} \subset \mathcal{U}$, it follows that $^*v_{\mathcal{U}}$ is trivial on k.

2) If $\omega_v \subset \mathcal{O}_v$ is a set of representatives for kv, then $^*\omega_{\mathcal{U}} := \prod_v \omega_v/\mathcal{U} \subset {^*\mathcal{O}_{\mathcal{U}}}$ is a system of representatives for the residue field $^*\kappa_{\mathcal{U}}$. Further, if ω_v are multiplicative, so is $^*\omega_{\mathcal{U}}$.

3) The value group $^*v_{\mathcal{U}}{^*k_{\mathcal{U}}} = {^*\mathbb{Z}_{\mathcal{U}}}$ is a $\mathbb{Z}$-group. Further, if $\pi_v \in k_v$ is a uniformizing parameter for $v \in \Sigma_k$, then $\pi_{\mathcal{U}} = (\pi_v)_v/\mathcal{U}$ is an element of minimal value in $^*v_{\mathcal{U}}{^*k_{\mathcal{U}}}$.

4) The field $^*k_{\mathcal{U}}$ is Henselian with respect to $^*v_{\mathcal{U}}$, and one has:

   a) Let $\mathrm{char}(k) = 0$. Recalling that $^*v_{\mathcal{U}}$ is trivial on k, hence $k = k{^*v_{\mathcal{U}}}$, let $\mathcal{T} \subset {^*\mathcal{O}_{\mathcal{U}}}$ be any lifting of a transcendence basis of $\kappa_{\mathcal{U}} \,|\, k$. Then by Hensel Lemma, the relative algebraic closure $\kappa_{\mathcal{U}} \subset {^*\mathcal{O}_{\mathcal{U}}}$ of $k(\mathcal{T})$ in $^*k_{\mathcal{U}}$ is a field of representatives for $^*\kappa_{\mathcal{U}}$.

b) Let $\mathrm{char}(k) = p > 0$. By hypothesis, kv is perfect $\forall\, v \in \Sigma_k$, hence the Teichmüller system of representatives $\mathbb{F}_v \subset k_v$ for kv is a field and $k_v = \mathbb{F}_v((\pi'_v))$ for any $\pi'_v \in k$ with $v(\pi'_v) = 1$. Therefore, one has: $\kappa_{\mathcal{U}} = \mathbb{F}_{\mathcal{U}} := \prod_v \mathbb{F}_v/\mathcal{U} \subset {}^*\mathcal{O}_{\mathcal{U}}$ is a perfect field and a system of representatives for ${}^*\kappa_{\mathcal{U}}$, the "Teichmüller system" of representatives.

- Note that in both cases a), b) above, the fields of representatives $\kappa_{\mathcal{U}} \subset {}^*\mathcal{O}_{\mathcal{U}}$ for $\kappa_{\mathcal{U}}$ defined there are relatively algebraically closed in ${}^*k_{\mathcal{U}}$.

5) Finally, for $\kappa_{\mathcal{U}} \subset {}^*k_{\mathcal{U}}$ as above, let $k_{\mathcal{U}} := \kappa_{\mathcal{U}}(\pi_{\mathcal{U}})^h \subset {}^*k_{\mathcal{U}}$ be the henselization of $\kappa_{\mathcal{U}}(\pi_{\mathcal{U}})$ with respect to the $\pi_{\mathcal{U}}$-adic valuation, and $v_{\mathcal{U}} := ({}^*v_{\mathcal{U}})|_{k_{\mathcal{U}}}$.

- Note that $k_{\mathcal{U}} \subset {}^*k_{\mathcal{U}}$ is the relative algebraic closure of $\kappa_{\mathcal{U}}(\pi_{\mathcal{U}})$ in ${}^*k_{\mathcal{U}}$.

2.2.2 Hypothesis $(\mathrm{H})_k$ revisited

Let k be as in Hypothesis $(\mathrm{H})_k$ from the Introduction, i.e., $\mathrm{char}(k) = 0$ and k satisfies one of the hypotheses:

(i) k is of finite type. (ii) k is a function field $k|k_0$ with k_0 pseudo-finite.

Recall the basic definitions/facts from Introduction: First, $\mathbf{\Omega}(k) \subset \mathrm{Val}(k)$ is the set of all discrete valuations v of k such that the residue field kv is finite in case (i), respectively, finite over k_0 in case (ii). Second, for models S of k, $\mathbf{\Omega}_S(k) \subset \mathbf{\Omega}(k)$ is the set of all $v \in \mathbf{\Omega}(k)$ which have a center x_v on S. In particular, the center $x_v \in S$ of $v \in \mathbf{\Omega}_S(k)$ is a closed point of S, and conversely, every closed point $x \in S$ is the center of some $v \in \mathbf{\Omega}_S(k)$.

Let $\mathbf{\Omega}^0_S(k) \subset \mathbf{\Omega}_S(k)$ be the set of all $v \in \mathbf{\Omega}_S(k)$ such that $kv = \kappa(x_v)$. Recall that if $x \in S_{\mathrm{reg}}$ is closed, then $\exists\, v_x \in \mathbf{\Omega}_S(k)$ having center x on S and $kv_x = \kappa(x)$, hence $v_x \in \mathbf{\Omega}^0_S(k)$.

Next, for arbitrary non-empty subsets $\Sigma_k \subset \mathbf{\Omega}(k)$, we denote:

$$S_{\Sigma_k} := \big\{x \in S \,|\, \exists\ v \in \Sigma_k \text{ such that } x \text{ is the center of } v \text{ on } S\big\}.$$

Fact 2.5 (Hypothesis $(\mathrm{H})_k$ revisited/Basics). *Let k satisfy $(\mathrm{H})_k$, S denote models of k, and $\Sigma_k \subset \mathbf{\Omega}(k)$ be non-empty. Then the following hold:*

1) *Letting $U \subset S$ denote open dense subsets, the following are equivalent:*

   (a) *Σ_k satisfies $(\mathcal{P})$;* (b) *S_{Σ_k} is Zariski dense in S;* (c) *$U_{\Sigma_k} \neq \emptyset\ \forall\, U$.*

2) *The same holds correspondingly for subsets $\Sigma^0_k \subset \mathbf{\Omega}^0_S(k)$.*

3) *In case (b), let S be geometrically integral over k_0. Then $S_{\mathrm{reg}}(k_0)$ is Zariski dense, hence, one can choose Σ_k such that $kv = k_0$ for all $v \in \Sigma_k$.*

Let us further suppose that $\Sigma_k \subset \mathbf{\Omega}(k)$ satisfies condition $(\mathcal{P})$ from Notation/Remark 2.4, and for ultrafilters $\mathcal{U}$ on Σ_k satisfying $\mathcal{U} \supset \mathcal{P}_{\Sigma_k}$, we denote/consider:

a) The field of representatives $\kappa_{\mathcal{U}} \subset {}^*\mathcal{O}_{\mathcal{U}}$ for ${}^*\kappa_{\mathcal{U}} = {}^*k_{\mathcal{U}}{}^*v_{\mathcal{U}}$ from loc. cit, 4).

b) The k-embedding of valued fields $k_{\mathcal{U}} = \kappa_{\mathcal{U}}(\pi_{\mathcal{U}})^h \hookrightarrow {}^*k_{\mathcal{U}}$ from loc. cit., 5).

Fact 2.6 (Hypothesis $(\mathrm{H})_k$/Residue fields). *By* [9], [13, Chapter 11], *one has that* $\kappa_{\mathcal{U}}$ *is a pseudo-finite field, which moreover, is* $\aleph_1$*-saturated in case* (i), *respectively* $\aleph_\bullet = \max(\aleph_1, \aleph_{|k|^+})$*-saturated in case* (ii).

Fact 2.7 (Hypothesis $(\mathrm{H})_k$/AKE). *The canonical* k*-embedding of valued fields* $k_{\mathcal{U}}, v_{\mathcal{U}} \hookrightarrow {}^*k_{\mathcal{U}}, {}^*v_{\mathcal{U}}$ *satisfies:*

(i) ${}^*v_{\mathcal{U}}$ *is trivial on* $\kappa_{\mathcal{U}}$ *giving canonical* k*-identifications* $\kappa_{\mathcal{U}} = k_{\mathcal{U}}v_{\mathcal{U}} = {}^*k_{\mathcal{U}}{}^*v_{\mathcal{U}}$.

(ii) $v_{\mathcal{U}}k_{\mathcal{U}} = \mathbb{Z} \hookrightarrow {}^*\mathbb{Z}_{\mathcal{U}} = {}^*v_{\mathcal{U}}{}^*k_{\mathcal{U}}$ *are* $\mathbb{Z}$*-groups with* $1_{\mathbb{Z}} = v_{\mathcal{U}}(\pi_{\mathcal{U}}) = {}^*v_{\mathcal{U}}(\pi_{\mathcal{U}}) = 1_{{}^*\mathbb{Z}_{\mathcal{U}}}$.

In particular, if $\mathrm{char}(k) = 0$*, by the AKE Principle one has:*

$(*)$ $k_{\mathcal{U}} \hookrightarrow {}^*k_{\mathcal{U}}$ *is an elementary* k*-embedding of (valued) fields.*

2.2.3 Pseudo-splitness revisited

Before discussing the more specific situation over fields satisfying Hypothesis $(\mathrm{H})_k$ from the Introduction, we make the following general definition, which is at the core of the *generalizations* of the results from the Introduction. Further, for every field, say F, we identify its absolute Galois group $G_F := \mathrm{Gal}(F^{\mathrm{s}}|F)$ with $\mathrm{Aut}_F(\overline{F})$ under $F^{\mathrm{s}} \hookrightarrow \overline{F}$. We say that a subextension $F'|F \hookrightarrow F^{\mathrm{s}}|F$ is *co-procyclic* if $G_{F'}$ is procyclic, or equivalently, $F' \subset F^{\mathrm{s}}$ is the fixed field $F' = (F^{\mathrm{s}})^\sigma$ of some $\sigma \in G_F$.

Definition/Remark 2.8. Let $\lambda|\kappa$ be a field extension, and $\kappa'|\kappa$ be an algebraic extension. We say that $\lambda|\kappa$ is κ'*-reduced-pseudo-split,* for short r.p.s. or *reduced-pseudo-split above* κ', if the κ'-algebra $(\lambda \otimes_\kappa \kappa')_{\mathrm{red}}$ has a factor λ' such that $\lambda'|\kappa'$ is a regular field extension.

Notice that in the case $\mathrm{char}(\kappa) = 0$, the κ'-algebra $\lambda \otimes_\kappa \kappa'$ is reduced, hence the notions of "reduced-pseudo-split" and "pseudo-split" are identical.

In the remaining of this subsection, we consider the following situation:

– k satisfies hypothesis $(\mathsf{H})_k$ from Introduction, in particular, $\mathrm{char}(k) = 0$.

– $\Sigma_k \subset \mathbf{\Omega}(k)$ satisfies condition $(\mathcal{P})$, as introduced in Notations/Remarks 2.4. Further, in the case (ii), i.e., k is the function field over a pseudo-finite field k_0, we fix a generator σ_0 of G_{k_0}, and for finite extensions $l_0|k_0$, we define $\mathrm{Frob}_{l_0} := \sigma_0^n$ with $n = [l_0 : k_0]$.

