APEXMATHS
TO THE TOP

SELECTION PROBLEMS '24

Below you will find 5 problems of various difficulty, some of which are open ended[1]. It is certainly not necessary to solve all the problems in order to be selected to participate in the Apex Math summer program ! We are more interested in seeing your unique way of approaching the problem and in the methods which you come up with than in seeing a full solution. We encourage you to write up (and develop to the extent that you feel inspired) any ideas that you have, even if it only concerns a special case or if it does not lead to complete solutions. Your curiosity in exploring the patterns that you notice will be much valued.

PROBLEM 1.

Suppose that we have two connected dented wheels W_1 and W_2 as in Figure 1 below. The wheel W_1 has 7 dents, numbered 1 to 7, while the second wheel has 9 dents, numbered 1 to 9. At the start, the dent nr. 1 of W_1 is in contact with the dents nr. 4 and 5 of W_2. Of course, this changes when you turn the wheels; for instance, if you turn W_1 one dent clockwise then W_2 turns one dent counterclockwise and now dent 7 on W_1 is in contact with dents nr. 5 and nr. 6 on W_2.

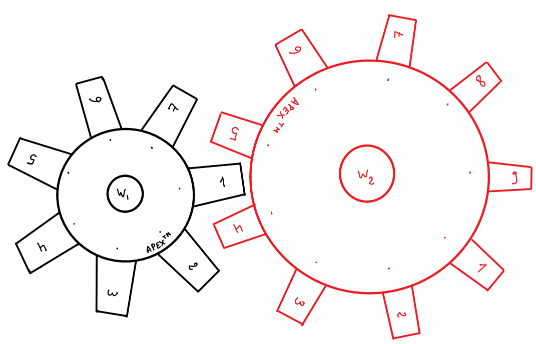

Figure 1: A pair of arithmetic dented wheels.

Which dents of W_1 and W_2 are in contact after W_1 makes a full turn? And after two full turns? After n full turns?

Will there ever be a point at which two dents with the same number will be in contact? If so, when and which numbers arise ? How often (in proportion) does this happen?

Can you analyse the situation if the wheel W_1 has d_1 dents and the wheel W_2 has d_2 dents? What if we add more wheels?

Did you notice some other interesting phenomena? are there other interesting questions that you came up with while thinking about the problem?

[1] Open-ended problems are those which have many solutions or no solutions as defined; they challenge us to ask side questions, to seek better questions and to consider many different angles, each shedding light on a different aspect of the problem studied.

PROBLEM 2.

Let $\Delta = ABC$ be a triangle. We define a sequence of triangles $\Delta_1, \Delta_2, \ldots$ inductively as follows. The vertices of Δ_{n+1} are the middle of the sides of Δ_n. What does Δ_n 'converge' to ? Note that there are several interesting ways to make sense of this ! What if we consider instead the intersections of the sides of Δ_n with the bisector of the opposite angle ? And what happens if we now consider the orthogonal projection of the vertices of Δ_n on the opposite side ? Can you think of similar problems ?

PROBLEM 3.

Imagine a large disc-shaped cake (or a pizza if you prefer !). How many pieces can one obtain after n cuts along a straight line ? Note that the pieces do not have to be of the same area, and the cuts do not need to go through the center. Does anything change if we replace 'disc' by 'rectangle' ? Or another type of polygon ? Or by an annulus ? What would be the three-dimensional analog of that problem ?

PROBLEM 4.

Discuss the existence of an infinite sequence $n_0, n_1, \ldots$ of positive integers such that the following two conditions are satisfied :

(a) Every integer appears infinitely many times in the sequence,

(b) For every positive integer N, if we reduce the sequence modulo N then it is periodic.

Does anything change if we omit the term 'positive' ?

PROBLEM 5.

Let $\mathbb{Z}^2$ be the set of points in the plane with integer coefficients. Let us color some points of $\mathbb{Z}^2$ in black and the others in white. Is it always possible to find three points in $\mathbb{Z}^2$, say A, B, C such that the center of gravity G of the triangle ABC is on $\mathbb{Z}^2$, and all of the points A, B, C and G are of the same color ?

What if we use 3 colors ? n colors ? If we replace 'triangle' by 'quadrilateral' ? Or even by 'polygon' ?