Simion Stoilow Institute of Mathematics of the Romanian Academy

Ph. D. Thesis

ABSTRACT

Potential Theory in Infinite Dimensional Spaces

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In this thesis we study with analytic and probabilistic potential theoretical methods nonlinear partial differential equations associated with measure-valued branching type processes.

We deal with the probabilistic approach to a nonlinear operator Λ of the form

$$\Lambda u = \Delta u + \sum_{k=1}^{\infty} q_k u^k, \tag{1}$$

in connection with the works of M. Nagasawa, N. Ikeda, S. Watanabe, and M.L. Silverstein on the discrete branching processes. Instead of the Laplace operator we may consider the generator of a right (Markov) process, called base process, with a general (not necessarily locally compact) state space. It turns out that solutions of the nonlinear equation $\Lambda u = 0$ are produced by the harmonic functions with respect to the (linear) generator of a discrete branching type process. The consideration of the general state space allows to take as base process a measure-valued superprocess (in the sense of E.B. Dynkin). The probabilistic counterpart is a Markov process which is a combination between a *continuous* branching process (e.g., associated with a nonlinear operator of the form $\Delta u - u^{\alpha}$, $1 < \alpha \leq 2$) and a *discrete* branching type one, on the space of configurations of finite measures.

We solve equations of the type

$$\Lambda u = f$$

in an L^p-weak sense, involving positive definite extensions of the solution from E to the set of all finite positive measures on E. The extensions to the set of measures are related to the classical connection between this nonlinear equation and the discrete branching Markov processes. We also investigate the existence of the bounded solutions for the Dirichlet problem associated to the above equation, with bounded boundary data.

Our approach is based on probabilistic and analytic potential theoretical methods, used on both spaces E and the set of all finite configurations of E, like the Revuz formula for the continuous additive functionals, subordination operators, and the perturbation with kernels of the transition functions and resolvents. We give an example of a subordination operator (in the sense of G. Mokobodzki) which occurs in the potential theoretical view on the discrete branching processes. It turns out that it is an operator which is neither of potential type nor of hitting distribution type. We complete the study of the probabilistic aspect for the inverse subordinate resolvent in the sense of [1], in the non-transient case.

The thesis is organized as follows.

In Chapter 2 we present preparatory results on the analytic and probabilistic potential theory of the resolvents of kernels, the versions we shall use further in the thesis: resolvent of kernels, excessive functions (Section 2.1), the fine topology, polar and semipolar sets (Section 2.2), Markov processes constructed from resolvents of kernels, continuous additive functionals, and multiplicative functionals, Revuz formula, moderate dual (Section 2.3). We follow essentially the monographs [1], [10], [12], and [23].

The author's results are presented in chapters 3, 4, and 5 and are included in the articles [6], [7], [8]. The results from Section 3.3 are not yet published.

In Chapter 3 we construct a subordination operator related to the potential theory of the discrete branching processes. The subordination operators have been considered in potential theory by G. Mokobodzki as an analytic counterpart of the classical modification of a Markov process with a right continuous multiplicative functional, founded by P.A. Meyer and E.B. Dynkin.

There are two basic examples of subordination operators. The first one is the hitting distribution kernel and corresponds to the killing of the process at the first entry time in the complement of an open set. The second one is the so called "potential type" subordination operator and corresponds to the zero order perturbation of the infinitesimal generator of the process, or equivalently to the perturbation given by a Feynmann-Kac formula with a continuous additive functional of the process.

In Section 3.2 we present, in a general frame given by a Borel right (Markov) process on a Lusin topological space E, an example of a subordination operator which is neither of potential type nor of hitting distribution type. The natural framework is given by a disjoint union of measurable spaces and the basic, motivating example is the space of all finite sums of Dirac measures of the given topological space E, called the space of *finite configurations* of E. In fact, we prove (Section 3.1) a more general result on the summation of a sequence of subordination operators. The origin of our approach is an analytic perturbation method proposed by K. Janssen (cf. [19]) in the frame of the balayage spaces (on metrizable locally compact spaces).