– Hence, if $l|k$ is finite Galois and $v \in \Sigma_k$ is unramified in $l|k$, say $w|v$ prolongs v to $l|k$, then $\mathrm{Frob}(v) := \mathrm{Frob}_{lw} \in \mathrm{Gal}(l|k)$ is well-defined up to conjugation in both cases (i) and (ii) of hypothesis $(\mathsf{H})_k$.

Definition 2.9. Let k, Σ_k be as above, $\sigma \in G_k$, and $F|k$ be a function field.

1) The co-procyclic extension $\overline{k}^{\sigma}\,|\,k$ of k is called Σ_k-*definable*, if for all finite Galois extensions $l\,|\,k$, and all $\Sigma_A \in \mathcal{P}_{\Sigma_k}$, one has:

$$\Sigma_{A,\,l|k}(\sigma) := \big\{v \in \Sigma_A \,|\, v \text{ unramified in } l\,|\,k \text{ and } \mathrm{Frob}(v) := \sigma|_l\big\} \neq \emptyset.$$

2) And algebraic extension $F'|F$ is *co-procyclic* Σ_k-*definable,* if $F' = \overline{F}^{\sigma_F}$ for some $\sigma_F \in G_F$ such that $\sigma_k := (\sigma_F)|_{\overline{k}} \in G_k$ is Σ_k-definable.

Remarks 2.10. Let k be of finite type, S be a model of k, $\Sigma_k \subset \mathbf{\Omega}_S(k)$.

1) If $S_{\Sigma_k} \subset S$ has the Dirichlet density $\delta(S_{\Sigma_k}) = 1$, e.g., if $S_{\Sigma_k} \subset S$ is open dense, then all elements $\sigma \in G_k$ are Σ_k-definable (apply the Chebotarev Density Theorem, e.g., [19], etc.).

2) If $S_{\Sigma_k} \subset S$ is *Frobenian*, [20, Theorem 3.3], say defined by a finite Galois extension $k_1|k$ and a set of conjugacy classes $\Phi \subset \mathrm{Gal}(k_1|k)$, then $\sigma \in G_k$ is Σ_k-definable iff $\sigma|_{k_1} \in \Phi$.

Proposition 2.11. *In the context and notation from Definition* 2.9, *let further $E|F$ be an extension of k-function fields. The following hold:*

1) *$\sigma \in G_k$ is Σ_k-definable iff $\overline{k}^{\sigma} = {}^*k_{\mathcal{U}} \cap \overline{k} = \kappa_{\mathcal{U}} \cap \overline{k}$ for some $\mathcal{U} \supset \mathcal{P}_{\Sigma_k}$.*

2) *$F'|F$ is co-procyclic Σ_k-definable iff there is an ultrafilter $\mathcal{U} \supset \mathcal{P}_{\Sigma_k}$ on Σ_k and a k-embedding $\jmath_F : F \hookrightarrow \kappa_{\mathcal{U}}$ such that $F' = \overline{F} \cap \kappa_{\mathcal{U}}$.*

3) *Let $F' = \overline{F} \cap \kappa_{\mathcal{U}}$ be as at* 2) *above. Then $E|F$ is reduced-pseudo-split above F' iff $\jmath_F : F \hookrightarrow \kappa_{\mathcal{U}}$ prolongs to a field embedding $\jmath_E : E \hookrightarrow \kappa_{\mathcal{U}}$.*

Proof. To 1): For the implication $\Rightarrow$, notice that $\mathcal{P}_{\Sigma_k}(\sigma) := \{\Sigma_{A,l|k}\}_{A,\,l|k}$ is a prefilter on Σ_k such that any ultrafilter $\mathcal{U}$ containing $\mathcal{P}_{\Sigma_k}(\sigma)$ contains $\mathcal{P}_{\Sigma_k}$. Let $l|k$ be a finite Galois extension. Then for $v \in \Sigma_{A,l|k}(\sigma) \in \mathcal{U}$, setting $l_v := lk_v$ one has: $l_v|k_v$ is unramified and $l^{\sigma} = l \cap k_v$. Hence $l^{\sigma} = l \cap {}^*k_{\mathcal{U}}$, and finally $\overline{k}^{\sigma} = \overline{k} \cap {}^*k_{\mathcal{U}}$.

For the converse implication, let $\mathcal{U}$ be such that $\overline{k}^\sigma = {}^*k_\mathcal{U} \cap \overline{k}$. To show that σ is Σ_k-definable, we have to show that for all finite Galois extensions $l|k$, the set $\Sigma_{A,\,l|k}(\sigma)$ is non-empty. First, since $\overline{k}^\sigma = {}^*k_\mathcal{U} \cap \overline{k}$, it follows that $l^\sigma = {}^*k_\mathcal{U} \cap l$. Hence, there exists a set $V_l \in \mathcal{U}$ such that for all $v \in V_l$ one has $l^\sigma = k_v \cap l$. Further, let $\Sigma_A \subset \Sigma_k$ be given. Since $\mathcal{P}_{\Sigma_k} \subset \mathcal{U}$, hence $\Sigma_A \in \mathcal{U}$, w.l.o.g., we can suppose that $V_l \subset \Sigma_A$. Second, let $B \subset k^\times$ be a finite set such that all discrete valuations w of k with $w(B) = 0$ are unramified in $l|k$. (Note that such sets B exist: If $S_l \to S$ is the normalization of S in the finite Galois extension $l|k$, then there exists an affine open dense subset $S' \subset S$ such that S_l is étale above S'. Hence, if w has its center in S', then w is unramified in $l|k$, etc.) Then setting $A_l := A \cup B$, one has: $V_l \cap \Sigma_{A_l} \in \mathcal{U}$, and all $v \in V_l \cap \Sigma_{A_l}$ are unramified in $l|k$. Hence $\Sigma_{A_l,\,l|k}$ is non-empty, thus $\Sigma_{A,k|l} \supset \Sigma_{A_l,\,l|k}$ is non-empty as well, concluding that σ is Σ_k-definable.

To 2): To $\Rightarrow$: Since $\kappa_\mathcal{U}$ is a perfect pseudo-finite field, $k \hookrightarrow F \hookrightarrow \kappa_\mathcal{U}$ gives rise to an embedding of perfect fields $k' = \overline{k} \cap \kappa_\mathcal{U} \hookrightarrow \kappa' = \overline{F} \cap \kappa_\mathcal{U} \hookrightarrow \kappa_\mathcal{U}$ and to surjective projections $\widehat{\mathbb{Z}} \cong G_{\kappa_\mathcal{U}} \twoheadrightarrow G_{F'} \twoheadrightarrow G_{k'}$. Hence, $F'|F$ is by mere definitions co-procyclic and Σ_k-definable. For the converse implication, let $F'|F$ be co-procyclic and Σ_k-definable. Then $k' := \overline{k} \cap F'$ is co-procyclic and Σ_k-definable. Hence, there is some $\mathcal{U}$ such that $k' = \overline{k} \cap \kappa_\mathcal{U}$, and obviously, $F'|k'$ is a regular field extension. We claim that there is a k-embedding $\jmath_F : F \hookrightarrow \kappa_\mathcal{U}$ such that $F' = \overline{F} \cap \kappa_\mathcal{U}$, hence $k' \subset F'$. First, $F'_0 := Fk' \subset F'$ is a regular function field over k', and setting $\tilde{F}_0 = F'_0$, there is an increasing sequence of cyclic field subextensions $(\tilde{F}_i|F'_i)_{i\in\mathbb{N}}$ of $\overline{F}|F'$ such that $F' = \cup_{i\in\mathbb{N}} F'_i$, $\overline{F} = \cup_{i\in\mathbb{N}} \tilde{F}_i$, and $\tilde{F}_i|F'_i$ is the maximal subextension of $\overline{F}|F'$ of degree $\leqslant i$. By algebra general non-sense, the sequence $(\tilde{F}_i|F'_i)_i$ and the conditions it satisfies are expressible by a type $p(\boldsymbol{t})$ over k', where $\boldsymbol{t}$ is a transcendence basis of $F_0|k'$; and since $\kappa_\mathcal{U}$ is a perfect PAC pseudo-finite field, the type $p(\boldsymbol{t})$ is finitely satisfiable. Since $\kappa_\mathcal{U}$ is $\aleph_1$-saturated in case (i), and $\aleph_{|k|^+}$-saturated in case (ii), the type $p(\boldsymbol{t})$ is satisfiable in $\kappa_\mathcal{U}$, thus $F = F_0$ has a k'-embedding $\jmath_F : F \hookrightarrow \kappa_\mathcal{U}$ such that $F' = \overline{F} \cap \kappa_\mathcal{U}$.

To 3): For the direct implication, let $\jmath_F : F \to \kappa_\mathcal{U}$ be a k-embedding, $F' = \overline{F} \cap \kappa_\mathcal{U}$, and E' be a factor of $(E \otimes_F F')_{\rm red}$ such that $E'|F'$ is a regular field extension. Then $E' = F'(Z')$ for a geometrically integral F'-variety Z'. Since $\kappa_\mathcal{U}$ is a PAC field which is $\aleph_1$-saturated in case (i), respectively $\aleph_{|k|^+}$-saturated in case (ii), it follows that $Z'(\kappa_\mathcal{U})$ contains "generic points" of Z', that is, points $z' \in Z'(\kappa_\mathcal{U})$ which are defined by an F'-embedding $\imath_{z'} : E' \to \kappa_\mathcal{U}$ of the function field $E' := F'(Z')$ into $\kappa_\mathcal{U}$. Hence, if $\jmath' : E \to E \otimes_F F'' \to E'$ is the canonical k-embedding, then $\jmath_E := \imath_{z'} \circ \jmath' :\to \kappa_\mathcal{U}$ prolongs $\jmath_F$ to E.

For the converse implication, let $\imath_E : E \to \kappa_\mathcal{U}$ be the given prolongation of $\jmath_F : F \to \kappa_\mathcal{U}$, and consider the compositum $E' \subset \kappa_\mathcal{U}$ of F' and $\imath_E(E)$ over

F inside $\kappa_{\mathcal{U}}$. Since $F' = \overline{F} \cap \kappa_{\mathcal{U}}$ and $\kappa_{\mathcal{U}}$ is perfect, it follows that F' is perfect and relatively algebraically closed in $\kappa_{\mathcal{U}}$. Hence, F' is perfect and relatively algebraically closed in the subfield $E' \subset \kappa_{\mathcal{U}}$ of $\kappa_{\mathcal{U}}$. Therefore, $E'|F'$ is a regular field extension. Finally, since E' is generated by F' and $\imath_{\mathcal{U}}(E)$ over $\jmath_{\mathcal{U}}(F)$ inside $\kappa_{\mathcal{U}}$, it follows that E' is a factor of the F-algebra $E \otimes_F F'$, thus of $(E \otimes_F F')_{\rm red}$ as well. Conclude that E is reduced-pseudo-split above F'. □

3. SETUP FOR GENERALIZATIONS OF THEOREM 1.4 AND THEOREM 1.5

The generalizations of Theorem 1.4 and Theorem 1.5 we aim at are based on generalizing $(\mathsf{Srj})_{\boldsymbol{\Omega}(k)}$ and both the pseudo-splitness of prime divisors in function field extensions over k and of morphisms of k-varieties as defined in the Introduction. These generalizations are obtained by considering arbitrary base fields k endowed with sets $\Sigma_k \subset \mathrm{Val}(k)$ of discrete valuations of k satisfying Hypothesis $(\mathcal{P})$ from Notations/Remarks 2.4 above, and defining/introducing $(\mathsf{Srj})_{\Sigma_k}$ and the Σ_k-pseudo-splitness of both prime divisors in function field extension over k and of morphisms of arbitrary k-varieties.

This being said, Theorem 1.4 and Theorem 1.5 from the Introduction are consequence of Theorems 4.1 and Theorem 5.1 below, which are a kind of general non-sense type results.

Finally, if not explicitly otherwise stated, *all fields in this section have characteristic equal to* 0 (although some facts discussed below hold in characteristic $p > 0$ as well). Recall that in this case, reduced-pseudo-splitness coincides with pseudo-splitness.

3.1. $\boldsymbol{\Sigma_k}$-pseudo-splitness ($\boldsymbol{\Sigma_k}$-p.s.)

Let k with $\mathrm{char}(k) = 0$ and Σ_k be a set of valuations v of k satisfying hypothesis $(\mathcal{P})$ from Notations/Remarks 2.4, but otherwise be arbitrary. Then Proposition 2.11 hints at the following *generalizations* of pseudo-splitness (p.s.).

Definition 3.1. Let k with $\mathrm{char}(k) = 0$, Σ_k and $\mathcal{U} \supset \mathcal{P}_{\Sigma_k}$ be as in Notations/Remarks 2.4, and $\kappa|k \hookrightarrow \lambda|k$ be an extension of k-field extensions.

1) Let a k-embedding $\jmath_\kappa : \kappa \to \kappa_{\mathcal{U}}$ be given. We say that:

   a) A field extension $\kappa'|\kappa$ is $\jmath_\kappa$*-definable*, if $\kappa' = \overline{\kappa} \cap \kappa_{\mathcal{U}}$ as κ-field extension.

   b) $\lambda|\kappa$ is $\jmath_\kappa$*-p.s.*, if $\jmath_\kappa$ prolongs to a κ-embedding $\jmath_\lambda : \lambda \hookrightarrow \kappa_{\mathcal{U}}$.

2) We say that $\lambda|\kappa$ is:

a) $\mathcal{U}$-*p.s.*, if $\lambda|\kappa$ is $\jmath_\kappa$-p.s. for all k-embeddings $\jmath_\kappa : \kappa \to \kappa_{\mathcal{U}}$.

b) Σ_k-*p.s.*, if $\lambda|\kappa$ is $\mathcal{U}$-p.s. for all $\mathcal{U} \supset \mathcal{P}_{\Sigma_k}$.

Remarks 3.2. In the above notation, the *transitivity* of pseudo-splitness holds as follows: Let $\lambda_\mu|\kappa_\mu$ be $\jmath_{\kappa_\mu}$-p.s., say via $\jmath_{\lambda_\mu} : \lambda_\mu \to \kappa_{\mathcal{U}}$, $\mu = 1, 2$. Then:

1) Suppose that $\lambda_1|\kappa_1 \hookrightarrow \lambda_2|\kappa_2$, and $(\jmath_{\lambda_2})|_{\lambda_1} = \jmath_{\lambda_1}$. Then $\lambda_2|\kappa_1$ is $\jmath_{\kappa_1}$-p.s.

2) In particular, if $\lambda_0|\kappa_1 \hookrightarrow \lambda_1|\kappa_1$ is a k-subextension, then $\lambda_0|\kappa_1$ is $\jmath_{\kappa_1}$-p.s.

Obviously, the same holds correspondingly for $\mathcal{U}$-p.s. and Σ_k-p.s.

Proposition 3.3. *Let k with* $\mathrm{char}(k) = 0$, *Σ_k and $\mathcal{U} \subset \mathcal{P}_{\Sigma_k}$ be as above, $E|F$ be an extension of k-function fields, and Z be an F-variety with function field $F(Z) = E$. Let $\jmath_F : F \hookrightarrow \kappa_{\mathcal{U}}$ be a k-embedding, and $F' = \overline{F} \cap \kappa_{\mathcal{U}}$ be the resulting $\jmath_F$-definable extension of F. One has:*

1) *$E|F$ is $\jmath_{\mathcal{U}}$-p.s. if and only if $Z(\kappa_{\mathcal{U}})$ is Zariski dense.*

2) *Suppose that $\kappa_{\mathcal{U}}$ is PAC. Then $E|F$ is $\jmath_F$-p.s. iff $E \otimes_F F'$ has a factor $E'|F'$ with $E'|F'$ a regular field extension.*

Proof. To 1): Let $Z' := Z_{F'} = Z \times_F F'$ be the base change under $F'|F$, Z'_μ be the irreducible components of Z'. Then reasoning as in the proof of assertion 2) from Proposition 2.11 one has: $Z(\kappa_{\mathcal{U}})$ is Zariski dense iff $Z'_\mu(\kappa_{\mathcal{U}})$ is Zariski dense for some Z'_μ iff Z'_μ is F'-geometrically integral and $E' = F'(Z'_\mu)$ is F'-embeddable in $\kappa_{\mathcal{U}}$ iff $E|F$ is $\jmath_{\mathcal{U}}$-p.s.

To 2): Argue as in the proof of assertion 1) using that $\kappa_{\mathcal{U}}$ is PAC. □

Corollary 3.4 (Proposition 2.11 revisited). *Let k satisfy the Hypothesis* (H)$_k$*, hence,* $\mathrm{char}(k) = 0$*, Σ_k satisfy condition $(\mathcal{P})$, and $\mathcal{U} \supset \mathcal{P}_{\Sigma_k}$ be an ultrafilter on Σ_k. Let $E|F$ be an extension k-function fields, $\jmath_F : F \hookrightarrow \kappa_{\mathcal{U}}$ be a k-embedding, and $F' := \overline{F} \cap \kappa_{\mathcal{U}}$. Then $E|F$ is p.s. above F' iff $\jmath_F : F \hookrightarrow \kappa_{\mathcal{U}}$ has a prolongation $\jmath_E : E \hookrightarrow \kappa_{\mathcal{U}}$ to E.*

3.2. Σ_k-pseudo-splitness and the properties $(\mathsf{Srj})_{\mathcal{U}}$ and $(\mathsf{Srj})_{\Sigma_k}$

For the moment, let $F|k$ be a function field, where $\mathrm{char}(k) = 0$ as usual.

1) For given $w \in \mathcal{D}(F|k)$, let $\boldsymbol{t} = (t_1, \dots, t_r)$ be a system of w-units in F whose image in Fw (which we denote by $\boldsymbol{t}$ as well), is a transcendence basis of $Fw|\,k$. Then $k(\boldsymbol{t}) \subset F$ is a rational function subfield, and by Lemma 4.4, the relative algebraic closure $k_{\boldsymbol{t}} \subset F^h$ of $k(\boldsymbol{t})$ in the henselization F^h is a field of representatives for $Fw = F^h w^h$. Further, if $\pi \in F'$ is a uniformizing parameter,

then $F_0 = k(\pi, \boldsymbol{t})$ is a rational function field with $F|F_0$ finite, $F^h = k_{\boldsymbol{t}}(\pi)^h$. Hence if $w_0 := w|_{F_0}$, then $e(w|w_0) = 1$ and $f(w|w_0) = [k_{\boldsymbol{t}} : k(\boldsymbol{t})]$.

2) Let $E|F$ be an extension of k-function fields, $pr_{EF} : \mathcal{D}(E|k) \to \mathcal{D}(F|k)$, $v \mapsto w := v|_F$. For $w \in \mathcal{D}(F|k)$, let $\mathcal{D}_w(E|k) := \{v \in \mathcal{D}(E|k) | w = v|_F\}$ be the fiber of pr_{EF} at $w \in \mathcal{D}(F|k)$.

3) Next, let $f : X \to Y$, $x \mapsto y$ be a morphism of k-varieties. In particular, f gives rise to restriction maps $\mathcal{D}(k_x|k) \to \mathcal{D}(k_y|k)$, $v_x \mapsto v_y := (v_x)|_{k_y}$. Thus, if X_y is the reduced fiber of f above y, one has a canonical restriction map

$$(\mathcal{D}) \qquad \mathcal{D}(X) := \cup_{x \in X_y} \mathcal{D}\big(k_x|k\big) \to \mathcal{D}\big(k_y|k\big), \quad v_x \mapsto v_y := (v_x)|_{k_y}$$

Notice: For $v_x \mapsto v_y$ as above, $\kappa_{v_x} := k_x v_x$, $\kappa_{v_y} := k_y v_y$ are k-function fields and since $\mathcal{O}_{v_x}$ dominates $\mathcal{O}_{v_y}$, one has a canonical residue field extension $\kappa_{v_x} | \kappa_{v_y}$.

Definition 3.5. Let k with $\mathrm{char}(k) = 0$, $\Sigma_k \subset \mathrm{Val}(k)$, and $\mathcal{U} \supset \mathcal{P}_{\Sigma_k}$ be as in Notations/Remarks 2.4. In the above notation and context, define/consider:

1) *Valuations.* We say that $w \in \mathcal{D}(F|k)$ is *$\mathcal{U}$-p.s.* in $\mathcal{D}(E|k)$ if for every k-embedding $\jmath_w : Fw \hookrightarrow \kappa_{\mathcal{U}}$ $\exists\, v \in \mathcal{D}(E|k)$ with $w = v|_L$ such that $Ev|Fw$ is $\jmath_w$-p.s. in the sense of Definition 3.1, and $e(v|w) = 1$ if w is non-trivial.

- We say that $\mathcal{D}(F|k)$ is *Σ_k-p.s.* in $\mathcal{D}(E|k)$, or under $\mathcal{D}(E|k) \to \mathcal{D}(F|k)$, if all $w \in \mathcal{D}(F|k)$ are $\mathcal{U}$-p.s. in $\mathcal{D}(E|k)$ for all $\mathcal{U} \supset \mathcal{P}_{\Sigma_k}$ in the above sense.

2) *Morphisms.* Let $f : X \to Y$, $x \mapsto y$, be given. We say that $v_y \in \mathcal{D}(k_y|k)$ is *$\mathcal{U}$-p.s. under f*, if for every k-embedding $\jmath_y : \kappa_{v_y} \to \kappa_{\mathcal{U}}$ $\exists\, v_x \in \mathcal{D}(X)$ such that $\kappa_{v_x} | \kappa_{v_y}$ is $\jmath_y$-p.s., and $e(v_x|v_y) = 1$ if v_y is non-trivial.

- We say that: (a) v_y is *Σ_k-p.s. under f,* if v_y is $\mathcal{U}$-p.s. for all $\mathcal{U} \supset \mathcal{P}_{\Sigma_k}$.

  (b) f is $\mathcal{U}$-p.s. above $y \in Y$, if all $v_y \in \mathcal{D}(k_y|k)$ are $\mathcal{U}$-p.s. under f.

  (c) f is Σ_k-p.s. if f is $\mathcal{U}$-p.s. above every $y \in Y$ for all $\mathcal{U} \supset \mathcal{P}_{\Sigma_k}$.