Section 3.3 is devoted to the so called "inverse subordination" for sub-Markovian resolvents of kernels. We complete and extend to the non-transient case the results from [1], Section 5.2. If the initial resolvent is associated to a right (Markov) process with state space E, then the inverse subordinate one has the same property. In Section 3.4 we present subordinations and inverse subordinations induced by potential kernels, e.g., generated by continuous additive functionals. This investigation was continued in [9], by proving regularity properties of the perturbed process.

In Chapter 4 we start the study of the nonlinear partial differential equations associated with measure-valued branching type processes. In Section 4.1 we first introduce the setup, following essentially [17]. Then we give the appropriate completion results, demanded by the consideration of a more general topological base space E (of Radon or Lusin type instead of a locally compact one). In Section 4.2 we collected some basic facts on the space S of all finite configurations of E, while Section 4.3 is devoted to the branching kernels and absolute monotonic maps.

In Chapter 5 we continue the study of the nonlinear partial differential equations associated with measure-valued branching type processes.

The classical works of M. Nagasawa, N. Ikeda, S. Watanabe, and M. Silverstein (cf. [21], [17], [18], and [24]) emphasized a natural connection between the discrete branching processes and nonlinear partial differential operators Λ given by (1), on an open subset of an Euclidean space, where the coefficients q_k are positive, Borelian functions with $\sum_{k=1}^{\infty} q_k \leq 1$. More precisely, the semigroup of nonlinear operators generated by Λ is used to prove existence results for the branching processes. One can replace Δ with the generator of a standard (Markov) process (called base process) with state space a metrizable locally compact space. Furthermore, instead of $\sum_{k=1}^{\infty} q_k u^k$ it is possible to consider a more general nonlinear part for Λ, generated by a "branching" kernel.

We investigate operators Λ of the above type, by means of the associated branching type Markov processes, using potential theoretical tools, with applications to infinite dimensional situations. In particular, we show that solutions of the nonlinear equation

$$\Delta u + \sum_{k=1}^{\infty} q_k u^k = 0$$

are produced by harmonic functions with respect to the generator of the associated branching type process. We complete results of M. Nagasawa from [22], where a similar probabilistic approach was used for solving the Dirichlet problem associated with the nonlinear operator Λ.

We avoid imposing secondary hypotheses (like the local compactness of the base space) which would limit the domain of applicability and therefore we can construct discrete branching type processes, starting from a continuous branching process (i.e., a superprocess in the sense of E.B. Dynkin) as base process. The generator of a superprocess is rather a second order integro-differential operator than a differential one and the state space is the set of all finite measures on the initial space E. Consequently, it is crucial for the treatment of this application the consideration of a general state space in the above mentioned investigation of the discrete branching processes. The final result of the construction will be a Markov process on the finite configurations of positive finite measures on E.

The probabilistic counterpart of our approach is a revisit of the discrete branching process theory as developed by Ikeda-Nagasawa-Watanabe. Our main potential theoretical arguments are two successive perturbations of the resolvent family of a right (Markov) process. The first one is associated with a subordination operator in the sense of G. Mokobodzki and it corresponds to the killing of a process with a multiplicative functional (see Section 3.1). This necessary transformation is followed by an inverse subordination (as we described in Section 3.3), induced by a branching kernel and it corresponds to a "renaissance" of the process (in the sense of [20]).

In Section 5.1 we consider the applications of the perturbations induced by potential kernels from Section 3.4 in the construction of branching type Markov process on the space of finite configurations. These processes are generated by the given base process and a pair of killing and branching kernels.

In Section 5.2 we expose the claimed application to the study of a combination between a continuous branching process and a discrete branching one. Recall that in [5] it is developed a method of proving existence and path regularity for both continuous and discrete measure-valued branching Markov processes.