3) *Properties* $(\mathsf{Srj})_{\Sigma_k}$ *and* $(\mathsf{Srj})_{\mathcal{U}}$ for k-morphisms $f : X \to Y$. We say that:

(a) f has the property $(\mathsf{Srj})_{\Sigma_k}$ if there is $\Sigma_A \in \mathcal{P}_{\Sigma_k}$ such that:

$$f^{k_v} : X(k_v) \to Y(k_v) \text{ is surjective for all } v \in \Sigma_A.$$

(b) f has the property $(\mathsf{Srj})_{\mathcal{U}}$ if $f^{k_{\mathcal{U}}} : X(k_{\mathcal{U}}) \to Y(k_{\mathcal{U}})$ is surjective.

Remarks 3.6. Let $f : X \to Y$ be a morphism of k-varieties, $\mathrm{char}(k) = 0$. If X, Y are integral and f is dominant, let $L = k(Y) \hookrightarrow k(X) = K$ be corresponding k-embedding of function fields. One has:

1) In notations from Introduction, let k satisfy $(\mathsf{H})_k$, and $\Sigma_k = \mathbf{\Omega}(k)$. Then:

   (a) f has property $(\mathsf{Srj})_{\mathbf{\Omega}(k)}$ iff f has property $(\mathsf{Srj})_{\Sigma_k}$.

   (b) If f is g.p.s. as in Definition 1.1, then f is Σ_k-p.s.

   (c) If $\mathcal{D}(L|k)$ is g.p.s. in $\mathcal{D}(K|k)$, then $\mathcal{D}(L|k)$ is Σ_k-p.s. in $\mathcal{D}(K|k)$.

1)′ In particular, this is so for k a number field. Further, since $\mathrm{char}(k) = 0$, the AKE Principle holds for $k_{\mathcal{U}} \hookrightarrow {}^*k_{\mathcal{U}}$ for each $\mathcal{U}$.

2) For k, Σ_k as in Notations/Remarks 2.4, by Fact 2.7, TFAE:

   – $f : X \to Y$ has the property $(\mathsf{Srj})_{\Sigma_k}$.

   – $f^{*k_{\mathcal{U}}} : X({}^*k_{\mathcal{U}}) \to Y({}^*k_{\mathcal{U}})$ is surjective $\forall\, \mathcal{U} \supset \mathcal{P}_{\Sigma_k}$ ultrafilters on Σ_k.

2)′ Since $\mathrm{char}(k) = 0$, the AKE Principle holds for $k_{\mathcal{U}} \hookrightarrow {}^*k_{\mathcal{U}}$, and therefore:

f has property $(\mathsf{Srj})_{\Sigma_k}$ iff f has the property $(\mathsf{Srj})_{\mathcal{U}}$ for all $\mathcal{U} \supset \mathcal{P}_{\Sigma_k}$.

4. PROOF OF THEOREM 1.4 (REVISITED)

In the notation/context from Section 3, Theorem 1.4 follows from:

THEOREM 4.1 (Theorem 1.4, revisited). *In the context of Notations/Remarks 2.4 and Definition 3.5 above, let $\mathrm{char}(k) = 0$ and $f : X \to Y$ be a morphism of k-varieties. Then one has:*

1) *f has property $(\mathsf{Srj})_{\Sigma_k}$ iff f is Σ_k-pseudo-split.*

2) *f has property $(\mathsf{Srj})_{\mathcal{U}}$ iff f is $\mathcal{U}$-pseudo-split.*

Proof. To 1): Since $\mathrm{char}(k) = 0$, by Remark 3.6, 2)′, the property $(\mathsf{Srj})_{\Sigma_k}$ is equivalent to the property $(\mathsf{Srj})_{\mathcal{U}}$ for all ultrafilters $\mathcal{U} \supset \mathcal{P}_{\Sigma_k}$. Further, by mere definition, f being Σ_k-pseudo-split is the same as f being $\mathcal{U}$-pseudo-split for all $\mathcal{U} \supset \mathcal{P}_{\Sigma_k}$. Thus, 1) follows from 2).

To 2): Let $\mathcal{U} \supset \mathcal{P}_{\Sigma_k}$ be fixed. Giving $y_{\mathcal{U}} \in Y(k_{\mathcal{U}})$ is equivalent to giving $(y, \imath_y)$ with $y \in Y$ and $\imath_y : k_y = \kappa(y) \to k_{\mathcal{U}}$ is a field k-embedding. In particular, if $f_y : X_y \to y$ is the (reduced) fiber of f at y, and letting $y(k_{\mathcal{U}})$ be the points defined by $(y, \imath_y)$ for all possible $\imath_y : k_y \to k_{\mathcal{U}}$, one has: $f^{k_{\mathcal{U}}} : X(k_{\mathcal{U}}) \to Y(k_{\mathcal{U}})$ is surjective iff the maps $f_y^{k_{\mathcal{U}}} : X_y(k_{\mathcal{U}}) \to y(k_{\mathcal{U}})$ are surjective for all $y \in Y$. Further, f is $\mathcal{U}$-pseudo-split iff f is $\mathcal{U}$-p.s. above every $y \in Y$. Therefore, the proof of Theorem 4.1, 2) is reduced to the Key Lemma 4.2 below. □

KEY LEMMA 4.2. *In the notation from Theorem 4.1, the following holds:*

$$f_y^{k_{\mathcal{U}}} : X_y(k_{\mathcal{U}}) \to y(k_{\mathcal{U}}) \textit{ is surjective iff } f \textit{ is } \mathcal{U}\textit{-p.s. above } y.$$

Proof of Key Lemma 4.2. We begin by recalling two basic facts about valuations without (transcendence) defect, see [7, Chapter VI], and [15], for some/more details on (special cases of) this. Let Ω, w be a valued field with v trivial on some subfield κ of Ω, hence on the prime field of Ω. One says that w has no (*transcendence*) *defect* on $\Omega|\kappa$ if there exists a transcendence basis of $\Omega\,|\,\kappa$ of the form $\boldsymbol{t}_w \cup \boldsymbol{t}$ satisfying the following: First, $w\boldsymbol{t}_w$ is a *basis* of the $\mathbb{Q}$-vector space $w\Omega \otimes \mathbb{Q}$, and second, $\boldsymbol{t}$ consists of w-units such that its image in the residue field Ωw, which we denote again by $\boldsymbol{t}$, is a *transcendence basis* of $\Omega w\,|\,\kappa$. In particular, if $\kappa_{\boldsymbol{t}} \subset \Omega$ is the relative algebraic closure of $\kappa(\boldsymbol{t})$ in Ω, then $\kappa_{\boldsymbol{t}}$ is a maximal subfield of Ω such that w is trivial on $\kappa_{\boldsymbol{t}}$, and further, Ωw is algebraic over $\kappa_{\boldsymbol{t}} w$. Moreover, if w is Henselian, then Hensel Lemma implies that Ωw is purely inseparable over $\kappa_{\boldsymbol{t}} w$. Hence, if $\boldsymbol{t}$ is a separable transcendence basis of of $\Omega w\,|\,\kappa$, then $\kappa_{\boldsymbol{t}} \subset \Omega$ is a field of representatives of Ωw.

One of the main properties of valuations w without defect is that for any subfield $F \subset \Omega$, the restriction of w to F is a valuation without defect as well, see [15]. In particular, if $l \subset \Omega$ is any subfield such that $w|_l$ is trivial, and $F\,|\,l$ is a function field, then $w|_F$ is a prime divisor of the function field $F|l$ if and only if $w|_F$ is a discrete valuation. Specifically, for $k_{\mathcal{U}} = \kappa_{\mathcal{U}}(\pi_{\mathcal{U}})^h$ endowed with $v_{\mathcal{U}}$ as in Notations/Remarks 2.4, 5), the discussion above implies:

Fact 4.3. *Let $l \subset k_{\mathcal{U}}$ be a subfield with $v_{\mathcal{U}}$ trivial on l. Let $F\,|\,l$ be a function field and $F \hookrightarrow k_{\mathcal{U}}$ be an l-embedding. Then $v := (v_{\mathcal{U}})|_F$ is either trivial, or a prime divisor of $F\,|\,l$.*

Lemma 4.4. *Let $F|k$ be a function field, F^h be the henselization of F with respect to $w \in \mathcal{D}(F|k)$, and $\pi \in F$ with $w(\pi) = 1$. The following hold:*

1) *Let $\kappa' \subset F^h$ be a field of representatives for $\kappa_w := Fw$. Then $F^h = \kappa'(\pi)^h$.*

2) *If $Fw\,|\,k$ is separably generated, fields of representative $\kappa' \subset F^h$ for the residue field $\kappa_w = Fw$ exist.*

Proof. To 1): Consider the henselization $F_1 := \kappa'(\pi)^h$. Then $F_1 \subset F^h$ satisfies $F_1 w = F^h w$, $wF_1 = wF$, thus $f(F^h|F_1) = 1 = e(F^h|F_1)$. Since w has no defect, the fundamental equality holds, i.e., $[F^h\colon F_1] = e(F^h|F_1)f(F^h|F_1)$. Thus, finally $[F^h : F_1] = 1$, hence $F^h = \kappa'(\pi)^h$.

To 2): Let $\boldsymbol{t}$ be the lifting to F of a separable transcendence basis of $Fw|k$, also denoted by $\boldsymbol{t}$. Then $Fw|k$ is finite separable over $k(\boldsymbol{t})$, and conclude by applying Hensel Lemma. □

Coming back to the proof of Key Lemma 4.2, we begin with two general remarks about the relationship between $k_{\mathcal{U}}$-rational points and prime divisors as follows. Let Z be a k-variety, and $k_z = \mathcal{O}_z/\mathfrak{m}_z$ be the residue field at $z \in Z$.

Recall the canonical field embeddings $k \hookrightarrow k_{\mathcal{U}}$ and the canonical valuation $v_{\mathcal{U}}$ of $k_{\mathcal{U}}$, its canonical uniformizing parameter $\pi_{\mathcal{U}}$, and its residue field $\kappa_{\mathcal{U}} = k_{\mathcal{U}} v_{\mathcal{U}}$. For $z \in Z$, let $k_z = \mathcal{O}_z/\mathfrak{m}_z$ be the residue field at $z \in Z$.

I) *Rational points*: Recall that every $z_{\mathcal{U}} \in Z(k_{\mathcal{U}})$ is given by $(z, \imath_z)$ with $z \in Z$ and $\imath_z : k_z \hookrightarrow k_{\mathcal{U}}$ a k-embedding. Let $v_z := (v_{\mathcal{U}})|_{k_z}$ be the restriction of $v_{\mathcal{U}}$ to k_z under $\imath_z : k_z \to k_{\mathcal{U}}$. Then $\imath_z : k_z, v_z \hookrightarrow k_{\mathcal{U}}, v_{\mathcal{U}}$ is a k-embeddings of valued fields, which defines the k-embedding of the residue fields $\jmath_z : \kappa_{v_z} \hookrightarrow \kappa_{\mathcal{U}}$. Finally, by Fact 4.3, one has that $v_z \in \mathcal{D}(k_z|k)$.

II) *Prime divisors*: Let $v_z \in \mathcal{D}(k_z|k)$ be a prime divisor, π_z a fixed unifomizing parameter, and $\kappa_{v_z} := k_z v_z$ be the residue function field over k. Let $\jmath_z : \kappa_{v_z} \to \kappa_{\mathcal{U}}$ be given. Then by Lemma 4.4, the v_z-henselization k_z^h of k_z contains a field of representatives $\kappa' \subset k_z^h$ for κ_{v_z} and $k_z^h = \kappa'(\pi_z)^h$. Hence, $\jmath_z : \kappa_{v_z} \to \kappa_{\mathcal{U}}$ and π_z give rise to a k-embedding of valued fields $\imath_z^0 : \kappa'(\pi) \hookrightarrow k_{\mathcal{U}}$ via $\pi \mapsto \pi_{\mathcal{U}}$ and $\kappa' \cong \kappa_{v_z} \to \kappa_{\mathcal{U}}$ defined by $\jmath_z$. Since $k_{\mathcal{U}}$ is henselian, $\imath_z^0$ extends to a k-embedding of the henselization $\kappa'(\pi)^h = k_z^h$, say $\imath_z^h : \kappa'(\pi)^h = k_z^h \to k_{\mathcal{U}}$, hence by restriction, to a k-embedding $\imath_z : k_z, v_x \to k_{\mathcal{U}}, v_{\mathcal{U}}$ of valued fields.

Conclude: Every k-embedding $\jmath_z : \kappa_{v_z} \to \kappa_{\mathcal{U}}$ together with a uniformizing parameter $\pi_z \in k_z$ define a k-embedding of valued fields $\imath_z : k_z, v_z \to k_{\mathcal{U}}, v_{\mathcal{U}}$ such that $\pi_z \mapsto \pi_{\mathcal{U}}$ and $\kappa_{v_z} \to \kappa_{\mathcal{U}}$ equals $\jmath_z : \kappa_{v_z} \to \kappa_{\mathcal{U}}$. Thus, $\jmath_z$ together with π_z give rise to the $k_{\mathcal{U}}$-rational point $z_{\mathcal{U}} \in Z(k_{\mathcal{U}})$ defined by $(z, \imath_z)$, thus by restriction, to $x_{\mathcal{U}} \in Z(k_{\mathcal{U}})$, such that $v_z = (v_{\mathcal{U}})|_{k_z}$ under $\imath_z : k_z \to k_{\mathcal{U}}$.

Back to the proof of the Key Lemma 4.2, proceed as follows:

The direct implication: $f_y^{k_{\mathcal{U}}}\big(X_y(k_{\mathcal{U}})\big) = y(k_{\mathcal{U}}) \ \Rightarrow\ f$ *is $\mathcal{U}$-p.s. above y.*

Given $v_y \in \mathcal{D}(k_y|k)$ and $\jmath_y : \kappa_{v_y} = k_y v_y \to \kappa_{\mathcal{U}}$, prove: $\exists\, v_x \in \mathcal{D}(X|k)$ with $v_y = (v_x)|_{k_y}$ such that $\kappa_{v_x}|\kappa_{v_y}$ is $\mathcal{U}$-p.s. and $e(v_x|v_y) = 1$ if v_y is non-trivial.

Case 1. v_y is the trivial valuation, hence $k_y = \kappa_{v_y}$. Then $y \in Y$ together with the k-embedding $\imath_y = \jmath_y : k_y \to \kappa_{\mathcal{U}} \subset k_{\mathcal{U}}$ define a $k_{\mathcal{U}}$-rational point $y_{\mathcal{U}} \in Y(k_{\mathcal{U}})$ such that the restriction of $v_{\mathcal{U}}$ to k_y under $\imath_y$ is the trivial valuation $v_y = v_{\mathcal{U}}|_{k_y}$. Since $f_y^{k_{\mathcal{U}}} : X(k_{\mathcal{U}}) \to y(k_{\mathcal{U}})$ is surjective, there is $x_{\mathcal{U}} \in X(k_{\mathcal{U}})$ with $f^{k_{\mathcal{U}}}(x_{\mathcal{U}}) = y_{\mathcal{U}}$. Hence, $x_{\mathcal{U}}$ is defined by a point $x \in X$ with $f(x) = y$ and a k-embedding $\imath_x : k_x \to k_{\mathcal{U}}$ whose restriction to k_y equals $\imath_y$. Then if v_x is the restriction of the valuation $v_{\mathcal{U}}$ to k_x under the k-embedding $\imath_x$, one has: First, $(v_x)|_{k_y} = v_y$, and second, by Fact 4.3, $v_x \in \mathcal{D}(k_x|k)$. Further, the residue field k-embedding $\jmath_x : \kappa_{v_x} \to \kappa_{\mathcal{U}}$ prolongs the residue field k-embedding $\jmath_y : \kappa_{v_y} = k_y \to \kappa_{\mathcal{U}}$. Conclude that $v_y \in \mathcal{D}(k_y|k)$ is $\mathcal{U}$-p.s. in $\mathcal{D}(k_x|k)$.

Case 2. v_y is non-trivial. Let $\pi_y \in k_y$ be a uniformizing parameter at v_y. By the discussion at *Prime divisors* above, any k-embedding $\jmath_y : \kappa_{v_y} \to \kappa_{\mathcal{U}}$ and a uniformizing parameter $\pi_y \in k_y$ of v_y gives rise to a k-embedding $\imath_y : k_y \to k_{\mathcal{U}}$ of valued fields, i.e., $v_y = (v_{\mathcal{U}})|_{k_y}$ under $\imath_y$, such that $\pi_y \mapsto \pi_{\mathcal{U}}$ and inducing

$\jmath_y : \kappa_{v_y} \to \kappa_{\mathcal{U}}$ on the residue field. Among other things, $(y, \imath_y)$ defines a $k_{\mathcal{U}}$-rational point $y_{\mathcal{U}} \in y(k_{\mathcal{U}})$. Since $f_y^{k_{\mathcal{U}}} : X_y(k_{\mathcal{U}}) \to y(k_{\mathcal{U}})$ is surjective, there is a $k_{\mathcal{U}}$-rational point $x_{\mathcal{U}} \in X(k_{\mathcal{U}})$ such that $f^{k_{\mathcal{U}}}(x_{\mathcal{U}}) = y_{\mathcal{U}}$. Let $x_{\mathcal{U}}$ be defined by $(x, \imath_x)$ with $x \in X_y$ and $\imath_x : k_x \to k_{\mathcal{U}}$ a k-embedding. Then $f(x_{\mathcal{U}}) = y_{\mathcal{U}}$ implies $f(x) = y$ and $\imath_x : k_x \to k_{\mathcal{U}}$ prolongs $\imath_y : k_y \to k_{\mathcal{U}}$, or equivalently, $(\imath_x)|_{k_y} = \imath_y$. Hence setting $v_x := (v_{\mathcal{U}})|_{k_x}$, and recalling that $v_y := (v_{\mathcal{U}})|_{k_y}$ by the definition of v_y, one gets: $v_y = (v_x)|_{k_y}$, and the residue field k-embedding $\jmath_x : \kappa_{v_x} \to \kappa_{\mathcal{U}}$ prolongs $\jmath_y : \kappa_{v_y} \to \kappa_{\mathcal{U}}$. Second, since v_y is non-trivial, if follows that v_x is non-trivial. Third, since $v_x = (v_{\mathcal{U}})|_{k_x}$ under $\imath_x : k_x \to k_{\mathcal{U}}$, it follows by Fact 4.3 that v_x is a prime divisor of k_x, which restricts to the prime divisor v_y of k_y under the field extension $k_x|k_y$. Finally, since $\pi_y \in k_y$ maps to $\pi_{\mathcal{U}} \in k_{\mathcal{U}}$, one has $e(v_{\mathcal{U}}|v_y) = 1$. Hence since $e(v_x|v_y)$ divides $e(v_{\mathcal{U}}|v_y) = 1$, we get $e(v_x|v_y) = 1$. Conclude that v_y is $\mathcal{U}$-p.s. in $\mathcal{D}(k_x|k)$, hence in $\mathcal{D}(X)$.

The converse implication: *f is $\mathcal{U}$-p.s. above y* $\Rightarrow$ $f_y^{k_{\mathcal{U}}}\big(X_y(k_{\mathcal{U}})\big) = y(k_{\mathcal{U}})$

Given $y_{\mathcal{U}} \in y(k_{\mathcal{U}})$, say defined by a k-embedding $\imath_y : k_y \to k_{\mathcal{U}}$, prove: $\exists\, x \in X_y$ and a k-embedding $\imath_x : k_x \to k_{\mathcal{U}}$ which prolongs $\imath_y$ to k_x.

To proceed, consider the restriction $v_y := (v_{\mathcal{U}})|_{k_y}$ of $v_{\mathcal{U}}$ to k_y under $\imath_y$. Then, v_y is either trivial or, by Fact 4.3, a prime divisor of $k_y|k$. Therefore, $v_y \in \mathcal{D}(k_y|k)$, and one has a k-embedding of valued fields $\imath_y : k_y, v_y \to k_{\mathcal{U}}, v_{\mathcal{U}}$, thus the k-embedding of residue fields $\jmath_y : \kappa_{v_y} = k_y v_y \to \kappa_{\mathcal{U}}$.

Case 1. v_y is trivial, hence $k_y = k_y v_y = \kappa_{v_y}$ and $\imath_y = \jmath_y : k_y = \kappa_{v_y} \to \kappa_{\mathcal{U}}$. Since $v_y \in \mathcal{D}(k_y|k)$ is $\mathcal{U}$-p.s., there is $x \in X$ such that setting $k_x = \kappa(x)$ one has: $f(x) = y$, hence $x \in X_y$, and there is $v_x \in \mathcal{D}(k_x|k)$ such that $v_y = (v_x)|_{k_y}$ and $\jmath_x : \kappa_{v_x} = k_x v_x \to \kappa_{\mathcal{U}}$ prolonging $\jmath_y$. First, if v_x is trivial, the k-embedding $\imath_x = \jmath_x : k_x \to \kappa_{\mathcal{U}}$ defines a k-rational point $x_{\mathcal{U}}$ such that $f^{\mathcal{U}}(x_{\mathcal{U}}) = y_{\mathcal{U}}$. Second, if v_x is non-trivial, the residue field $\kappa_{v_x} = k_x v_x$ is a function field over k. Thus, by the discussion at *Prime divisors* above, it follows that choosing a uniformizing parameter $\pi_x \in k_x$, one gets a k-embedding $\imath_x : k_x \to k_{\mathcal{U}}$ by $\pi_x \mapsto \pi_{\mathcal{U}}$ and having $\jmath_x : \kappa_{v_x} \to \kappa_{\mathcal{U}}$ as residue field k-embedding. Finally, the resulting $(x, \imath_x)$ define a $k_{\mathcal{U}}$-rational point $x_{\mathcal{U}}$ such that $f^{\mathcal{U}}(x_{\mathcal{U}}) = y_{\mathcal{U}}$.