Section 5.3 is devoted to the bounded solutions of the nonlinear equation of the form $\Lambda u = 0$. It turns out that the solutions of this nonlinear equation are produced by the harmonic functions with respect to the branching type processes on S.

In Section 5.4 we investigate nonlinear Schrödinger type equations of the form $\mathcal{L}u - \mu u + \mu B\overline{u} = f$, where μ is a positive measure and the nonlinear term $B\overline{u}$ is given by a kernel B, acting on a positive definite extension $\overline{u}$ of u from E to the set $M(E)$ of all positive finite measures on E.

The linear Schrödinger type equation $\Delta u + \mu u = f$ has been studied by using both probabilistic (Markov processes) and analytic (Dirichlet forms and L^p-semigroups) methods; see [13], [25], [14], [15], [2], and [3]. Actually, we consider the generator $\mathcal{L}$ of a Borel right Markov process instead of the Laplace operator; here μ is a signed measure on the underlying state space E of the process. One of the strategy of the proof is to kill first the process with the multiplicative functional associated with the negative part μ^- of μ, or equivalently, to replace $\mathcal{L}$ by $\mathcal{L} - \mu^-$, and to continue with a Kato type perturbation of this operator with the positive part of μ. In particular, the existence of the L^p-weak solutions was proven for a wide class of measures μ and a main tool for the approach was the Revuz formula, which might be considered an infinite dimensional version of the classical Green formula.

To solve this nonlinear equation we use first the connection between the discrete branching processes and nonlinear partial differential operators of the type considered here. As we already mentioned (and proved in Section 5.3), any harmonic functions from the domain of the (linear) generator of a Markov process with state space S produces a solution for our nonlinear equation on E; this Markov process is obtained by perturbing a canonical diagonal one (let $\boldsymbol{\mathcal{L}}$ be its generator) with a branching kernel $\widehat{B}$ on S, induced by the given kernel B and it has the infinitesimal generator of the form $\boldsymbol{\mathcal{L}} - \boldsymbol{\mu} + \boldsymbol{\mu}\widehat{B}$.

A main step of acting here is to extend the above mentioned L^p-semigroup approach for solving on S the linear Schrödinger type equation associated with $\boldsymbol{\mathcal{L}} - \boldsymbol{\mu} + \boldsymbol{\mu}\widehat{B}$. It turns out that the restriction to E of such a solution solves in an L^p-weak sense the given nonlinear equation on E, accepting extensions of the functions from E to positive definite functions on $M(E)$. A key point is to consider convenient test functions on E and then on S, which involves the moderate dual process technique developed in [11].

Our approach for the nonlinear equation on E has similarities with the method of solving the linear Schrödinger type equation we mentioned: a Kato type nonlinear perturbation is applied to the linear operator $\mathcal{L} - \mu$ (obtained by the killing induced by the measure μ). In fact, the connection is much deeper because we actually first solve in the weak sense the linear equation on S, using the Kato type perturbations developed in the already indicated articles [14], [15], [2], and [3].

Section 5.4 treats the Dirichlet problem associated with the nonlinear operator Λ. Following the approach of M. Silverstein [24], we first outline the construction of a branching semigroup on S by solving an appropriate integral equation; it was proved in [4] that this branching semigroup is the transition function of a branching process with state space E. As in [22], it turns out that the restriction to E of an invariant function with respect to this semigroup is a solution of the Dirichlet problem associated with Λ. Then, we solve the Dirichlet problem associated to the above nonlinear equation, with bounded boundary data. This answers to a question of P. Hsu. In [16] it was treated the case of the Laplace operator in an

Euclidean domain and of the branching kernel associated to a sequence $(q_k)_{k\geqslant 1}$ with $\sum_{k\geqslant 1} q_k = 1$; the existence of this extension to bounded boundary data is related to the absence of the explosion of the branching near the boundary of the domain.

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