Case 2. v_y is non-trivial, hence a prime divisor of $k_y|k$. Since v_y is $\mathcal{U}$-p.s. above y, there is $x \in X$ with $f(x) = y$ and $v_x \in \mathcal{D}(k_x|k)$ such that $v_y = (v_x)|_{k_y}$, $\jmath_x : \kappa_{v_x} \to \kappa_{\mathcal{U}}$ prolonging $\jmath_y : \kappa_{v_y} \to \kappa_{\mathcal{U}}$ and $e(v_x|v_y) = 1$. In particular, if $\pi = \pi_y \in k_y$ is a uniformizing parameter of v_y, then $\pi \in k_x$ is a uniformizing parameter of v_x. Hence, arguing as in *Prime divisors* above, π together with $\jmath_x : \kappa_{v_x} \to \kappa_{\mathcal{U}}$ give rise to a k-embedding of valued fields $\imath_x : k_x, v_x \to k_{\mathcal{U}}, v_{\mathcal{U}}$ with $\pi \mapsto \imath_y(\pi_y)$ and having $\jmath_x : \kappa_{v_x} \to \kappa_{\mathcal{U}}$ as k-embedding of the residue fields. Since $\jmath_x$ prolongs $\jmath_y$ to κ_{v_x}, it follows that $v_y = (v_x)|_{k_y}$.

Therefore, $(x, \imath_x)$ defines a rational point $x_{\mathcal{U}} \in X(k_{\mathcal{U}})$ such that $f^{\mathcal{U}}(x_{\mathcal{U}}) = y_{\mathcal{U}}$.

This completes the proof of Key Lemma 4.2, thus of Theorem 4.1. $\square$

5. PROOF OF THEOREM 1.5 (REVISITED)

In the notation/context from Section 3, Theorem 1.5 follows from:

THEOREM 5.1 (Theorem 1.5, revisited). *Let k with $\mathrm{char}(k) = 0$, Σ_k and $\mathcal{U} \supset \mathcal{P}_{\Sigma_k}$ be as in Notations/Remarks 2.4 and Definition 3.5. Given a dominant morphism $f : X \to Y$ of proper smooth k-varieties, let $K = k(X)$, $L = k(Y)$ be their function fields. Then one has:*

1) *f satisfies $(\mathsf{Srj})_{\Sigma_k}$ iff $\mathcal{D}(L|k)$ is Σ_k-p.s. in $\mathcal{D}(K|k)$.*

2) *f satisfies $(\mathsf{Srj})_{\mathcal{U}}$ iff $\mathcal{D}(L|k)$ is $\mathcal{U}$-p.s. in $\mathcal{D}(K|k)$.*

Hence, $(\mathsf{Srj})_{\Sigma_k}$ and $(\mathsf{Srj})_{\mathcal{U}}$ are fully birational properties of dominant morphisms of proper smooth k-varieties, i.e., these properties depend on properties of the corresponding function field extensions only.

Proof. First, by Theorem 4.1, f has property $(\mathsf{Srj})_{\Sigma_k}$ iff f is Σ-p.s., and correspondingly for $(\mathsf{Srj})_{\mathcal{U}}$. Hence, 1), 2) from Theorem 5.1 are equivalent to/can be reformulated as follows:

1)′ f is Σ_k-p.s. iff $\mathcal{D}(L|k)$ is Σ_k-p.s. in $\mathcal{D}(K|k)$.

2)′ f is $\mathcal{U}$-p.s. iff $\mathcal{D}(L|k)$ is $\mathcal{U}$-p.s. in $\mathcal{D}(K|k)$.

Second, by mere definitions, 2)′ $\Rightarrow$ 1)′, hence, we need to prove assertion 2)′ only. We begin by recalling a few facts, all of which follow by mere definition (and are well known to experts). $\square$

FACT 5.2. (I) *Let Z be a proper k-variety with function field $F = k(Z)$. Then there are "many" surjective projective systems $(Z_\mu)_{\mu\in I}$ of proper k-models of $F|k$ w.r.t. the domination relation $\succ$. If $\mathrm{char}(k) = 0$, one can choose Z_μ to be projective smooth k-varieties. Finally, the projective limit of any such system $(Z_\mu)_\mu$ is the Riemann–Zariski space $\mathrm{Val}_k(F)$. Precisely:*

(a) *If $v \in \mathrm{Val}_k(F)$ has center $z_\mu \in Z_\mu$, then $\mathcal{O}_v = \cup_\mu \mathcal{O}_{z_\mu}$, $v = (z_\mu) \in \varprojlim_\mu Z_\mu$.*

(b) *If $(z_\mu)_\mu \in \varprojlim_\mu Z_\mu$, then $\exists v \in \mathrm{Val}_k(F)$ having center $z_\mu \in Z_\mu$, etc. Further, $v \leftrightarrow (z_\mu)_\mu$ iff $\mathfrak{m}_v = \cup_\mu \mathfrak{m}_{z_\mu}$, and if so, then $Fv = \cup_\mu \kappa(z_\mu)$.*

(II) *Given a dominant morphism $f : X \to Y$ of proper k-varieties, with function field extension $K = k(X) \hookleftarrow k(Y) = L$, there are "many" co-final systems $f_\mu : X_\mu \to Y_\mu$, $\mu \in I$ of modifications of f. Further, one has:*

(a) *If $\mathrm{char}(k) = 0$, one can choose $f_\mu : X_\mu \to Y_\mu$ to be smooth modifications.*

(b) *Let* $v \in \mathrm{Val}(K|k)$, $w := v|_L \in \mathrm{Val}(L)$ *have centers* $x_\mu \in X_\mu$, $y_\mu \in Y_\mu$. *Then* $f_\mu(x_\mu) = y_\mu$, *and* $L \hookrightarrow K$ *gives rise to canonical* k*-embeddings:*

$$\mathfrak{m}_w = \cup_\mu \mathfrak{m}_{y_\mu} \subset \cup_\mu \mathcal{O}_{y_\mu} = \mathcal{O}_w \hookrightarrow \mathcal{O}_v = \cup_\mu \mathcal{O}_{x_\mu} \supset \cup_\mu \mathfrak{m}_{x_\mu} = \mathfrak{m}_v$$
$$Lw = \cup_\mu \kappa(y_\mu) \hookrightarrow \cup_\mu \kappa(x_\mu) = Kv.$$

(c) *If* $v \in \mathcal{D}(K|k)$, $w = v|_L$, $\exists\, I_v \subset I$ *cofinal s.t.* $\mathcal{O}_v = \mathcal{O}_{x_\mu}$, $\mathcal{O}_w = \mathcal{O}_{y_\mu}$ *for all* $\mu \in I_v$.

Definition/Remark 5.3. Let Z be an integral k-variety, $F = k(Z)$, $z \in Z$ be a regular point, $\boldsymbol{t} = (t_1, \dots, t_d)$ be a regular system of parameters at z. Define/consider the following:

1) The *deg-valuation* w of $\mathcal{O}_z$, defined by $w(t) = 1$ for $t \in \mathfrak{m}_z \backslash \mathfrak{m}_z^2$, satisfies: $w \in \mathcal{D}(F|k)$ and $Fw = k_z(t_i/t_d)_{i<d}$ is the rational function field in $(t_i/t_d)_{i<d}$.

2) The *lex-valuation* $\tilde{w} \in \mathrm{Val}_k(K)$ of $\mathcal{O}_z$ is defined via the k-embedding $F \hookrightarrow k_z((t_1))\dots((t_d))$, and has residue field $F\tilde{w} = \kappa(z) = k_z$.

Back to the proof of Theorem 5.1, recall that $f : X \to Y$ being a dominant morphism of proper k-varieties, the fiber $X_y \subset X$ at any $y \in Y$ is a proper (not necessarily) integral k_y-variety. And if $y = \eta_Y \in Y$ is the generic point, thus $k_{\eta_Y} = L = k(Y)$, then $X_{\eta_Y} = X_L$ is a proper integral L-variety. Finally, the same holds, correspondingly, for all modifications $f' : X' \to Y'$ of $f : X \to Y$, thus for all $f_\mu : X_\mu \to Y_\mu$, etc.

The direct implication: *f is $\mathcal{U}$-p.s. $\Rightarrow$ $\mathcal{D}(L|k)$ is $\mathcal{U}$-p.s. in $\mathcal{D}(K|k)$*

We have to show that every $w \in \mathcal{D}(L|k)$ is $\mathcal{U}$-p.s. in $\mathcal{D}(K|k)$.

Case 1. w is the trivial valuation of $L|k$. Then the center of w on Y is the generic point η_Y, X_{η_Y} is a proper smooth L-variety and w has residue field $\kappa_w = L$. Since η_Y is $\mathcal{U}$-p.s. under f, for each k-embedding $\jmath_w : L \hookrightarrow \kappa_{\mathcal{U}}$ there is $x \in X_L$ and $v_x \in \mathcal{D}(k_x|k)$ with $w = (v_x)|_L$ trivial and a prolongation of $\jmath_w : L \to \kappa_{\mathcal{U}}$ to a k-embedding $\jmath_x : \kappa_{v_x} \to \kappa_{\mathcal{U}}$. Since X_L is proper, the valuation v_x has center z on X, and the following hold: First, since $\eta_Y \in Y$ is the center of $w = (v_x)|_L$ on Y, one has $f(z) = \eta_Y$. Second, $\kappa(z) \subset \kappa_{v_x}$, thus $\jmath_z := (\jmath_x)|_{\kappa(z)}$ is a k-embedding prolonging $\jmath_w$ to $\kappa(z)$. Finally, if $v \in \mathcal{D}(K|k)$ is the *deg-valuation* of $\mathcal{O}_z$, then the residue field κ_v of v is a rational function field over $\kappa(z)$. Hence, since $\mathrm{td}(\kappa_{\mathcal{U}}|k)$ is infinite, $\jmath_z$ prolongs to a k-embedding $\jmath_v : Kv \to \kappa_{\mathcal{U}}$. Thus, w is $\mathcal{U}$-p.s. in $\mathcal{D}(K|k)$.

Case 2. w is non-trivial, hence $w \in \mathcal{D}(L|k)$ is a prime divisor of $L|k$. Since $L = \kappa(\eta_Y)$, and f is $\mathcal{U}$-p.s., there is $x \in X_{\eta_Y}$ and $v_x \in \mathcal{D}(k_x|k)$ such that $w = (v_x)|_L$, $e(v_x|w) = 1$ and the k-embedding $\jmath_w : Lw \to \kappa_{\mathcal{U}}$ prolongs to a k-embedding $\jmath_x : \kappa_{v_x} \to \kappa_{\mathcal{U}}$. Let $\tilde{v} \in \mathrm{Val}_k(K)$ be the lex-valuation of $\mathcal{O}_x$, thus

$K\tilde{v} = \kappa(x) = k_x$. By Fact 5.2, there is a smooth modification $f' : X' \to Y'$ of f such that the center $y'_0 \in Y'$ of w satisfies $\mathcal{O}_w = \mathcal{O}_{y'_0}$, hence $\mathfrak{m}_w = \mathfrak{m}_{y'_0}$ and $Lw = \kappa(y'_0)$. Further, if $x' \in X'$ is the center of $\tilde{v}$ on X', then $x' \mapsto x$ under $X' \to X$, and $\kappa(x) \subset \kappa(x') \subset K\tilde{v} = \kappa(x)$, thus $k_x = \kappa(x) = \kappa(x') =: k_{x'}$. Therefore, $v_x \in \mathcal{D}(k_{x'}|k)$, and $\jmath_{x'} = \jmath_x$ prolongs $\jmath_w$ to k'_x. Hence *mutatis mutandis,* w.l.o.g., we can suppose that $Y = Y'$, $X = X'$, and $y_0 \in Y$ satisfies $\mathcal{O}_w = \mathcal{O}_{y_0}$, etc. Finally, let $v_0 := v_x \circ \tilde{v}$ be the valuation theoretical composition. Since $w = (v_x)|_L$, one has: $\tilde{v}|_L$ is trivial, and $(v_0)|_L = (v_x)|_L = w$. Thus, if $x_0, x \in X$ are the centers of v_0, $\tilde{v}$ on X, then $f(x) = \eta_Y$ and $f(x_0) = y_0$.

Let $Z \subset X$ be the Zariski closure of x, hence $x_0 \in Z$, and $\mathfrak{p}_x \in \mathrm{Spec}(\mathcal{O}_{x_0})$ be such that $\mathcal{O}_{Z,x_0} = \mathcal{O}_{x_0}/\mathfrak{p}_x$, and $\mathfrak{m}_{Z,x_0} = \mathfrak{m}_{x_0}/\mathfrak{p}_x$ is the center of v_x in Z. In particular, if $\pi \in \mathcal{O}_w = \mathcal{O}_{y_0}$ is a uniformizing parameter, then $\pi \in \mathfrak{m}_{Z,x_0} \backslash \mathfrak{m}^2_{Z,x_0}$. Hence, if $\pi_0 \in \mathcal{O}_{x_0}$ is a preimage of π under $\mathcal{O}_{x_0} \twoheadrightarrow \mathcal{O}_{x_0}/\mathfrak{p}_x = \mathcal{O}_{Z,x_0}$, then one has $\pi_0 \in \mathfrak{m}_{x_0} \backslash \mathfrak{m}^2_{x_0}$. Further, $Z \hookrightarrow X \xrightarrow{f} Y$ defines an injective k-morphism $\mathcal{O}_{Z,x_0} \twoheadleftarrow \mathcal{O}_{x_0} \hookleftarrow \mathcal{O}_{y_0}$ such that $\mathfrak{m}_{Z,x_0} \twoheadleftarrow \mathfrak{m}_{x_0} \hookleftarrow \mathfrak{m}_{y_0} = \mathfrak{m}_{x_0} \cap \mathcal{O}_{y_0}$, thus residue field k-embeddings $\kappa(x_0) = \kappa(x_0) \hookleftarrow \kappa(y_0)$. And since v_x has center x_0 on Z, it follows that $\kappa(x_0) \subset \kappa_{v_x}$, and therefore, $\jmath_0 := (\jmath_x)|_{\kappa(x_0)}$ is a k-prolongation of $\jmath_w$ to $\kappa(x_0)$. Next, let v be the *deg-valuation* of $\mathcal{O}_{x_0}$. Then $v \in \mathcal{D}(K|k)$, $v(\pi) = 1$, $\mathcal{O}_{x_0} \prec \mathcal{O}_v$, and Kv is a $\kappa(x_0)$-rational function field. Thus $\mathrm{td}(\kappa_{\mathcal{U}}|k)$ being infinite, $\jmath_0 : \kappa(x_0) \to \kappa_{\mathcal{U}}$ prolongs to a k-embedding $\jmath_v : Kv \to \kappa_{\mathcal{U}}$. Finally, $\mathcal{O}_w = \mathcal{O}_{y_0} \prec \mathcal{O}_{x_0}$, thus $\mathcal{O}_w \prec \mathcal{O}_v$, implying $w = v|_L$.

Conclude that $w \in \mathcal{D}(L|k)$ is $\mathcal{U}$-p.s. in $\mathcal{D}(K|k)$.

The converse implication: $\mathcal{D}(L|k)$ *is* $\mathcal{U}$*-p.s. in* $\mathcal{D}(K|k) \Rightarrow f$ *is* $\mathcal{U}$*-p.s.*

Given $y \in Y$ and $k_y = \kappa(y)$, we show that every $v_y \in \mathcal{D}(k_y|k)$ is $\mathcal{U}$-p.s. under f. First, if $y = \eta_Y$ is the generic point, then the generic point $x = \eta_X$ of X is in X_y, $L = k_y \hookrightarrow k_x = K$ under f, and $w := v_y \in \mathcal{D}(k_y|k) = \mathcal{D}(L|k)$ is $\mathcal{U}$-p.s. in $\mathcal{D}(K|k) = \mathcal{D}(k_x|k)$ by hypothesis. Next suppose that $y \neq \eta_Y$. Since $f : X \to Y$ is proper, the fiber X_y is a proper k_y-variety.

Case 1. v_y is the trivial valuation of k_y, i.e., $k_y = \kappa_{v_y}$. Let $\jmath_y : \kappa_{v_y} \hookrightarrow \kappa_{\mathcal{U}}$ be a k-embedding, $w \in \mathcal{D}(L|k)$ be the deg-valuation of the local ring $\mathcal{O}_y$, thus Lw is a rational function field over $\kappa_{v_y} = k_y$. Hence, since $\mathrm{td}(\kappa_{\mathcal{U}}|k)$ is infinite and $Lw|\kappa_{v_y}$ is a rational function field, the k embedding $\jmath_y : \kappa_{v_y} \to \kappa_{\mathcal{U}}$ has k-prolongations to Lw. Since $\mathcal{D}(L|k)$ is $\mathcal{U}$-p.s. in $\mathcal{D}(K|k)$, there is $v \in \mathcal{D}(K|k)$ such that $e(v|w) = 1$ and $Kv|Lw$ is $\jmath_w$-p.s., that is, $\jmath_w$ has a prolongation $\jmath_v : Kv \to \kappa_{\mathcal{U}}$ to Kv. So, if $x \in X$ is the center of v, then $f(x) = y$ is the center of w on Y, thus $x \in X_y$, and $\kappa_{v_y} = k_y = \kappa(y) \hookrightarrow \kappa(x) \subset Kv$ canonically. Hence, setting $k_x = \kappa(x)$, the restriction $\jmath_x := (\jmath_v)|_{k_x}$ prolongs $\jmath_y : \kappa_{v_y} \hookrightarrow \kappa_{\mathcal{U}}$ to k_x. Thus the trivial valuation $v_x \in \mathcal{D}(k_x|k)$ satisfies $v_y = (v_x)|_{k_y}$, and $\kappa_{v_y} = k_y \hookrightarrow \kappa_{\mathcal{U}}$ prolongs to a k-embedding $\kappa_{v_x} = k_x \hookrightarrow \kappa_{\mathcal{U}}$.

Conclude: $v_y \in \mathcal{D}(k_y|k)$ is $\mathcal{U}$-p.s. in $\mathcal{D}(k_x|k)$ for some $x \in X_y$.

Case 2. $v_y \in \mathcal{D}(k_y|k)$ is non-trivial. The proof is a little bit involved, and takes place in two main steps: Namely, let $\jmath_y : \kappa_{v_y} \to \kappa_{\mathcal{U}}$ be given. Then, in Step 1, we find some point $x' \in X_y$, and a *discrete k-valuation* v' of $k_{x'} := \kappa(x')$ with $v_y = v'|_{k_y}$ and $e(v'|w_y) = 1$ and a k-embedding $\jmath_{x'} : \kappa_{v'} = k_{x'}v' \to \kappa_{\mathcal{U}}$ prolonging $\jmath_y : \kappa_{v_y} \to \kappa_{\mathcal{U}}$. Nevertheless, v' is not necessarily a prime divisor of $k_{x'}$. In Step 2, we use v' to finally find the "right" point $x \in X_y$ and $v_x \in \mathcal{D}(k_x|k)$ with the desired properties.

Step 1. Let $\tilde{w} \in \mathrm{Val}_k(L)$ be the *lex-valuation* of $\mathcal{O}_y$, thus $L\tilde{w} = k_y$, and $w := v_y \circ \tilde{w}$ be the valuation theoretical composition. Then $Lw = k_y v_y = \kappa_{v_y}$, $\mathcal{O}_w \subset \mathcal{O}_{\tilde{w}}$, $\mathfrak{m}_w \supset \mathfrak{m}_{\tilde{w}}$, and $\mathcal{O}_{v_y} = \mathcal{O}_w/\mathfrak{m}_{\tilde{w}}$, thus $wL/v_y k_y = \tilde{w}L$ canonically. Let $\pi_y \in k_y$ have $v_y(\pi_y) = 1$, and $\pi \in \mathcal{O}_w$ be a preimage of π_y. Then $w(\pi) \in wL$ is the unique minimal positive element, hence $\mathfrak{m}_w = \pi\mathcal{O}_w$, and the "canonical" coarsening $\mathcal{O}_{\tilde{w}} = \mathcal{O}_w[1/\pi]$ of $\mathcal{O}_w$ has valuation ideal $\mathfrak{m}_{\tilde{w}}$. We construct a valuation $v \in \mathrm{Val}_k(K)$ such that $w = v|_L$, $v(\pi)$ is the minimal element in vK, and $\jmath_w$ prongs to a k-embedding $\jmath_v : Kv \to \kappa_{\mathcal{U}}$. Hence, if $\mathcal{O}_{\tilde{v}} = \mathcal{O}_v[1/\pi]$ is the "canonical" coarsening of $\mathcal{O}_v$, one has: $\tilde{w} = \tilde{v}|_L$. Hence, if x' is the center of $\tilde{v}$ on X, then $f(x') = y$, and $\mathcal{O}_{v'} = \mathcal{O}_v/\mathfrak{m}_{\tilde{v}}$ is a DVR of $k_{x'}$ with residue field $\kappa_{v'_x} = Kv$, thus $\jmath_{x'} : \kappa_{v'_x} = Kv \to \kappa_{\mathcal{U}}$ prolongs $\jmath_y$.

Concretely: Since $k_y = \kappa(y)$ and $\kappa_{v_y} = k_y v_y$ are finitely generated over k, and further, $\mathcal{O}_{v_y} = \mathcal{O}_w/\mathfrak{m}_{\tilde{w}}$, $\mathcal{O}_{v_y}/(\pi) = \kappa_{v_y}$, by Fact 5.2, 2) and 3), there is a smooth modification $f_0 : X_0 \to Y_0$ of f such that the centers $\tilde{y}$ and y_w of $\tilde{w}$ and w on Y_0, and $\mathfrak{p}_{\tilde{y}} := \mathfrak{m}_{\tilde{w}} \cap \mathcal{O}_{y_w}$ satisfy $\pi \in \mathfrak{m}_{y_w} \backslash \mathfrak{m}^2_{y_w}$ and further:

$$(*)\qquad \kappa(\tilde{y}) = k_y = L\tilde{w},\ \kappa(y_w) = \kappa_{v_y} = Lw,\ \mathcal{O}_{v_y} = \mathcal{O}_{y_w}/\mathfrak{p}_{\tilde{y}},\ \mathfrak{m}_{v_y} = \mathfrak{m}_{y_w}/\mathfrak{p}_{\tilde{y}},$$

Next, let $\jmath_y : \kappa_{v_y} \hookrightarrow \kappa_{\mathcal{U}}$ be a given k-embedding. Since Y_0 is smooth, the *deg-valuation* $w_0 \in \mathcal{D}(L|k)$ of $\mathcal{O}_{y_w}$ satisfies $w_0(a) = 1$ for all $a \in \mathfrak{m}_{y_w} \backslash \mathfrak{m}^2_{y_w}$, hence $w_0(\pi) = 1$, and Lw_0 is a rational function field over $\kappa(y_w) = \kappa_{v_y} = Lw$. Hence, since $\mathrm{td}(\kappa_{\mathcal{U}}|k)$ is infinite, $\jmath_y : \kappa_{v_y} \to \kappa_{\mathcal{U}}$ has (many) k-prolongations $\jmath_{w_0} : Lw_0 \hookrightarrow \kappa_{\mathcal{U}}$. Finally, since $\mathcal{D}(L|k)$ is $\mathcal{U}$-p.s. in $\mathcal{D}(K|k)$, there is $v_0 \in \mathcal{D}(K|k)$ satisfying: $w_0 = (v_0)|_L$, $e(v_0|w_0) = 1$, and $Kv_0|Lw_0$ is $\jmath_{w_0}$-p.s., i.e., there is a k-embedding $\jmath_{v_0} : Kv_0 \hookrightarrow \kappa_{\mathcal{U}}$ prolonging $\jmath_{w_0}$. Hence, if $x_0 \in X_0$ is the center of v_0, one has $f_0(x_0) = y_0$ and k-embeddings $\kappa_{v_y} = \kappa(y_0) \hookrightarrow \kappa(x_0) \to Kv_0$, and $(\jmath_{v_0})|_{\kappa(x_0)}$ prolongs $\jmath_y$. Hence, one has that $\kappa(x_0)|\kappa_{v_y}$ is $\jmath_y$-pseudo-split, and second, $v_0(\pi) = 1 = w_0(\pi)$ implies $\pi \in \mathfrak{m}_{x_0} \backslash \mathfrak{m}^2_{x_0}$.

Finally, let $f_\mu : X_\mu \to Y_\mu$, $\mu \in I$ be a cofinal projective system of smooth modifications of $f_0 : X_0 \to Y_0$, and $\tilde{y}_\mu$, y_μ be the centers of $\tilde{w}$, w on Y_μ. Further, let $\tilde{f}_\mu : Z_\mu \to \mathrm{Spec}\,\mathcal{O}_{y_\mu}$, $\mu \in I$ be the projective system of the fibers of f_μ above $\mathrm{Spec}\,\mathcal{O}_{y_\mu}$. By Fact 5.2, 2), one has $\cup_\mu \mathcal{O}_{y_\mu} = \mathcal{O}_w$, and therefore,

$$\varprojlim_{\mu} Z_\mu = \mathrm{Val}_w(K) = \{v' \in \mathrm{Val}_k(K) \mid v'|_L = w\}.$$

LEMMA 5.4. *Let* $Z_{\mu,\pi,\jmath_y} \subset f_\mu^{-1}(y_\mu) \subset Z_\mu$ *be the set of points* x_μ *satisfying both* (j) $\pi \in \mathfrak{m}_{x_\mu} \backslash \mathfrak{m}_{x_\mu}^2$ *and* (jj) $\jmath_y : \kappa_{v_y} \to \kappa_{\mathcal{U}}$ *prolongs to some* $\jmath_{x_\mu} : \kappa(x_\mu) \hookrightarrow \kappa_{\mathcal{U}}$. *Then* $(Z_{\mu,\pi,\jmath_y})_\mu$ *is a projective system with the non-empty projective limit*

$$\mathrm{Val}_{\jmath_y}(K) = \big\{v \in \mathrm{Val}_w(K) | \pi \in \mathfrak{m}_v \backslash \mathfrak{m}_v^2, \jmath_y : \kappa_{v_y} \to \kappa_{\mathcal{U}} \text{ prolongs to } \jmath_v : Kv \hookrightarrow \kappa_{\mathcal{U}}\big\}.$$

Proof of Lemma 5.4. First, $(X_{\mu,\pi,\jmath_y})_\mu$ is a projective system, because (j), (jj) are compatible with the projections $X_{\mu'} \to X_\mu$, $x_{\mu'} \mapsto x_\mu$. Indeed, one has

$$x_{\mu'} \mapsto x_\mu \;\Rightarrow\; \mathcal{O}_{x_\mu} \prec \mathcal{O}_{x_{\mu'}} \;\Rightarrow\; \mathfrak{m}_{x_\mu} = \mathfrak{m}_{x_{\mu'}} \cap \mathcal{O}_{x_\mu} \text{ and } \kappa(x_\mu) \hookrightarrow \kappa(x_{\mu'}).$$

Hence, if $x_{\mu'}$ satisfies (j), (jj) and $x_{\mu'} \mapsto x_\mu$, then x_μ satisfies (j), (jj). Next, let $(x_\mu)_\mu = v$ be given with $v \in \mathrm{Val}_w(K)$. Then, by Fact 5.2, (I), it immediately follows that $\pi \in \mathfrak{m}_v \backslash \mathfrak{m}_v^2$. Finally, since $Kv = \cup_\mu \kappa(x_\mu)$, by the saturation property of $\kappa_{\mathcal{U}}$, the inductive system of prolongations $\jmath_{x_\mu} : \kappa(x_\mu) \hookrightarrow \kappa_{\mathcal{U}}$, $\mu \in I$ of $\jmath_y$ to each $\kappa(x_\mu)$ gives rise to a k-embedding $\jmath_v : Kv = \cup_\mu \kappa(x_\mu) \hookrightarrow \kappa_{\mathcal{U}}$ which k-prolongs $\jmath_y : \kappa_{v_y} \to \kappa_{\mathcal{U}}$ to Kv, i.e., $\jmath_y = (\jmath_v)|_{\kappa_{v_y}}$. □

In the notation from Lemma 5.4 above, let $v \in \mathrm{Val}_{\jmath_y}(K)$, thus $w = v|_L$, $\pi \in \mathfrak{m}_v \backslash \mathfrak{m}_v^2$, and $\jmath_y : \kappa_{v_y} \to \kappa_{\mathcal{U}}$ prolongs to a k-embedding $\jmath_v : \kappa(v) \to \kappa_{\mathcal{U}}$. Then $v(\pi)$ is the minimal positive element in vK, thus $\mathfrak{m}_w = \pi\mathcal{O}_w \hookrightarrow \pi\mathcal{O}_v = \mathfrak{m}_v$, and therefore: $\mathcal{O}_{\tilde v} := \mathcal{O}_v[1/\pi]$, is a valuation ring such that $\tilde w = \tilde v|_L$, and $\mathcal{O}_0 := \mathcal{O}_v/\mathfrak{m}_{\tilde v}$ is a DVR of $k_0 := K\tilde v$ with valuation ideal $\mathfrak{m}_0 = \pi\mathcal{O}_0 = \mathfrak{m}_v/\mathfrak{m}_{\tilde v}$. Let v_0 be the canonical valuation of $\mathcal{O}_0$, thus $v_0(\pi) = 1$.

Let $x' \in X$ be the center of $\tilde v$ on X. Then $\tilde w = \tilde v|_L$ implies $f(x') = y$, and consider the residue field embeddings $k_y = \kappa(y) \hookrightarrow \kappa(x') =: k_{x'} \hookrightarrow k_0 = K\tilde v$. Then $v' := (v_0)|_{k_{x'}}$ satisfies: First, $v'|_{k_y} = (v_0)|_{k_y} = v_y$, hence $e(v'|v_y)$ divides $e(v_0|v_y) = 1$, thus $e(v'|v_y) = 1$, i.e., $\mathcal{O}_{v'}$ is a DVR of $k_{x'}$ with $v'(\pi) = 1 = v_y(\pi)$.[4] Second, the residue field k-embeddings $\kappa_{v_y} \hookrightarrow \kappa_{v'} := k_{x'}v' \hookrightarrow Kv$ satisfy: $\jmath_y = (\jmath_v)|_{\kappa_{v_y}}$, hence $\jmath_{v'} := (\jmath_v)|_{\kappa_{v'}}$ prolongs $\jmath_y$ to $\kappa_{v'}$.

Step 2. To simplify notations, set $F := k_y$, $w := v_y \in \mathcal{D}(F|k)$, $\jmath_w := \jmath_{v_y}$, and $E' := k_{x'}$. Hence $w = v'|_F$, $v'(\pi) = 1 = w(\pi)$, and $\jmath_{v'} : E'v' \to \kappa_{\mathcal{U}}$ prolongs $\jmath_w : Fw \to \kappa_{\mathcal{U}}$ to $E'v'$. Since $\mathrm{char}(k) = 0$, there is a system of representatives $\lambda \subset F^h$ and let $\boldsymbol{t}$ be a system of v'-units whose image in $E'v'$ is a transcendence basis of $E'v'$ over Fw, thus $|\boldsymbol{t}| = \mathrm{td}(E'v'|Fw)$. Then $F^h = \lambda(\pi)^h$, and setting $E_0 := \lambda(\pi, \boldsymbol{t})$ and $E^h := E_0^h \subset E'^h$, $v^h := (v'^h)|_{E^h}$, one has: $v^h E^h = \mathbb{Z} = v'^h E'^h$ and $E^h v^h = E'v' = E'^h v'^h$ have characteristic zero.

Conclude: By the AKE Principle, one hast that $E^h \prec E'^h$.

To proceed, the canonical k-embeddings $k_y \hookrightarrow F^h \hookrightarrow E'^h$ and $k_{x'} \hookrightarrow E'^h$ define an F^h-rational point $y_F^h \in Y(F^h)$ and an E'^h-rational point $x_{E'}^h \in X(E'^h)$ such that $f(x_{E'}^h) = y_F^h$. Hence, since $E^h \prec E'^h$ and $f : X \to Y$ is defined over k, there is $x_E^h \in X(E^h)$ such that $f(x_E^h) = y_F^h$. Hence, if x_E^h is defined by a point

[4] Recall: we do not claim that v' is a prime divisor of $k_{x'}|k$, but rather a discrete valuation.

$x \in X$ and a k-embedding $k_x \hookrightarrow E^h$, then $f(x) = y$, and $v_x := (v')|_{k_x}$ satisfies $w = (v_x)|_F$. And since $E'v' = E'^h v'^h = E^h v^h$, one has that $\jmath_{v_x} := (\jmath_{v'})|_{\kappa_{v_x}}$ prolongs $\jmath_w$ to κ_{v_x}.

Claim. *v_x has no transcendence defect.*
Indeed, let $r_F = \mathrm{td}(F|k) - 1$. Since $w \in \mathcal{D}(F|k)$, $\mathrm{td}(Fw|k) = r_F = \mathrm{td}(\lambda|k)$, the latter equality following by the definition of $\lambda|k$. Second, by definition of $E_0|k$ one has $\mathrm{td}(E_0|k) = r_F + |\boldsymbol{t}| + 1$. Next, since $E^h|E_0$ is algebraic, one has $\mathrm{td}(E^h|k) = \mathrm{td}(E_0|k) = r_F + |\boldsymbol{t}| + 1$. Hence $E'v' = E^h v^h$ implies:

$$\mathrm{td}(E^h|k) - 1 = r_F + |\boldsymbol{t}| = \mathrm{td}(Fw|k) + \mathrm{td}(E'v'|Fw) = \mathrm{td}(E^h v^h|k).$$

Hence, by the discussion before Fact 4.3, $v_x = (v^h)|_{k_x}$ is discrete and has no transcendence defect, thus $v_x \in \mathcal{D}(k_x|k)$. Conclude that $w = v_y$ is $\mathcal{U}$-p.s. in $\mathcal{D}(k_x|k)$ for some $x \in X_y$. □

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